$K(x, y) = (x^T y)^2$ Q: is K a valid kernel? $\mathbb{R} \times \mathbb{R}$

$x = [x_1, x_2]$

$y = [y_1, y_2]$

By definiton, we need to prove $(\exists)\ \phi : \mathcal{X} \to \mathbb{R}^k$ such that $K(x, y) = \phi(x)^T \phi(y)$, $\forall\ x, y \in \mathcal{X}$

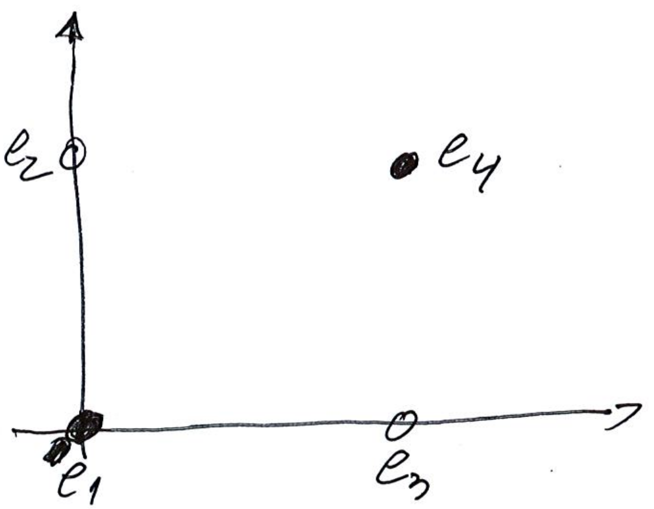
We need to find ϕ s.t. $\phi(x)^T \phi(y) = (x^T y)^2 = (x_1 y_1 + x_2 y_2)^2$

$$= x_1^2 y_1^2 + x_2^2 y_2^2 + 2 x_1 y_1 x_2 y_2$$

$$= x_1^2 y_1^2 + x_2^2 y_2^2 + (\sqrt{2}\, x_1 x_2)(\sqrt{2}\, y_1 y_2)$$

$$= \underbrace{[x_1^2, x_2^2, \sqrt{2}\, x_1 x_2]}_{\phi(x)^T} \underbrace{\begin{bmatrix} y_1^2 \\ y_2^2 \\ \sqrt{2}\, y_1 y_2 \end{bmatrix}}_{\phi(y)}$$

	raw		kernel		
	x_1	x_2	x_1^2	x_2^2	$\sqrt{2} x_1 x_2$
e_1	0	0	0	0	0
e_2	0	1	0	1	0
e_3	1	0	1	0	0
e_4	1	1	1	1	$\sqrt{2}$



This proves (by def) the $K(x, y) = (x^T y)^2$ is a valid kernel.

$$k(x,y) = k_1(x,y) + k_2(x,y), \qquad x,y \in \mathbb{R}^K$$

k_1: valid $\downarrow$ $\qquad$ k_2: valid $\downarrow$

k: valid? $\downarrow$

M_1+M_2

$$(\exists)\ \phi : \mathbb{R}^K \longrightarrow \mathbb{R}^M \text{ s.t. } k(x,y) = \underline{\phi}(x)^T \underline{\phi}(y), \quad \forall\, x,y \in \mathbb{R}^K.$$

k_1 is a valid kernel $\Rightarrow$ $(\exists)\ \psi : \mathbb{R}^K \longrightarrow \mathbb{R}^{M_1}$ s.t.

$$k_1(x,y) = \psi^T(x)\,\psi(y)$$

k_2 is a valid kernel $\Rightarrow$ $(\exists)\ \omega : \mathbb{R}^K \longrightarrow \mathbb{R}^{M_2}$ s.t.

$$k_2(x,y) = \omega^T(x)\,\omega(y).$$

$$k(x,y) = \underline{\psi^T(x)}\,\psi(y) + \underline{\omega^T(x)}\,\omega(y)$$

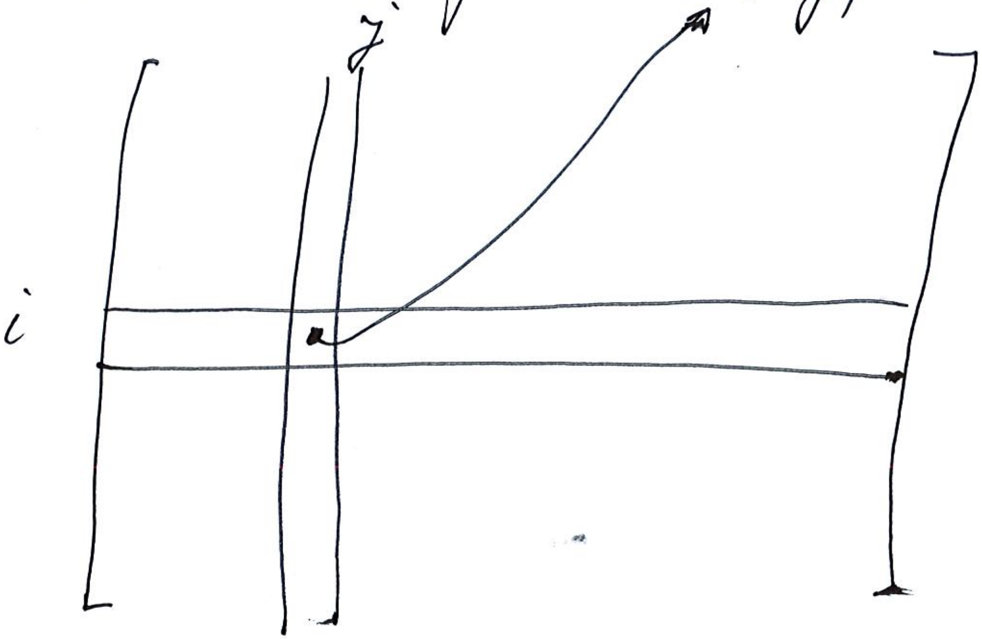
$$= \underbrace{\left[\psi(x)^T \,\vdots\, \omega(x)^T\right]}_{\substack{1\times(M_1+M_2) \\ \phi(x)^T}} \underbrace{\left[\begin{array}{c}\psi(y)\\ \hline \omega(y)\end{array}\right]_{(M_1+M_2)\times 1}}_{\phi(y)} = \phi(x)^T\phi(y)$$

where $\phi(x) = \left[\begin{array}{c}\psi(x)\\ \hline \omega(x)\end{array}\right]$

Dataset $\mathcal{D} = \{x_1, x_2, \dots, x_n, \dots, x_N\}$

Gram (kernel) matrix for $\mathcal{D}$ is an $N \times N$ matrix K

where $K[i,j] = k(x_i, x_j)$



$$
\begin{aligned}
k(x_i, x_j) &= \phi(x_i)^T \phi(x_j) \\
&= \phi(x_j)^T \phi(x_i) \\
&= k(x_j, x_i).
\end{aligned}
$$

K is symmetric matrix $\Rightarrow$ compute only $\frac{N(N+1)}{2}$ values.

if $K \succeq 0$ for any $\mathcal{D}$ $\Leftrightarrow$ k is a valid kernel.

(def: $x^T K x \geq 0$, $(\forall)$ vector $x \in \mathbb{R}^N$)

$x = \text{"ohio"}$ $\qquad n = 2$

$y = \text{"carolina"}$

$$k(x, y) = |\{oi\}| = 1.$$