Triangular norms

T norm is an equivalent to connective "**and**" **(*)** while S norm is related to "**or**" (+).
To calculate the meaning *M(a*b)* of a*b, triangular T-norms will be used.
To calculate the meaning *M(a+b)* of a+b, triangular S-norms will be used.

For any T-norm the following conditions apply:

- commutativity: $T(a,b) = T(b,a)$
- monotonicity: $T(a,b) \leq T(c,d) \; if \;\; a \leq c \wedge b \leq d$
- associativity: $T(a,T(b,c)) = T(T(a,b),c)$
- boundary: $T(a,1) = T(1,a) = a; \quad T(0,0) = 0$

For any S-norm the following conditions apply:

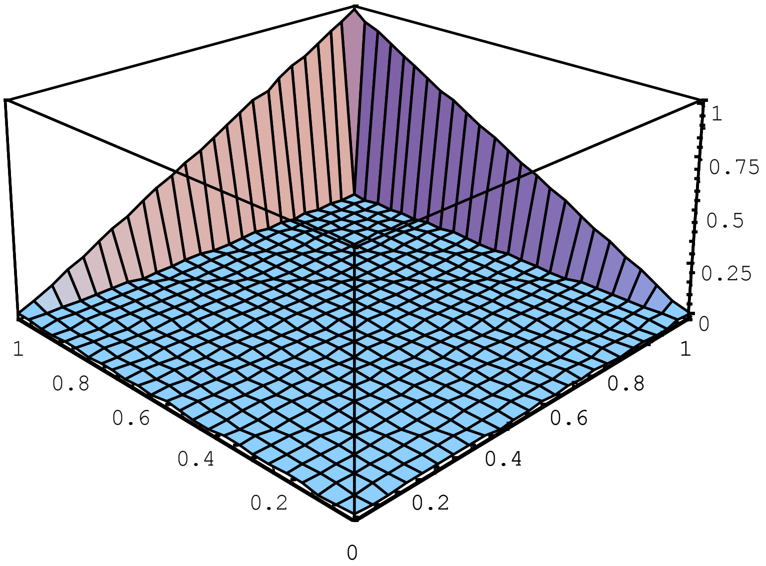
- commutativity: $S(a,b) = S(b,a)$
- monotonicity: $S(a,b) \leq S(c,d) \; if \;\; a \leq c \wedge b \leq d$
- associativity: $S(a,S(b,c)) = S(S(a,b),c)$
- boundary: $S(a,0) = S(0,a) = a; \quad S(1,1) = 1$

Both T and S norms are functions from $[0,1] \times [0,1]$ to $[0,1]$.
The following T and S norms are the most popular:

T_0

$$T_0(x, y) = \min(x, y) \quad when \quad \max(x, y) = 1$$
$$0 \quad otherwise$$

Norm T_0 is also called the Drastic T-Norm

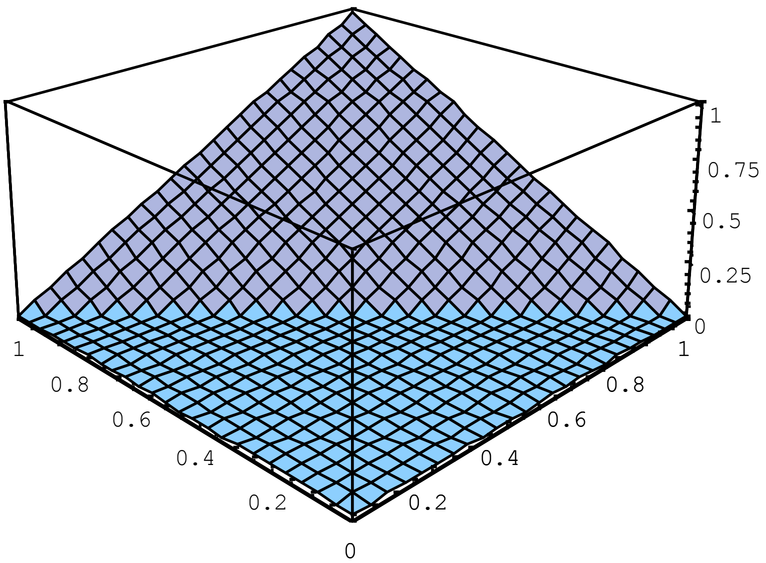


Graph 1. T_0 3D surface

T_1

$$T_1(x, y) = \max(0, x + y - 1)$$

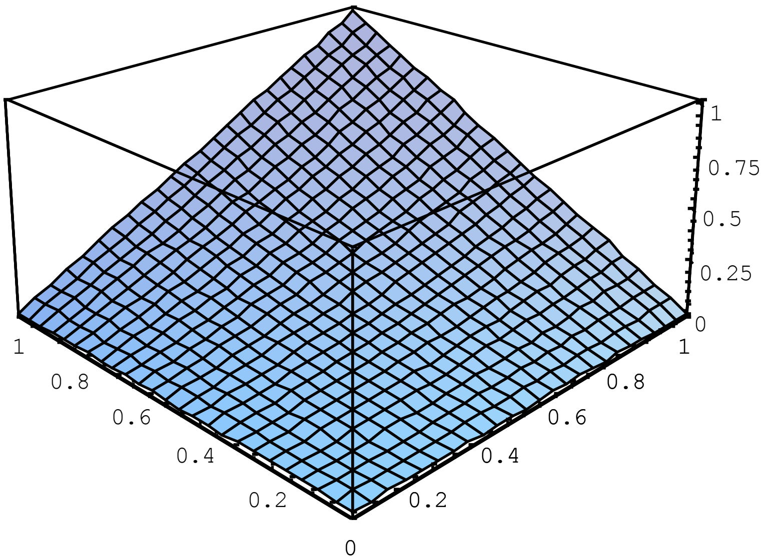
Norm T_1 is also called the Łukasiewicz T-Norm



Graph 2 – T_1 3D surface

T_2

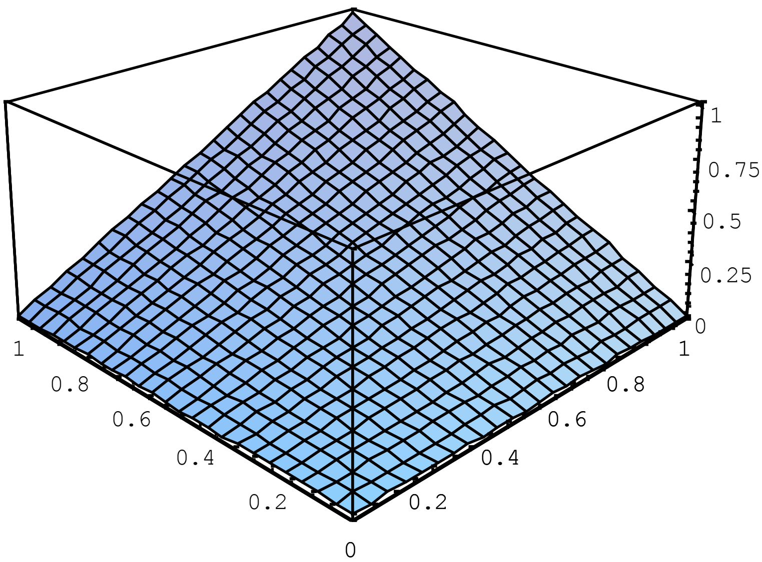
$$T_2(x,y) = \frac{x*y}{2-(x+y-x*y)}$$



Graph 1.3 – T_2 3D surface

T_3

$$T_3(x,y) = x*y$$

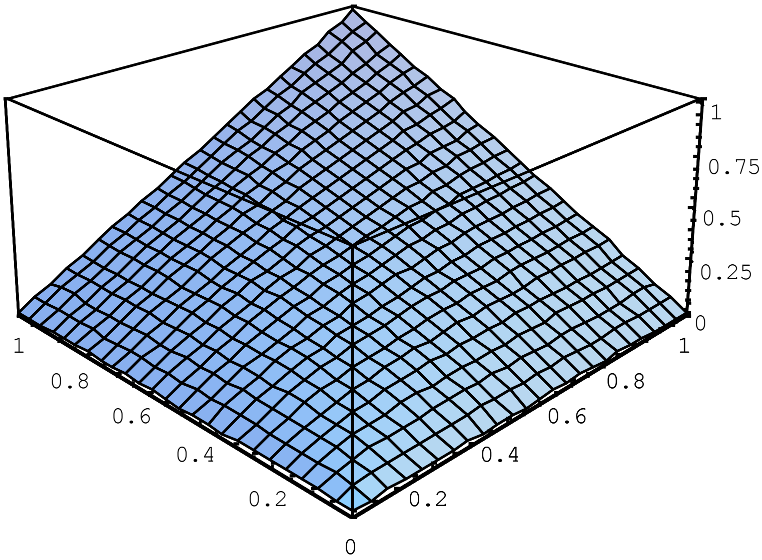


Graph 1.4 – T_3 3D surface

T_4

$$T_4(x,y) = \frac{x*y}{x+y-x*y} \quad when \quad x>0 \wedge y>0$$
$$0 \quad when \quad x=y=0$$

Norm T_4 is also called the Homacher product

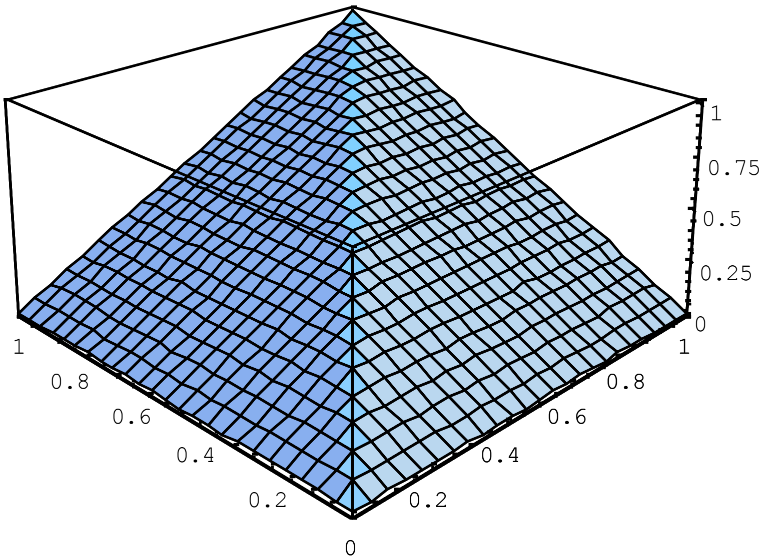


Graph 1.5 – T_4 3D surface

T_5

$$T_5(x,y) = \min(x,y)$$

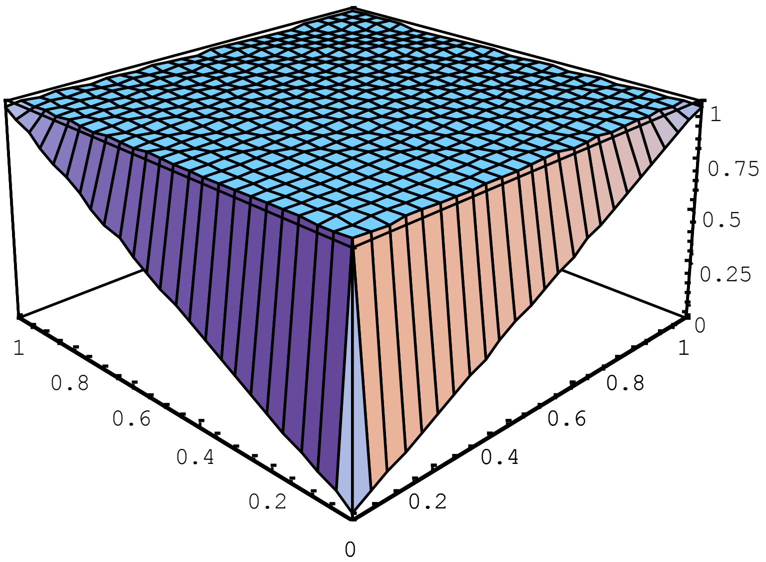
Norm T_4 is also called the Gōdel T-norm



Graph 1.6 – T_5 3D surface

S_0

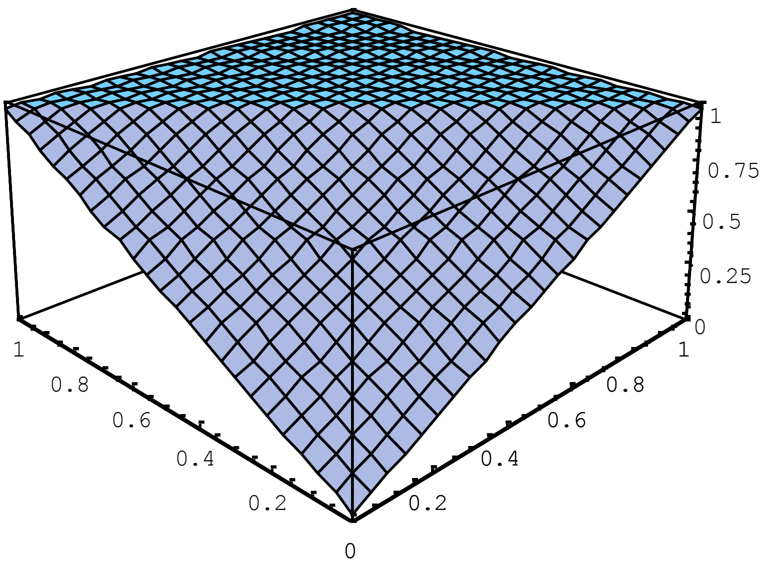
$$T_0(x,y) = \max(x,y) \quad when \quad \min(x,y)=0$$
$$1 \quad otherwise$$



Graph 1.7 – S_0 3D surface

S_1

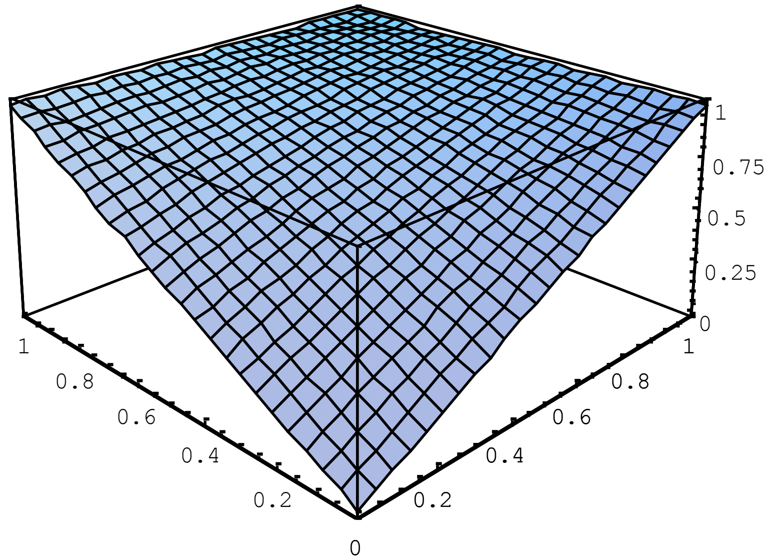
$$S_1(x,y) = \min(1, x+y)$$



Graph 1.8 – S_1 3D surface

S_2

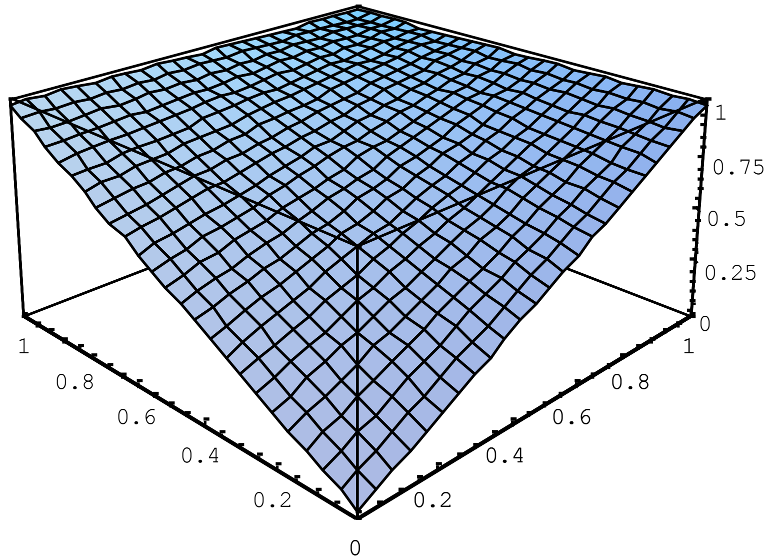
$$S_2(x,y) = \frac{x+y}{1+x*y}$$



Graph 1.9 – S_2 3D surface

S_3

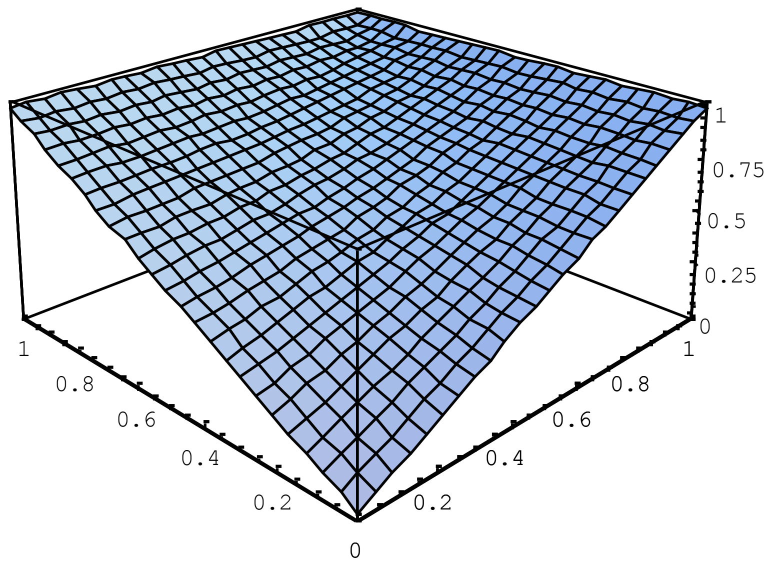
$$S_3(x,y) = x+y-x*y$$



Graph 1.10 – S_3 3D surface

S_4

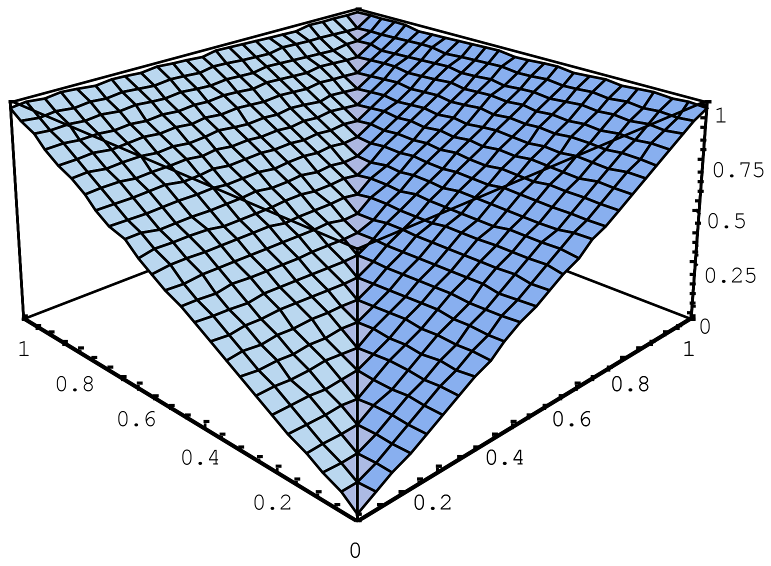
$$S_4(x,y) = \frac{x + y - 2 * x * y}{1 - x * y}$$



Graph 1.11 – S_4 3D surface

S_5

$$S_5(x,y) = \max(x,y)$$



Graph 1.12 – S_5 3D surface

Graphics generated with Wolfram Mathematica.5.1

1.3.2 Combinations of triangular norms

If there is a need to calculate a confidence of a given element in a probabilistic-type of set being a result of a combination involving both sum and intersection, we will use a pair (T_i, S_k), where T_i and S_k norms. But, not all combination of T_i and S_k norms can be used however.

Some combinations do not preserve the distributivity law and therefore, if possible, they should not be used.

Below is a list of allowed combinations together with properties related to the corresponding distributivity law.

T_0

$$a\,T_0\,(b\,S_2\,c) \;\le\; (a\,T_0\,b)\,S_2\,(a\,T_0\,c)$$

$$a\,T_0\,(b\,S_3\,c) \;\le\; (a\,T_0\,b)\,S_3\,(a\,T_0\,c)$$

$$a\,T_0\,(b\,S_4\,c) \;\le\; (a\,T_0\,b)\,S_4\,(a\,T_0\,c)$$

$$a\,T_0\,(b\,S_5\,c) \;=\; (a\,T_0\,b)\,S_5\,(a\,T_0\,c)$$

T_1

$$a\,T_1\,(b\,S_5\,c) \;=\; (a\,T_1\,b)\,S_5\,(a\,T_1\,c)$$

T_2

$$a\,T_2\,(b\,S_0\,c) \;\le\; (a\,T_2\,b)\,S_0\,(a\,T_2\,c)$$

$$a\,T_2\,(b\,S_3\,c) \;\le\; (a\,T_2\,b)\,S_3\,(a\,T_2\,c)$$

$$a\,T_2\,(b\,S_4\,c) \;\le\; (a\,T_2\,b)\,S_4\,(a\,T_2\,c)$$

$$a\,T_2\,(b\,S_5\,c) \;=\; (a\,T_2\,b)\,S_5\,(a\,T_2\,c)$$

T_3

$$a\,T_3\,(b\,S_0\,c) \;\le\; (a\,T_3\,b)\,S_0\,(a\,T_3\,c)$$

$$a\,T_3\,(b\,S_1\,c) \;\le\; (a\,T_3\,b)\,S_1\,(a\,T_3\,c)$$

$$a\,T_3\,(b\,S_2\,c) \;\le\; (a\,T_3\,b)\,S_2\,(a\,T_3\,c)$$

$$a\,T_3\,(b\,S_3\,c) \;\le\; (a\,T_3\,b)\,S_3\,(a\,T_3\,c)$$

$$a\,T_3\,(b\,S_4\,c) \;\le\; (a\,T_3\,b)\,S_4\,(a\,T_3\,c)$$

$$a\,T_3\,(b\,S_5\,c) \;=\; (a\,T_3\,b)\,S_5\,(a\,T_3\,c)$$

T_4

$$a\,T_4\;(b\,S_0\;c) \;\le\; (a\,T_4\;b)\;S_0\;(a\,T_4\;c)$$

$$a\,T_4\;(b\,S_1\;c) \;\le\; (a\,T_4\;b)\;S_1\;(a\,T_4\;c)$$

$$a\,T_4\;(b\,S_2\;c) \;\le\; (a\,T_4\;b)\;S_2\;(a\,T_4\;c)$$

$$a\,T_4\;(b\,S_3\;c) \;\le\; (a\,T_4\;b)\;S_3\;(a\,T_4\;c)$$

$$a\,T_4\;(b\,S_4\;c) \;\le\; (a\,T_4\;b)\;S_4\;(a\,T_4\;c)$$

$$a\,T_4\;(b\,S_5\;c) \;=\; (a\,T_4\;b)\;S_5\;(a\,T_4\;c)$$

T_5

$$a\,T_5\;(b\,S_0\;c) \;\le\; (a\,T_5\;b)\;S_0\;(a\,T_5\;c)$$

$$a\,T_5\;(b\,S_1\;c) \;\le\; (a\,T_5\;b)\;S_1\;(a\,T_5\;c)$$

$$a\,T_5\;(b\,S_2\;c) \;\le\; (a\,T_5\;b)\;S_2\;(a\,T_5\;c)$$

$$a\,T_5\;(b\,S_3\;c) \;\le\; (a\,T_5\;b)\;S_3\;(a\,T_5\;c)$$

$$a\,T_5\;(b\,S_4\;c) \;\le\; (a\,T_5\;b)\;S_4\;(a\,T_5\;c)$$

$$a\,T_5\;(b\,S_5\;c) \;=\; (a\,T_5\;b)\;S_5\;(a\,T_5\;c)$$

The above results have been obtained using heuristic approach.
For each combination of the norms, a million points $\{a,b,c\}$ were randomly chosen from $[0,1]^3$. The relation was then tested against each of them.
The results obtained from these tests are as follows:

$T_0\;S_2$

For 994143 points	$a\,T_0\;(b\,S_2\;c) \;=\; (a\,T_0\;b)\;S_2\;(a\,T_0\;c)$
For 5857 points	$a\,T_0\;(b\,S_2\;c) \;<\; (a\,T_0\;b)\;S_2\;(a\,T_0\;c)$
Conclusion	$a\,T_0\;(b\,S_2\;c) \;\le\; (a\,T_0\;b)\;S_2\;(a\,T_0\;c)$

$T_0\;S_3$

For 994188 points	$a\,T_0\;(b\,S_3\;c) \;=\; (a\,T_0\;b)\;S_3\;(a\,T_0\;c)$

For 5812 points	$a\,T_0\ (b\,S_3\ c)$	$<$	$(a\,T_0\ b)\,S_3\ (a\,T_0\ c)$
Conclusion	$a\,T_0\ (b\,S_3\ c)$	$\leq$	$(a\,T_0\ b)\,S_3\ (a\,T_0\ c)$

$T_0\ S_4$

For 994177 points	$a\,T_0\ (b\,S_4\ c)$	$=$	$(a\,T_0\ b)\,S_4\ (a\,T_0\ c)$
For 5823 points	$a\,T_0\ (b\,S_4\ c)$	$<$	$(a\,T_0\ b)\,S_4\ (a\,T_0\ c)$
Conclusion	$a\,T_0\ (b\,S_4\ c)$	$\leq$	$(a\,T_0\ b)\,S_4\ (a\,T_0\ c)$

$T_0\ S_5$

For 1000000 points	$a\,T_0\ (b\,S_5\ c)$	$=$	$(a\,T_0\ b)\,S_5\ (a\,T_0\ c)$
Conclusion	$a\,T_0\ (b\,S_5\ c)$	$=$	$(a\,T_0\ b)\,S_5\ (a\,T_0\ c)$

$T_1\ S_5$

For 1000000 points	$a\,T_1\ (b\,S_5\ c)$	$=$	$(a\,T_1\ b)\,S_5\ (a\,T_1\ c)$
Conclusion	$a\,T_1\ (b\,S_5\ c)$	$=$	$(a\,T_1\ b)\,S_5\ (a\,T_1\ c)$

$T_2\ S_0$

For 299635 points	$a\,T_2\ (b\,S_0\ c)$	$=$	$(a\,T_2\ b)\,S_0\ (a\,T_2\ c)$
For 700365 points	$a\,T_2\ (b\,S_0\ c)$	$<$	$(a\,T_2\ b)\,S_0\ (a\,T_2\ c)$
Conclusion	$a\,T_2\ (b\,S_0\ c)$	$\leq$	$(a\,T_2\ b)\,S_0\ (a\,T_2\ c)$

$T_2\ S_3$

For 299652 points	$a\,T_2\ (b\,S_3\ c)$	$=$	$(a\,T_2\ b)\,S_3\ (a\,T_2\ c)$
For 700348 points	$a\,T_2\ (b\,S_3\ c)$	$<$	$(a\,T_2\ b)\,S_3\ (a\,T_2\ c)$
Conclusion	$a\,T_2\ (b\,S_3\ c)$	$\leq$	$(a\,T_2\ b)\,S_3\ (a\,T_2\ c)$

$T_2\ S_4$

For 299687 points	$a\,T_2\ (b\,S_4\ c)$	$=$	$(a\,T_2\ b)\,S_4\ (a\,T_2\ c)$
For 700131 points	$a\,T_2\ (b\,S_4\ c)$	$<$	$(a\,T_2\ b)\,S_4\ (a\,T_2\ c)$

Conclusion $a\,T_2\,(b\,S_4\,c) \;\le\; (a\,T_2\,b)\,S_4\,(a\,T_2\,c)$

$T_2\;S_5$

For 1000000 points $a\,T_2\,(b\,S_5\,c) \;=\; (a\,T_2\,b)\,S_5\,(a\,T_2\,c)$

Conclusion $a\,T_2\,(b\,S_5\,c) \;=\; (a\,T_2\,b)\,S_5\,(a\,T_2\,c)$

$T_3\;S_0$

For 299455 points $a\,T_3\,(b\,S_0\,c) \;=\; (a\,T_3\,b)\,S_0\,(a\,T_3\,c)$

For 700545 points $a\,T_3\,(b\,S_0\,c) \;<\; (a\,T_3\,b)\,S_0\,(a\,T_3\,c)$

Conclusion $a\,T_3\,(b\,S_0\,c) \;\le\; (a\,T_3\,b)\,S_0\,(a\,T_3\,c)$

$T_3\;S_1$

For 588730 points $a\,T_3\,(b\,S_1\,c) \;=\; (a\,T_3\,b)\,S_1\,(a\,T_3\,c)$

For 411270 points $a\,T_3\,(b\,S_1\,c) \;<\; (a\,T_3\,b)\,S_1\,(a\,T_3\,c)$

Conclusion $a\,T_3\,(b\,S_1\,c) \;\le\; (a\,T_3\,b)\,S_1\,(a\,T_3\,c)$

$T_3\;S_2$

For 299176 points $a\,T_3\,(b\,S_2\,c) \;=\; (a\,T_3\,b)\,S_2\,(a\,T_3\,c)$

For 700824 points $a\,T_3\,(b\,S_2\,c) \;<\; (a\,T_3\,b)\,S_2\,(a\,T_3\,c)$

Conclusion $a\,T_3\,(b\,S_2\,c) \;\le\; (a\,T_3\,b)\,S_2\,(a\,T_3\,c)$

$T_3\;S_3$

For 299827 points $a\,T_3\,(b\,S_3\,c) \;=\; (a\,T_3\,b)\,S_3\,(a\,T_3\,c)$

For 700173 points $a\,T_3\,(b\,S_3\,c) \;<\; (a\,T_3\,b)\,S_3\,(a\,T_3\,c)$

Conclusion $a\,T_3\,(b\,S_3\,c) \;\le\; (a\,T_3\,b)\,S_3\,(a\,T_3\,c)$

$T_3\;S_4$

For 299406 points $a\,T_3\,(b\,S_4\,c) \;=\; (a\,T_3\,b)\,S_4\,(a\,T_3\,c)$

For 700594 points $a\,T_3\,(b\,S_4\,c) \;<\; (a\,T_3\,b)\,S_4\,(a\,T_3\,c)$

Conclusion $a\,T_3\,(b\,S_4\,c) \;\le\; (a\,T_3\,b)\,S_4\,(a\,T_3\,c)$

$T_3\ S_5$

For 1000000 points $a\,T_3\,(b\,S_5\,c) \;=\; (a\,T_3\,b)\,S_5\,(a\,T_3\,c)$

Conclusion $a\,T_3\,(b\,S_5\,c) \;=\; (a\,T_3\,b)\,S_5\,(a\,T_3\,c)$

$T_4\ S_0$

For 299858 points $a\,T_4\,(b\,S_0\,c) \;=\; (a\,T_4\,b)\,S_0\,(a\,T_4\,c)$

For 700142 points $a\,T_4\,(b\,S_0\,c) \;<\; (a\,T_4\,b)\,S_0\,(a\,T_4\,c)$

Conclusion $a\,T_4\,(b\,S_0\,c) \;\le\; (a\,T_4\,b)\,S_0\,(a\,T_4\,c)$

$T_4\ S_1$

For 299771 points $a\,T_4\,(b\,S_1\,c) \;=\; (a\,T_4\,b)\,S_1\,(a\,T_4\,c)$

For 700229 points $a\,T_4\,(b\,S_1\,c) \;<\; (a\,T_4\,b)\,S_1\,(a\,T_4\,c)$

Conclusion $a\,T_4\,(b\,S_1\,c) \;\le\; (a\,T_4\,b)\,S_1\,(a\,T_4\,c)$

$T_4\ S_2$

For 299587 points $a\,T_4\,(b\,S_2\,c) \;=\; (a\,T_4\,b)\,S_2\,(a\,T_4\,c)$

For 700413 points $a\,T_4\,(b\,S_2\,c) \;<\; (a\,T_4\,b)\,S_2\,(a\,T_4\,c)$

Conclusion $a\,T_4\,(b\,S_2\,c) \;\le\; (a\,T_4\,b)\,S_2\,(a\,T_4\,c)$

$T_4\ S_3$

For 299490 points $a\,T_4\,(b\,S_3\,c) \;=\; (a\,T_4\,b)\,S_3\,(a\,T_4\,c)$

For 700510 points $a\,T_4\,(b\,S_3\,c) \;<\; (a\,T_4\,b)\,S_3\,(a\,T_4\,c)$

Conclusion $a\,T_4\,(b\,S_3\,c) \;\le\; (a\,T_4\,b)\,S_3\,(a\,T_4\,c)$

$T_4\ S_4$

For 299888 points $a\,T_4\,(b\,S_4\,c) \;=\; (a\,T_4\,b)\,S_4\,(a\,T_4\,c)$

For 700112 points $a\,T_4\,(b\,S_4\,c) \;<\; (a\,T_4\,b)\,S_4\,(a\,T_4\,c)$

Conclusion $a\,T_4\ (b\,S_4\ c)\ \le\ (a\,T_4\ b)\ S_4\ (a\,T_4\ c)$

$T_4\ S_5$

For 1000000 points $a\,T_4\ (b\,S_5\ c)\ =\ (a\,T_4\ b)\ S_5\ (a\,T_4\ c)$

Conclusion $a\,T_4\ (b\,S_5\ c)\ =\ (a\,T_4\ b)\ S_5\ (a\,T_4\ c)$

$T_5\ S_0$

For 299987 points $a\,T_5\ (b\,S_0\ c)\ =\ (a\,T_5\ b)\ S_0\ (a\,T_5\ c)$

For 700013 points $a\,T_5\ (b\,S_0\ c)\ <\ (a\,T_5\ b)\ S_0\ (a\,T_5\ c)$

Conclusion $a\,T_5\ (b\,S_0\ c)\ \le\ (a\,T_5\ b)\ S_0\ (a\,T_5\ c)$

$T_5\ S_1$

For 395919 points $a\,T_5\ (b\,S_1\ c)\ =\ (a\,T_5\ b)\ S_1\ (a\,T_5\ c)$

For 604081 points $a\,T_5\ (b\,S_1\ c)\ <\ (a\,T_5\ b)\ S_1\ (a\,T_5\ c)$

Conclusion $a\,T_5\ (b\,S_1\ c)\ \le\ (a\,T_5\ b)\ S_1\ (a\,T_5\ c)$

$T_5\ S_2$

For 431394 points $a\,T_5\ (b\,S_2\ c)\ =\ (a\,T_5\ b)\ S_2\ (a\,T_5\ c)$

For 568606 points $a\,T_5\ (b\,S_2\ c)\ <\ (a\,T_5\ b)\ S_2\ (a\,T_5\ c)$

Conclusion $a\,T_5\ (b\,S_2\ c)\ \le\ (a\,T_5\ b)\ S_2\ (a\,T_5\ c)$

$T_5\ S_3$

For 444371 points $a\,T_5\ (b\,S_3\ c)\ =\ (a\,T_5\ b)\ S_3\ (a\,T_5\ c)$

For 555629 points $a\,T_5\ (b\,S_3\ c)\ <\ (a\,T_5\ b)\ S_3\ (a\,T_5\ c)$

Conclusion $a\,T_5\ (b\,S_3\ c)\ \le\ (a\,T_5\ b)\ S_3\ (a\,T_5\ c)$

$T_5\ S_4$

For 466996 points $a\,T_5\ (b\,S_4\ c)\ =\ (a\,T_5\ b)\ S_4\ (a\,T_5\ c)$

For 533004 points $a\,T_5\ (b\,S_4\ c)\ <\ (a\,T_5\ b)\ S_4\ (a\,T_5\ c)$

Conclusion $a\,T_5\,(b\,S_4\,c) \;\leq\; (a\,T_5\,b)\,S_4\,(a\,T_5\,c)$

$T_5\ S_5$

For 1000000 points $a\,T_5\,(b\,S_5\,c) \;=\; (a\,T_5\,b)\,S_5\,(a\,T_5\,c)$

Conclusion $a\,T_5\,(b\,S_5\,c) \;=\; (a\,T_5\,b)\,S_5\,(a\,T_5\,c)$

1.3.3 Ordering combinations of triangular norms

In [3], a partial order relation over pairs (T_i, S_k) has been proposed.

This ordering relation is defined as follows: $(T_{i1}, S_{k1}) \leq (T_{i2}, S_{k2}) \Leftrightarrow (i1 \leq i2) \wedge (k1 \leq k2)$

For any $i1, i2, k1, k2$ the following inequalities are true:

$(T_{\min(i1,i2)}, S_{\max(k1,k2)}) \leq (T_{i1}, S_{k1}) \leq (T_{\max(i1,i2)}, S_{\min(k1,k2)})$

$(T_{\min(i1,i2)}, S_{\max(k1,k2)}) \leq (T_{i2}, S_{k2}) \leq (T_{\max(i1,i2)}, S_{\min(k1,k2)})$

Described relation creates ordering of lattice type – not all elements are comparable.
There are however ordered chains, and all of them do have upper bounds.
Upon this properties, according to Kuratowski-Zorn lemma [4], the subject set contains maximal element, which is (T_5, S_5).