# On the Representation of Contact States Between Curved Objects

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***Abstract*— Information of high-level, topological contact states is useful and even necessary for a wide range of applications, including many robotic applications. A contact state between two polyhedral objects can be effectively represented as a contact formation in terms of a set of principal contacts between faces, edges, and vertices of the two objects. However, little is done to characterize and represent contact states between curved objects. In order to facilitate the representation of contact states between such objects, we introduce a novel approach to segment the boundary of curved objects based on monotonic changes of curvatures, which we call the curvature monotonic segmentation. We specifically apply this approach to curved 2D and 3D objects with boundary curves or surfaces represented by algebraic polynomials of degrees up to 2. The segmentation yields curvature monotonic faces and edges (or pseudo edges), and vertices (or pseudo vertices). With these faces, (pseudo) edges, and (pseudo) vertices, we effectively extend the concept of contact formation to curved objects to represent high-level, topological contact states between such objects with the same desirable characteristics as the contact formations between polyhedral objects.**

## I. Introduction

Proper characterization and representation of contacts between physical objects, including robots, is essential to many applications, from real-world robotic tasks involving compliant motion to dynamic simulation and haptic interaction in a virtual world. While contacts between two objects can be described by the relative contact configurations between them, a higher-level representation of contacts in terms of certain discrete “contact states” is often more descriptive of the common topological and physical contact characteristics shared by a set of contact configurations and is thus quite useful and even necessary for many tasks.

For contacting polyhedral objects, it is rather natural and common to describe a contact state as a set of primitive contacts, each of which is defined by a pair of contacting surface features in terms of faces, edges, and vertices. Different contact state representations essentially differ only in how primitive contacts are defined. One common representation [1], [2] defines primitive contacts as point contacts in terms of vertex-edge contacts for 2D polygons, and vertex-face and edge-edge contacts for 3D polyhedra. Another representation [3] defines primitive contacts in terms of any pair of surface features in contact. The highest-level definition of primitive contacts was introduced by [4] in the notion of principal contacts, which best enables the distinction between one contact state and another and facilitates robust identification of contact states. Each PC is associated with a single tangent plane of contact, thanks to the fact that every surface feature of a polyhedral object, i.e., every face, edge, or vertex, is a well-defined, convex feature.

However, how to describe and represent a high-level contact state properly between non-polyhedral, curved objects, i.e., objects whose boundary is composed by non-planar piecewise smooth surfaces, remains an open problem even though such objects are more common in the real world. We seek to address this problem in this paper. Our approach is to decompose a curved object in such a way that it yields meaningful surface features with desirable properties analogous to the flat surface features of a polyhedral object. Next, principal contacts between two curved objects can be defined in terms of those surface features so that a contact state can be described as a set of principal contacts.

In this paper we consider three dimensional objects with boundaries represented by algebraic polynomials of degrees up to 2, i.e., by planes and quadrics, and two dimensional objects with boundaries described by algebraic polynomials of degrees at most 2, i.e., lines and conics. Quadrics and conics just like planes and lines are common boundary primitives, especially for primitive components in a hierarchical representation of complex objects, such as in a constructive solid geometry (CSG) tree [5], [6], [7], [8], [9]. For example, mechanical parts and tools often use cylinders and rectangular parts as primitives, ellipsoids are often used in human body modelling, spheres are extremely popular in molecular modelling, and so on.

A previous approach to object decomposition is [10] in which the problem of representing 3D free form objects for object recognition is addressed, and shape-based description of objects is introduced. It is based on the concept of maximal surface patches having similar shape index. They define different classes for surfaces dividing the range of the shape index into nine levels. [11] proposes a natural decomposition of 3D surfaces into a graph very similar to the polyhedral representation of piecewise linear surfaces which is defined even for objects whose topology is arbitrarily complicated. This representation is called the extremal mesh of the surface. However, none of these approaches of surface decomposition can serve our purpose of representing contact states between two curved objects readily.

In this paper, we present our approach of decomposing or segmenting the surface of a curved object based on monotonic change of curvatures. The rest of the paper is organized as follows. In Section II we will review the notions of principal contacts and contact formation for polyhedral objects and describe how to extend this formalism to curved objects. In Sections III, IV, and V, we introduce our approach to decompose 2D curved objects and to extend the notions of PC and contact formation to describe contact states between such objects. In Sections VI, VII, and VIII, we extend our approach to 3D curved objects and surfaces. Section IX concludes the paper.

## II. Principal Contacts and Contact Formations for Curved Objects

### A. Principal contacts and contact formations between polyhedral objects

The notion *principal contacts* (PC) has been introduced in [4] to characterize the contact states between two arbitrary polyhedral objects. PCs are contact primitives defined in terms of contacting boundary elements of objects. Boundary elements are faces, edges, and vertices. The boundary of a face consists of the edges and vertices bounding it, and the boundary of an edge consists of the vertices bounding it.

Formally, a PC denotes a contact between a pair of boundary elements that do not bound other contacting boundary elements. This ensures that PCs are the highest level contact primitives to describe a contact state most concisely. For example, a face-face contact between two polyhedral objects is described just as a single face-face PC rather than in terms of a set of vertex-face, edge-face, or edge-edge contacts. Each PC defines a single contact region of a point, a straight-line segment, or a plane segment, associated with a single contact tangent plane, called a *contact plane*. There are four types of PCs between two arbitrary polygons and ten PCs between two arbitrary polyhedra, as shown in Fig. 1. An arbitrary contact state between two polyhedral solids can be described in terms of a set of PCs, called a *contact formation* (CF) [4].

Although PCs were first introduced for polyhedral objects, they can be extended to non-polyhedral objects after a suitable decomposition of the boundaries of such objects are performed, as explained in the next section.

### B. Principal contacts and contact formations between curved objects

Our approach is to decompose a curved object in such a way that it yields useful curved surface features with desirable properties for the description of a contact analogous to those of the flat boundary elements of a polyhedral object.

In this paper we consider three dimensional objects defined by algebraic surfaces of degree at most 2, that is planes and quadrics, and two dimensional objects defined by algebraic curves of degree at most 2, that is lines and conics.

We observe that by decomposing an arbitrary quadric surface (or an arbitrary conic curve in 2D) into segments having monotonic curvature (i.e., surface patches in 3D or curve segments in 2D with monotonic curvature), the resulting segments

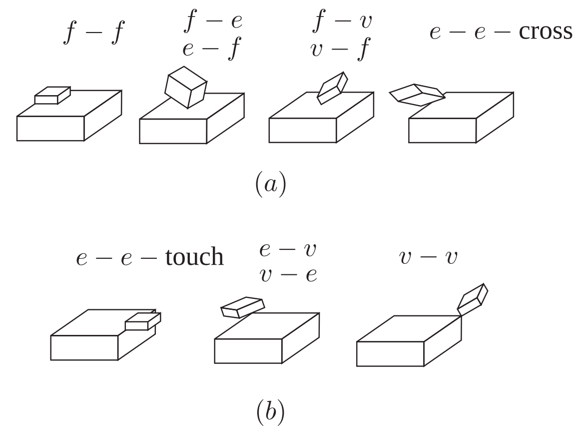


Fig. 1. (**a**) Principal contacts. (**b**) Degenerate principal contacts.

have the interesting property that when two of them are in contact at an interior point, the contact region consists of only a single contact point, independently of the type of contact, i.e., no matter if the contact is between two convex segments or between a convex and a concave segment. As a consequence the contact between two such curvature monotonic segments can be described in a way analogous to a single PC between two flat surface or edge elements of polyhedral objects, which can be further used as the building blocks to describe arbitrary contact states between curved objects.

Thus, we decompose 3D curved objects into smooth surface patches with monotonic curvature. This decomposition, which we call *curvature monotonic segmentation*, is done by analyzing the differential geometric properties of the surfaces and introducing additional features such as special curves and points on them where at least one of the curvatures has a local maximum or a minimum. As the result, the notion of PCs for polyhedral objects can be extended here for curved objects in terms of relationships between curvature monotonic patches or curve segments. Finally, the notion of CFs can again be defined as a set of PCs to describe a contact state between two curved objects.

We first describe how PCs and CFs can be defined for 2D curved objects and then extend our method to 3D curved objects.

## III. General properties of planar algebraic curves

We assume that the reader is familiar with the basic concepts related to planar algebraic curves such as smoothness, regularity, singularity, as well as with the notion of local convexity and local concavity [12, Chap. 1]. Two notions of particular interest here are inflection points and extreme points. Inflection points are stationary points of the curvature, i.e., points at which the first derivative of the curvature $\kappa$ vanishes and the sign of curvature changes. If curvature $\kappa$ of $C$ has local maximum or minimum value at $\mathbf{p}$, $\mathbf{p}$ is an *extreme point*.

Such information is important for understanding how to subdivide the curves that define the boundaries of planar

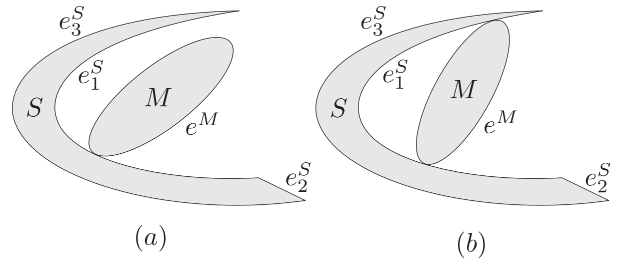


Fig. 2. Different contacts between the same pair of boundary elements of two planar curved objects.

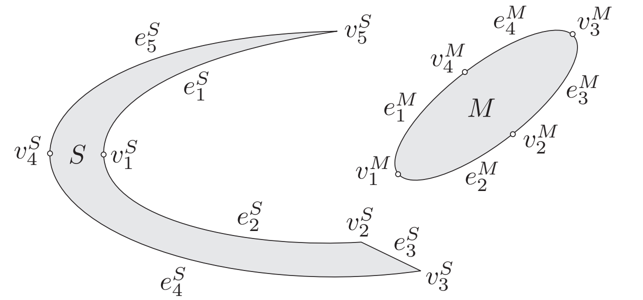


Fig. 3. Stationary points of the curvature function along the boundary of the two objects of Fig. 2 used for their curvature monotonic decomposition. For example, the curvature of the boundary of $M$ has a maximum in $v_1^M$ and a minimum in $v_2^M$.

objects into segments along which the curvature function is monotonic. As will be described below, monotonic segmentation of general curves requires adding inflection points and extreme points as extra vertices.

## IV. Curvature Monotonic Segmentation of Planar Curved Objects

Consider the situation represented in Fig. 2. A contact between the boundary curve $e_1^S$ of object $S$ and the curve $e^M$ of object $M$ can take place at one point or at two different points. These two different contact situations have different geometrical and physical properties, e.g., $M$ is more constrained in the situation of Fig. 2.b than in that of Fig. 2.a. Unfortunately both contacts take place between the same boundary elements of $M$ and $S$ so that we cannot distinguish these contact cases by means of a high level topological description. Therefore, we need to perform a finer segmentation of the boundaries of curved objects and use inflection points and extreme points in addition to vertices for segmentation.

For planar curved objects, we define edges and vertices as follows.

*Definition 4.1 (Vertices): Vertices* are the intersection points between two curves that form the boundary of a planar curved object.

*Definition 4.2 (Pseudo-vertices): Pseudo-vertices* are inflection points and extreme points on a planar curve.

*Definition 4.3 (Edges): Edges* are the curve between any two neighboring vertices or pseudo-vertices.

In other words, we use all the stationary points of the curvature function along the curve, i.e., points at which $\kappa' = 0$. Segments of curve along which the curvature is constant are considered as one single edge. It is easy to see that after the segmentation, an edge either has constant curvature or strictly monotonically increasing or decreasing curvature. For example, the objects represented in Fig. 3 have boundaries defined by conic curves. The vertices of $S$ are $v_2^S$, $v_3^S$ and $v_5^S$, whereas $M$ has no vertices. They don't have inflection points. The extreme points of $S$ as pseudo-vertices are $v_1^S$, $v_4^S$, and those of $M$ are $v_1^M$, $v_2^M$, $v_3^M$ and $v_4^M$.

It is well known that any smooth closed curve other than a circle has at least 4 pseudo-vertices. A circle, which has a constant curvature, is considered as one single edge without any pseudo-vertex.

## V. Contact formations between planar curved objects

In a way analogous to the description of PCs between polygons on a plane, we give the following definition of PCs between planar curved objects. First, we extend the notion of boundary elements to planar curved objects as the edges and (pseudo) vertices of the objects.

*Definition 5.1 (Principal contacts):* A PC between two planar curved objects is defined by the contact between a pair of boundary elements (edges, vertices, pseudo-vertices) and corresponds to a single contact region at which the tangent line is uniquely defined.

A general contact between two planar curved objects can then be described again by a *contact formation* defined as the set of PCs formed.

Since in a planar curved object there are pseudo-vertices and both convex and concave edges, there are more types of PCs between planar curved objects than between polygons (where there are only vertex-vertex, vertex-edge and edge-edge types of PCs). Thus, we can classify the types of PCs between planar curved objects considering all the possible combinations of boundary elements. For instance, after the introduction of the pseudo-vertices, the contact in Fig. 2.b can be regarded as a 2-PC CF $\{e_1^S - e_4^M, e_2^S - e_1^M\}$.

Now we need to show that at each PC, just as in the case of polygons, there is a single contact region, and the contact tangent line is uniquely defined.

If a contact takes place between an edge and vertex (Fig. 4.1) or pseudo-vertex, we always have a PC with a single contact point and the tangent line is uniquely defined. If a contact takes place between one convex (pseudo) vertex and one concave (pseudo) vertex (Fig. 4.2), it can be regarded as a CF with two (pseudo) vertex-edge PCs in which the convex (pseudo) vertex is in contact with two edges one on each side of the concave (pseudo) vertex. The two tangent lines at the PCs are uniquely defined.

In Fig. 4.3.a a PC between two convex edges is shown. It is easy to see that in this case the contact region can only be at a single internal point with a single tangent line.

Now consider one convex edge and one concave edge in contact. Depending on the relative dimensions of the two edges, several cases are possible.

In the case of Fig. 4.3.b. Two vertex-edge PCs are formed, and the contact state can be described by a CF formed by the two PCs. Each PC consists of a single contact point along with a unique contact tangent line.

If a convex edge contacts a concave edge, there can be two cases depending on whether the curvatures of the two edges are monotonically increasing in the same direction as shown in 4.3.c or in opposite directions as shown in 4.3.d.

The following theorem ensures that in both cases a contact between the interiors of the two edges defines a PC.

*Theorem 5.1:* Consider two planar curved objects $M$ and $S$ whose boundaries are segmented by curvature monotonic segmentation. Between the interiors of a concave edge of $S$ and a convex edge of $M$, there can only be a single contact point with a unique tangent line.

*Proof:* Let $e^S$ and $e^M$ be the two edges in contact and let $p$ denote a contact point between the interiors of $e^S$ and $e^M$. If both $e^S$ and $e^M$ are of constant curvatures (i.e., circles of different radii), then $p$ is the only contact point between them. Otherwise, let $e_1^S$ and $e_1^M$ be the segments of $e^S$ and $e^M$ on one side of $p$. If both $e_1^S$ and $e_1^M$ have strictly monotonically changing curvature and if the curvature of $e_1^M$ decreases more rapidly or increases more slowly than the curvature of $e_1^S$ does, in addition to the point $p$ they can meet at another point $q$. Due to the strictly monotonic nature of the curvatures, $q$ can only be a boundary vertex of either $e_1^S$ and $e_1^M$, or the two segments interpenetrate, which cannot happen at a valid contact state. Therefore, $p$ is the only contact point between the interiors of the edges. A similar proof can be conducted for the situation where one of the edges has a constant curvature and the other has a strictly monotonic curvature. ■

It is easy to see that between two concave edges of Fig. 4.3.e, two vertex-concave edge PCs always form, and they give rise to a CF containing two PCs.

A PC can also occur between two (pseudo) vertices (Fig. 4.4).

For an implementation of monotonic segmentation of planar curves based on resultants and root isolation see [13].

In the following sections the representation of CFs between 3D curved objects will be described.

## VI. General Properties of Surfaces

In this section the general properties of a surface relevant to its curvature monotonic segmentation are described. More details can be found in [14]. Other comprehensive references about properties of surfaces are [15], [12], [16], [17].

If $p$ is a point on a smooth surface $M$ it is convenient to study the local properties of $M$ at $p$ using a reference frame whose origin is $p$ and the plane $z = 0$ coincides with the tangent plane to $M$ at $p$. $M$ has a local equation of the form $z = \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2 + \cdots$. Rotating around the $z$ axis it is possible to eliminate the coefficient $b$ in this equation which takes the form $z = \frac{a}{2}x^2 + \frac{c}{2}y^2 + \cdots$. Now the coefficients $a$ and $c$ are the *principal curvatures* of $M$ at $p$, and the directions of the axes $x$ and $y$ are the *principal directions*.

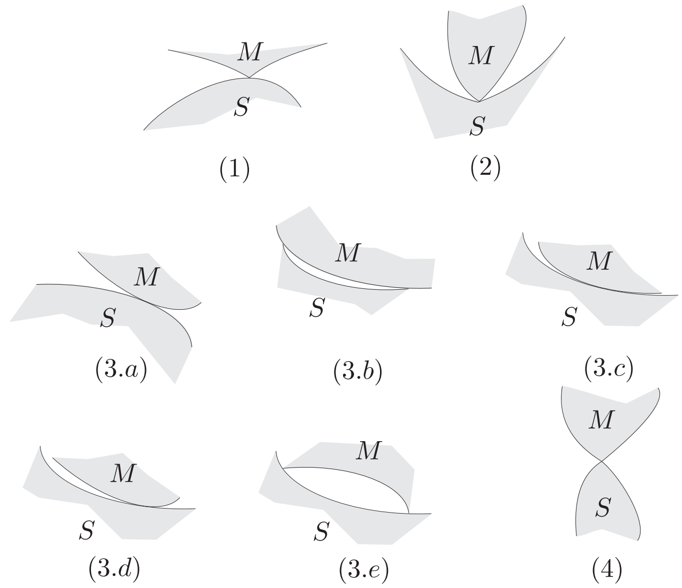


Fig. 4. Different contact formations between two planar curved objects.

### A. Surfaces in Monge form

In *Monge form*, which can be regarded as a special kind of parametrization, a surface has the following equation

$$\begin{aligned} z =& f(x,y) = \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 y^2 + \\ & \frac{1}{6}(b_0x^3 + 3b_1x^2y + 3b_2xy^2 + b_3y^3) + \\ & \frac{1}{24}(c_0x^4 + 4c_1x^3y + 6c_2x^2y^2 + 4c_3xy^3 + c_4y^4) + \\ & \cdots \end{aligned} \tag{1}$$

in which $\kappa_1$ and $\kappa_2$ are the principal curvatures and the axes $x$ and $y$ are the principal directions, as above. As long as $\kappa_1 \neq \kappa_2$ these directions are well defined. We assume that $\kappa_1 > \kappa_2$. Since the signs of the curvatures depend on the direction of the normal. We assume that $z > \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 y^2 + \cdots$ is inside the object. In this way $\kappa_1, \kappa_2 > 0$ means that $M$ is locally *convex* near $p$, and if $\kappa_1, \kappa_2 < 0$ $M$ is locally *concave*.

The *Gaussian* and the *mean* curvatures of $M$ are defined as $K = \kappa_1\kappa_2$ and $H = \frac{\kappa_1+\kappa_2}{2}$, respectively. If the $K > 0$ at $p$, it is said to be *elliptic*, whereas if $K < 0$, $p$ is said to be *hyperbolic*. If $K = 0$ at $p$, it is called *parabolic*.

The point $p$ is called an *umbilic* when $\kappa_1 = \kappa_2$. Unless $\kappa_1 = \kappa_2 = 0$ the Gaussian curvature $K$ is necessarily positive at an umbilic. This means that non-flat umbilics only occur in elliptic regions of a surface. The points at which $\kappa_1 = \kappa_2 = 0$ are called *planar points* of $M$.

So, if $\kappa_1 \geq \kappa_2 > 0$, the point $p$ is a *elliptic convex point* of $M$, if $\kappa_1 > 0 > \kappa_2$ it is an *hyperbolic point*, and if $0 > \kappa_1 \geq \kappa_2$, it is an *elliptic concave point*.

When $K = 0$ and $\kappa_1 > \kappa_2 = 0$, $p$ is said to be a *red parabolic point*, whereas if $0 = \kappa_1 > \kappa_2$, it is a *blue parabolic point* of $M$. The terms blue and red are used in the literature to indicate something special happening with the curvatures $\kappa_1$ and $\kappa_2$, respectively.

### B. Parabolic curves on a surface

For any surface $z = f(x, y)$, not necessarily described in Monge form, the set of parabolic points is given by $f_{xx}f_{yy} - f_{xy}^2 = 0$. For a generic surface they are smooth curves called *parabolic curves* which breaks up into disjoint red parabolic curves and blue parabolic curves.

A *blue parabolic curve* is a curve along which $\kappa_1 = 0$ and a *red parabolic curve* is a curve along which $\kappa_2 = 0$.

Red parabolic curves separate *elliptic convex regions* and *hyperbolic regions* of $M$, i.e., regions for which $\kappa_1 > \kappa_2 > 0$ and regions for which $\kappa_1 > 0 > \kappa_2$. Blue parabolic curves separate hyperbolic regions and *elliptic concave regions*, i.e., regions for which $\kappa_1 0 > \kappa_2$ and regions for which $0 > \kappa_1 > \kappa_2$.

Parabolic curves are of interest for curvature monotonic decomposition of surfaces because they are the analogous for surfaces of the inflection points for planar curves. Indeed, they are also called *curves of inflection* of a surface.

*Special Points on Parabolic Curves:* Although they are smooth curves, parabolic curves have special points. At a red parabolic point $p$ since $\kappa_1 > \kappa_2 = 0$, if $b_3 \neq 0$, the Monge form becomes

$$f(x,y) = \frac{\kappa_1}{2}(x + \frac{b_0}{6\kappa_1}x^2 + \frac{b_1}{2\kappa_1}xy + \frac{b_2}{2\kappa_1}y^2 + \cdots)^2 + \frac{b_3}{6}(y + \cdots)^3$$

and has an *ordinary cusp*. If $b_3 = 0$, it becomes

$$f(x,y) = \frac{\kappa_1}{2}(x + \frac{b_0}{6\kappa_1}x^2 + \frac{b_1}{2\kappa_1}xy + \frac{b_2}{2\kappa_1}y^2 + \cdots)^2 + \frac{1}{24\kappa_1}(\kappa_1 c_4 - 3b_2^2)y^4 + \cdots$$

As long as $b_2^2 \neq 3\kappa_1 c_4$, there are points special points called *cusps of Gauss* which can be divided into 4 groups. More specifically, if $\kappa_2 = b_3 = 0, \kappa_1 > 0, \kappa_1 c_4 - 3b_2^2 > 0$, $p$ is said to be a *red elliptic* cusp of Gauss, if $\kappa_2 = b_3 = 0, \kappa_1 > 0, \kappa_1 c_4 - 3b_2^2 < 0$, $p$ is said to be a *red hyperbolic* cusp of Gauss, if $\kappa_1 = b_0 = 0, \kappa_2 < 0, \kappa_2 c_0 - 3b_1^2 > 0$, $p$ is said to be a *blue elliptic* cusp of Gauss, and if $\kappa_1 = b_0 = 0, \kappa_2 < 0, \kappa_2 c_0 - 3b_1^2 < 0$, $p$ is said to be a *blue hyperbolic* cusp of Gauss. This terminology comes from the fact that red and blue cusps of Gauss lie on red and blue parabolic curves, respectively.

The importance of cusps of Gauss for the curvature monotonic segmentation of surfaces will be described in the next section.

### C. Ridges on a surface

Ridge points can be characterized by means of contact with spheres. For this characterization one uses spheres centered at $(0, 0, r)$ and passing through the point $p$, hence tangent to the surface $M$ at $p$.

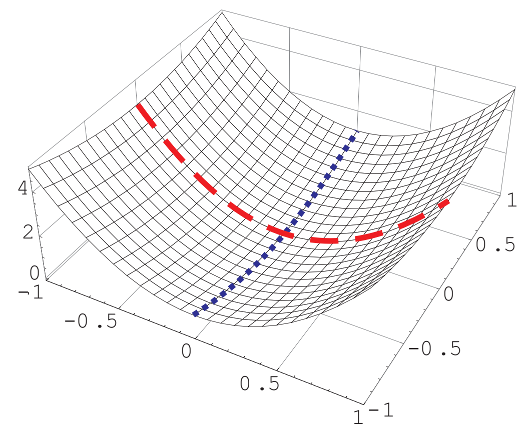


Fig. 5. The dotted curve on this surface is an elliptic blue ridge along which the largest principal curvature has a maximum. The dashed curve on this surface is a hyperbolic red ridge along which the smaller principal curvature has a maximum.

The curve of intersection between $M$ and the sphere has the following expression

$$\begin{aligned} g(x,y) = & x^2(1 - r\kappa_1) + y^2(1 - r\kappa_2) \\ & - \frac{r}{3}(b_0x^3 + 3b_1x^2y + 3b_2xy^2 + b_3y^3) \\ & - \frac{r}{12}(c_0x^4 + \cdots) + (\frac{\kappa_1}{2}x^2 + \frac{\kappa_2}{2}y^2)^2 \\ & + \cdots \end{aligned} \tag{2}$$

Ridge points of a smooth surface are points at which one of the spheres of curvature has a more degenerate contact with the surface than the usual contact [14, Sect. 6.4]. A ridge point is *elliptic* if locally the intersection between the surface and its sphere of curvature is an isolated point and *hyperbolic* otherwise.

If $b_0 = 0$ and $3b_1^2 + (\kappa_1 - \kappa_2)(c_0 - 3\kappa_1^3) < 0$, $p$ is said to be a *blue elliptic ridge point*, if $b_0 = 0$ and $3b_1^2 + (\kappa_1 - \kappa_2)(c_0 - 3\kappa_1^3) > 0$, $p$ is said to be a *blue hyperbolic ridge point*, if $b_0 = 0$ and $3b_2^2 + (\kappa_2 - \kappa_1)(c_4 - 3\kappa_2^3) < 0$, $p$ is said to be a *red elliptic ridge point*, and if $b_0 = 0$ and $3b_1^2 + (\kappa_2 - \kappa_1)(c_4 - 3\kappa_2^3) > 0$, $p$ is said to be a *red hyperbolic ridge point*.

Since the term blue and red always refers to the larger and the smaller principal curvatures respectively, the conditions for $p$ to be a blue ridge are obtained from (2), assuming $\kappa_1 \neq \kappa_2$, setting $r = \frac{1}{\kappa_1}$. Similarly the conditions for $p$ to be a red ridge are obtained by setting $r = \frac{1}{\kappa_2}$. Ridge points form curves on a surface are called *curves of ridge points* or simply *ridges*.

Ridges are of interest to the curvature monotonic segmentation of surfaces because at a blue elliptic ridge point, $\kappa_1$ has a maximum along the blue line of curvature. Similarly, at a blue hyperbolic ridge point, $\kappa_1$ has a minimum along its line of curvature. At a red elliptic ridge point, $\kappa_2$ has a minimum and at a red hyperbolic ridge point, $\kappa_2$ has a maximum in each case along the corresponding line of curvature.

It is easy to see that ridges are for surfaces the analogous of the points of a planar curve at which the curvature has a maximum or a minimum and together with inflection points, form the set of pseudo-vertices of a planar curve.

Ridges of opposite colors in general may cross each other transversally. Their points of intersection are called *purple points* of the surface. It is easy to see that they do not have a counterpart on planar curves.

Cusps of Gauss are points of interest for the curvature monotonic segmentation of surfaces because they are the points at which parabolic curves cross ridges of the same color. A cusp of Gauss is elliptic if and only if the ridge is elliptic.

Although ridge points have been defined as those points on a surface in which the corresponding sphere of curvature has a degenerate contact with the surface, in practice ridge curves on surfaces can be computed by determining the extrema of the principal curvatures [11].

### D. Umbilics on a surfaces

We said before that the principal curvatures are equal at an umbilic point. An umbilic is called *elliptic* if $\kappa_1$ has a local minimum and $\kappa_2$ has a local maximum. An umbilic is called *hyperbolic* if there are curves through the umbilic where $\kappa_1 = k$ and $\kappa_2 = k$. At an elliptic umbilic, there are three ridge curves passing through it. All of them are hyperbolic ridges. Furthermore, the color of any ridge curve through the umbilic changes at the umbilic from a blue minimum ($\kappa_1$ has a local minimum along the corresponding lines of curvature) to a red maximum ($\kappa_2$ has a local maximum along the corresponding lines of curvature). Other types of umbilics are the so-called *hyperbolic stars* and *hyperbolic lemons*. In both cases there is one hyperbolic ridge curve passing through the umbilic which changes color at the umbilic, but the principal curvatures do not have local extrema at hyperbolic star and hyperbolic lemon umbilics. Since only at elliptic umbilics the principal curvatures have extrema, only this kind of umbilics will be taken into account for the monotonic segmentation of 3D objects.

## VII. Curvature Monotonic Segmentation of 3D Curved Objects

It has been shown in Section IV that the boundary of a planar curved object whose boundary is described by conic curves can be decomposed into curvature monotonic pieces using stationary points of the curvature. The goal of this section is to generalize this decomposition to 3D curved objects.

First, we define the boundary elements of a 3D curved object.

*Definition 7.1 (Vertices):* The intersection points of three or more surface patches that bound a 3D curved object are called *vertices*.

*Definition 7.2 (Edges):* The intersection curves between two surface patches that define the boundary of a 3D curved object are called *edges*.

The boundary of a 3D curved object is decomposed into a set of smooth surfaces bounded by edges and vertices, based

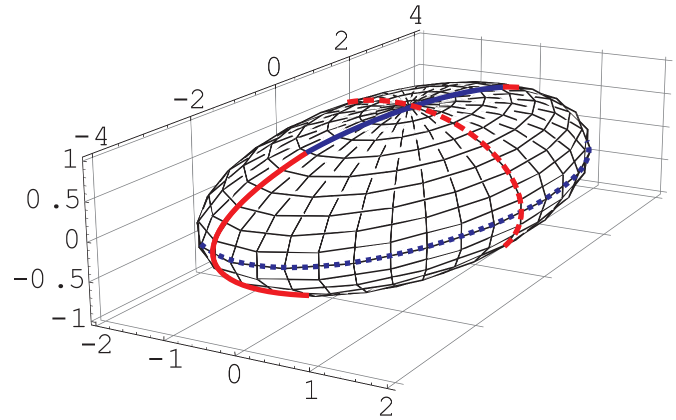


Fig. 6. Curvature monotonic segmentation of an ellipsoid. The ellipsoid has three ridges the major and minor sections of symmetry (dotted and dashed curves, respectively) and the intermediate section of symmetry (continuous curve) which contains four umbilical points. The ridges through the umbilics change color at each one.

on that the curvature monotonic decomposition can be further carried out on the surface patches and edges.

The points of a generic surface can be subdivided into three open sets: the elliptic convex points, hyperbolic points and elliptic concave points, and there are two types of parabolic curves, the red and the blue, that separate them.

Moreover, there are four types of ridges on them, red and blue, elliptic and hyperbolic, and ridges also have special points such as cusps of Gauss and umbilic points. These elements of a surface coincide with extrema of at least one of the principal curvatures of the surface and therefore will be used to create a partition of a surface into regions in which both the principal curvatures are monotonic. Note that the described decomposition does not apply to edges with non-zero torsion (i.e., space curves). How to decompose space curves is still under investigation.

*Definition 7.3 (Pseudo-vertex):* Purple points, cusps of Gauss, elliptic umbilics of a surface, and the intersection points between ridge and parabolic curves and edge curves are called *pseudo-vertices*.

*Definition 7.4 (Pseudo-edges):* The portion of a ridge or of a parabolic curve comprised between two pseudo-vertices or one pseudo-vertex and one vertex of a surface is called *pseudo-edge*.

*Definition 7.5 (Face):* Each portion of a surface patch bounded by edges and pseudo-edges is called a *face*.

A general 3D curved object can be decomposed as follows: first, a general curved object is decomposed into piecewise smooth surface patches bounded by edges that represent the intersection of at least two of them. Each edge is bounded by two vertices that are the intersection points of at least three surfaces. Then each smooth surface patch is considered separately and its points are classified into elliptic convex, elliptic concave, hyperbolic and parabolic. Then pseudo-edges are found and used to decompose each surface patch into faces. The adjacency relationship between these elements is represented by a graph. A spherical surface patch will be considered as one face.

## VIII. Contact Formations Between 3D Curved Objects

Now we can extend the concept of PCs to 3D curved objects in terms of faces, (pseudo) edges, and (pseudo) vertices.

It is easy to see that in this case the number of possible types of PCs is higher than in the polyhedral case, and we do not enumerate them for brevity.

*Lemma 8.1:* Each planar section of a quadric surface is a conic curve.

*Proof:* The proof can be easily obtained by solving the set composed by the general equations of a quadric surface and of a plane. ■

We now show that PCs defined for 3D curved objects share the same characteristic as PCs for polyhedral objects by the following theorem.

*Theorem 8.1:* A PC between a pair of faces has only one tangent plane of contact.

*Proof:* Consider a pair of faces of two 3D objects having non-zero principal curvatures (i.e. convex or concave faces) that are in contact at a point $p$ and an arbitrary plane $T_p$ through $p$ that intersects both faces. Lemma 8.1 ensures that intersection of $T_p$ and each face is a segment of conic curve and since each face is a curvature monotonic patch, this curve will have monotonic curvature. The two resulting segments of planar curves are in contact at the point $p$. Theorem 5.1 ensures that each pair of different curvature monotonic segments of planar curves do not have more than one contact at an interior point. Therefore the tangent line is uniquely defined. Since the plane $T_p$ has been arbitrarily chosen, we can say that it will be the same for two faces. If at least one of the two faces have one zero principle curvature, it is easy to see that the contact region can be a point or a straight line segment (e.g., a contact between two cylinders). In this case, there is still a single tangent plane of contact. If both faces have two zero principle curvatures, they are flat planes, and the contact between them still uniquely defines a single tangent plane. ■

The above theorem can be extended to the other types of PCs as well except for certain PCs involving a concave edge. Unlike in the case of polyhedral objects, where straight-line edges can only cross at one point, two curves in 3D space may cross at either one or two points, even if both curves are monotonic in curvature. These kinds of special cases are under investigation.

## IX. Conclusion

In order to characterize and represent contact states between 2D and 3D curved objects we have introduced a novel approach to curve and surface segmentation based on monotonicity of the curvature, which we call curvature monotonic segmentation. We have described the properties of general planar curves and surfaces with respect to the curvature and how to perform the curvature segmentation. We have proven that if the boundary curve of a 2D planar curved object is a conic and the boundary surface of a 3D curved object is a quadric the curvature monotonic decompositions gives rise to curve segments and surface patches that permit a high level topological description of contact states in a manner similar to the case of polyhedral objects. This restriction of our approach to algebraic curves and surfaces of order less that 2 should not be regarded as a limitation. We plan to continue this research by developing strategies for segmentation and contact state representation of curves and surfaces of higher degree bearing in mind that due to the great variety of shape of higher order curves and surfaces a decomposition based only on the curvature probably will not be sufficient.

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