Longest Common Subsequence

Let's look at another dynamic programming example:

Longest Common Subsequence (LCS):
Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ be a sequence. Then, another sequence $Z = \langle z_1, \ldots, z_k \rangle$ is a *subsequence* of X if there exists a strictly increasing sequence of indices $i_1, \ldots, i_k$ such that

$$z_j = x_{i_j} \text{ for all } 1 \le j \le k.$$

Given two sequences X and Y, a third sequence Z is a *common subsequence* of both X and Y if it is a subsequence of X and a subsequence of Y.

Examples

Consider the sequence $X = \langle A, B, C, B, D, A, B \rangle$. Then, the sequence $Z = \langle B, B, A, B \rangle$ is a subsequence of X.

Similarly, let

$$X = \langle A, B, C, B, D, A, B, C, D \rangle$$

and

$$Y = \langle B, A, C, A, D, B, C, A, A, A \rangle.$$

Then, $Z = \langle A, C, D \rangle$ is a common subsequence of X and Y.

What is the longest common subsequence of X and Y?

Step (i): Optimal Substructure

Let X $= < x_1, x_2, ..., x_m >$ and Y $= < y_1, y_2, ..., y_n >$ be two sequences, and let Z $= < z_1, z_2, ..., z_k >$ be a LCS of X and Y.

Then:

- if $x_m = y_n$, then $z_k = x_m = y_n$ and Z$_{k-1}$ is a LCS of X$_{m-1}$ and Y$_{n-1}$ *(z_k has to be equal to x_m/y_n, otherwise Z won't be a LCS)*.

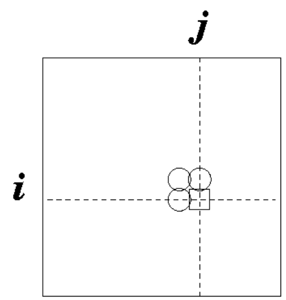
- if $x_m \neq y_n$, then:
  - $z_k \neq x_m \Rightarrow$ Z is an LCS of X$_{m-1}$ and Y.
  - $z_k \neq y_n \Rightarrow$ Z is an LCS of X and Y$_{n-1}$.

Step (ii): A recursive solution

Definition: Let $c[i,j]$ be the length of the longest common subsequence between $X_i = \langle x_1, \ldots, x_i \rangle$ and $Y_j = \langle y_1, \ldots, y_j \rangle$.

Then c[n, m] contains the length of an LCS of X and Y, and:

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max(c[i-1, j], c[i, j-1]) & \text{otherwise.} \end{cases}$$



Step (iii): Bottom-up iterative computation

```
LCS(X,Y,m,n)   /*X has m elements, Y has n elements */
      for i = 1 to m      c[i,0] := 0;
      for j = 1 to n      c[0,j] := 0;
      for i = 1 to m
            for j = 1 to n
              if X[i] == Y[j]
                c[i,j] := c[i-1,j-1]+1;
                b[i,j] := "↖";
              else
                if c[i-1,j] ≥ c[i,j-1]
                  c[i,j] := c[i-1,j];
                  b[i,j] := "↑";
                else
                  c[i,j] := c[i,j-1];
                  b[i,j] := "←";
```

Step (iv): Figuring out the LCS

Use a recursive algorithm: $b[i,j]$ points to the table entry corresponding to the optimal subproblem solution chosen when computing $c[i,j]$.

```
Print-LCS(b, X, i, j)
    if i = 0 return;
    if j = 0 return;

    if b[i, j] = "↖"
        Print-LCS(b, X, i − 1, j − 1);
        print X[i];
    else if b[i, j] = "↑"
          Print-LCS(b, X, i − 1, j);
        else
          Print-LCS(b, X, i, j − 1);
```

An Example

Consider the following example:

$$X = \langle A, B, C, B, D, A, B \rangle$$

and

$$Y = \langle B, D, C, A, B, A \rangle.$$

Let's compute c and b on the board. Then, we'll compute the LCS.