Machine Learning: ITCS 4156

k-Nearest Neighbor Algorithms for Classification and Regression

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Nonparametric Methods: k-Nearest Neighbors

Input:

- A training dataset $(\mathbf{x}_1, t_1), (\mathbf{x}_2, t_2), \ldots (\mathbf{x}_n, t_n)$.
- A test instance $\mathbf{x}$.

Output:

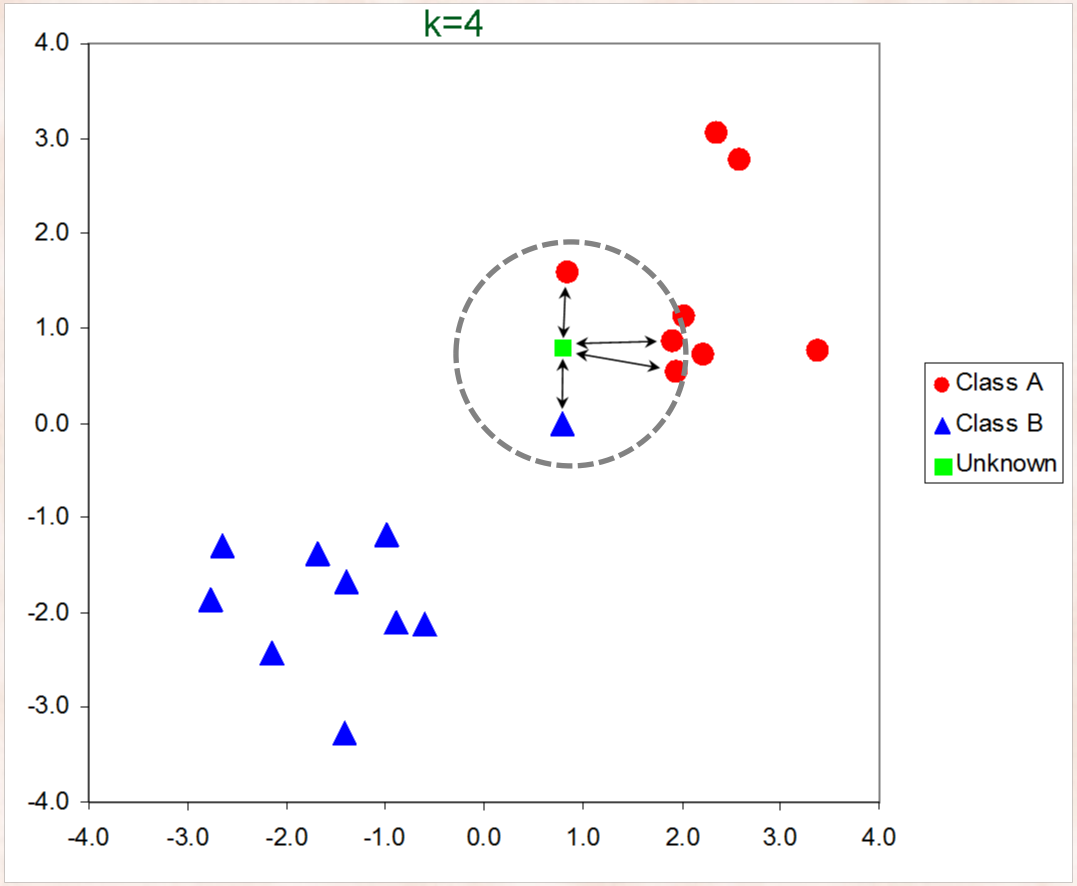
- Estimated class label $y(\mathbf{x})$.

1. Find k instances $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ *nearest* to $\mathbf{x}$.
2. Let $y(x) = \arg\max_{t \in T} \sum_{i=1}^{k} \delta_t(t_i)$

where $\delta_t(x) = \begin{cases} 1 & x = t \\ 0 & x \neq t \end{cases}$ is the *Kronecker delta* function.

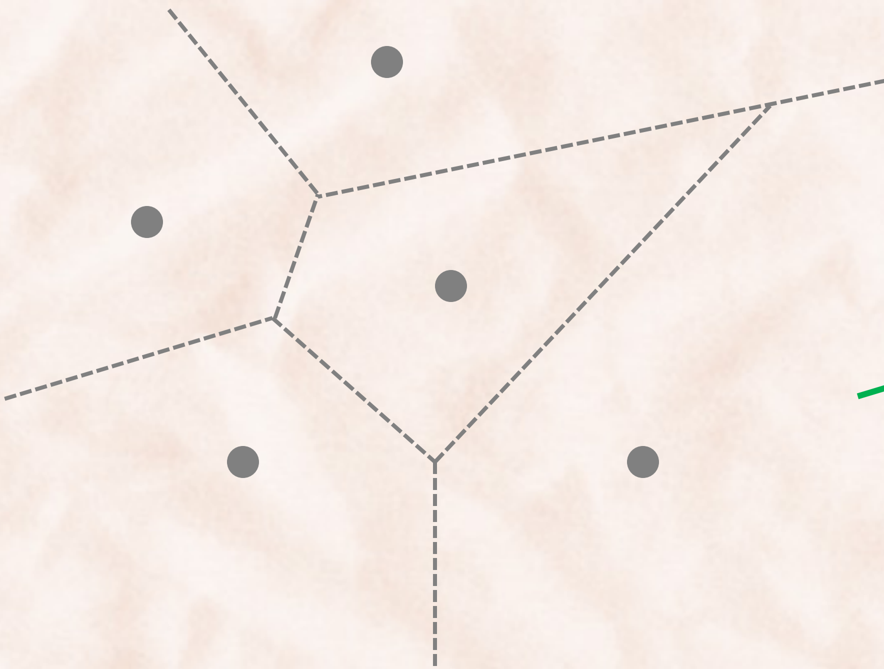
k-Nearest Neighbors (k-NN)

- Euclidean distance, $k = 4$

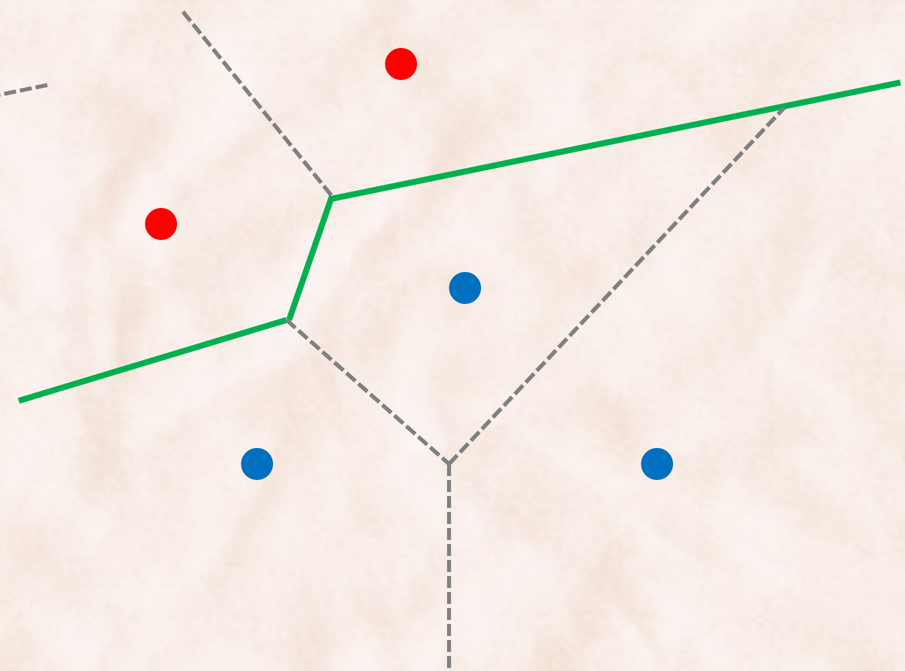


k-Nearest Neighbors (k-NN)

- Euclidian distance, k = 1.



Voronoi diagram



decision boundary

k-NN for Classification: Probabilistic Justification

- Assume a dataset with N_j points in class C_j.

  $\Rightarrow$ total number of points is $N = \sum_j N_j$

- Draw a sphere centered at **x** containing K points:
  - sphere has volume V.
  - sphere contains K_j points from class C_j.
- If V sufficiently small and K sufficiently large, we can estimate [2.5.1]:

$$p(\mathbf{x} \mid C_j) = \frac{K_j}{N_j V} \qquad p(\mathbf{x}) = \frac{K}{NV} \qquad p(C_j) = \frac{N_j}{N}$$

- Bayes' theorem $\Rightarrow p(C_j \mid \mathbf{x}) = \frac{K_j}{K} \Rightarrow$ choose class C_j with most neighbors.

Distance Metrics

- Euclidean distance:

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_2 = \sqrt{(\mathbf{x}-\mathbf{y})^T(\mathbf{x}-\mathbf{y})}$$

- Hamming distance:

  # of (discrete) features that have different values in **x** and **y**.

- Mahalanobis distance:

(sample) covariance matrix

$$d(\mathbf{x},\mathbf{y}) = \sqrt{(\mathbf{x}-\mathbf{y})^T S^{-1}(\mathbf{x}-\mathbf{y})}$$

  - scale-invariant metric that normalizes for variance.
  - if $S = I \Rightarrow$ Euclidean distance.
  - if $S = diag(\sigma_1^{-2}, \sigma_2^{-2}, \ldots \sigma_K^{-2}) \Rightarrow$ *normalized* Euclidean distance.

Distance Metrics

- Cosine similarity:

$$d(\mathbf{x},\mathbf{y}) = 1 - \cos(\mathbf{x},\mathbf{y}) = 1 - \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

  - used for text and other high-dimensional data.

- Levenshtein distance (Edit distance):
  - distance metric on strings (sequences of symbols).
  - min. # of basic edit operations that can transform one string into the other (delete, insert, substitute).

$$\left.\begin{array}{l}\mathbf{x} = \text{“athens”} \\ \mathbf{y} = \text{“hints”}\end{array}\right\} \Rightarrow d(\mathbf{x},\mathbf{y}) = 4$$

  - used in bioinformatics.

Efficient Indexing

- Linear searching for *k*-nearest neighbors is not efficient for large training sets:
  - O(N) time complexity.
- For Euclidean distance use a **kd-tree**:
  - instances stored at leaves of the tree.
  - internal nodes branch on threshold test on individual features.
  - expected time to find the nearest neighbor is O(log N)
- Indexing structures depend on distance function:
  - **inverted index** for text retrieval with cosine similarity.

k-NN and The Curse of Dimensionality

- Standard metrics weigh each feature equally:
  - Problematic when many features are irrelevant.

- One solution is to weigh each feature differently:
  - Use measure indicating ability to discriminate between classes, such as:
    - Information Gain, Chi-square Statistic
    - Pearson Correlation, Signal to Noise Ration, T test.
  - “Stretch” the axes:
    - lengthen for relevant features, shorten for irrelevant features.
  - Equivalent with Mahalanobis distance with diagonal covariance.

Distance-Weighted k-NN

For any test point **x**, weight each of the *k* neighbors according to their distance from **x**.

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1. Find k instances $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ nearest to $\mathbf{x}$.
2. Let $y(x) = \arg\max_{t \in T} \sum_{i=1}^{k} w_i \delta_t(t_i)$

   where $w_i = \|\mathbf{x} - \mathbf{x}_i\|^{-2}$ measures the similarity between $\mathbf{x}$ and $\mathbf{x}_i$

Kernel-based Distance-Weighted NN

For any test point **x**, weight all training instances according to their similarity with **x**.

1. Assume binary classification, T = {+1, −1}.

2. Compute weighted majority:

$$y(\mathbf{x}) = sign\left(\sum_{i=1}^{N} K(\mathbf{x}, \mathbf{x}_i) t_i\right)$$

Regression with k-Nearest Neighbor

Input:

- A training dataset $(\mathbf{x}_1, t_1)$, $(\mathbf{x}_2, t_2)$, ... $(\mathbf{x}_n, t_n)$.
- A test instance $\mathbf{x}$.

Output:

- Estimated function value $y(\mathbf{x})$.

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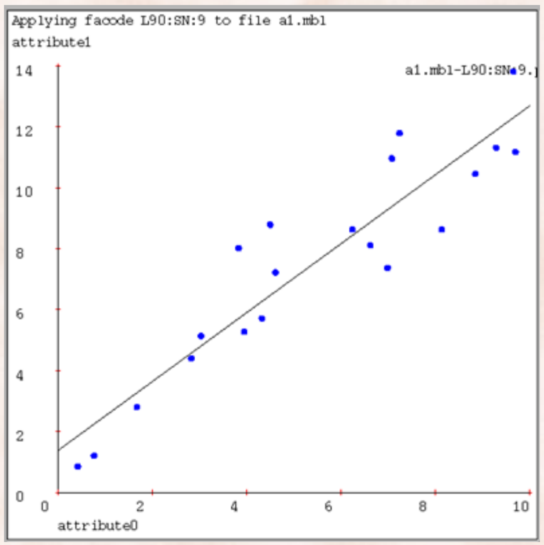
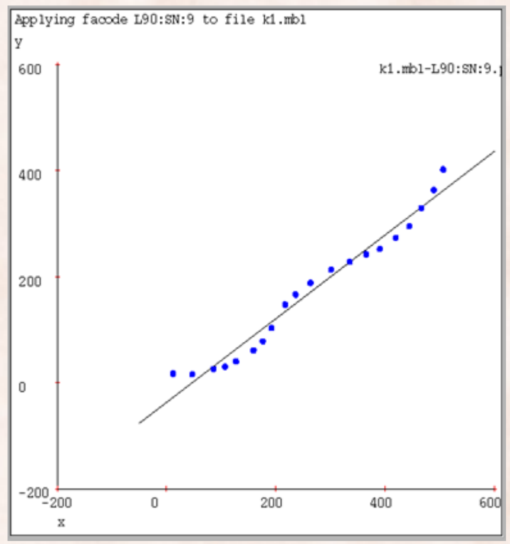
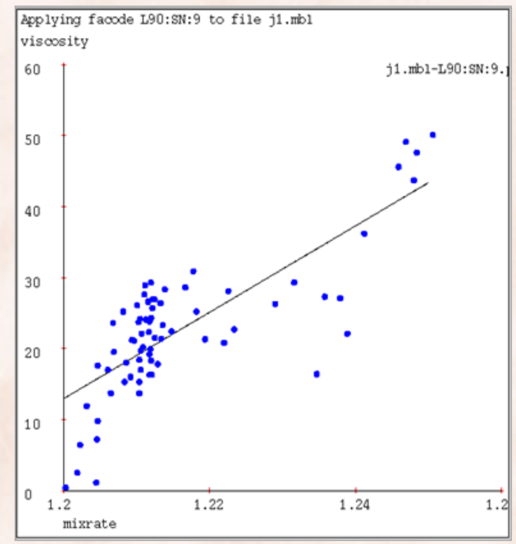
1. Find k instances $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_k$ nearest to $\mathbf{x}$.
2. Let $y(x) = \frac{1}{k}\sum_{i=1}^{k} t_i$

3 Datasets & Linear Interpolation

[http://www.autonlab.org/tutorials/mbl08.pdf]

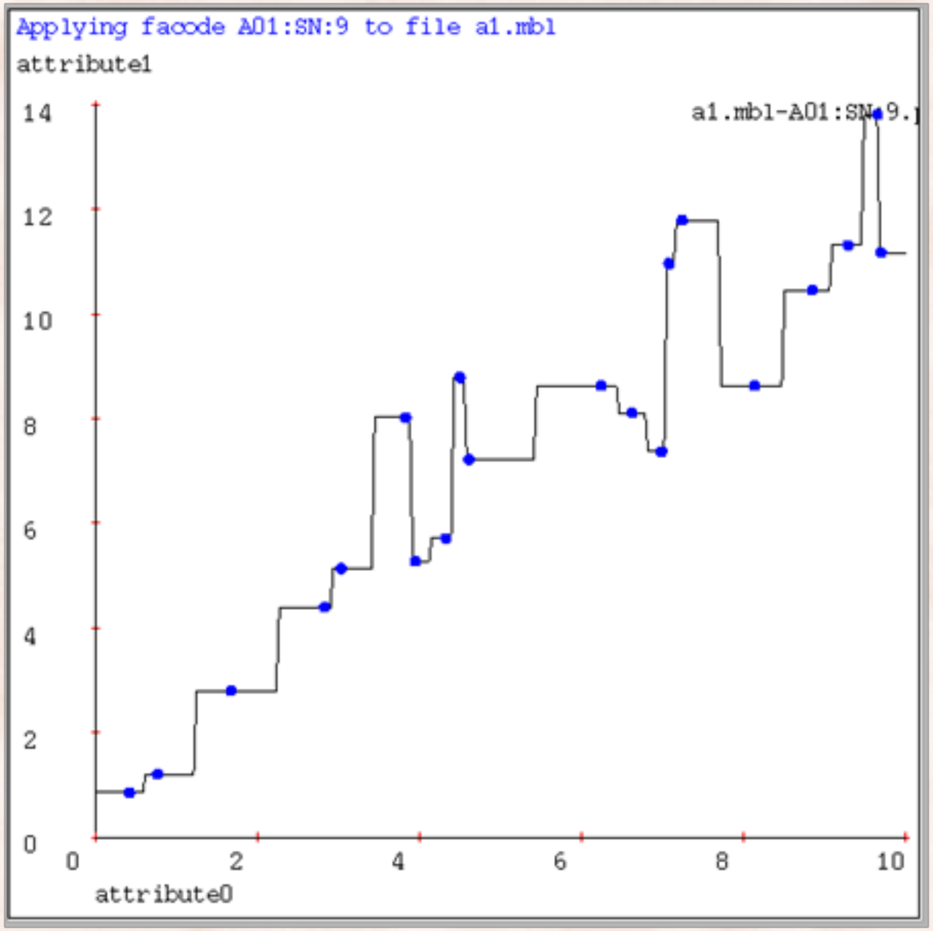




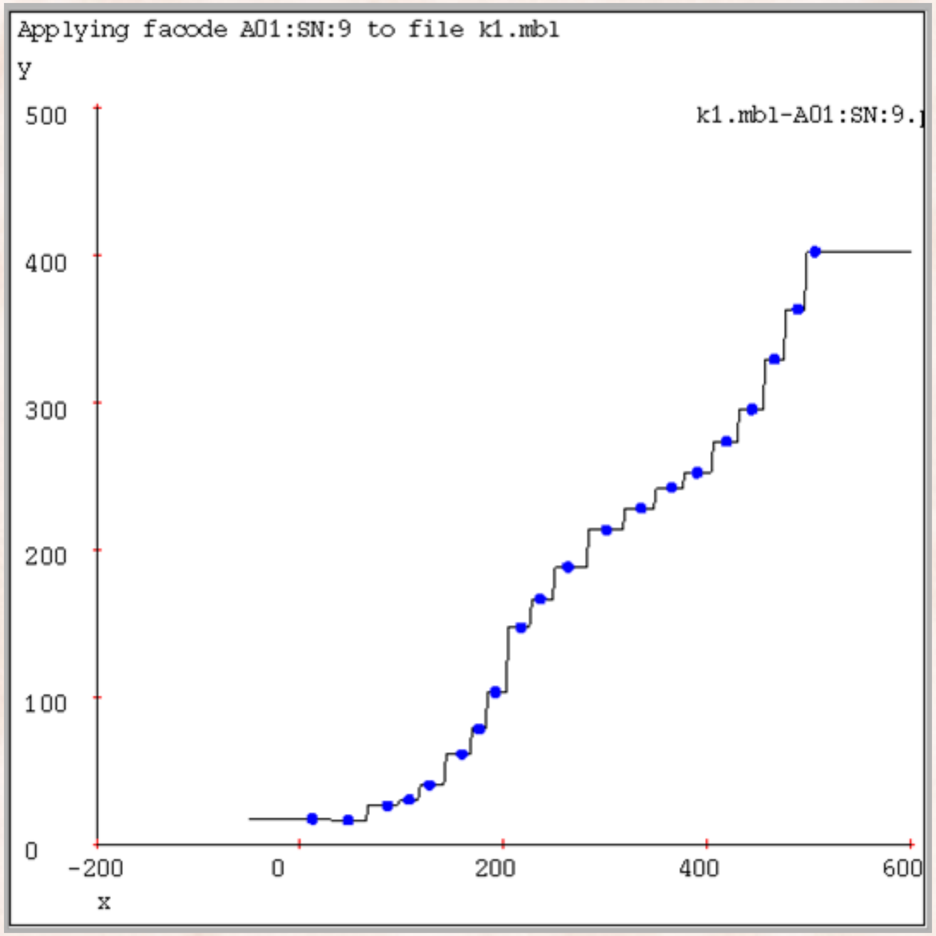


Linear interpolation does not always lead to good models of the data.

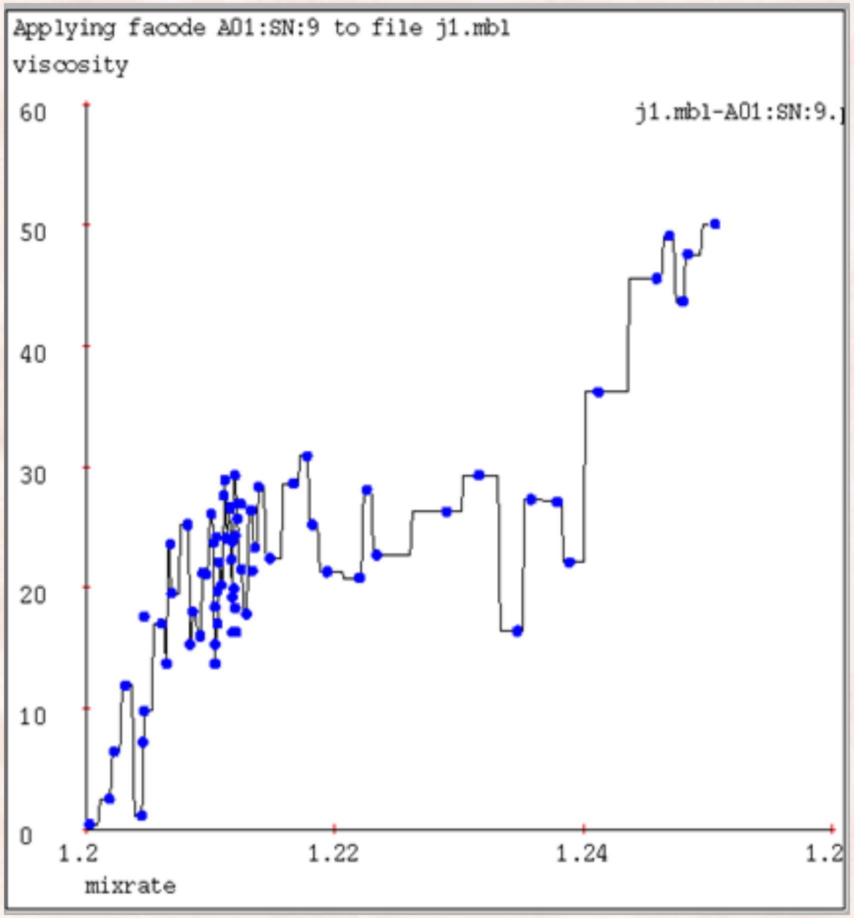
Regression with 1-Nearest Neighbor



Regression with 1-Nearest Neighbor



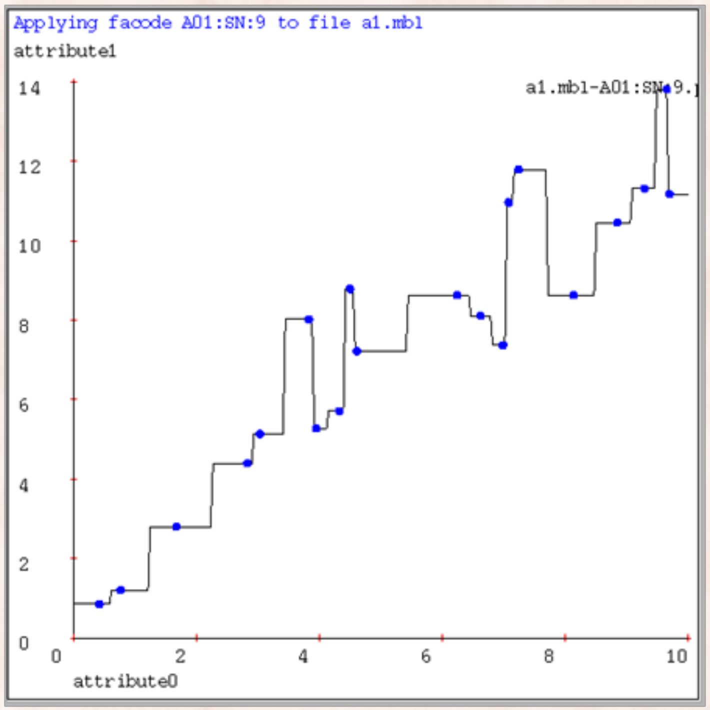
Regression with 1-Nearest Neighbor



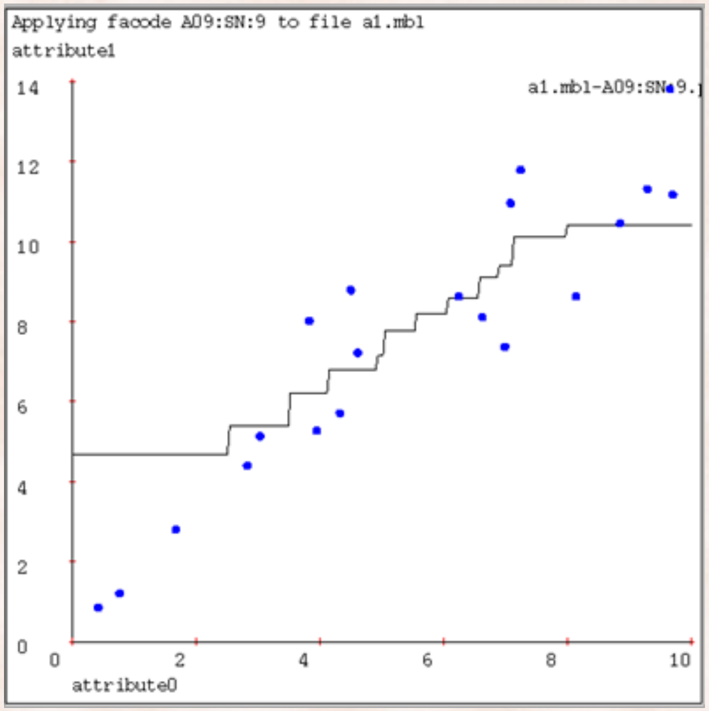
⇒ *1-NN has high variance*

Regression with 9-Nearest Neighbor

$k = 1$

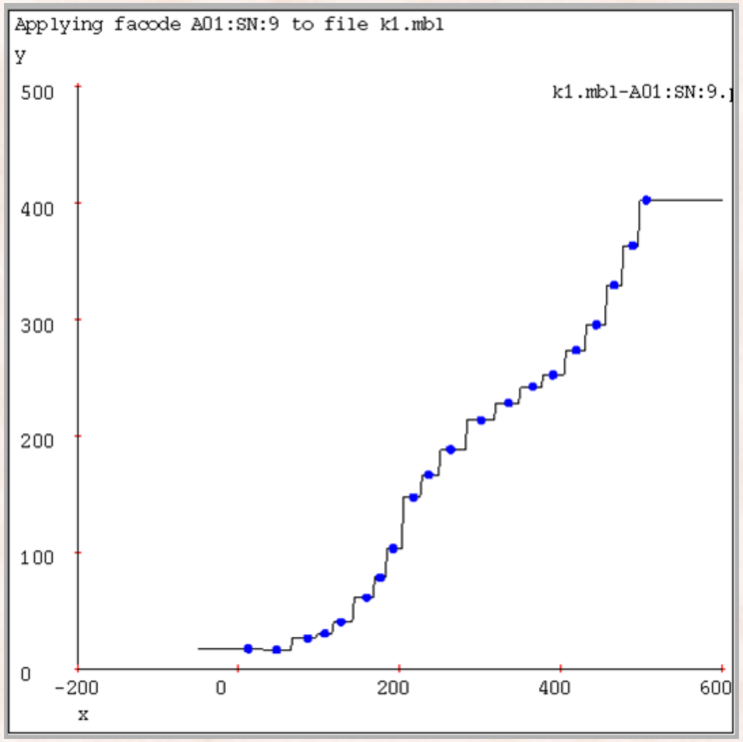


$k = 9$

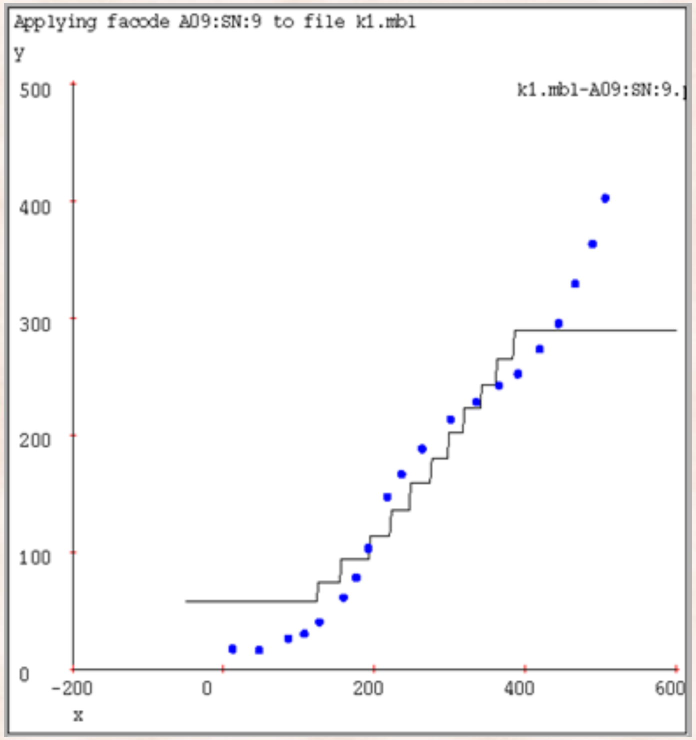


Regression with 9-Nearest Neighbor

$k = 1$

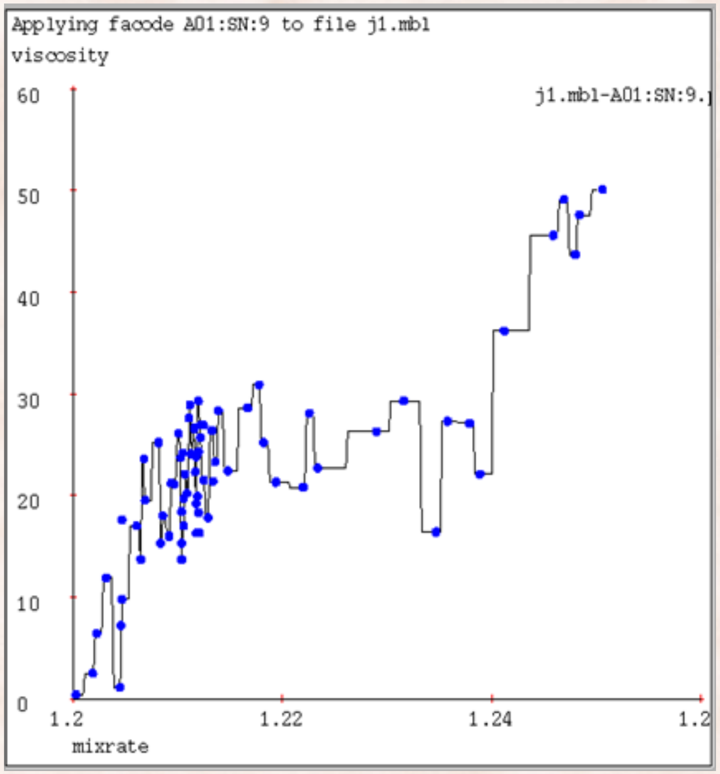


$k = 9$

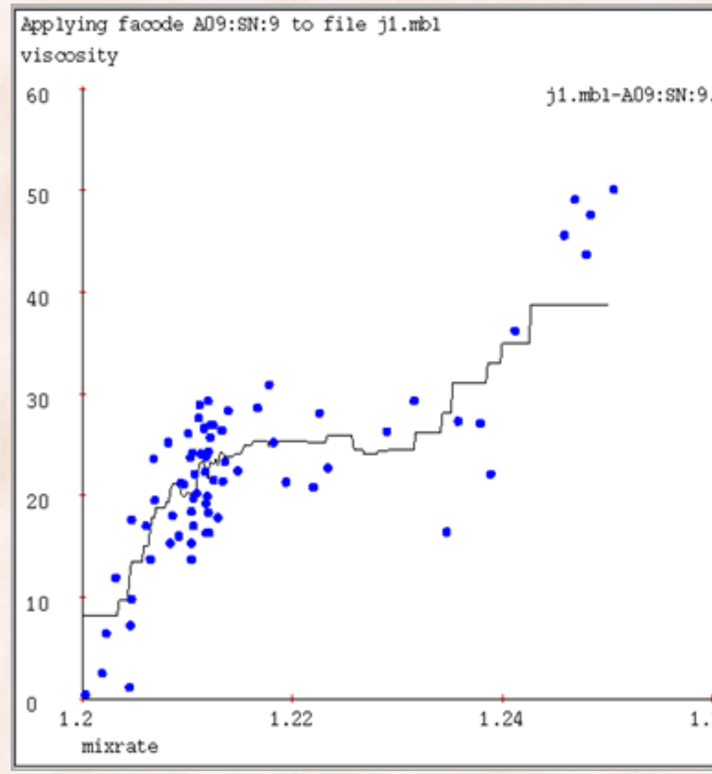


Regression with 9-Nearest Neighbor

$k = 1$



$k = 9$



Distance-Weighted k-NN for Regression

For any test point **x**, weight each of the *k* neighbors according to their similarity with **x**.

1. Find *k* instances $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ nearest to **x**.
2. Let $y(x) = \sum_{i=1}^{k} w_i t_i \Big/ \sum_{i=1}^{k} w_i$

   where $w_i = \|\mathbf{x} - \mathbf{x}_i\|^{-2}$

For $k = N \Rightarrow$ Shepard's method [Shepard, ACM '68].

Kernel-based Distance Weighted NN Regression

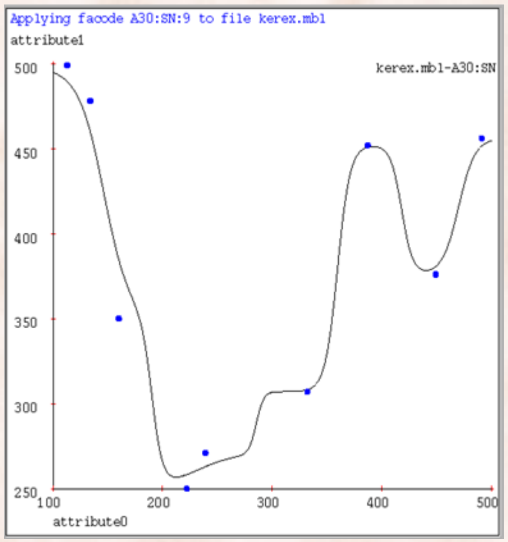
For any test point **x**, weight all training instances according to their similarity with **x**.

1. Return weighted average:

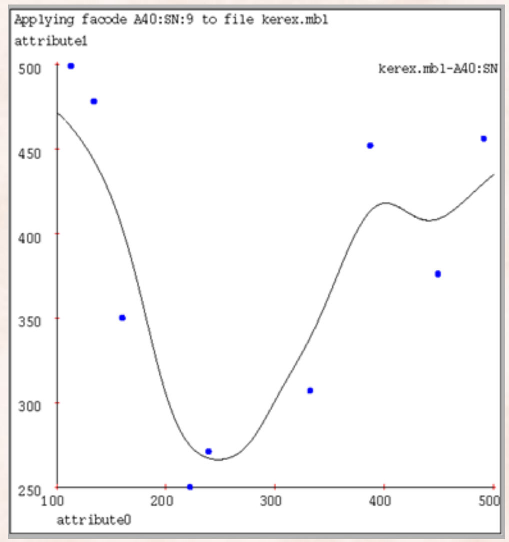
$$y(\mathbf{x}) = \frac{\sum_{i=1}^{N} K(\mathbf{x}, \mathbf{x}_i) t_i}{\sum_{i=1}^{N} K(\mathbf{x}, \mathbf{x}_i)}$$

NN Regression with Gaussian Kernel

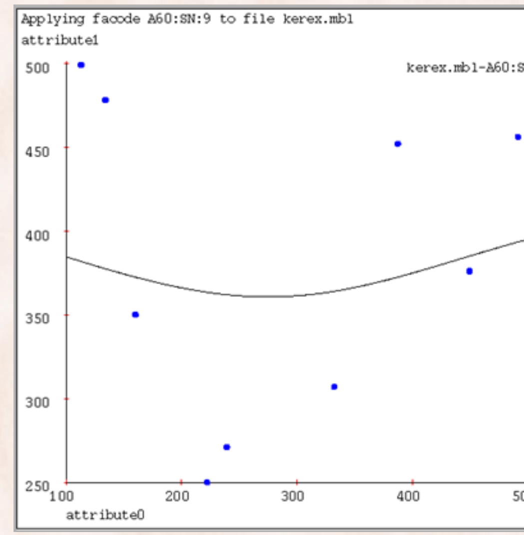
$2\sigma^2=10$



$2\sigma^2=20$



$2\sigma^2=80$

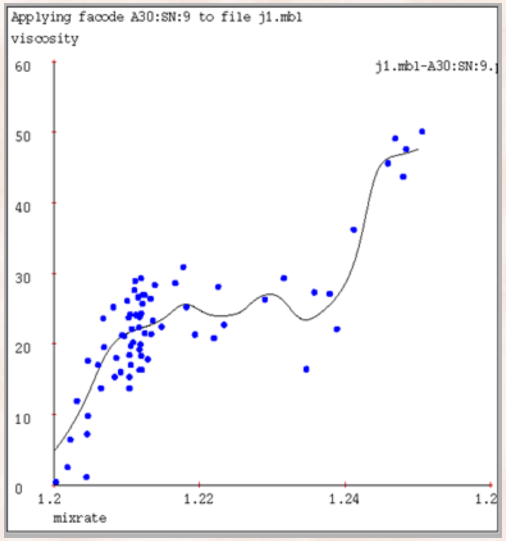


$$K(\mathbf{x},\mathbf{x}_i)=e^{-\frac{\|\mathbf{x}-\mathbf{x}_i\|^2}{2\sigma^2}}$$

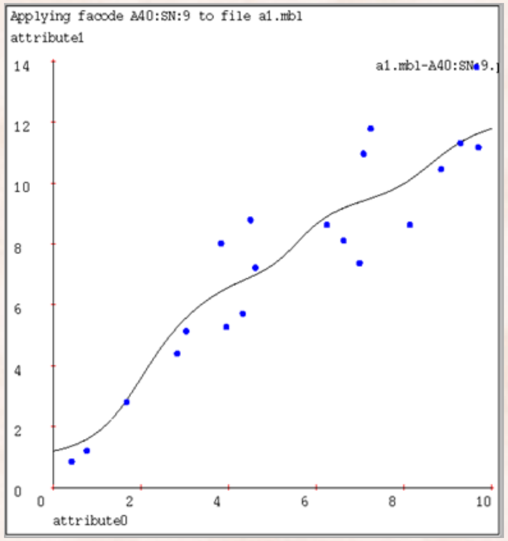
Increased kernel width means more influence from distant points.

NN Regression with Gaussian Kernel

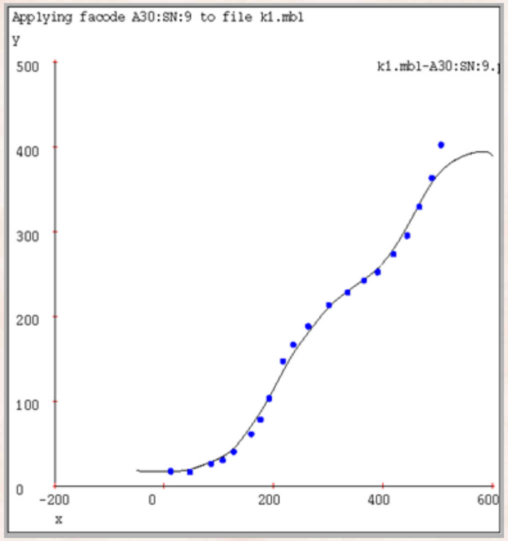
$2\sigma^2$=1/16 of x axis



$2\sigma^2$=1/32 of x axis



$2\sigma^2$=1/32 of x axis



$$K(\mathbf{x}, \mathbf{x}_i) = e^{-\frac{\|\mathbf{x}-\mathbf{x}_i\|^2}{2\sigma^2}}$$

k-Nearest Neighbor Summary

- Training: memorize the training examples.
- Testing: compute distance/similarity with training examples.
- Trades decreased training time for increased test time.
- Use kernel trick to work in implicit high dimensional space.
- Needs feature selection when many irrelevant features.
- An Instance-Based Learning (IBL) algorithm:
  - Memory-based learning
  - Lazy learning
  - Exemplar-based
  - Case-based