All-Pairs Shortest Paths

Input: A directed graph G = (V, E) where each edge (v_i, v_j) has a weight $w(i, j)$.

Output: A "shortest" path from u to v, for all $u, v \in V$.

Weight of path: Given a path $p =< v_1, ..., v_k >$, its weight is:

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}) \tag{1}$$

"shortest" path = path of minimum weight. We use $\sigma(u, v)$ to denote this minimum weight.

Different variants of shortest path problems

- **Single-pair shortest path (SPSP):**
  Find a shortest path from u to v.

- **Single-source shortest paths (SSSP)**:
  Find a shortest path from source s to all vertices $v \in V$.
  – solved with a Greedy algorithm (Dijkstra's).

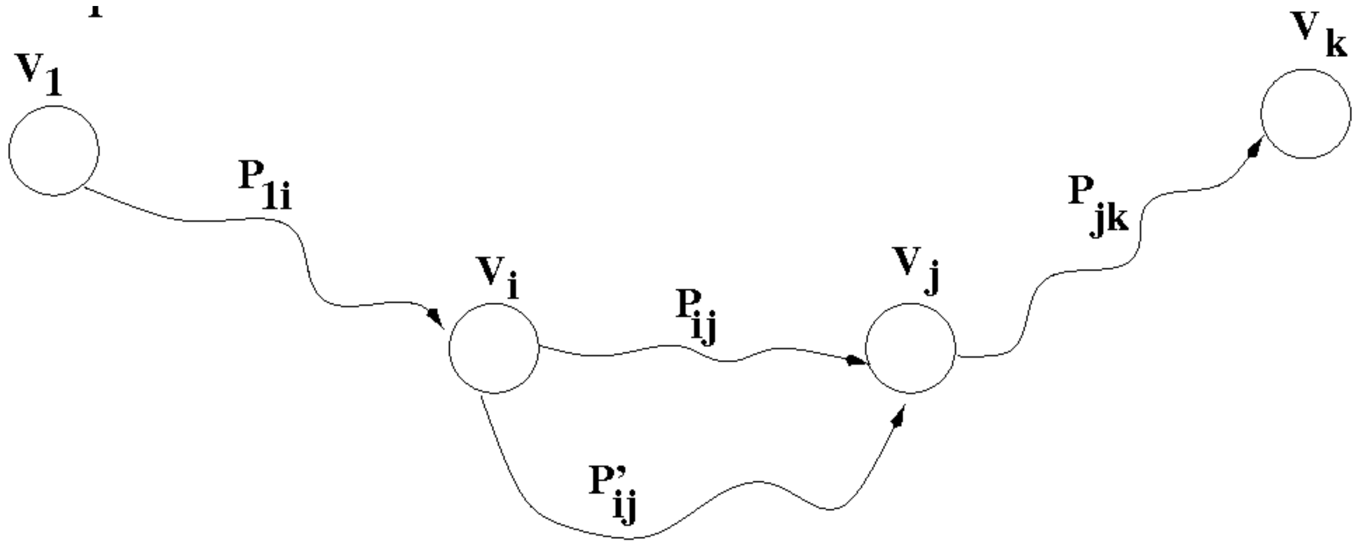
- **All-pairs shortest paths (APSP):**
  Find a shortest path from u to v for all $u, v \in V$.
  – solved with a Dynamic Programming algorithm (Floyd-Warshall).

• Both algorithms need the Optimal Substructure property.

Properties of shortest paths: Optimal Substructure

Lemma 24.1: **Subpaths of shortest paths are shortest paths.**

Proof: Cut and paste:



If some subpath were not a shortest path, we could substitute the shorter subpath and create an even shorter total path.

All-Pairs Shortest Paths (APSP)

- **All-pairs shortest paths (APSP):** Find a shortest path from u to v for all $u, v \in V$.
  - The output has size $O(V^2)$, so we cannot hope to design a better than $O(V^2)$-time algorithm.
  - We can solve the problem simply by running Dijkstra's algorithm $|V|$ times. It takes $O(VElgV)$ time, if the min-priority queue is implemented using a binary heap.

The Floyd-Warshall algorithm: Step (i)

Step (i): *Characterize the structure of the APSP solution.*

- Definition: An **intermediate vertex** of a simple path $p =< v_1, v_2, ..., v_l >$ is any vertex of p other than v_1 and v_l, i.e., any vertex in the set $\{v_2, v_3, ..., v_{l-1}\}$.

- Define $d_{ij}^{(k)}$ to be the weight of a shortest path p from i to j for which all intermediate vertices are in the set $\{1, 2, ..., k\}$ *(similar to second DP approach to the Coin-Changing problem)*.

- Depending on whether or not k is an intermediate vertex on p, we have two cases:

The Floyd-Warshall algorithm: Step (i)

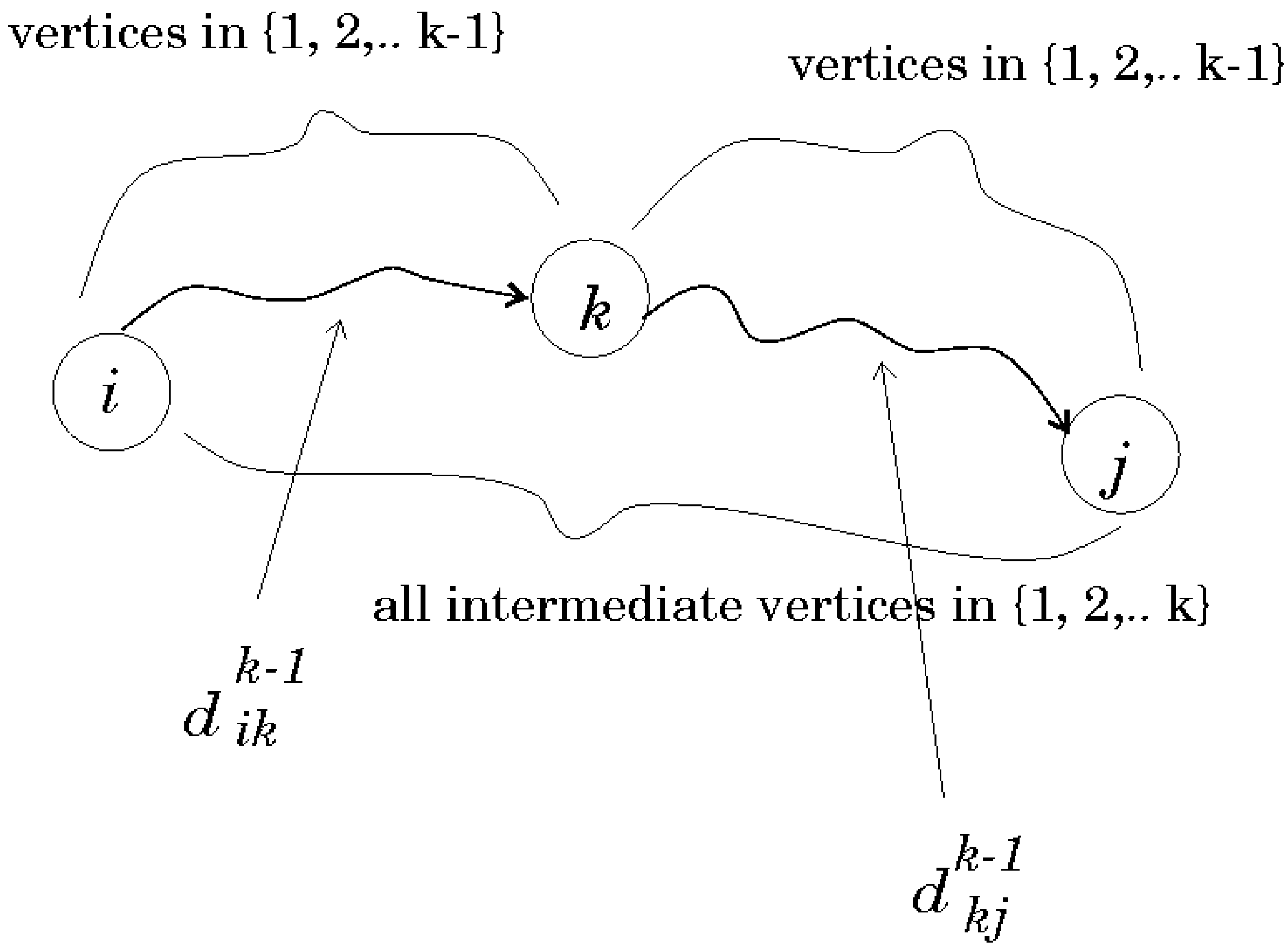
Two cases:

Case (1): If the shortest path p (from i to j going through vertices with indeces $\leq k$) does not go through the vertex k, then:
$d_{ij}^{(k)} = d_{ij}^{(k-1)}$.

Case (2): If the shortest path p goes through vertex k, then:
$d_{ij}^{(k)} = d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$.

Therefore, $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$.

The Floyd-Warshall algorithm: Step (i)



The Floyd-Warshall algorithm: Step (ii)

Step (ii): *Recursively define the value of an optimal solution*.

- **Boundary conditions:** for $k = 0$, a path from vertex i to j with no intermediate vertex numbered higher than 0 has no intermediate vertices at all, hance $d_{ij}^{(0)} = w_{ij}$.
- **Recursive formulation:**

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}) & \text{if } k \geq 1 \end{cases}$$

$D^{(n)} = (\ d_{ij}^{(n)})$ is the solution for this APSP problem:

$d_{ij}^{(n)} = \sigma(i, j)$.

The Floyd-Warshall algorithm: Step (iii)

Step (iii): *Compute the shortest-path weights bottom up.*

```
FLOYD-WARSHALL(W, n)
{
    D^(0) = W;

    for k := 1 to n
        for i := 1 to n
            for j := 1 to n
                d_ij^(k) := min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))

    return D^(n);
}
```

Complexity: $\Theta(n^3)$.

The Floyd-Warshall algorithm: Step (iv)

Step (iv): *Constructing the shortest paths.*

Need to compute the **predecessor matrix** Π, in which π_{ij} is the predecessor of vertex j on a shortest path from vertex i.

- Compute predecessor matrix Π from the weights matrix D (Exercise 25.1-6).
- Compute Π online, at the same time with D:
  - Compute a sequence $\Pi^{(0)}, \Pi^{(1)}, ..., \Pi^{(n)}$, where $\pi_{ij}^{(k)}$ is defined as the predecessor of vertex j on a shortest path from vertex i with all intermediate vertices in $\{1, 2, ..., k\}$.
  - $\Pi = \Pi^{(n)}$.

The Floyd-Warshall algorithm: Step (iv)

Recursive formulation of $\pi_{ij}^{(k)}$:

- When $k = 0$, a shortest path from i to j has no intermediate vertices at all. Hence:

$$\pi_{ij}^{(k)} = \begin{cases} NIL & \text{if } i = j \text{ or } w_{ij} = \infty \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty \end{cases}$$

- When $k \geq 1$:
  - If we take the path $i \rightsquigarrow k \rightsquigarrow j$, then $\pi_{ij}^{(k)}$ is the same as the predecessor of j on the shortest path from k with intermediate vertices in $1, 2, ..., k-1$.

$$\pi_{ij}^{(k)} = \pi_{kj}^{(k-1)} \quad \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$$

The Floyd-Warshall algorithm: Step (iv)

- When $k \geq 1$:
  - Otherwise, $\pi_{ij}^{(k)}$ is the same as the predecessor of j on the shortest path from i with intermediate vertices in $1, 2, ..., k-1$.

$$\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)} \quad \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$$

  - Putting these two cases together:

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \end{cases}$$

The Floyd-Warshall algorithm: Step (iv)

FLOYD-WARSHALL(W, n)

$D^{(0)} =$ W;

INIT-PREDECESSORS($\Pi^{(0)}$)

for $k := 1$ to n

for $i := 1$ to n

for $j := 1$ to n

if $d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$) then

$d_{ij}^{(k)} := d_{ij}^{(k-1)}$

$\pi_{ij}^{(k)} := \pi_{ij}^{(k-1)}$

else

$d_{ij}^{(k)} := d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

$\pi_{ij}^{(k)} := \pi_{kj}^{(k-1)}$

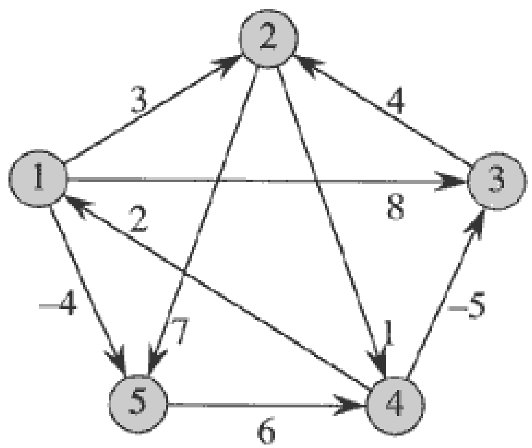
return $D^{(n)}$;

Printing Shortest Paths with Π

The predecessor matrix is $\Pi = \Pi^{(n)}$. The following recursive procedure prints the shortest path between vertices i and j, using Π:

PRINT-ALL-PAIRS-SHORTEST-PATHS(Π, i, j)
    if $i = j$ then
        print i
    else
        if $\pi_{ij} = NIL$ then
            print "no path from" i " to " j
        else
            PRINT-ALL-PAIRS-SHORTEST-PATHS(Π, i, π_{ij})
            print j

APSP: Example

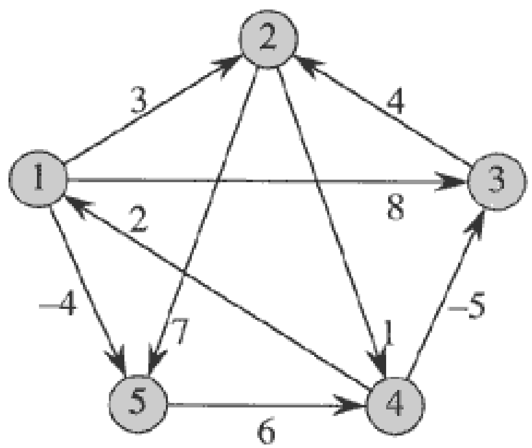


$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(0)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

APSP: Example



$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$