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A New Formalism to Characterize Contact States Involving Articulated Polyhedral Objects *

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***Abstract*— In this paper a novel formalism to characterize contact states between an articulated polyhedral object and a polyhedral environment for the generation of the graph of feasible contact states between them is presented. This formalism is based upon a particular representation of the stratification of the configuration space of the articulated object by means of oriented matroid theory. A stratification is a decomposition of a set into a collection of manifolds which in our case correspond to the different contact states between the articulated object and the environment. In the representation of the stratification of the configuration space using oriented matroid theory the topological properties of the different strata are represented at a purely combinatorial level. An algorithm to enumerate the existing strata and to find the adjacency relationships among them is proposed. It will be shown that the symbolic computation based on oriented matroids simplifies and in some cases even replaces the computation with coordinates.**

***Index Terms*— Articulated objects, graph of contact states, motion planning, oriented matroid theory, stratification of the configuration space.**

I. INTRODUCTION

Information about contact states is important for many robotics tasks involving contact with the physical environment such as grasp planning, dexterous manipulation planning, assembly planning involving articulated parts, motion planning of articulated robots like snake robots, whose motion relies on contact with the environment, and accurate haptic rendering involving articulated objects.

While there is considerable study on how to represent and analyze contact states between non-articulated polyhedral objects [1], there is no systematic study on general representation of discrete contact states involving articulated objects. This paper represents a first study of the characteristics of contact states involving articulated objects to construct the contact state graph between an articulated polyhedral object and a polyhedral environment, i.e., the exhaustive list of the feasible contact states between them and their adjacency relationships. Specifically, we focus on contact states between an articulated object M with polyhedral links and a polyhedral environment S. We suppose that the joints of the articulated objects are revolute, prismatic or spherical, that the objects are rigid, and the contacts between them are frictionless. The extra degrees of freedom introduced by the joints of M make it insufficient to characterize a contact state between M and S in terms of only the surface features of M that are in contact with S, as in the case of a contact state between two non-articulated polyhedral bodies. Therefore, in this paper we propose to represent such contact states using a novel concept called topological configuration, which not only characterizes the contacting links of M but also the configuration of the links of M that are not in contact with S. It will be shown that this information is required to check the existence of a certain contact state and the feasibility of the transition between two of them.

Consider the simple planar situation represented in Fig. 1 in which a moveable object M, formed by only one edge e_1^M, is in different contact states with another static object S, which is composed by two edges e_1^S and e_2^S. Suppose that we are interested in determining the feasible contact states between M and S and in finding their adjacency relationships. We use elementary contacts (ECs) to describe the contact states between M and S. ECs are the vertex-face (v-f) and the edge-edge (e-e) contacts between polyhedra in the 3D space and the vertex-edge (v-e) contacts between polygons in the 2D plane [2, Chapter 3]. If another edge e_2^M is rigidly attached to e_1^M as in Fig. 2, then the number of possible ECs between e_1^M and e_1^S reduces. If e_2^M is connected to e_1^S through a revolute joint, as represented in Fig. 3, the ECs that e_1^M can make with e_1^S depends on the configuration of e_2^M with respect to e_1^M. In addition, some transitions between contact states e_1^M and e_1^S are possible only if M is in certain configurations. For example, the transition between the configurations of Fig. 3.d and Fig. 3.e is possible if and only if e_2^M is placed above the line that supports the edge e_1^S. However, as long as this condition is satisfied, the actual configuration of e_2^M does not matter.

Therefore, to study feasibility of contact states and their adjacency, it is necessary to group configurations of the articulated object M into certain equivalence sets such that, at every configuration inside each of them, surface features of M and S have the same relationships of incidence and relative position. Since incidence and relative position are topological relationships, we call such equivalence sets

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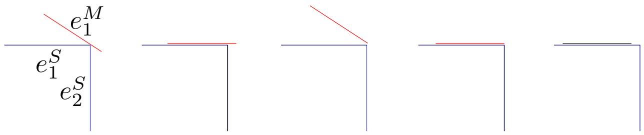

Fig. 1. Some of the possible contact states between two rigid objects M and S. M is formed by one edge e_1^M whereas S is composed by two edges e_1^S and e_2^S.

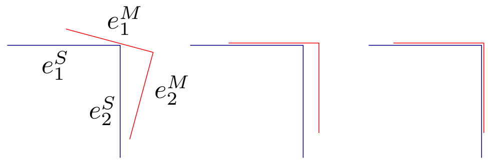

Fig. 2. If another edge is rigidly attached to e_1^M the number of ECs it can make with e_1^S reduces with respect to the case of Fig 1.

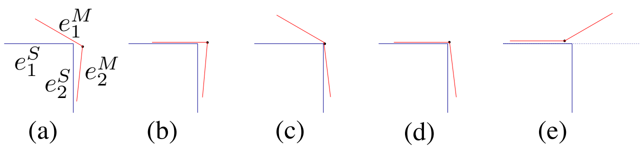

Fig. 3. If another edge is connected to e_1^M through a revolute joint, the number of ECs it can make with e_1^S does not reduce with respect to the case of Fig 1 if the topological configuration of the articulated body M is suitably modified.

topological configurations. The adjacency between contact states depends on the topological configuration of the articulated object M, that is, some contact states and some transitions between contact states are possible if and only only if M is in some topological configuration independently of the actual configuration. The simple example of Fig. 3 also shows that the topological configuration of the articulated object M alone is not enough. It is necessary to take into account the topological configuration of M with respect to a set of locally relevant features of S.

From the above considerations it should be clear that the concept of topological configuration is crucial to characterize the nature of the problem but it needs a mathematical formalism to be represented. Additionally, this mathematical formalism should also incorporate some mathematical tool to facilitate geometric reasoning in the enumeration process (e.g., inclusion and adjacency between contact states).

Oriented matroid theory fulfills all these requirements. It is a broad setting in which the combinatorial properties of geometrical configurations can be described and analyzed. It provides a common generalization of a large number of different mathematical objects such as arrangements of points and vectors, arrangements of hyperplanes, convex polytopes and directed graphs. Consider the vertices of the polyhedra that form the links of the articulated object M. Whereas the matrix that represents these points in homogeneous coordinates is coordinate dependent, a set of signs of determinants that encode the relative position of the elements of M and S, which, as will be shown in this paper, corresponds to the notion of oriented matroid, is a coordinate free representation that is also a topological invariant, that is, an invariant under homeomorphisms. Roughly speaking, this means that the oriented matroid that represents the geometrical objects M and S only changes when a vertex of one link of the articulated object M crosses one of the planes passing through three vertices of another link of the articulated object or one of the planes passing through three vertices of the static environment. This abstract representation permits a deeper understanding of geometric problems and in some cases the symbolic computation based on it can even replace the computation with coordinates. In the abstraction process from the vertices of M and S to the oriented matroid, metric information is lost but the structural properties of their configuration are represented at a purely combinatorial level.

Exhaustive enumeration of the existing contact states between M and S entails an iterative hypothesize-and-test procedure in which contact states between the articulated object and the environment are hypothesized by expressing a set topological constraints between features of the links of M and features of S and tested by checking the existence of at least one configuration that satisfies the constraints. In this paper this is done by expressing the topological constraints using oriented matroids and testing their feasibility in the workspace.

The use of oriented matroids to study the motion planning problem for a non-articulated polyhedral object in a polyhedral environment has been introduced in [3]. Oriented matroids can also be regarded as a qualitative representation of spatial relationships. For another framework for qualitative representation and reasoning about spatial relationships with applications to compliant motion planning of articulated objects see [4] in which the importance of a topological representation of the configuration the objects is implicitly recognized through the concept of "qualitative reasoning" but the development is carried out without the underpinning of a mathematical formalism for its representation.

Recent works [5], [6] propose motion planning methods for articulated objects that rely on the concept of stratified configuration space. A stratification is a decomposition of a set into a collection of manifolds which in this case correspond to the different contact states between the articulated object and the environment. In particular in [7] the motion planning problem on non-smooth manifolds is considered. This problem arises, for instance, when planning the motion of a legged robot on a non-smooth terrain or in manipulation planning for non-smooth objects. This work shows that the knowledge of the topological structure of the stratification of the configuration space is essential for planning and, since polyhedra are particular non-smooth objects, this confirms the relevance of the study of the topological properties of the stratification of the configuration space described in the following sections, in which the stratification of the configuration space of the articulated object M is characterized using oriented matroid theory.

The paper is organized as follows. In Section II the basic notions of oriented matroid theory are introduced. In Section III it is explained how topological configurations of M can be represented using oriented matroids and how this representation generates a stratification of the configuration space of M. In Section IV the algorithm for the enumeration of the contact states between M and S is described together with the advantages introduced by the oriented matroid structure. Finally, Section V contains the conclusions.

II. Oriented Matroids

Oriented matroids can be introduced over several models such as arrangement of vectors, arrangements of hyperplanes and arrangements of pseudolines which are usually treated at the level of usual coordinates [8], [9], [10], [11]. In this section oriented bases of arrangement of vectors will be used to define them.

Definition 2.1 (Arrangement of points): An *arrangement of points* is a set of points $P = \{p_1, p_2, \ldots, p_n\}$ of $\Re^3$. The matrix $\mathbf{P} = (p_1, p_2, \ldots, p_n)$ of $(\Re^3)^n$ that contains them as column vectors is assumed to have full rank 3.

Definition 2.2 (Associated arrangement of vectors): The arrangement of vectors X associated to an arrangement of points is the set of vectors obtained from the arrangement of points by representing them in homogeneous coordinates, i.e., setting $x_i = \binom{p_i}{1}$. This corresponds to the embedding of the space $\Re^3$ into the linear vector space $\Re^4$. Let $\mathbf{X}$ be the matrix of $(\Re^4)^n$ that contains the set of vectors as column vectors.

A. Chirotope of an arrangement of vectors

Consider an arrangement of vectors X in $\Re^4$

Definition 2.3 (Chirotope): The *chirotope* of X is the map

$$\begin{aligned} \chi_X : &\{1, 2, \ldots, n\}^4 \rightarrow \{+, 0, -\} \\ &(i, j, k, l) \mapsto \text{sign}\left([x_i, x_j, x_k, x_l]\right) \end{aligned}$$

that assigns to each 4-tuple (x_i, x_j, x_k, x_l) of vectors of the finite configuration X with $i < j < k < l$ a sign $+$ or $-$ depending on whether it forms a positively oriented basis of $\Re^4$ or a basis with negative orientation. This function assigns the value 0 to those 4-tuples that do not constitute a basis of $\Re^4$.

B. Set of cocircuits of an arrangement of vectors

Definition 2.4 (Set of cocircuits): The set of *cocircuits* of an arrangement of vectors X is the set

$$\mathcal{C}^*(X) = \{(\text{sign}(a^T x_1), \text{sign}(a^T x_2), \ldots, \text{sign}(a^T x_n)) \in \{+, -, 0\}^n : a \in \Re^4 \text{ a is orthogonal to a plane spanned by vectors in } X\}.$$

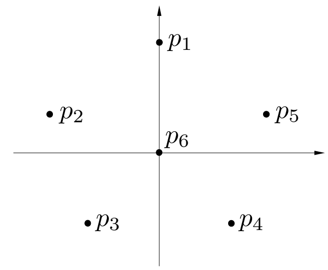

Fig. 4. Planar arrangement of points

This means that the set of cocircuits of X is the set of all partitions generated by special planes: those spanned by the vectors of the configuration X.

C. Oriented matroids

Other data structures can be defined to encode the combinatorial properties of a vector configurations [11, Chapter 3]. The following theorem gives an important result that relates chirotopes and sets of cocircuits. It states that

Theorem 2.1: The sets of data χ_X and $\mathcal{C}^*(X)$ are equivalent. Whenever one of them is given it is possible to reconstruct the other.

Proof: See [11, Chapter 3] ■

A similar result exists to relate all the data structures that can be used to describe an arrangement of vectors.

Definition 2.5 (Oriented matroid): These combinatorial structures, equivalent in the sense of the previous theorem, are referred to as *oriented matroids*.

Example 2.1: Consider the arrangement of vectors X represented in Fig. 4. The homogeneous coordinates of the points in X are listed in Table I which is the arrangement of vectors X that corresponds to the arrangement of points of Fig. 4.

$p_1 = (0, 3, 1)$	$p_2 = (-3, 1, 1)$	$p_3 = (-2, -2, 1)$
$p_4 = (2, -2, 1)$	$p_5 = (3, 1, 1)$	$p_6 = (0, 0, 1)$

TABLE I

Arrangement of vectors that corresponds to the planar arrangement of points represented in Fig. 4

The chirotope χ_X of this arrangement of vectors is given by the following orientations The element $\chi(1, 2, 3) = +$

$\chi(1,2,3) = +$	$\chi(1,2,4) = +$	$\chi(1,2,5) = +$	$\chi(1,2,6) = +$
$\chi(1,3,4) = +$	$\chi(1,3,5) = +$	$\chi(1,3,6) = +$	$\chi(1,4,5) = +$
$\chi(1,4,6) = -$	$\chi(1,5,6) = -$	$\chi(2,3,4) = +$	$\chi(2,3,5) = +$
$\chi(2,3,6) = +$	$\chi(2,4,5) = +$	$\chi(2,4,6) = +$	$\chi(2,5,6) = -$
$\chi(3,4,5) = +$	$\chi(3,4,6) = +$	$\chi(3,5,6) = +$	$\chi(4,5,6) = +$

TABLE II

Chirotope of the planar arrangement of points represented in Fig. 4

indicates that the basis formed by the vectors p_1, p_2, and p_3 has positive orientation. Geometrically, this is equivalent to say that in the triangle formed by p_1, p_2, and p_3 these points are counterclockwise ordered.

Half of the cocircuits of the arrangement of points X represented in Fig. 4 are listed in Table III. The other half can be obtained by negating the data. In this case, the set

$(0,0,+,+,+,+)$	$(0,-,0,+,+,+)$	$(0,-,-,0,+,-)$
$(0,-,-,-,0,-)$	$(0,-,-,+,+,0)$	$(+,0,0,+,+,+)$
$(+,0,-,0,+,+)$	$(+,0,-,-,0,-)$	$(+,0,-,-,+,0)$
$(+,+,0,0,+,+)$	$(+,+,0,-,0,+)$	$(+,+,0,-,-,0)$
$(+,+,+,0,0,+)$	$(-,+,+,0,-,0)$	$(-,-,+,+,0,0)$

TABLE III

HALF OF THE COCIRCUITS OF THE PLANAR ARRANGEMENT OF POINTS REPRESENTED IN FIG. 4

of cocircuits of X is the set of all partitions generated by lines passing through two points of the configuration X. For example, $(0,0,+,+,+,+)$ means that the points p_3, p_4, p_5, and p_6 lie on the half plane determined by the line through the points p_1 and p_2. Changing the sign that denotes that half planes we obtain an equivalent combinatorial description of the planar arrangement of points.

III. Topological Configurations of Polyhedral Articulated Objects and Oriented Matroids

Let M be an articulated polyhedral object with l links $M_1, M_2, \ldots, M_l$ and m vertices. Suppose that each link moves in a static polyhedral environment S composed by one or more polyhedra with n vertices. Let $\{x_1, \ldots, x_m\}$ be the set of vertices of M and $\{x_{m+1}, \ldots, x_{m+n}\}$ be the set vertices of S, and consider the arrangement of vectors X formed by the vertices of M and S which are represented in homogeneous coordinates as column vectors of the matrix $\mathbf{X} = (x_1, x_2, \ldots, x_{m+n})$ of $(\Re^4)^{m+n}$. The union of the sets of vertices of M and S is called the *underlying point arrangement* of M and S. We can give a more intuitive interpretation to the concept of oriented bases of arrangement of vectors using the elementary geometric concept of *simplex*. We call simplex each ordered 4-tuple (i,j,k,l) of indices of the vertices of the underlying point arrangement with $i,j,k,l \in \{1,2,\ldots,m+n\}$ and $i < j < k < l$. The determinant $f = [x_i\, x_j\, x_k\, x_l]$ is called the *oriented volume* of the simplex (i,j,k,l). Since the orientation of a basis corresponds to the sign of the oriented volume of the simplex formed by the vectors of the basis, the map χ_X that assigns to each simplex (i,j,k,l) the sign of the corresponding oriented volume is called the *orientation* of the simplex.

A. Topological Configurations

Suppose that only the links $M_i, M_{i+1}, \ldots M_l$ of the articulated object M move with respect to the links $M_1, M_2, \ldots, M_{i-1}$ and with respect to the static environment S. There is a sign change in the chirotope when one of the vertices of the moving links touches or crosses one of the planes passing through at least three vertices of the static objects (i.e., the links of M that are not moving and S) and vice versa. Similarly, any EC between $M_i, M_{i+1}, \ldots M_l$ and $M_1, M_2, \ldots, M_{i-1}$ and S corresponds to a zero entry in the chirotope (see Fig. 5). Therefore there is a sign change in the chirotope when there is a change in the

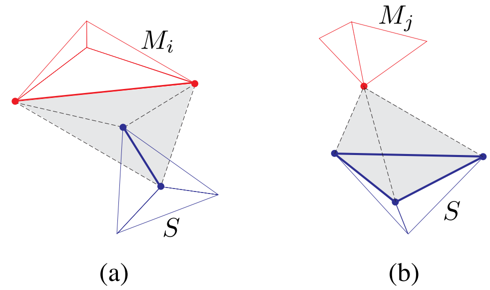

Fig. 5. Elementary contacts between the polyhedra can be characterized by means of oriented volumes of simplices (shaded regions). (a) e-e contact. (b) v-f contact.

topological relationships of incidence or relative location between vertices and planes passing through at least three vertices of the underlying point arrangement, i.e., when there is a change in the topology of the configuration of M. Therefore we can assert that oriented matroids characterize the topology of the configuration of M. Thus it make sense to give the following definition.

Definition 3.1 (Topological configuration): We define topological configuration of an articulated object M with respect to a static environment S the oriented matroid associated to the underlying point arrangement.

In particular, we call *topological contact configurations* those topological configuration at which at there is an EC between M and S.

Since both the moving object and the polyhedral environment are supposed to be rigid bodies, we are particularly interested in those simplices that contain at least one of the vertices of M because, actually, only their orientations depend on the relative location of M and S.

Example 3.1: The chirotope that characterizes the contact configuration of Fig. 6.a is listed in Table IV. In this case it is the sign vector $(++-++-\cdots+0-)$ having 35 elements.

Remark 3.1: In the following sections the topological configurations of the articulated polyhedral object M will be represented by means of the data structure of chirotope.

B. Stratification of the configuration space of the articulated object

Oriented volumes of simplices can be expressed as polynomials in the configuration variables of M. It will be shown and these polynomials can be used to generate a stratification of the configuration space of M.

Definition 3.2 (Stratification of a set): A stratification of a set S is a partition of S into a finite number of disjoint subsets S_i called strata such that each S_i is a manifold.

There are several ways to construct regular stratifications [12], [13] one of which makes use of the preimage of a *transversal map* [14, Chap. 3].

Definition 3.3 (Transversal map): Let $F : X \to Y$ be a smooth map between two manifolds X and Y and let $D \subseteq Y$ be a smooth submanifold of Y. The map F is said to be transversal to D if for every $y \in D$ and every $x \in F^{-1}(y)$, $dF_x(TX_x) + TD_y = TY_y$, i.e., the basis

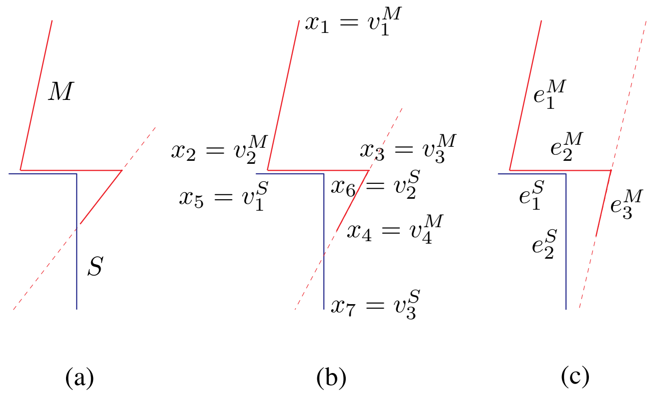

Fig. 6. (a) A contact configuration between an articulated object M and a static environment S whose topological configuration is represented by the chirotope of Table IV. In particular at this configuration $\chi(4,6,7) = 0$. This means that this configuration lies on the contact manifold $\chi(4,6,7) = 0$ which corresponds to the v-e contact between x_4 and e_2^S. (b) At this configuration the v-e contact between x_4 and e_2^S is broken. This change in the topological configuration is reflected by a change in the chirotope. Now $\chi(4,6,7) = +$. (c) The line that supports the edge e_3^M crossed the vertex x_7, i.e. the configuration of M crossed the manifold $\chi(3,4,7) = 0$. At this topological configuration $\chi(3,4,7) = -$.

$\chi(1,2,3) = +$	$\chi(1,2,4) = +$	$\chi(1,2,5) = -$	$\chi(1,2,6) = +$
$\chi(1,2,7) = +$	$\chi(1,3,4) = -$	$\chi(1,3,5) = -$	$\chi(1,3,6) = -$
$\chi(1,3,7) = -$	$\chi(1,4,5) = -$	$\chi(1,4,6) = +$	$\chi(1,4,7) = -$
$\chi(1,5,6) = +$	$\chi(1,5,7) = +$	$\chi(1,6,7) = -$	$\chi(2,3,4) = +$
$\chi(2,3,5) = 0$	$\chi(2,3,6) = 0$	$\chi(2,3,7) = +$	$\chi(2,4,5) = -$
$\chi(2,4,6) = +$	$\chi(2,4,7) = -$	$\chi(2,5,6) = 0$	$\chi(2,5,7) = +$
$\chi(2,6,7) = -$	$\chi(3,4,5) = +$	$\chi(3,4,6) = +$	$\chi(3,4,7) = +$
$\chi(3,5,6) = 0$	$\chi(3,5,7) = +$	$\chi(3,6,7) = +$	$\chi(4,5,6) = -$
$\chi(4,5,7) = +$	$\chi(4,6,7) = 0$	$\chi(5,6,7) = -$	

TABLE IV
CHIROTOPE THAT CHARACTERIZES THE TOPOLOGICAL CONFIGURATION OF M REPRESENTED IN FIG. 6.A.

vectors of the image of the tangent space to X at x under the derivative together with the basis vectors of the tangent space to D at $y = F(x)$ span the tangent space to Y at y.

If the map $F : X \to Y$ is transversal to a regular stratification of a subset $D \subseteq Y$, then the preimage of the regular stratification of D is a regular stratification.

Let $f_i, i = 1, \ldots, p$ be the collection of polynomials obtained by considering all the oriented volumes of the p simplices that contain at least one vertex of M. If $\Re^q$ is the space in which the configuration space of M is embedded [2, Chapter 2], they define a map $F = (f_1, f_2, \ldots, f_n) : \Re^q \to \Re^p$. Given a p-dimensional *sign vector*, i.e. a p-tuple of signs and zeros, σ, i.e., an element of $\{+, 0, -\}^p$, the sets $F^{-1}(\sigma)$ are called *sign invariant sets* of F.

Definition 3.4 (Semi-algebraic set): A *semi-algebraic set*, is a set described by means of first order logic sentences whose predicates are $= \neq > < \geq \leq$, whose variables are real numbers, and whose terms are multivariate polynomials with rational coefficients.

Then, a semi-algebraic set can be viewed as a finite union of sign-invariant sets for some polynomial map F. It is well known [2, Chapter 3] that, if the objects M and S can be represented as semi-algebraic sets, the C-obstacle, i.e., the set that represent configurations at which M collides with or intersects S, can be represented as semi-algebraic sets of $\Re^q$, the space in which the configuration space of M is represented. But also the set of configurations at which M and S have a given set of topological relationships expressed in terms of incidence, and relative position, can be described as a semi-algebraic set of $\Re^q$.

From the above considerations it follows that any partition of $\Re^p$ induced by all the p-dimensional sign vectors is a regular stratification of $\Re^p$. Since the map $F = (f_1, f_2, \ldots, f_p) : \Re^q \to \Re^p$ is transversal to the set $\{+, 0, -\}^p$ [12], [13], the preimage of the above regular stratification of $\Re^p$ is a regular stratification of $\Re^q$ and, as a consequence, a regular stratification of the configuration space of M.

Let S_0 denote the configuration manifold of M. It is easy to see that the i-th elementary contact EC_i between M and S corresponds to one codimension 1 stratum of $S_i \subset S_0$. Note that the opposite is not true. For example, the value zero of an oriented volume of a simplex formed by a vertex of M and three vertices of S that do not lie on the same face does not correspond to a EC between M and S. Similarly, the value zero of an oriented volume of a simplex formed by a vertex of M and three vertices of S occurring at a configuration where the vertex of M is in contact with the plane supporting the face of S, and not with the face itself, does not correspond to an EC between M and S.

In other words, our stratification of the configuration space is composed by three types of strata. The first one corresponds to ECs between M and S, which therefore characterize the topological relationship of incidence between vertices of M and faces of S and vice versa and between edges of M and S. These strata represent constraints on the motion of M. The second type of strata is represented by those that characterize the relationship of incidence between vertices and planes that do not support faces or between lines that do not support edges. Finally the third type of strata characterizes incidence between vertices of M and planes that support faces of S and vice versa (i.e., in which the contacts take place in the regions of the planes outside the faces) or between edges of M and lines that support edges of S and vice versa (i.e., in which the contacts take place in the part of the lines outside the edges). It is easy to see that these two types of strata do not constrain the motion of M. Together these different types of strata characterize the topological configuration of M.

When M makes two ECs with S, say EC_i and EC_j, the configuration of M lies on a codimension 2 stratum $S_{i,j} = S_i \cap S_j \subset S_0$ formed by the intersection of S_i and S_j and so forth.

The key point of the formalism presented in this paper is that each stratum represents the set of configuration having equivalent topological properties and the stratification can be represented by oriented matroids and the oriented matroid representation of the stratum is its topological invariant, i.e., all the configuration inside the same stratum are topologically characterized by the same oriented matroid.

Note that oriented matroids cannot distinguish among different connected components of the same stratum of the configuration space, i.e., components characterized by the same sign vector for which it is impossible to move to each other without changing any sign.

In the next section an algorithm to determine the feasible contact states between M and S and the adjacencies among them which takes advantage of the characterization of the stratification of the configuration space by means of oriented matroids will be presented.

IV. Generation of the Graph of Topological Contact States

Generating the graph of topological contact states entails enumerating the existing strata that represent feasible contact topological configurations of M with respect to S and determining their adjacency relationships. Feasible topological contact configurations will be represented by means of oriented matroids and the adjacency between them will be represented by means of a graph G, which we suppose to be a connected graph.

To do that an algorithm that is based on the oriented matroid representation of the strata will be presented. It is an extension of the *reverse search algorithm* [15] which has been successfully applied to a several enumeration problems such as vertex enumeration in polyhedra and enumeration of cells in arrangements of hyperplanes.

With the oriented matroid representation of the stratification it is straightforward to characterize inclusion between strata

Definition 4.1 (Inclusion between strata): Let S' and S'' be strata of the configuration space of an articulated object M. The stratum S' is said to be included in S'' if in the chirotope that represents S' there is at least one 0 that corresponds to a $+$ or a $-$ in the chirotope that represents S''.

Using the notion of dimension of a stratum we can compare the mobility of M [16, Chapter 13] at different topological contact configurations. However we are interested in characterizing the mobility of M in each stratum included in a given stratum. It is clear that the inclusion between strata ensures that the dimension of the including strata will be higher than that of the included strata, i.e., including strata will correspond to *less constrained strata* for M and vice versa. Thus, we give the following definition.

Definition 4.2 (Less and more constrained strata): Let S' and S'' be strata of the configuration space of an articulated object M that correspond to a contact state between M and S and let S' be included in S''. Then S'' it is called a less constrained stratum with respect to S' that includes S'. On the contrary, S' is called a more constrained stratum with respect to S'' included in S''.

According to the above definition, for instance, the stratum represented by the chirotope $(+00+)$ is a more constrained stratum with respect to $(+0-+)$ included in it.

A. Hypothesis generation

For technical reasons that will be clear soon, to systematically generate p-dimensional sign vectors that represent strata is needed an objective function to be maximized over the set of p-dimensional sign vectors, $\{+,0,-\}^p$. This phase, which can be thought of as a hypothesis generation for the nodes of G, will be followed by a feasibility check as described in the next section.

The set of nodes of G is unknown at this stage of the algorithm but, since each feasible and not feasible topological configuration, and, in particular, each feasible and not feasible topological contact configuration can be characterized by a chirotope, i.e., by a p-dimensional sign vector, an objective function can be defined based on these p-dimensional sign vectors regardless the existence of the corresponding stratum. It is important to point out that the choice of this objective function is crucial for the efficiency of the enumeration because the order in which the strata are hypothesized depends on this function. Since, given an existing stratum, checking the existence of a less constrained stratum in which it is included is easier than checking the existence of a more constrained stratum that it includes, this objective function should be chosen in such a way that it induces the generation of a less constrained stratum that includes one of the existing strata. An objective function that fulfills these requirement is obtained by interpreting the three symbols of the chirotope that represent each stratum as the symbols of a ternary numeric representation in which the symbols $+$,$-$, and 0 correspond to the numerical decimal values 2, 1, and 0, respectively. For instance, the value of the objective function for the stratum $(--0+)$ is $1\times3^3+1\times3^2+0\times3^1+2\times3^0=38$. Consider Fig. 7.a in which the stratification generated by four curves in the plane is represented. Each stratum of this stratification of the plane is thus described by a 4-dimensional signs vector. In Fig. 7.b the numerical values of such an objective function are represented for each stratum represented in Fig. 7.a.

B. Feasibility check

The enumeration procedure starts from an existing stratum that corresponds to a feasible topological contact configuration. Then, a sequence of less constrained strata in which it is included are hypothesized and their existence tested. The key point of the enumeration algorithm is that, using the objective function previously described, the less constrained strata that include this stratum are characterized by those p-dimensional sign vectors for which the objective function is greater. Thus, another component of our enumeration algorithm is a finite deterministic procedure to establish, given an existing stratum, the existence of strata in which it is included i.e., for which the value of the objective function in greater until there exists no better strata with respect to the value of the objective function.

For consistency with [15] we call it *local search algorithm* on G. A node without a better neighboring node is called *local optimum* and corresponds to a codimension 0

stratum i.e., a stratum without contacts between M and S. Once a node corresponding to a local optimum of the objective function has been reached, another existing stratum has to be determined (see Section IV-B.5) and the above procedure repeated.

As shown in Fig. 7.a, in general there are several local optima of the objective function. As a consequence, the local search algorithm generates several trees $T_1, T_2, \ldots, T_l$. Note that some of them can be single-node trees as the node $(- - --)$. Suppose that T_1 is the tree having the stratum $(+ + -+)$ as root and that it is the only existing local optimum of the objective function. T_1 is composed by the strata $(++-+)$, $(+0-+)$, $(++-0)$, and $(+0-0)$. Thus if we visit T_1 from $(++-+)$ systematically, say by depth-first search, we can enumerate all the existing strata. The strata whose corresponding nodes are neighbors in T_i will be adjacent in G_i and, as will be explained in Section IV-B.5, some other adjacency relationships result from the reverse search algorithm. The oriented matroid representation of the existing strata simplifies the exhaustive enumeration of the adjacencies between strata of T_i. Therefore, given T_i, is rather simple to generate a graph G_i in which arcs are added to connect adjacent nodes of the tree T_i. The graphs G_i are then merged together form the graph G. The characterization of each existing node with oriented matroids makes the merging process of the graphs G_i straightforward.

It is easy to see that many topological configurations can corresponds to the same topological contact configuration, as in the case represented in Fig. 6. To give a more compact representation in G, the different topological configuration associated to with each feasible topological contact configuration will be grouped together. The oriented matroid characterization makes this process very simple. Indeed the three topological configurations of Fig. 6 can be characterized by the chirotope of one of them, e.g, that of Fig. 6.a represented in Table IV, together with the signs to be changed to represent the other two, i.e. $\chi(4, 6, 7)$ for that of Fig. 6.b, $\chi(4, 6, 7)$ and $\chi(3, 4, 7)$ for the configuration represented in Fig. 6.c.

Feasibility or existence of hypothesized strata can be checked in several steps. Most of them can be based on topological consideration and therefore fully exploit the oriented matroid structure used to represent the stratification of the configuration space of M.

1) Checking the sign consistency: Not all the possible combinations of signs and zeros in the p-dimensional sign vectors that characterize the stratification of the configuration space are admissible. Indeed they must fulfill the Grassmann-Plücker relations [11, Chapter 3]. Only the sign vectors that fulfill these relations represent a chirotope. However, there exist strategies to generate consistent p-dimensional sign vectors [17]. This fact greatly simplifies the feasibility check.

2) Checking the redundancy: Redundancy in the stratification of the configuration space occurs when two or more strata are equivalent because correspond to the same constraint. Eliminating redundant contact constraints of the configuration space can be solved in the workspace using simple geometric considerations [18].

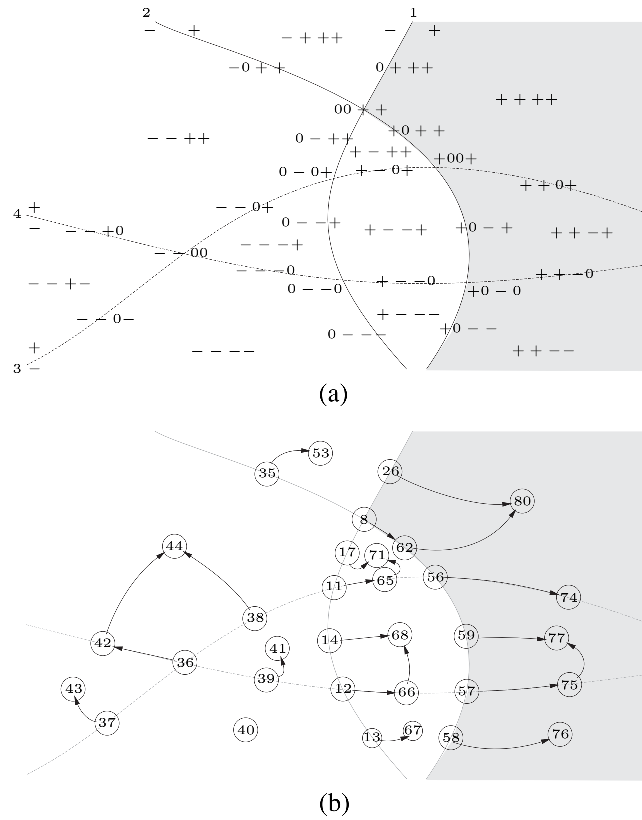

Fig. 7. (a) A simple stratification of the plane generated by four curves together with the oriented matroid representation of each stratum. (b) Trees generated by the local search algorithm. Tracking each edge against its orientation and merging the results leads to the enumeration of the existing strata. In correspondence with each stratum the value of the objective function used in enumeration procedure is reported.

3) Checking the intersection: A chirotope does not correspond to any feasible contact topological configuration if the corresponding topological configuration gives rise to intersection between M and S or to a self intersection of M. Since the intersection between an edge and a face is a topological relationship, it can be detected using topological information that can be read off from the oriented matroid representation of the stratification of the configuration space of M. It is easy to see that there is an intersection between M and S if at least one edge of M intersects at least one face of S or vice versa. Consider Fig. 8 in which an edge of M and a non-convex face of S are represented. Let Π be an arbitrary plane containing the face f^S and let Γ be an arbitrary plane containing the edge e^M. Consider the half plane Γ' determined by the line that supports e^M. There is an intersection between e^M and f^S if the endpoints of e^M lie in different half spaces with respect to Π and the number of intersections between the edges that bound f^S and Γ' is odd [3] It is easy to see that these intersection conditions can be implemented comparing signs of the oriented matroid representation of M and S. Moreover this simple method does not require decomposition of non-

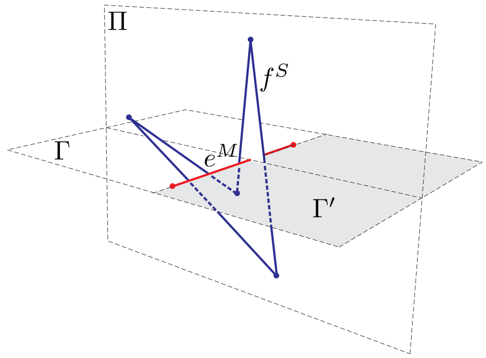

Fig. 8. Intersection detection between polyhedra can be implemented comparing the signs of the oriented matroid representation of the configuration of M without decomposing of non-convex polyhedra in convex parts.

convex polyhedra in convex parts.

If a topological configuration is consistent, non-redundant, and does not give rise to intersections, its feasibility is checked in the workspace using geometric methods. This corresponds to checking the existence of the corresponding stratum.

4) Checking the existence of a less constrained stratum that includes a given stratum: Given a certain stratum of the configuration space of M one has to establish the existence of another stratum described by a consistent, non-redundant, p-dimensional sign vector that does not correspond to an intersection between M and S in which one or more contact constraints have been relaxed. The problem can be solved in most part using instantaneous kinematics after modeling each contact between M and S by means of an instantaneously equivalent kinematic chain [19]. This entails solving a system of linear equations to check whether it does exist a solution that corresponds to an instantaneous velocity in the direction in which the above contact constraints are relaxed.

5) Checking the existence of a more constrained stratum included in a given stratum: One has to generate a feasible topological configuration whenever the objective function of the reverse search algorithm reaches a local optimum. However, generating a feasible topological contact state that fulfills a set of topological and geometrical conditions expressed by a chirotope "from scratch" is not a simple issue [8, Chapter IV]. It is possible to overcome this difficulty using a feasible topological contact configuration, i.e., starting from a stratum whose existence has already been determined. Since its less constrained strata in which it is included have already been enumerated by the reverse search algorithm we need to hypothesize and test the existence of more constrained strata included in it. We study the problem using a motion planning technique to check whether the configuration of M can be steered on the existing stratum to a more constrained stratum included in it using an approach similar to that described in [1, Section 3] for non-articulated objects.

V. Conclusions

In this paper a new formalism to characterize contact states involving articulated objects has been presented. It has been shown that to solve the problem of enumerating the existing contact states between an articulated polyhedral object M and a polyhedral environment S it is fundamental to take into account the topological properties of the configuration space of M. Therefore, the concept of topological configuration has been introduced and its interpretation in terms of oriented matroid theory has been described. It has been shown that this representation generates a stratification of the configuration space of M and that its combinatorial representation by means of oriented matroids facilitates the exhaustive enumeration of feasible contact states and the determination the adjacency relationships among them.

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