Modal logic is extremely important both for its philosophical applications and in order to clarify the terms and conditions of arguments.

The label "modal logic" refers to a variety of logics:

- **alethic** modal logic, dealing with statements such as "It is necessary that *p*", "It is possible that *p*", etc.

- **epistemic** modal logic, that deals with statements such as "I know that *p*", "I believe that *p*", etc.

- **deontic** modal logic, dealing with statements such as "It is compulsory that *p*", "It is forbidden that *p*", etc.

- **temporal** modal logic, dealing with statements such as "It is always true that *p*", "It is sometimes true that *p*", etc.
- **ethical** modal logic, dealing with statements such as "It is good that *p*", "It is bad that *p*"

The starting point, once again, is Aristotle, who was the first to study the relationship between modal statements and their validity.

However, the great discussion it enjoyed in the Middle Ages notwithstanding, the official birth date of modal logic is 1921, when Clarence Irving Lewis wrote a famous essay on implication.

We saw that $p \rightarrow q$ is true, by definition, for all possible combinations of the truth-values of p and q, except when p is true and q is false.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The truth-table defining $\rightarrow$ may raise some doubts, especially when we "compare" it with the intuitive notion of implication.

In order to clarify the issue, Lewis introduced the notion of strict implication, and with it the symbol of a new logical connective: $\Rightarrow$

According to Lewis – but also to the Stoic philosopher Chrysippus of Soli (III century BC) –

$p \Rightarrow q$ when the negation of the consequent is incompatible with the antecedent: that is, when

it is *impossible* that both p and $\neg q$ are true

or

it is *necessary* that $p \rightarrow q$

This definition introduces two **modal terms** such as ***impossible*** and ***necessary***.

In order to define strict implication, that is, we need two new symbols, $\square$ and $\diamond$.

Given a statement *p*,

by "$\square p$" we mean "**It is necessary that *p***"

and

by "$\diamond p$" we mean "**It is possible that *p***"

Now we can define strict implication:

$$p \Rightarrow q := \neg \diamond (p \land \neg q)$$

that is

it is not possible that *both* p and ¬q are true

Both operators, that of necessity □ and that of possibility ◇, can be reciprocally defined.

If we take ◇ as primitive, we have:

$$\Box p := \neg \Diamond \neg p$$

that is

"it is necessary that *p*" means
"it is not possible that non-*p*"

Therefore, we can define strict implication as:

$$p \Rightarrow q := \Box(\neg p \land q)$$

but since $p \rightarrow q$ is logically equivalent to $\neg(p \land \neg q)$, or $(\neg p \land q)$, we have

$$p \Rightarrow q := \Box(p \rightarrow q)$$

Analogously, if we take $\Box$ as primitive, we have:

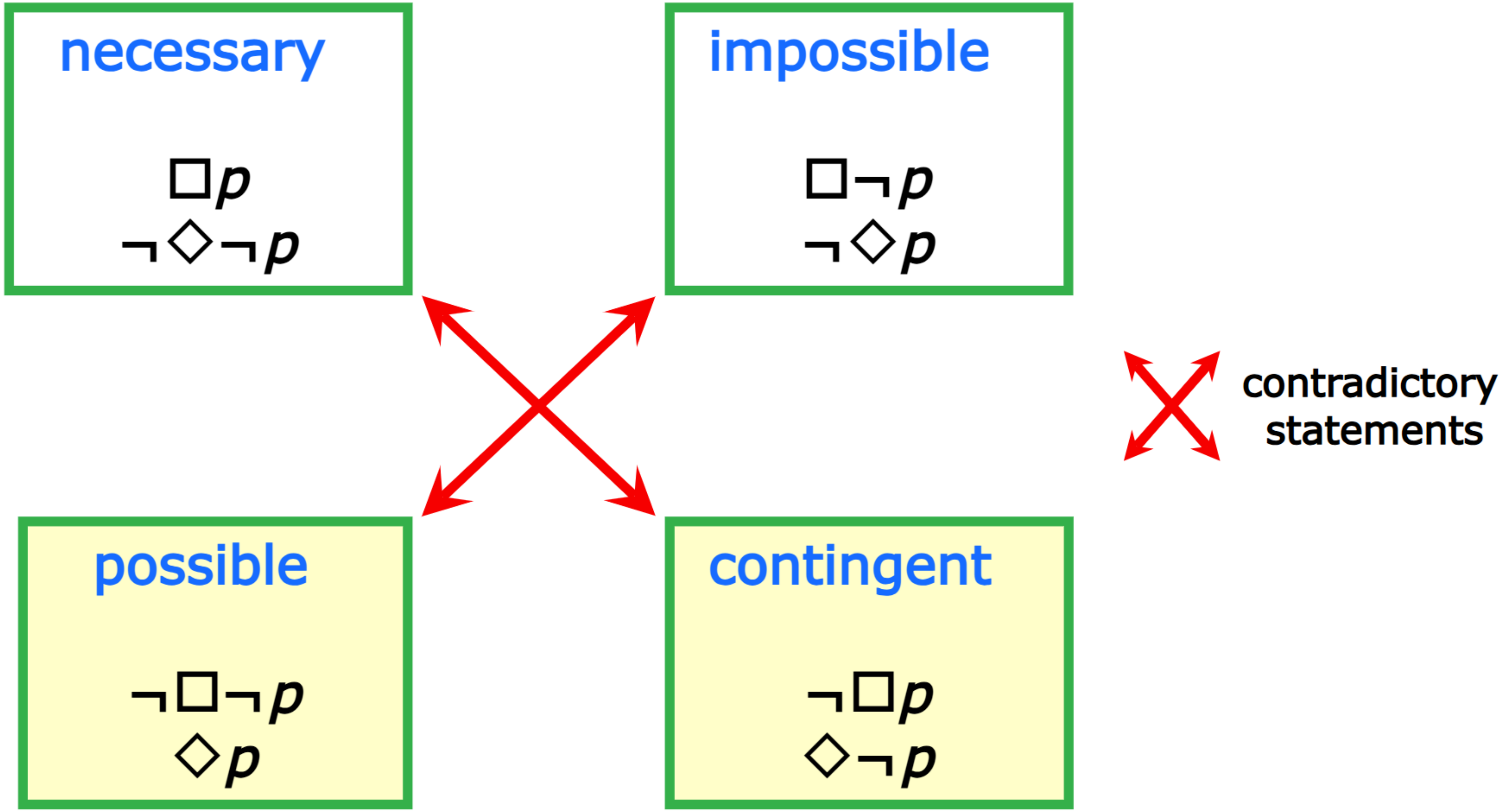
$$\Diamond p := \neg\Box\neg p$$

that is

"it is possible that *p*" means
"it is not necessary that non-*p*"

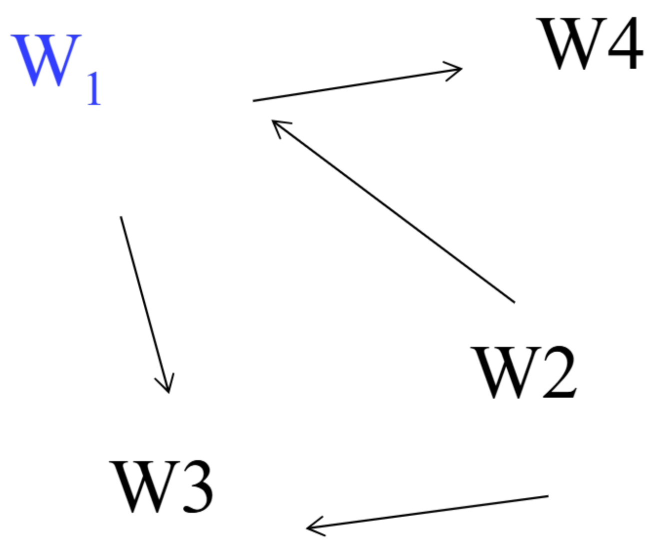
And again, from the definition of strict implication and the above definition, we can conclude that

$$p \Rightarrow q := \Box(p \rightarrow q)$$


necessary
$\Box p$
$\neg\Diamond\neg p$
impossible
$\Box\neg p$
$\neg\Diamond p$
contradictory statements
possible
$\neg\Box\neg p$
$\Diamond p$
contingent
$\neg\Box p$
$\Diamond\neg p$

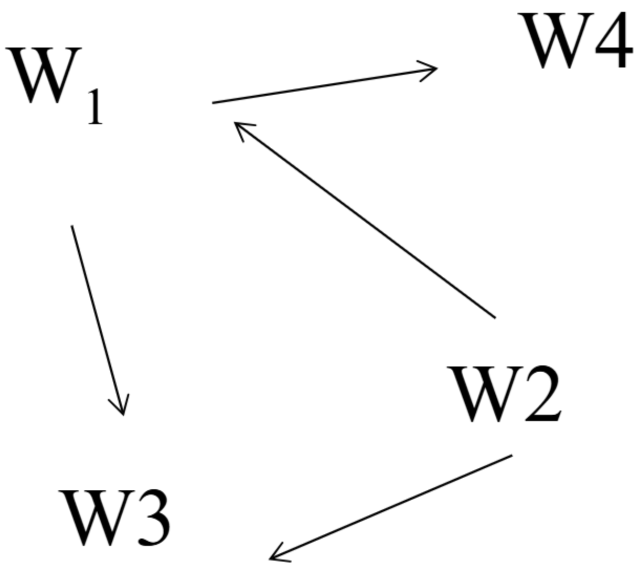
Kripke Semantics of Modal Logic

- The "universe" seen as a collection of worlds.
- Truth defined "in each world".
- Say U is the universe.
- I.e. each $w \in U$ is a prepositional or predicate model.



Kripke Semantics of Modal Logic

- W_1 satisfies **X** if X is satisfied in each world accessible from W_1.
  - If W_3 and W_4 satisfy X.
  - Notation:
    - W_1 |= X if and only if W_3 |= X and W_4 |= X
- W_1 satisfies ◊ X if X is satisfied in <u>at least one</u> world accessible from W_1.



–**Notation:**

•W_1 |= ◊ X if and only if

–W_3 |= X or W_4 |= X

Meaning of Entailment

Entailment says what we can deduce about state of world, what is true in them.

Part of the Definition of entailment relation :

1. $M,w \models \varphi$ **if** φ is true in w
2. $M,w \models \varphi \wedge \psi$ **if** $M,w \models \varphi$ and $M,w \models \psi$

Given Kripke model with state w

state w

formula

If there are two formulas that are true in **some world w** than a logic AND of these formulas is also true in this world.

Semantics of Modal Logic: Definition of Entailment

Kripke Structure

A world

Definition of Kripke Model

- A Kripke model is a pair M, w where
  - $M = (W, R)$ is a Kripke structure and
  - $w \in W$ is a world

Definition of Entailment Relation in Kripke Model

- **The entailment relation** is defined as follows:
  1. $M,w \models \varphi$ **if** φ is true in w
  2. $M,w \models \varphi \wedge \psi$ **if** $M,w \models \varphi$ and $M,w \models \psi$
  3. $M,w \models \neg\varphi$ **if and only if** we do not have $M,w \models \varphi$
  4. $M,w \models \Box\varphi$ **if and only if** $\forall w' \in W$ such that $R(w,w')$ we have $M,w' \models \varphi$

It is true in every word that is accessible from world w

accessibility relation over W

Satisfiable formulas in Kripke models for modal logic

1. In classical logic we have the concept of valid formulas and satisfiable formulas.

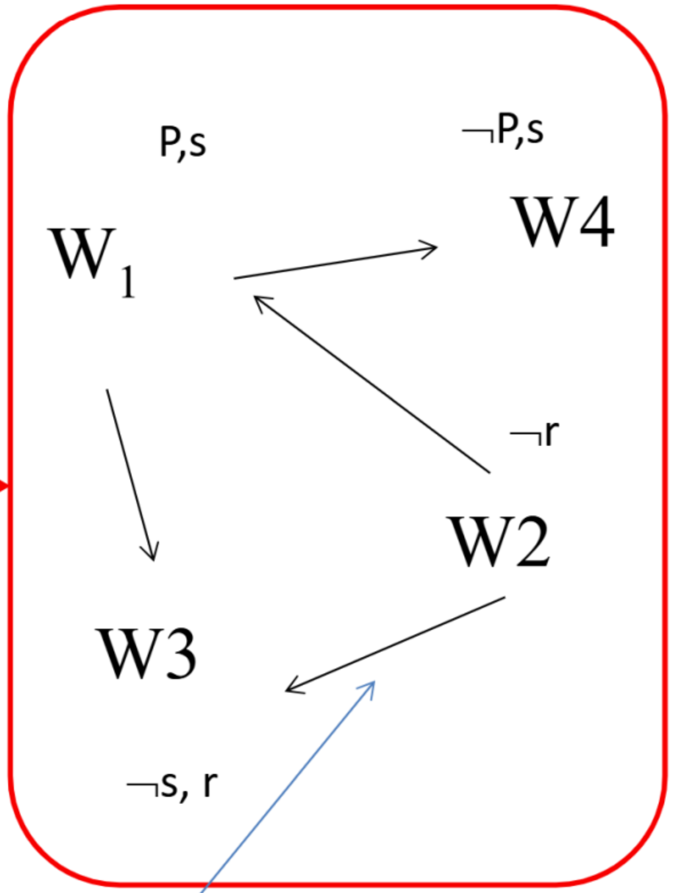
2. In modal logic it is as in classical logic:
   - Any formula φ is valid (written $|= \varphi$) if and only if φ is true in all Kripke models

     E.g. ⇨$\varphi \vee \neg$⇨φ is valid
   - Any formula φ is satisfiable **<u>if and only if</u>** φ is true in some Kripke models

3. We write *M*, $|= \varphi$ if φ is true in all worlds of *M*

Relation to classical satisfiability and entailment

1. For a particular set of propositional constants P, a Kripke model is a three-tuple **<W, R, V> .**
   1. **W** is the set of worlds.
   2. **R** is a subset of $W \times W$, which defines a directed graph over W.
   3. **V** maps each propositional constant to the set of worlds in which it is true.
2. Conceptually, a Kripke model is a directed graph where each node is a propositional model.
3. Given a Kripke model M = <W,R, V> , each world $w \in W$ corresponds to a propositional model:
   1. it says which propositions are true in that world.
   2. In each such world, satisfaction for propositional logic is defined as usual.
4. Satisfaction is also defined at each world **for and ◊ , and this is where R is important.**
5. A sentence is possibly true at a particular world whenever the sentence is true in one of the worlds adjacent to that world in the graph defined by R.



Modal Logic: Axiomatics of system K

- Is there a set of minimal axioms that allows us to derive precisely all the valid sentences?

- Some well-known axioms of basic modal logic are:
  1. **Axiom(Classical)** All propositional tautologies are valid
  2. **Axiom (K)** $(\text{⇨}\varphi \wedge \text{⇨}(\varphi \rightarrow \psi)) \rightarrow \text{⇨}\psi$ is valid
  3. **Rule (Modus Ponens)** if φ and $\varphi \rightarrow \psi$ are valid, infer that ψ is valid
  4. **Rule (Necessitation)** if φ is valid, infer that $\text{⇨}\varphi$ is valid

These are enough, but many other can be added for convenience

Sound and complete sets of inference rules in Modal Logic Axiomatics

- **Refresher:**

  remember that

  1. A set of inference rules (i.e. an inference procedure) is sound if everything it concludes is true
  2. A set of inference rules (i.e. an inference procedure) is complete if it can find all true sentences

- **Theorem:**

  System **K** is sound and complete for the class of all Kripke models.