1

$$Score(x) = z(x) = W^T x = w_0 + w_1 x_1 + w_2 x_2$$

(*) Suppose $\boxed{(\nexists)\ W = [w_0, w_1, w_2] \text{ s.t. } W^T x \geq 0 \text{ iff } y(x) = +1}$ (i.e. the examples are not linearly separable w/ features x_1 & x_2).

Add $x_3 = x_1 + x_2$ as a feature, $x' = [x_0 = 1, x_1, x_2, x_3 = x_1 + x_2]$

(**) Assume $(\exists)$ $W' = [w_0', w_1', w_2', w_3']$ s.t. $W^T x' \geq 0$ iff $y = +1$ (i.e. the data is now linearly separable w/ features x_1, x_2, and $\underbrace{x_1 + x_2}_{x_3}$)

this means $\underbrace{w_0' + w_1' x_1 + w_2' x_2 + w_3'(x_1 + x_2)}$ ≥ 0 iff $y = +1$.

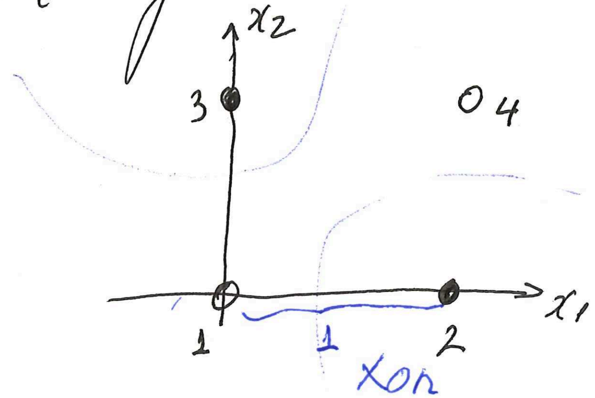
$\Rightarrow$ $w_0' + (w_1' + w_3') x_1 + (w_2' + w_3') x_2 \geq 0$ iff $y = +1$ $\Rightarrow$ ok.

Let
$$\left.\begin{aligned} w_0 &= w_0' \\ w_1 &= w_1' + w_3' \\ w_2 &= w_2' + w_3' \end{aligned}\right\} \Rightarrow w_0 + w_1 x_1 + w_2 x_2 \geq 0 \text{ iff } y(x) = +1.$$

But this is in contradiction with (*) above $\Rightarrow$

$\Rightarrow$ the assumption (**) ~~must be false~~ $\Rightarrow$ we proved by contradiction that adding a 3rd feature $x_3 = x_1 + x_2$ does not make a dataset linearly separable.

(2) The original XOR dataset is $X = \begin{array}{c|ccc} & x_0 & x_1 & x_2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 \end{array}$ $\longrightarrow$ $\begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}$ $N=4$



$y = [-1, +1, +1, -1]$
1 2 3 4

train(X, y): perceptron

converge! 100%.

Add a third feature $x_3 = x_1 x_2$

$x_0 \; x_1 \; x_2$

$X_2 = [1, 1, 0]$ $y_2 = +1$

$X[1]$ $y[1]$

Add a third feature $x_3 = x_1 + x_2$

perceptron update $\underline{W = W + y[1] * X[1]}$

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{array}{c} -1 \\ +1 \\ +1 \\ -1 \end{array}$$

$w_0 \; w_1 \; w_2 \; w_3$

train(X, y) $\longrightarrow$ not converge!

$w_0 + w_1 x_1 + w_2 x_2 + w_3 x_3 \geq 0$ for
≤ 0 for

X

3 Decision boundaries in a 2D feature space:

$X = [x_0 = 1, x_1, x_2]$

$W = [w_0, w_1, w_2]$

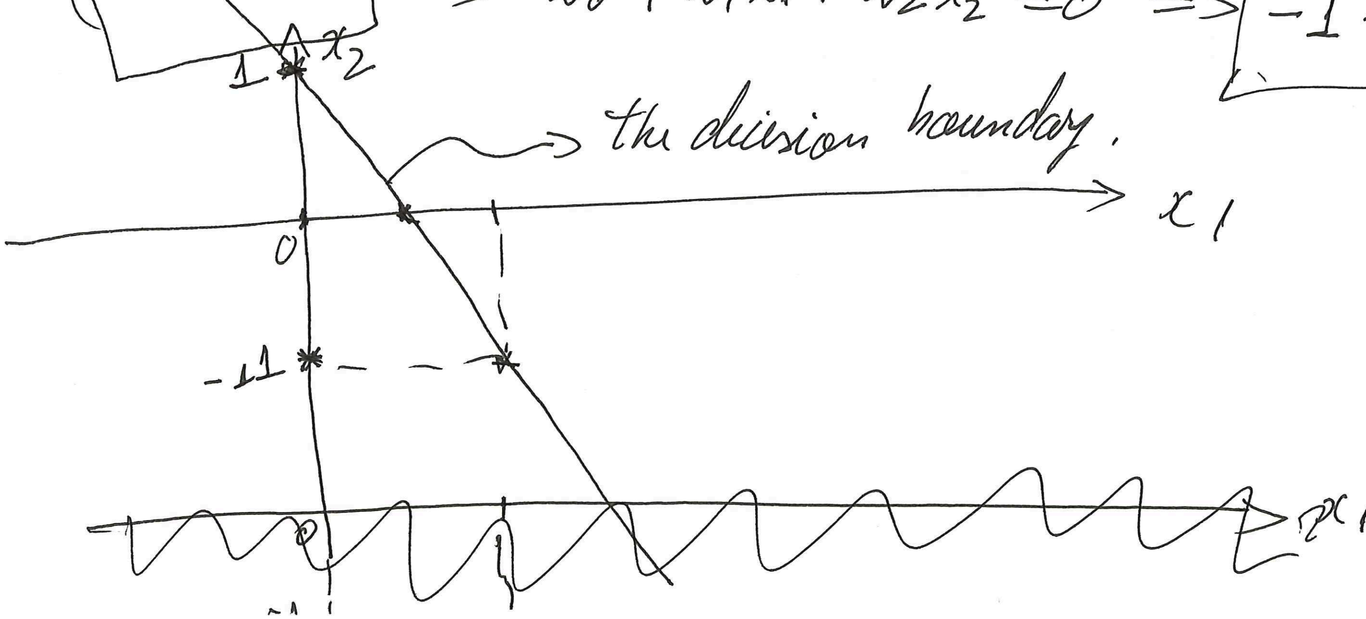
Dec. boundary is the set of points X in the 2D feature space such that $W^T X = 0$.

$\mathcal{D} = \{ X \in \mathbb{R}^2 \text{ s.t. } W^T [1, x_1, x_2] = 0 \}$

$X \to [x_1, x_2]$

For example, we trained a model $\Rightarrow$ $W = [\underbrace{-1, 2, 1}]$ (w_0, w_1, w_2)

$\boxed{W^T X = 0} \Rightarrow w_0 + w_1 x_1 + w_2 x_2 = 0 \Rightarrow \boxed{-1 + 2x_1 + x_2 = 0} \Rightarrow x_2 = -2x_1 + 1$



$X = (x_1, x_2)$

x_1	x_2
1	0
-1	1
	0.5

(0, 1)

(1, -1)

(0.5, 0)

④ X and w are two $k \times 1$ column vectors.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}_{k\times 1} \quad \text{and} \quad w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}_{k\times 1}$$

$$w^T x = \underset{1\times k}{\begin{bmatrix} w_1 & w_2 & \cdots & w_k \end{bmatrix}} \cdot \underset{k\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}} = \underbrace{w_1 x_1 + w_2 x_2 + \cdots + w_k x_k}.$$

$\cancel{x^T w}$

$$x^T w = \underset{1\times k}{\begin{bmatrix} x_1 & x_2 & \cdots & x_k \end{bmatrix}} \underset{k\times 1}{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}} = \underbrace{x_1 w_1 + x_2 w_2 + \cdots x_k w_k}$$

$$=$$

$$\underline{(x w^T)^T} = (w^T)^T \cdot x^T = \underline{w x^T}$$