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The algebraic structure of the isomorphic types of tally, polynomial time computable sets

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Abstract. We investigate the polynomial time isomorphic type structure of $\mathbf{P}_T \subset \{A : A \subseteq \{0\}^*\}$ (the class of tally, polynomial time computable sets). We partition $\mathbf{P}_T$ into six parts: D^-, $\hat{D}^-$, C, S, F, $\hat{F}$, and study their p-isomorphic properties separately. The structures of $\langle \mathbf{deg}_1(F); \leq\rangle$, $\langle \mathbf{deg}_1(\hat{F}); \leq\rangle$, and $\langle \mathbf{deg}_1(C); \leq\rangle$ are obvious, where F, $\hat{F}$, and C are the class of tally finite sets, the class of tally co-finite sets, and the class of tally bi-dense sets respectively. The following results for the structures of $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$ and $\langle \mathbf{deg}_1(S); \leq\rangle$ will be proved, where $\hat{D}$ is the class of tally, co-dense, polynomial time computable sets and S is the class of tally, scatted (i.e., neither dense nor co-dense), polynomial time computable sets.

1. $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$ is a countable distributive lattice with the greatest element.
2. Infinitely many intervals in $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$ can be distinguished by first order formulas.
3. There exist infinitely many nontrivial automorphisms for $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$.
4. $\langle \mathbf{deg}_1(S); \leq\rangle$ is not distributive, but any interval in $\langle \mathbf{deg}_1(S); \leq\rangle$ is a countable distributive lattice.

1. Introduction

Cook [10] and Karp [12] introduced the polynomial time bounded counterparts to the two most important recursive reducibility notions respectively, namely polynomial time Turing and polynomial time many-one reducibilities. The importance of the stronger, conceptually simpler, p-m-reducibility (Karp) stems from the fact that many interesting problems turned out to be equivalent with respect to this stronger notion, that is, to be p-m-reduced to each other. Many important problems in the class **NP** of nondeterministically polynomial time computable sets have been shown to be p-m-equivalent, in fact to be p-m-complete for **NP**.

Like the recursive reducibilities which give a natural classification of the unsolvable problems according to their relative difficulty, the polynomial time reducibilities provide a natural tool for classifying any specific complexity class.

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Structural questions about the recursive reducibilities have been first studied by Post in 1940s, and up to now the analysis of these questions remained one of the major areas in recursion theory. The study of these questions in the polynomial setting was initiated by Ladner, Lynch, and Selman [14] in 1973. Then the polynomial degree structures of recursive sets and other complexity classes (e.g., **NP** and **P**) under polynomial time reductions were studied by Ambos-Spies [1–5], Schöning [18,19], Chew [9], Schmidt [17] and others in a series of papers.

Berman and Hartmanis [7] made a careful analysis of the p-m-reducibility in the complexity class **NP** and then conjectured that all p-m-complete sets for **NP** are polynomial time isomorphic. This conjecture has been one of the most important questions of structural complexity theory in the last two decades. In order to solve this conjecture, the analysis of the stronger degree structure inside a weaker degree, especially the structure of polynomial time isomorphic types inside a p-m-degree, becomes more and more attractive. Many results have been obtained in this area (for more details of the corresponding questions in recursive reducibilities, it is referred to Odifreddi [16]).

A p-m-degree is a collection of sets which are equivalent under p-m-reductions, a p-m-degree is collapsing if and only if its members are p-isomorphic, that is, equivalent under polynomial time, one-one, onto, polynomial time invertible reductions. So the Berman and Hartmanis conjecture is equivalent the following statement: The p-m-complete degree for **NP** is collapsing. Mahaney and Young [15] showed that any p-m-degree is either collapsing or has infinitely many isomorphic types inside it. And Kurtz, Mahaney and Royer [13] have showed the existence of a collapsing p-m-degree in **EXP**.

It is well known (see [16]) that the degree structure of recursive sets under the recursively many-one reduction is $\{\mathbf{deg}_m(\emptyset), \mathbf{deg}_m(\Sigma^*), \mathbf{deg}_m(\{0\})\}$, and the degree structure of recursive sets under the recursively one-one reduction is $\{\mathbf{deg}_1(\emptyset), \mathbf{deg}_1(\Sigma^*)\} \cup \{\mathbf{deg}_1(i) : i \in \omega\} \cup \{\mathbf{deg}_1(co\text{-}i) : i \in \omega\} \cup \{\mathbf{c}\}$, where we use $\mathbf{deg}_1$ (i) to denote the recursive 1-degree of all sets which have exactly i elements, $\mathbf{deg}_1$ (co-i) to denote the recursive 1-degree of all sets whose complement sets have exactly i elements and **c** to denote the recursive 1-degree of all infinite sets whose complement sets are also infinite.

The p-m-degree structure of **P** is the same as the recursive m-degree structure of recursive sets. But up to now, nothing is known about the p-1-degree structure of **P** or p-isomorphic type structure of **P**.

In order to understand more about the polynomial time reductions and the general questions regarding collapsing degrees, it is necessary to study extensively the p-1-degree structure of **P** and the p-isomorphic type structure of **P**. As a first step for the research in this area, we study the p-1-degree structure and p-isomorphic type structure of $\mathbf{P}_T \subset \{A : A \subseteq \{0\}^*\}$ (the class of tally, polynomial time computable sets). And we show the relations between the p-1-degree structure of **P** and the p-1-degree structure of $\mathbf{P}_T$.

In section 2, we introduce some notation we need later and show that, for $\mathbf{P}_T$, the p-1-degree structure is the same as the p-isomorphic type structure. We then partition $\mathbf{P}_T$ into six parts: D^-, $\hat{D}^-$, C, S, F, $\hat{F}$ and study their p-isomorphic properties separately. The following results are straightforward.

1. The degree structure of the class C of tally, polynomial time computable, dense and co-dense sets consists of a single element **1** which is the greatest element of $\langle \mathbf{deg}_1(\mathbf{P}_T); \leq\rangle$. Note that we call a set *dense* if $\{0\}^*$ can be polynomial time one-one reduced to it.
2. The degree structure of the class F (respectively $\hat{F}$) of tally, finite (respectively co-finite) sets is a linear ordering of order type ω where the nth degree in this ordering contains the sets of cardinality $n-1$ (respectively the sets whose complement set has exactly $n-1$ elements).
3. $\langle \mathbf{deg}_1(D); \leq\rangle$ and $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$ are isomorphic, where D (respectively $\hat{D}$) is the class of polynomial time computable, tally, dense (respectively co-dense) sets.

Hence it suffices to study the structures of $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$ and $\langle \mathbf{deg}_1(S); \leq\rangle$. Note that $D^- = D - C$ and $\hat{D}^- = \hat{D} - C$.

In Section 3, we investigate the p-isomorphic type structure of $\hat{D}$. The class $PTGF$ of polynomial time constructible growth functions are used to study $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$, and it is shown that $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle \cong \langle \mathbf{PTGF}; \leq\rangle$, where **PTGF** is the degree structure of $PTGF$ under the reduction $\leq^{\#}$. Then we show that $\langle \mathbf{PTGF}; \leq\rangle$ is a countable distributive lattice with the greatest element. The cap, cup and lattice embedding properties of $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$ are discussed in this section. For the product property, we show that, for any n, any interval in $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$ is a subdirect product of n-chains. We also show that there are infinitely many intervals in $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$ which can be distinguished by first order formulas. In the last part of this section, we show that there are infinitely many nontrivial automorphisms for $\langle \mathbf{deg}_1(\hat{D}); \leq\rangle$.

Section 4 is devoted to the study of the p-isomorphic type structure of S, which is the class of scatted sets (i.e., neither dense nor co-dense). It is shown that $\langle \mathbf{deg}_1(S); \leq\rangle$ is not distributive, but any interval in $\langle \mathbf{deg}_1(S); \leq\rangle$ is a countable distributive lattice. The cap, cup and lattice embedding properties of $\langle \mathbf{deg}_1(S); \leq\rangle$ are discussed also.

2. Preliminaries

ω is the set of natural numbers. $\Sigma = \{0, 1\}$ is the binary alphabet and Σ^* is the set of (finite) binary strings. The length of a string x is denoted by $|x|$. For strings $x, y \in \Sigma^*$, xy is the concatenation of x and y. A subset of Σ^* is called a language or simply a set. For a language A, $\bar{A}$ is the complement set of A and $\|A\|$ denotes the cardinality of A. Let $\mathbf{TALLY} = \{A : A \subseteq \{0\}^*\}$ be the class of tally sets and let $\mathbf{P}_T = \mathbf{P} \cap \mathbf{TALLY}$ be the class of tally, polynomial time computable sets. $A^{\leq n} = A \cap \{x : |x| \leq n\}$, $A^{[n_1,n_2]} = A \cap \{x : n_1 \leq |x| \leq n_2\}$ and A^{α} has the corresponding meaning for any kinds of interval α. All sets in this paper will be in $\mathbf{P}_T$ unless stated otherwise. For a lattice $\langle L; \leq\rangle$ and an element $a \in L$, $(\cdot, a]$ is the set of elements $b \in L$ such that $b \leq a$.

We fix a standard polynomial time computable and invertible pairing function $\lambda x, y\langle x, y\rangle$ on ω, and fix a recursive enumeration $\{f_k : k \geq 0\}$ of the polynomial time computable functions with corresponding time bounds n^k ($k \geq 0$).

Definition 1. *Given $A, B \in \mathbf{P}_T$, we say that A can be* polynomial time one-one reduced to *B in $\mathbf{P}_T$, if there exists a total, polynomial time computable one-one function $f: \{0\}^* \to \{0\}^*$ such that $f(A) \subseteq B$ and $f(\bar{A}) \subseteq \bar{B}$, written: $A \leq_1^p B$. If $A \leq_1^p B$ and $B \not\leq_1^p A$, then we write $A <_1^p B$, and write $A \equiv_1^p B$ for both $A \leq_1^p B$ and $B \leq_1^p A$.*

By the above definition of $\leq_1^p$, $\langle \mathbf{P}_T; \leq_1^p \rangle$ is a partial preordering. So we can define a corresponding degree structure $\langle \mathbf{deg}_1(\mathbf{P}_T); \leq \rangle$ by factorizing over the equivalence relation $\equiv_1^p$ induced by $\leq_1^p$ and by letting $\leq$ be the induced partial ordering on the equivalence classes. Formally, for $A \in \mathbf{P}_T$, let $\mathbf{deg}_1(A) = \{B : B \equiv_1^p A\}$.

In the following, $A, B, C, \ldots$ denote elements of $\mathbf{P}_T$ and **a, b, c,** ... are the corresponding p-1-degrees. Let $A \oplus B = \{0^{2n} : 0^n \in A\} \cup \{0^{2n+1} : 0^n \in B\}$.

Definition 2. *For $A, B \in \mathbf{P}_T$, $A \leq^{\#} B$* via a polynomial *p if $\|A^{\leq n}\| \leq \|B^{\leq p(n)}\|$ for all n.*

It is easy to check that $\leq^{\#}$ is a partial preordering on $\mathbf{P}_T$ with the greatest element $\{0\}^*$.

Lemma 1. *Let $A, B \in \mathbf{P}_T$. Then the following properties hold.*

1. *$A \leq_1^p B$ if and only if both $A \leq^{\#} B$ and $\bar{A} \leq^{\#} \bar{B}$.*
2. *$A \equiv_1^p B$ if and only if A is polynomial time isomorphic to B.*

Proof. 1. Assume that $A \leq_1^p B$ via $f(x)$ and let $p(n) = \max\{|f(x)| : x \in \{0\}^* \;\&\; |x| \leq n\}$. Then both $A \leq^{\#} B$ and $\bar{A} \leq^{\#} \bar{B}$ via p. For the other direction, assume that both $A \leq^{\#} B$ and $\bar{A} \leq^{\#} \bar{B}$ via p. Then the following function is a polynomial time one-one reduction from A to B.

$$f(x) = \begin{cases} \text{the } n\text{th element of } B & \text{if } x \text{ is the } n\text{th element of } A, \\ \text{the } n\text{th element of } \bar{B} & \text{if } x \text{ is the } n\text{th element of } \bar{A}. \end{cases}$$

2. If A is polynomial time isomorphic to B, then it is straightforward that $A \equiv_1^p B$. For the other direction, let f be defined as above, and g be defined similarly by changing the positions of A and B. Then $f(g(x)) = x$ for all $x \in \{0\}^*$. That is, f is a polynomial time isomorphism between A and B. □

Corollary 1. *For $A \in \mathbf{P}_T$, $\mathbf{deg}_1(A) = \{B : B$ is polynomial time isomorphic to $A\}$.*

The above corollary shows that, in $\mathbf{P}_T$, the polynomial time one-one degree structure is the same as the polynomial time isomorphic type structure.

Now we use $\leq^{\#}$ to give a partition of $\mathbf{P}_T$.

Definition 3. *For an infinite and co-infinite set $A \in \mathbf{P}_T$, we have the following definitions.*

1. *A is* dense *if $\{0\}^* \leq^{\#} A$.*
2. *A is* co-dense *if $\{0\}^* \leq^{\#} \bar{A}$ (that is, $\bar{A}$ is dense).*
3. *A is* bi-dense *if $\{0\}^* \leq^{\#} A$ and $\{0\}^* \leq^{\#} \bar{A}$ (that is, both A and $\bar{A}$ are dense).*
4. *A is* scattered *if A is neither dense nor co-dense.*

Let D, $\hat{D}$, C, S, F, $\hat{F}$ be the classes of dense, co-dense, bi-dense, scattered, finite, co-finite sets in $\mathbf{P}_T$, respectively, and let $D^- = D - C$, $\hat{D}^- = \hat{D} - C$. The corresponding degree classes are denoted by the corresponding boldface letters.

Lemma 2. *The classes D^-, $\hat{D}^-$, C, S, F, $\hat{F}$ partition $\mathbf{P}_T$. Moreover, these classes are closed under $\equiv_1$. Whence $\mathbf{D}^-$, $\hat{\mathbf{D}}^-$, $\mathbf{C}$, $\mathbf{S}$, $\mathbf{F}$, $\hat{\mathbf{F}}$ is a partition of $\mathbf{deg}_1(\mathbf{P}_T)$.*

Proof. Straightforward. □

By Lemma 2, in order to characterize $\mathbf{deg}_1(\mathbf{P}_T)$, it suffices to study the classes $\mathbf{D}^-$, $\hat{\mathbf{D}}^-$, $\mathbf{C}$, $\mathbf{S}$, $\mathbf{F}$, $\hat{\mathbf{F}}$ and the relations to each other. We first reduce this task by exploiting a symmetry in $\mathbf{P}_T$ and by eliminating the trivial classes.

Lemma 3. *1. For $A, B \in \mathbf{P}_T$, $A \leq_1^p B$ (via f) if and only if $\bar{A} \leq_1^p \bar{B}$ (via f).*
2. The function which maps a degree of a set to the degree of its complement set is an automorphism of $\langle \mathbf{deg}_1(\mathbf{P}_T); \leq \rangle$.

Proof. Straightforward. □

Lemma 3 implies the following lemma.

Lemma 4. *1. Let $A \in \mathbf{P}_T$. Then $A \in C$ (D, D^-, S, F) if and only if $\bar{A} \in C$ ($\hat{D}, \hat{D}^-, S, \hat{F}$).*
2. $\langle \mathbf{D}; \leq \rangle \cong \langle \hat{\mathbf{D}}; \leq \rangle$; $\langle \mathbf{D}^-; \leq \rangle \cong \langle \hat{\mathbf{D}}^-; \leq \rangle$ and $\langle \mathbf{F}; \leq \rangle \cong \langle \hat{\mathbf{F}}; \leq \rangle$.

Proof. Straightforward. □

The trivial classes are $\mathbf{C}$, $\mathbf{F}$, $\hat{\mathbf{F}}$.

Lemma 5. *For $A \in \mathbf{P}_T$, $A \in C$ if and only if A is $\leq_1^p$-complete for $\mathbf{P}_T$.*

Proof. Straightforward. □

By Lemma 5, $\mathbf{C}$ consists of a single element $\mathbf{1}$ which is the greatest element of $\langle \mathbf{deg}_1(\mathbf{P}_T); \leq \rangle$. In case of the other trivial classes $\mathbf{F}$ and $\hat{\mathbf{F}}$, by Lemma 4, it suffices to consider $\mathbf{F}$.

Lemma 6. $\mathbf{F}$ *is a linear ordering of order type ω where the n-th degree in this ordering contains the sets of cardinality $n - 1$.*

Proof. Straightforward. □

Note that Lemma 6 implies that $\langle \mathbf{F}; \leq \rangle$ is a distributive sublattice of $\langle \mathbf{deg}_1(\mathbf{P}_T); \leq \rangle$ (in particular closed under joins and meets). The relations of $\langle \mathbf{F}; \leq \rangle$ and $\langle \hat{\mathbf{F}}; \leq \rangle$ to the other parts of $\langle \mathbf{deg}_1(\mathbf{P}_T); \leq \rangle$ are completely described by the following lemma.

Lemma 7. *1. For $\mathbf{a} \in \mathbf{F}$ and $\mathbf{b} \notin \mathbf{F}$, $\mathbf{a} \leq \mathbf{b}$ if and only if $\mathbf{b} \in \hat{\mathbf{D}}$.*
2. For $\mathbf{a} \in \hat{\mathbf{F}}$ and $\mathbf{b} \notin \hat{\mathbf{F}}$, $\mathbf{a} \leq \mathbf{b}$ if and only if $\mathbf{b} \in \mathbf{D}$.

3. *For* $\mathbf{a} \in \mathbf{F}$*,* $\hat{\mathbf{a}} \in \hat{\mathbf{F}}$ *and* $\mathbf{b} \in \mathbf{P}_T$*, both* $\mathbf{a} \leq \mathbf{b}$ *and* $\hat{\mathbf{a}} \leq \mathbf{b}$ *if and only if* $\mathbf{b}=\mathbf{1}$.
4. *For* $\mathbf{a} \in \mathbf{F}$ *and* $\mathbf{b} \in \mathbf{P}_T$*,* $\mathbf{b} \leq \mathbf{a}$ *implies* $\mathbf{b} \in \mathbf{F}$.
5. *For* $\hat{\mathbf{a}} \in \hat{\mathbf{F}}$ *and* $\mathbf{b} \in \mathbf{P}_T$*,* $\mathbf{b} \leq \hat{\mathbf{a}}$ *implies* $\mathbf{b} \in \hat{\mathbf{F}}$.

Proof. Straightforward. □

This leaves the classes $\mathbf{D}$, $\hat{\mathbf{D}}$ (or equivalently $\mathbf{D}^-$, $\hat{\mathbf{D}}^-$) and $\mathbf{S}$ (and their relations to each other) for analysis. In the next section, we study the structure of $\langle \hat{\mathbf{D}}; \leq \rangle$, which by Lemma 4 is isomorphic to $\langle \mathbf{D}; \leq \rangle$. The structure of $\langle \mathbf{S}; \leq \rangle$ is analyzed in Section 4.

3. The Class $\hat{\mathbf{D}}$

In this section, we investigate the p-isomorphic type structure of $\hat{D}$. The class $PTGF$ of polynomial time constructible growth functions are used to study $\langle \mathbf{deg}_1(\hat{D}); \leq \rangle$, and it is shown that $\langle \mathbf{deg}_1(\hat{D}); \leq \rangle \cong \langle \mathbf{PTGF}; \leq \rangle$, where $\mathbf{PTGF}$ is the degree structure of $PTGF$ under the reduction $\leq^{\#}$. Then we show that $\langle \mathbf{PTGF}; \leq \rangle$ is a countable distributive lattice with the greatest element. The cap, cup and lattice embedding properties of $\langle \mathbf{deg}_1(\hat{D}); \leq \rangle$ are discussed in this section. For the product property, we show that, for any n, any interval in $\langle \mathbf{deg}_1(\hat{D}); \leq \rangle$ is a subdirect product of n-chains. We also show that there are infinitely many intervals in $\langle \mathbf{deg}_1(\hat{D}); \leq \rangle$ which can be distinguished by first order formulas. In the last part of this section, we show that there are infinitely many nontrivial automorphisms for $\langle \mathbf{deg}_1(\hat{D}); \leq \rangle$.

3.1. *Basic facts*

In this section we show that $\langle \hat{\mathbf{D}}; \leq \rangle$ is a countable distributive lattice. First we note that the structure of $\langle \hat{\mathbf{D}}; \leq \rangle$ is closely related to the structure of $\langle \mathbf{P}_T; \leq^{\#} \rangle$. To show this we use the following observations.

Lemma 8. *1. For* $A \in \mathbf{P}_T$*,* $A \in \hat{D}$ *if and only if there exists* $B \in \mathbf{P}_T - F$ *such that* $A \equiv_1^p B \oplus \emptyset$.
2. *For* $A, B \in \mathbf{P}_T$*,* $A \oplus \emptyset \leq_1^p B \oplus \emptyset$ *if and only if* $A \leq^{\#} B$.
3. *For* $A, B \in \hat{D}$*,* $A \leq_1^p B$ *if and only if* $A \leq^{\#} B$.

Proof. Straightforward. □

Lemma 9. $\langle \hat{D}; \leq_1^p \rangle \cong \langle \mathbf{P}_T - F; \leq^{\#} \rangle$.

Proof. Follows from Lemma 8. □

By Lemma 9, we may analyze the structure of $\langle \mathbf{P}_T - F; \leq^{\#} \rangle$ instead of $\langle \hat{D}; \leq_1^p \rangle$. Since the relation $\leq^{\#}$ depends on the census functions of the sets which are compared, it is useful to introduce the type of census functions which may occur and the inverse functions describing the growth of the sets.

Definition 4. *1. A function* $f: \omega \to \omega$ *is* an (eligible) census function *if*
– f *is polynomial time computable w.r.t. the unary representation of numbers.*

– $f(0) = 0$.
– $\lim_n f(n) = \infty$.
– *For all n,* $f(n) \leq f(n+1) \leq f(n) + 1$.

2. *For census functions* f *and* g, $f \leq^{\#} g$ *if and only if there is a polynomial* p *such that* $f(n) \leq g(p(n))$ *for all* n.

Let $\langle PTCF; \leq^{\#} \rangle$ be the class of census functions with the relation $\leq^{\#}$ defined as above and **PTCF** be the quotient structure of $PTCF$ under $\equiv^{\#}$. For $A \in \mathbf{P}_T$, let $cens_A(n) = \|A^{\leq n}\|$. We call the “inverse" function of a census function a growth function.

Definition 5. *1. A function* $f : \omega \to \omega$ *is* an eligible growth function *if*

– f *is polynomial time constructible w.r.t. the unary representation of numbers, that is, there exists a polynomial p such that* $f(n)$ *is computable in time* $p(f(n))$.
– f *is strictly increasing.*
– $f(0) = 0$.

2. *For growth functions* f *and* g, $f \leq^{\#} g$ *if and only if there is a polynomial* p *such that* $g(n) \leq p(f(n))$ *for all* n.

Let $\langle PTGF; \leq^{\#} \rangle$ be the class of growth functions with the relation $\leq^{\#}$ defined as above and **PTGF** be the quotient structure of $PTGF$ under $\equiv^{\#}$. For $A \in \mathbf{P}_T - F$, let $grow_A(n) = \mu m \ (cens_A(m) = n)$.

Lemma 10. *1. For* $A \in \mathbf{P}_T - F$, $cens_A \in PTCF$ *and* $grow_A \in PTGF$.

2. *For* $f \in PTCF$, *there is a unique set* $A \in \mathbf{P}_T - F$ *such that* $f = cens_A$. *For* $f \in PTGF$, *there is a unique set* $A \in \mathbf{P}_T - F$ *such that* $f = grow_A$.
3. *For* $A \in \mathbf{P}_T - F$, $cens_A(grow_A(n)) = n$ *and* $grow_A(cens_A(n)) = \mu m \ (cens_A (m) = cens_A(n))$.
4. $\langle PTGF; \leq^{\#} \rangle \cong \langle PTCF; \leq^{\#} \rangle \cong \langle \mathbf{P}_T - F; \leq^{\#} \rangle (\cong \langle \hat{D}; \leq_1^p \rangle)$.

Proof. Straightforward. □

Now the first question to ask about $\langle \hat{\mathbf{D}}; \leq \rangle$ (or equivalently, $\langle \mathbf{PTGF}; \leq \rangle$) is whether this structure is a lattice.

Theorem 1. $\langle \mathbf{PTGF}; \leq \rangle$ *(hence* $\langle \hat{\mathbf{D}}; \leq \rangle$*) is a countable distributive lattice with the greatest element.*

To prove the theorem we first introduce representations of the meet and join operator.

Definition 6. *For* $f, g \in PTGF$, *let* $\min_{f,g}(n) = \min\{f(n), g(n)\}$ *and* $\max_{f,g}(n) = \max\{f(n), g(n)\}$.

Lemma 11. *Let* $f, g, h \in PTGF$. *Then we have the following properties.*

1. $\min_{f,g} \in PTGF$ *and* $\max_{f,g} \in PTGF$.
2. $f \leq^{\#} \min_{f,g}$ *and* $g \leq^{\#} \min_{f,g}$.

3. $\max_{f,g} \leq^{\#} f$ *and* $\max_{f,g} \leq^{\#} g$.
4. If $h \leq^{\#} f$ *and* $h \leq^{\#} g$*, then* $h \leq^{\#} \max_{f,g}$.
5. If $f \leq^{\#} h$ *and* $g \leq^{\#} h$*, then* $\min_{f,g} \leq^{\#} h$.

Proof. Straightforward. □

Proof of Theorem 1. By Lemma 11, the join and meet operation (modulo factorization) in $\langle PTGF; \leq^{\#}\rangle$ is given by the max and min operators of Definition 6. To show that the lattice is distributive we have to verify that, for $f, g, h \in PTGF$, $\min_{\max_{f,g},h} = \max_{\min_{f,h},\min_{g,h}}$. That is, by definition,

$$\min\{\max\{f(n), g(n)\}, h(n)\} = \max\{\min\{f(n), h(n)\}, \min\{g(n), h(n)\}\}.$$

But the latter is immediate by distributivity of the maximum and minimum functions on the natural numbers. Finally, $f(n) = n$ is the greatest element of $PTGF$. □

Remark. Later in this paper, $f, g, \ldots$ denote elements of $PTGF$, and $\mathbf{f}, \mathbf{g}, \ldots$ denote the corresponding elements of **PTGF**. For $f, g \in PTGF$, $f \cup g$ and $f \cap g$ denote $\min_{f,g}$ and $\max_{f,g}$, respectively.

We close this section with some definitions which will be used in the following sections.

Definition 7. *For* $\mathbf{f}, \mathbf{g} \in \mathbf{PTGF}$ *with* $\mathbf{f} < \mathbf{g}$*, we have the following definitions.*

1. $\mathbf{f}$ *is* weakly below $\mathbf{g}$*, written* $\mathbf{f} \leq_w \mathbf{g}$*, if there exist* $f \in \mathbf{f}, g \in \mathbf{g}$ *such that* $f(n) = g(n)$ *infinitely often.*
2. $\mathbf{f}$ *is* strongly below $\mathbf{g}$*, written* $\mathbf{f} <_s \mathbf{g}$*, if for all* $f \in \mathbf{f}, g \in \mathbf{g}$*,* $f(n) \neq g(n)$ *almost everywhere.*

Lemma 12. *1. For* $\mathbf{f} \leq_w \mathbf{g}$ *and* $\mathbf{h} \in [\mathbf{f}, \mathbf{g}]$*,* $\mathbf{f} \leq_w \mathbf{h} \leq_w \mathbf{g}$.
2. If $\mathbf{f} <_s \mathbf{h} <_s \mathbf{g}$*, then* $\mathbf{f} <_s \mathbf{g}$.
3. For $\mathbf{f} <_s \mathbf{g}$*, there exists* $\mathbf{h} \in [\mathbf{f}, \mathbf{g}]$ *such that* $\mathbf{f} \leq_w \mathbf{h}$ *and* $\mathbf{h} \leq_w \mathbf{g}$.

Proof. Straightforward. □

3.2. Cap and cup properties

In this section, we study the cap and cup properties of $\langle \mathbf{PTGF}; \leq\rangle$.

Definition 8. *Given* $\mathbf{u}, \mathbf{v} \in \mathbf{PTGF}$ *and* $\mathbf{f}, \mathbf{g} \in [\mathbf{u}, \mathbf{v}]$ *such that* $\mathbf{f} < \mathbf{g}$*, we have the following definitions.*

1. $\mathbf{f}$ *is* cuppable *to* $\mathbf{g}$ *(in the interval* $(\mathbf{u}, \mathbf{v})$*) if there exists* $\mathbf{w} \in (\mathbf{u}, \mathbf{g})$ *such that* $\mathbf{w} \cup \mathbf{f} = \mathbf{g}$.
2. $\mathbf{g}$ *is* cappable *to* $\mathbf{f}$ *(in the interval* $(\mathbf{u}, \mathbf{v})$*) if there exists* $\mathbf{w} \in (\mathbf{f}, \mathbf{v})$ *such that* $\mathbf{w} \cap \mathbf{g} = \mathbf{f}$.

Lemma 13. *Given* $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbf{PTGF}$*, the following properties hold.*

1. $\mathbf{f} \leq \mathbf{g}$ *if and only if there exist* $f \in \mathbf{f}$ *and* $g \in \mathbf{g}$ *such that* $f(n) \geq g(n)$ *for all* n.
2. $\mathbf{h} \in [\mathbf{f}, \mathbf{g}]$ *if and only if there exist* $f \in \mathbf{f}, g \in \mathbf{g}$, *and* $h \in \mathbf{h}$ *such that* $f(n) \geq h(n) \geq g(n)$ *for all* n.
3. $\mathbf{f} \cup \mathbf{g} = \mathbf{h}$ *if and only if there exist* $f \in \mathbf{f}, g \in \mathbf{g}$ *and* $h \in \mathbf{h}$ *such that* $h = \min_{f,g}$.
4. $\mathbf{f} \cap \mathbf{g} = \mathbf{h}$ *if and only if there exist* $f \in \mathbf{f}, g \in \mathbf{g}$, *and* $h \in \mathbf{h}$ *such that* $h = \max_{f,g}$.
5. *For* $\mathbf{u}, \mathbf{v} \in [\mathbf{f}, \mathbf{g}]$, $\mathbf{u} \cup \mathbf{v} = \mathbf{g}$ *and* $\mathbf{u} \cap \mathbf{v} = \mathbf{f}$ *if and only if there exist* $f \in \mathbf{f}, g \in \mathbf{g}, u \in \mathbf{u}$, *and* $v \in \mathbf{v}$ *such that* $f = \max_{u,v}$ *and* $g = \min_{u,v}$.

Proof. Straightforward (see also Lemma 11 and Definition 4). □

Theorem 2. *Let* $\mathbf{f}, \mathbf{g} \in \mathbf{PTGF}$ *such that* $\mathbf{f} < \mathbf{g}$. *Then there exists* $\mathbf{h} \in (\mathbf{f}, \mathbf{g})$ *such that* $\mathbf{h}$ *is not cuppable to* $\mathbf{g}$ *in the interval* $(\mathbf{f}, \mathbf{g})$.

Proof. Let $f \in \mathbf{f}$ and $g \in \mathbf{g}$ such that $f(n) \geq g(n)$ for all n. Define $h \in PTGF$ by letting

$$h(n) = \left\lceil 2^{\sqrt{\log(f(n)) \cdot \log(g(n))}} \right\rceil.$$

Then $\mathbf{h} \in (\mathbf{f}, \mathbf{g})$. In the following we show that $\mathbf{h}$ is not cuppable to $\mathbf{g}$ in the interval $(\mathbf{f}, \mathbf{g})$.

It suffices to show that for any $\mathbf{u} \in (\mathbf{f}, \mathbf{g})$ we have $g \not\leq^{\#} \min_{h,u}$. Let $u \in \mathbf{u}$ with the property that $f(n) \geq u(n) \geq g(n)$ for all n. For any $k > 0$, by the property that $\mathbf{u} < \mathbf{g}$, we have

$$f(n) \geq u(n) > (g(n))^{k^2}$$

for infinitely many n. Whence there are infinitely n satisfying the following property:

$$\begin{aligned} \min_{h,u}(n) &= \min\left\{\left\lceil 2^{\sqrt{\log(f(n)) \cdot \log(g(n))}} \right\rceil, u(n)\right\} \\ &> \min\left\{\left\lceil 2^{\sqrt{\log(g(n))^{k^2} \cdot \log(g(n))}} \right\rceil, u(n)\right\} \\ &\geq \min\left\{(g(n))^k, u(n)\right\} \\ &\geq (g(n))^k. \end{aligned}$$

The above inequality implies that $g \not\leq^{\#} \min_{h,u}$. This completes the proof of the theorem. □

Corollary 2. *There exist* $f, g, h \in PTGF$ *such that* $f <^{\#} g <^{\#} h$ *and* $\mathbf{g}$ *is cuppable to* $\mathbf{h}$, *but* $\mathbf{g}$ *is not cuppable to* $\mathbf{h}$ *in the interval* $[\mathbf{f}, \mathbf{h}]$.

In the same way as in the proof of Theorem 2, we can prove the following theorem.

Theorem 3. *Let* $\mathbf{f} < \mathbf{g}$. *Then there exists a countably infinite set of independent degrees which are not cuppable to* $\mathbf{g}$ *in the interval* $(\mathbf{f}, \mathbf{g})$.

Proof. The proof is left as an exercise. □

Theorem 4. 1. *Let* $\mathbf{f} < \mathbf{1}$. *Then there exists* $\mathbf{g}$ *such that* $\mathbf{f} <_w \mathbf{g}$ *and* $\mathbf{g}$ *is cappable to* $\mathbf{f}$.
2. *Let* $\mathbf{f} < \mathbf{1}$. *Then there exists* $\mathbf{g}$ *such that* $\mathbf{f} < \mathbf{g}$ *and* $\mathbf{g}$ *is not cappable to* $\mathbf{f}$.

Proof. 1. We construct two polynomial time constructible growth functions g and h by stages. It suffices to satisfy for all e the following requirements:

$R_e : \exists n > e\ (f(n) > (g(n))^e)$.

$S_e : \exists n > e\ (f(n) > (h(n)^e)$.

$T : \forall n\ (f(n) = \max\{g(n), h(n)\})$.

The construction is standard, and we will omit the details.

2. Let $f \in \mathbf{f}$ such that $f(n) \geq n$ for all n. Define $g \in PTGF$ by letting

$$g(n) = \left\lceil 2^{\sqrt{\log(f(n))\cdot\log n}} \right\rceil.$$

It is straightforward to check that $\mathbf{g} \in (\mathbf{f}, \mathbf{1})$. Now a similar argument as in the proof of Theorem 2 can be used to show that $\mathbf{g}$ is not cappable to $\mathbf{f}$, the details are omitted here. □

We close this section by introducing some notation which will be used later.

Definition 9. *Let* $\mathbf{f} \leq \mathbf{g}$. *Then we have the following definitions.*

1. $[\mathbf{f}, \mathbf{g}]_{cap} = \{\mathbf{u} : \mathbf{u} \in (\mathbf{f}, \mathbf{g}), \mathbf{u}$ *is cappable to* $\mathbf{f}$ *in the interval* $(\mathbf{f}, \mathbf{g})\}$.
2. $[\mathbf{f}, \mathbf{g}]_{cup} = \{\mathbf{u} : \mathbf{u} \in (\mathbf{f}, \mathbf{g}), \mathbf{u}$ *is cuppable to* $\mathbf{g}$ *in the interval* $(\mathbf{f}, \mathbf{g})\}$.
3. $[\mathbf{f}, \mathbf{g}]_{cc} = [\mathbf{f}, \mathbf{g}]_{cap} \cap [\mathbf{f}, \mathbf{g}]_{cup}$.
4. $[\mathbf{f}, \mathbf{g}]_{comp} = \{\mathbf{u} :$ *there exists* $\mathbf{v} \in (\mathbf{f}, \mathbf{g})$ *such that* $\mathbf{u} \cap \mathbf{v} = \mathbf{f}$ *and* $\mathbf{u} \cup \mathbf{v} = \mathbf{g}\}$.

Using the same technique as in the proofs of Theorem 2 and Theorem 3, we can prove the following theorem.

Theorem 5. *Let* $\mathbf{f} <_w \mathbf{g}$. *Then* $\|[\mathbf{f}, \mathbf{g}]_{cc} - [\mathbf{f}, \mathbf{g}]_{comp}\| = \|[\mathbf{f}, \mathbf{g}]_{cap} - [\mathbf{f}, \mathbf{g}]_{cc}\| = \|[\mathbf{f}, \mathbf{g}]_{cap}\| = \|[\mathbf{f}, \mathbf{g}]_{cup}\| = \|[\mathbf{f}, \mathbf{g}]_{comp}\| = \infty$

The proofs are left as an exercise.

3.3. Lattices embedding

In this section, we show that any countable distributive lattice can be embedded into any intervals of $\langle \mathbf{PTGF}; \leq \rangle$ preserving either the least or the greatest element. At first, we introduce some concepts.

Definition 10. *For* $f, g \in PTGF$ *such that* $f(n) \geq g(n)$ *for all n, let*

$$I_{fg}(n) = \Sigma_{i=1}^{n} |sign(f(i) - g(i)) - sign(f(i-1) - g(i-1))|,$$

$$meet'_{fg}(n) = \mu m\ (1 + I_{fg}(m) = 2n),$$

$$meet''_{fg}(n) = \mu m\ (I_{fg}(m) = 2n),$$

where $sign(x) = \begin{cases} 1 & if\, x \neq 0, \\ 0 & if\, x = 0. \end{cases}$

Definition 11. *Given* $A \in \mathbf{TALLY}$ *and* $f, g \in PTGF$ *such that* $f(n) = g(n)$ *for infinitely many* n*,* A *is called* polynomial time computable relative to f, g *if there exists a polynomial* p *such that* A *is computable in time* $p(f(meet'_{fg}))$*, written* $A \in \mathbf{P}_T^{[f,g]}$*. For the set* $A \in \mathbf{P}_T^{[f,g]}$*, we define a growth function* u_A *by letting*

$$u_A(n) = \begin{cases} f(n) & \text{if } n \in [meet'_{fg}(m), meet''_{fg}(m)] \text{ for some } 0^m \notin A, \\ g(n) & \text{otherwise.} \end{cases}$$

Definition 12. *For* $A \in \mathbf{P}_T^{[f,g]}$*,* A *is called* semi-good *if* $\mathbf{f} < \mathbf{u}_A$*.* A *is called* good *if both* A *and* $\bar{A}$ *are semi-good.*

Lemma 14. *Let* $f, g \in PTGF$ *and* $A \in \mathbf{P}_T^{[f,g]}$ *be semi-good. Then there exists* $B \in \mathbf{P}_T^{[f,g]}$ *such that* B *is good and* $\mathbf{u}_B \leq \mathbf{u}_A$*.*

Proof. It suffices to construct a set $B \in \mathbf{P}_T^{[f,g]}$ satisfying for all e the following requirements:

R_e : There exists $n > e$ such that $u_B(n) > (g(n))^e$.

S_e : There exists $n > e$ such that $f(n) > (u_B(n))^e$.

T: $u_B(n) \geq u_A(n)$ for all n.

The construction is standard, which is omitted here. □

Theorem 6. *Let* $\mathbf{f} \leq_w \mathbf{g} \in \mathbf{PTGF}$*. Then the following hold.*

1. $[\mathbf{f}, \mathbf{g}]_{comp} = \{\mathbf{u}$: *there exist* $f \in \mathbf{f}$*,* $g \in \mathbf{g}$ *and* $A \in \mathbf{P}_T^{[f,g]}$ *such that* $\mathbf{u} = \mathbf{u}_A$ *and* A *is good* $\}$.
2. *If there exist* $f \in \mathbf{f}$, $g \in \mathbf{g}$ *such that* $f(n) \leq g(n+1)$ *for all* n*, then* $[\mathbf{f}, \mathbf{g}]_{cup} = \{\mathbf{u} : \mathbf{u} \in [\mathbf{v}, \mathbf{g}]$ *for some* $\mathbf{v} \in [\mathbf{f}, \mathbf{g}]_{comp}\}$.

Proof. 1. The inclusion $\supseteq$ is straightforward. Now we show the inclusion $\subseteq$ holds. For any $\mathbf{u} \in [\mathbf{f}, \mathbf{g}]_{comp}$, by Lemma 13, there exist u, v, f, g such that $f = \max_{u,v}$ and $g = \min_{u,v}$. Let A be defined by

$$0^n \in A \iff g(meet'_{fg}(n)) = u(meet'_{fg}(n)).$$

Then it is easily checked that $\mathbf{u} = \mathbf{u}_A$, $\mathbf{v} = \mathbf{u}_{\bar{A}}$ and A is good.

2. For the nontrivial implication, let $\mathbf{u}, \mathbf{v} \in [\mathbf{f}, \mathbf{g}]_{cup}$, and let $u \in \mathbf{u}$, $v \in \mathbf{v}$, $g \in \mathbf{g}$ such that $g = \min_{u,v}$. Define $B \in \mathbf{P}_T^{[f,g]}$ by

$$0^n \in B \text{ if and only if } f(meet'_{fg}(n)) = v(meet'_{fg}(n)).$$

Then it is easily checked that $\mathbf{u} \in [\mathbf{u}_B, \mathbf{g}]$ and B is semi-good. By Lemma 14, there exists a good set $A \in \mathbf{P}_T^{[f,g]}$ such that $\mathbf{u} \in [\mathbf{u}_A, \mathbf{g}]$. □

Let $f, g \in PTGF$ such that $f(n) = g(n)$ for infinitely many n, and let $A, B \in \mathbf{P}_T^{[f,g]}$. We say $A \equiv_{good} B$ if $(A - B) \cup (B - A)$ is not good.

Lemma 15. *Let $f, g \in PTGF$ such that $f(n) = g(n)$ for infinitely many n. Then $\{A : A \in \mathbf{P}_T^{[f,g]}, A$ is good$\}$ forms a countable infinite free Boolean algebra under the inclusion relation module $\equiv_{good}$.*

Proof. Straightforward. □

Theorem 7. *Let $\mathbf{f} <_w \mathbf{g}$. Then $[\mathbf{f}, \mathbf{g}]_{comp}$ is a countable infinite free Boolean algebra.*

Proof. Follows from Theorem 6 and Lemma 15. □

Corollary 3. *Let $\mathbf{f} < \mathbf{g}$. Then the following are equivalent.*

1. $\mathbf{f} <_w \mathbf{g}$.
2. *The four element Boolean algebra (diamond) can be embedded into the interval $[\mathbf{f}, \mathbf{g}]$ preserving both the least and the greatest elements.*
3. *$\mathcal{L}_\omega$ can be embedded into the interval $[\mathbf{f}, \mathbf{g}]$ preserving both the least and the greatest elements, where $\mathcal{L}_\omega$ is the countable atomless Boolean algebra.*

Corollary 4. *For $\mathbf{f} < \mathbf{g}$, $\mathcal{L}_\omega$ can be embedded into the interval $[\mathbf{f}, \mathbf{g}]$ preserving the least or the greatest element.*

Proof. Follows from Lemma 12 and Theorem 7. □

Corollary 5. *Let $\mathbf{f} < \mathbf{g}$. Then the 2-diamond can be embedded into the interval $[\mathbf{f}, \mathbf{g}]$ preserving both the least and the greatest elements.*

Definition 13. *Given two lattices $\mathcal{L}_1 = \langle L_1; \leq_1\rangle$ and $\mathcal{L}_2 = \langle L_2; \leq_2\rangle$, let $\mathcal{L}_1 \cdot \mathcal{L}_2 = \langle L_1 \cup L_2; \leq_3\rangle$, where for $x, y \in L_1 \cup L_1$, $x \leq_3 y$ if and only if one of the following holds.*

1. *$x, y \in L_1$ and $x \leq_1 y$.*
2. *$x, y \in L_2$ and $x \leq_2 y$.*
3. *$x \in L_1$ and $y \in L_2$.*

Corollary 6. *For $\mathbf{f} < \mathbf{g}$, $\mathcal{L}_\omega \cdot \mathcal{L}_\omega$ can be embedded into the interval $[\mathbf{f}, \mathbf{g}]$ preserving both the least and the greatest elements.*

3.4. Subdirect product of chains

In this section, we show that, for any n, any interval in $\langle \mathbf{PTGF}; \leq\rangle$ is a subdirect product of n-chains. We start with some definitions and theorems from algebra and lattice theory. For more details about the algebra and lattice theory, it is referred to [6], [8], and [11].

Definition 14. *An algebra A of type τ is said to be* a subdirect product *of a family $(A_s)_{s\in S}$ of type τ if there exists an embedding $f : A \to \Pi_{s\in S} A_s$ such that for each $s \in S$, $p_s \circ f$ (the composition of f and g) is onto (where p_s is the projection function on the sth index).*

Definition 15. *Given a chain K, let $C_K = \{L$: there is a subdirect embedding f from L to $\Pi_{s \in S} K_s$ where $K_s \cong K$ and S is any set$\}$.*

Theorem 8. *([6])*

1. *$C_M \subset C_\infty \subset \ldots \subset C_3 \subset C_2 = DL$, where DL is the class of distributive lattices and C_M is the class of distributive lattices which have neither maximal nor minimal prime ideals.*
2. *$B \subset R \subset C_2 - C_3$, where R is the class of all relatively complemented distributive lattices and B is the class of Boolean algebras.*

Definition 16. *For $L \in DL$, a chain $a_0 \leq a_1 \leq \ldots \leq a_{n-1}$ in L is* a locally separated n-chain *if the system*

$$\begin{aligned} a_1 \wedge x_0 &\leq a_0 \\ &\ldots \\ a_{i+1} \wedge x_i &\leq a_i \vee x_{i-1} \ (i = 1, 2, \ldots, n-3) \\ &\ldots \\ a_{n-1} &\leq a_{n-2} \vee x_{n-3} \end{aligned}$$

has no solution for $x_0, x_1, \ldots, x_{n-3}$ in L.

Theorem 9. *([6]) Let $L \in DL$. Then $L \in C_n$ if and only if, for each pair $a, b \in L$ with $a < b$, there is a locally separated n-chain containing both a and b.*

Theorem 10. *Let $\mathbf{f} < \mathbf{g}$. Then $\langle [\mathbf{f}, \mathbf{g}]; \leq, \cap, \cup \rangle \in C_n$ for all n.*

Proof. By Theorem 9, it suffices to prove that for each pair $\mathbf{f}_0, \mathbf{g}_0 \in [\mathbf{f}, \mathbf{g}]$ with $\mathbf{f}_0 < \mathbf{g}_0$ there exists a locally separated n-chain $\mathbf{f}_0 \leq \mathbf{f}_1 \leq \ldots \leq \mathbf{f}_{n-1} = \mathbf{g}_0$.

A standard construction can be used to construct $f_0, f_1, \ldots, f_{n-1}$ such that the following requirements are satisfied for all e.

$R_e : \exists m > e \ ((\forall i < n \ (f_{i-1}(m) > (f_i(m))^e))$.

Now, for a contradiction, assume that the above constructed $\mathbf{f}_0, \ \mathbf{f}_1, \ldots, \mathbf{f}_{n-1} = \mathbf{g}_0$ is not a locally separated n-chain. Then there exists a sequence $h_0, \ldots, h_{n-3} \in PTGF$ such that

$$\begin{aligned} \mathbf{f}_1 \cap \mathbf{h}_0 &\leq \mathbf{f}_0, \\ &\ldots \\ \mathbf{f}_{i+1} \cap \mathbf{h}_i &\leq \mathbf{f}_i \cup \mathbf{h}_{i-1} \ (i = 1, 2, \ldots, n-3), \\ &\ldots \\ \mathbf{f}_{n-1} &\leq \mathbf{f}_{n-2} \cup \mathbf{h}_{n-3}. \end{aligned}$$

Hence there exists $k_1 \geq 2$ such that for all m,

$$\begin{aligned} &(\max\{f_1(m), h_0(m)\})^{k_1} \geq f_0(m), \\ &\ldots \\ &(\max\{f_{i+1}(m), h_i(m)\})^{k_1} \geq \min\{f_i(m), h_{i-1}(m)\} \ (i = 1, 2, \ldots, n-3), \\ &\ldots \\ &(f_{n-1}(m))^{k_1} \geq \min\{f_{n-2}(m), h_{n-3}(m)\}. \end{aligned}$$

By the requirement $R_{k_1^2}$, there exists m_0 such that $f_{i-1}(m_0) > (f_i(m_0))^{k_1^2}$ for all $i < n$. By the above inequalities, we have,

$$\begin{array}{rcl} h_0(m_0) & > & f_1(m_0) \\ h_1(m_0) & > & f_2(m_0) \\ & \cdots & \\ h_i(m_0) & > & f_{i+1}(m_0) \\ & \cdots & \\ h_{n-3}(m_0) & > & f_{n-2}(m_0). \end{array}$$

Hence

$$\begin{aligned} (f_{n-1}(m_0))^{k_1} &\geq \min\{f_{n-2}(m_0), h_{n-3}(m_0)\} \\ &\geq f_{n-2}(m_0) \\ &> f_{n-1}(m_0)^{k_1^2}. \end{aligned}$$

This is a contradiction. □

3.5. Types of intervals

Now we turn our attention to the question of distinguishing intervals of $\langle \mathbf{PTGF}; \leq \rangle$ with first order formulas. We start with an example.

Example 1. There exist $\mathbf{f}, \mathbf{g}, \mathbf{u} \in \mathbf{PTGF}$ such that $\mathbf{u} \in [\mathbf{f},\ \mathbf{g}]_{cup}$ and $(\mathbf{f}, \mathbf{u}] \cap [\mathbf{f}, \mathbf{g}]_{comp} = \emptyset$.

Proof. Define the growth functions f, g, u, u' by

$$\begin{array}{l} f(0) = 0,\ \ f(2n+1) = 2^{2^{f(2n)}},\ \ f(2n+2) = f(2n+1) + 1, \\ g(i) = i\ (i \leq 2), g(2n+1) = f(2n) + 1, g(2n+2) = 2^{f(2n)}, \\ u(i) = i\ (i \leq 2), u(2n+1) = g(2n+1), u(2n+2) = f(2n+2), \\ u'(i) = i\ (i \leq 2), u'(2n+1) = g(2n+2), u'(2n+2) = g(2n+2) + 1. \end{array}$$

It is straightforward to check that $\mathbf{u} < \mathbf{g}, \mathbf{u}' < \mathbf{g}$ and $\mathbf{u} \cup \mathbf{u}' = \mathbf{g}$. That is, $\mathbf{u} \in [\mathbf{f}, \mathbf{g}]_{cup}$.

In the following we show that $(\mathbf{f}, \mathbf{u}] \cap [\mathbf{f}, \mathbf{g}]_{comp} = \emptyset$. For a contradiction, assume that there exist $\mathbf{v} \in (\mathbf{f}, \mathbf{u}]$, $\mathbf{v}' \in [\mathbf{f}, \mathbf{g}]$ such that $f = \max_{v,v'}$ and $g = \min_{v,v'}$. Then $f(2n+2) \geq v(2n+2) \geq u(2n+2) = f(2n+2)$, that is, $f(2n+2) = v(2n+2)$. This implies that $g(2n+2) = v'(2n+2)$. Since $g(2n+2) < f(2n+1)$, we must have $f(2n+1) = v(2n+1)$. This again implies that $g(2n+1) = v'(2n+1)$. That is, we have $f(n) = v(n)$ and $g(n) = v'(n)$ for all n. A contradiction with the assumption that $\mathbf{v} \in (\mathbf{f}, \mathbf{u}]$. □

Theorem 11. *There exist two intervals* $[\mathbf{u}, \mathbf{v}]$ *and* $[\mathbf{f}, \mathbf{g}]$ *such that*

$$Th(\langle [\mathbf{f}, \mathbf{g}]; \leq, \cap, \cup \rangle) \neq Th(\langle [\mathbf{u}, \mathbf{v}]; \leq, \cap, \cup \rangle),$$

where $Th(\mathcal{M})$ *denotes the first order theory of* $\mathcal{M}$.

Proof. This follows from Example 1 and (5) of Theorem 6. □

Theorem 11 shows that the property "$[\mathbf{f}, \mathbf{g}]_{cup} \supset \{\mathbf{u} : \mathbf{u} \in [\mathbf{v}, \mathbf{g}]$ for some $\mathbf{v} \in [\mathbf{f}, \mathbf{g}]_{comp}\}$" can be used to distinguish intervals. Indeed this property can be used to distinguish more intervals. First we define some formulas.

$\varphi_1(x, y) = \forall u\ (u \in [x, y]_{cup} \rightarrow (\exists v \in [x, y]_{comp}\ (v \leq u)))$,

$\varphi_2(x, y) = (\neg\varphi_1(x, y))\ \wedge\ (\exists u \in [x, y]\ (\varphi_1(x, u) \wedge \varphi_1(u, y)))$,

$\varphi_3(x, y) = (\neg\varphi_1(x, y)\ \wedge\ \neg\varphi_2(x, y))\ \wedge\ (\exists u \in [x, y]\ (\varphi_1(x, u)\ \wedge\ \varphi_2(u, y)))$,

......

$\varphi_{n+1}(x, y) = (\neg\varphi_1(x, y)\ \wedge\ \ldots\ \wedge\ \neg\varphi_n(x, y))\ \wedge\ (\exists u \in [x, y]\ (\varphi_1(x, u)\ \wedge\ \varphi_n(u, y)))$,

......

Lemma 16. *For any intervals* $[\mathbf{f}_1, \mathbf{g}_1]$ *and* $[\mathbf{f}_2, \mathbf{g}_2]$, *if* $i \neq j$ *and both* $\varphi_i(\mathbf{f}_1, \mathbf{g}_1)$ *and* $\varphi_j(\mathbf{f}_2, \mathbf{g}_2)$ *are true, then*

$$Th(\langle[\mathbf{f}_1, \mathbf{g}_1];\ \leq, \cap, \cup\rangle) \neq Th(\langle[\mathbf{f}_1, \mathbf{g}_2];\ \leq, \cap, \cup\rangle).$$

Proof. Straightforward. □

Lemma 17. *There exist* $\mathbf{f}_1 \leq \mathbf{g}_1 \leq \mathbf{g}_2 \leq \ldots$ *such that* $\varphi_i(\mathbf{f}_1, \mathbf{g}_i)$ *is true for all* i.

Proof. The construction of the sequence $\mathbf{f}_1 \leq \mathbf{g}_1 \leq \mathbf{g}_2 \leq \ldots$ is the same as the construction in the proof of Example 1, the details are omitted here. □

Theorem 12. *There exist infinitely many intervals that can be distinguished by first order properties.*

Proof. This follows from Lemmas 16 and 17. □

3.6. Nontrivial automorphisms

Up to now, we have only considered the first order theory of $\langle\hat{\mathbf{D}}; \leq\rangle$. As a first step of a detailed study of the second order theory of $\langle\hat{\mathbf{D}}; \leq\rangle$, in this section we construct infinitely many nontrivial automorphisms for $\langle\hat{\mathbf{D}}; \leq\rangle$. The outline of the proof is as follows: In Theorem 13, we will construct isomorphism $\mathcal{F}_A$ from $\langle\hat{\mathbf{D}}; \leq\rangle$ to $\langle(\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}}; \leq\rangle$ for certain set A. In Theorem 14, we will construct isomorphism $\mathcal{G}_A$ from $\langle\hat{\mathbf{D}}; \leq\rangle$ to $\langle(\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}}; \leq\rangle$ for certain set A. Then we will show that for infinitely many A, $\mathcal{G}_A^{-1}\mathcal{F}_A$ is a nontrivial isomorphism of $\langle\hat{\mathbf{D}}; \leq\rangle$.

Theorem 13. *Let* $A \in \hat{D}$ *such that, for each polynomial* $p(n)$, *there exist polynomials* p_1 *and* p_2 *satisfying*

1. $p(cens_A(n)) \leq cens_A(p_1(n))$.
2. $cens_A(p(n)) \leq p_2(cens_A(n))$.

Then $\langle\mathbf{PTGF}; \leq, \cup, \cap\rangle \cong \langle(\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}}; \leq, \cup, \cap\rangle$.

Proof. Define the mapping $\mathcal{F}_A$ from $PTGF$ to $\mathcal{B} = \{C : C \subseteq A\} \cap \mathbf{P}_T$ by

$$\mathcal{F}_A : f \mapsto C_f$$

such that $grow_{C_f}(n) = grow_A(f(n))$.

It is straightforward to check that $\mathcal{F}_A$ is an onto mapping and $cens_{C_f}(n) = cens_f(cens_A(n))$ for all n. It suffices to show that $\mathcal{F}_A$ induces an isomorphism between $\langle \mathbf{PTGF}; \leq, \cup, \cap\rangle$ and $\langle(\cdot, \mathbf{deg}_1(A)]\cap\hat{\mathbf{D}}; \leq, \cup, \cap\rangle$. We do this by establishing a series of claims.

Claim 1. *Let $f(n) = n$. Then $\mathcal{F}_A(f) = A$.*

Proof. Straightforward.

Claim 2. *The mapping $\mathcal{F}_A$ induces an onto mapping from* **PTGF** *to* $(\cdot, \mathbf{deg}_1(A)]\cap \hat{\mathbf{D}}$.

Proof. It suffices to show that for all $C \in \hat{D}$ with $C \leq_1^p A$, there exists $C_1 \in \mathcal{B}$ such that $C \equiv_1^p C_1$.

Construction of C_1 from C.
$C_1(0) = 0$.
Stage s.
Let $x = 0^s$. For the definition of $C_1(x)$, we distinguish the following two cases.

1. $x \notin C$.
If $C_1(x)$ has been defined then do nothing, else let $C_1(x) = 0$. Go to stage $s + 1$.
2. $x \in C$.
Fix $x_0 = \min\{y : y \in A, C_1(y) \uparrow\}$, let $C_1(x_0) = 1$ and, for all $y < x_0$ such that $C_1(y)$ is not defined yet, let $C_1(y) = 0$. Go to stage $s + 1$.

End of construction.

It is straightforward to check that $C_1 \in \mathcal{B}$ and $cens_{C_1}(n) \leq cens_C(n)$ for all n.

Now we show that $cens_C(n) \leq cens_{C_1}(p_1(p_3(n)))$, where $p_3(n)$ is the polynomial satisfying $cens_C(n) \leq cens_A(p_3(n))$ and $p_1(n)$ is the polynomial satisfying $grow_A(2cens_A(n)) \leq p_1(n)$, the existence of p_1 is insured by the first condition. Since

$$\begin{aligned}\|A^{[n,p_1(p_3(n))]}\| &\geq cens_A(p_1(p_3(n))) - cens_A(n)\\ &\geq cens_A(grow_A(2cens_A(p_3(n)))) - cens_A(n)\\ &= 2cens_A(p_3(n)) - cens_A(n)\\ &\geq cens_A(p_3(n))\\ &\geq cens_C(n),\end{aligned}$$

by the construction, $cens_C(n) \leq cens_{C_1}(p_1(p_3(n)))$. Hence, $C \equiv^{\#} C_1$. By (3) of Lemma 8, $C \equiv_1^p C_1$.

Claim 3. $f_1 \leq^{\#} f_2$ *if and only if* $C_{f_1} \leq_1^p C_{f_2}$.

Proof. First assume that $f_1 \leq^{\#} f_2$. Then there exists a polynomial p such that $cens_{f_1}(n) \leq cens_{f_2}(p(n))$. Let p_1 be the polynomial such that $p(cens_A(n)) \leq cens_A(p_1(n))$. Then

$$\begin{aligned} cens_{C_{f_1}}(n) &= cens_{f_1}(cens_A(n)) \\ &\leq cens_{f_2}(p(cens_A(n))) \\ &\leq cens_{f_2}(cens_A(p_1(n))) \\ &= cens_{C_{f_2}}(p_1(n)). \end{aligned}$$

That is, $C_{f_1} \leq^{\#} C_{f_2}$. Hence $C_{f_1} \leq_1^p C_{f_2}$.

For the other direction, assume that $C_{f_1} \leq_1^p C_{f_2}$. Then $C_{f_1} \leq^{\#} C_{f_2}$ via p for some polynomial p. Let p_2 be the polynomial such that $cens_A(p(n)) \leq p_2(cens_A(n))$. Then

$$\begin{aligned} f_2(n) &= cens_A(grow_A(f_2(n))) \\ &= cens_A(grow_{C_{f_2}}(n)) \\ &\leq cens_A(p(grow_{C_{f_1}}(n))) \\ &\leq p_2(cens_A(grow_{C_{f_1}}(n))) \\ &= p_2(cens_A(grow_A(f_1(n)))) \\ &= p_2(f_1(n)). \end{aligned}$$

Claim 4. *For $f_1, f_2 \in PTGF$, $\mathbf{deg}_1(C_{f_1 \cup f_2}) = \mathbf{deg}_1(C_{f_1}) \cup \mathbf{deg}_1(C_{f_2})$.*

Proof.

$$\begin{aligned} grow_{C_{f_1 \cup f_2}}(n) &= grow_A((f_1 \cup f_2)(n)) \\ &= \min\{grow_A(f_1(n)), grow_A(f_2(n))\} \\ &= \min\{grow_{C_{f_1}}(n), grow_{C_{f_2}}(n)\} \end{aligned}$$

Hence, by the proof of Lemma 10, $\mathbf{deg}_1(C_{f_1 \cup f_2}) = \mathbf{deg}_1(C_{f_1}) \cup \mathbf{deg}_1(C_{f_2})$.

Claim 5. *For $f_1, f_2 \in PTGF$, $\mathbf{deg}_1(C_{f_1 \cap f_2}) = \mathbf{deg}_1(C_{f_1}) \cap \mathbf{deg}_1(C_{f_2})$.*

Proof. The same as the proof of Claim 4.. □

Corollary 7. *Given $m \in \omega$, let A be defined by $grow_A(n) = n^{(\log n)^m}$. Then*

$$\langle \hat{\mathbf{D}}; \leq, \cap, \cup \rangle \cong \langle (\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}}; \leq, \cap, \cup \rangle \quad \text{via } \mathcal{F}_A.$$

Proof. First, it is obvious that $(\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}} = (\cdot, \mathbf{deg}_1(A)]$.

Let $p = n^k$. Then

$$\begin{aligned} p(cens_A(n)) &\leq 2^{k \sqrt[m+1]{\log n}} \\ &= 2^{\sqrt[m+1]{\log(n^{k^{1+m}})}} \\ &\leq cens_A(n^{k^{1+m}}) + 1 \\ &\leq cens_A(n^{(2k)^{m+1}}) \end{aligned}$$

and

$$\begin{aligned} cens_A(p(n)) &= [2^{\sqrt[m+1]{k \log n}}] \\ &\leq (2^{\sqrt[m+1]{\log n}})^{\sqrt[m+1]{k}} \\ &\leq (cens_A(n) + 1)^{\sqrt[m+1]{k}}. \end{aligned}$$

Now the corollary follows from Theorem 13. □

Theorem 14. *Let $A \in \hat{D}$ such that there exists k_0 satisfying*

$$\frac{\log(grow_A(n))}{\log(grow_A(n+i))} \leq \frac{k_0 \log n}{\log(n+i)}$$

for all n and i. Then $\langle \mathbf{PTGF};\ \leq, \cup, \cap\rangle$ can be embedded into $\langle (\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}};\ \leq, \cup, \cap\rangle$ preserving the greatest element. Moreover, if there exists a number series $\{a_n\}$ such that $\Sigma_{n=0}^{\infty} a_n$ converges and $grow_A(n+1) \leq (grow_A(n))^{1+a_{n+1}}$ for all n, then $\langle \mathbf{PTGF};\ \leq, \cup, \cap\rangle \cong \langle (\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}};\ \leq, \cup, \cap\rangle$.

Proof. Define the mapping $\mathcal{G}_A : PTGF \to \hat{D}$ by letting $\mathcal{G}_A(f) = C_f \oplus \emptyset$ such that

$$grow_{C_f}(n+1) = \max\{(grow_A(n+1))^{k^f_{n+1}}, grow_{C_f}(n) + 1\}$$

where $k^f_n = \mu k\ (n^k \geq f(n))$.

Now we show that $\mathcal{G}_A$ induces a lattice embedding from $\langle \mathbf{PTGF};\ \leq, \cup, \cap\rangle$ to $\langle (\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}};\ \leq, \cup, \cap\rangle$ by establishing a series of claims. Note that, usually, $\mathcal{G}_A$ does not induce an isomorphism.

Claim 1. *Let $f(n) = n$. Then $\mathcal{G}_A(f) = A \oplus \emptyset$.*

Proof. Straightforward.

Claim 2.
1. *If $x \geq 1$ and $y \geq 1$, then $x + y \leq 2xy$.*
2. *For all n and i, $\frac{k^f_n \log n}{\log(n+i)} \leq k^f_{n+i} + 1$.*

Proof. 1. $2xy = xy + xy \geq x + y$.

2.
$$\begin{aligned} n^{k^f_n} &= n \cdot n^{k^f_n - 1} \\ &< n \cdot f(n) \\ &< (n+i) f(n+i) \\ &\leq (n+i)^{k^f_{n+i}+1}. \end{aligned}$$

Hence $k^f_n \log n \leq (k^f_{n+i} + 1)\log(n+i)$, that is, $\frac{k^f_n \log n}{\log(n+i)} \leq k^f_{n+i} + 1$.

Claim 3. *Let $C_f \leq^{\#} C_g$ via n^k. For $n \in \omega$, if $(grow_A(n+1))^{k^f_{n+1}} < grow_{C_f}(n) + 1$, then $k^g_{n+1} \leq 5kk_0k^f_{n+1}$.*

Proof. Let

$$n_0 = \max\{s : (grow_A(s))^{k^f_s} \geq grow_{C_f}(s-1) + 1, s \leq n\}$$

and $i = n + 1 - n_0$. By the definition of C_f and C_g,

$$\begin{aligned} (grow_{C_f}(n+1))^k &= ((grow_A(n_0))^{k^f_{n_0}} + i)^k \geq grow_{C_g}(n+1) \\ &\geq (grow_A(n_0+i))^{k^g_{n_0+i}}. \end{aligned}$$

Hence

$$k^g_{n_0+i}\log(grow_A(n_0+i)) \le k\log((grow_A(n_0))^{k^f_{n_0}}+i) \\ \le 2kk^f_{n_0}\log(grow_A(n_0)) + k\log i.$$

By the hypothesis and Claim 2.,

$$\begin{aligned} k^g_{n_0+i} &\le k + 2k^f_{n_0}k\frac{\log(grow_A(n_0))}{\log(grow_A(n_0+i))} \\ &\le k + 2kk_0k^f_{n_0}\frac{\log n_0}{\log(n_0+i)} \\ &\le k + 2kk_0 + 2kk_0k^f_{n_0+i} \\ &\le 5kk_0k^f_{n_0+i}. \end{aligned}$$

That is, $k^g_{n+1} \le 5kk_0k^f_{n+1}$.

Claim 4. *$f \le^\# g$ if and only if $C_f \le^\# C_g$ (that is, $C_f \oplus \emptyset \le^p_1 C_g \oplus \emptyset$).*

Proof. First, assume that $f \le^\# g$. Then there exists k such that $(f(n))^k \ge g(n)$ for all n. So $n^{k^f_n k} \ge g(n)$, hence $k^f_n k \ge k^g_n$.

Now we prove that $(grow_{C_f}(n))^k \ge grow_{C_g}(n)$ (i.e. $C_f \le^\# C_g$) by induction on n.

$$\begin{aligned} (grow_{C_f}(0))^k &\ge grow_{C_g}(0), \\ (grow_{C_f}(n+1))^k &= \max\{(grow_A(n+1))^{k^f_{n+1}k}, (grow_{C_f}(n)+1)^k\} \\ &\ge \max\{(grow_A(n+1))^{k^g_{n+1}}, (grow_{C_f}(n))^k+1\} \\ &\ge \max\{(grow_A(n+1))^{k^g_{n+1}}, grow_{C_g}(n)+1\} \\ &= grow_{C_g}(n+1). \end{aligned}$$

For the other direction, assume that $C_f \le^\# C_g$ via n^k. Then

$$\max\{(grow_A(n+1))^{kk^f_{n+1}}, (grow_{C_f}(n)+1)^k\} \\ \ge \max\{(grow_A(n+1))^{k^g_{n+1}}, grow_{C_g}(n)+1\}.$$

We distinguish the following two cases:

Case 1. $(grow_A(n+1))^{k^f_{n+1}} \ge grow_{C_f}(n)+1$.

Then

$$(grow_A(n+1))^{kk^f_{n+1}} \ge (grow_A(n+1))^{k^g_{n+1}},$$

that is, $k^f_{n+1}k \ge k^g_{n+1}$. So

$$\begin{aligned} (f(n+1))^{2k} &\ge (n+1)^k(f(n+1))^k \\ &\ge (n+1)^k(n+1)^{(k^f_{n+1}-1)k} \\ &\ge (n+1)^{k^g_{n+1}} \\ &\ge g(n+1). \end{aligned}$$

Case 2. $(grow_A(n+1))^{k^f_{n+1}} < grow_{C_f}(n)+1$.
Let n_0 and i be defined as in the proof of Claim 3.. Then, by Claim 3.,

$$\begin{aligned}(f(n+1))^{10kk_0} &= (f(n_0+i))^{10kk_0} \\ &\geq (n_0+i)^{5kk_0}(f(n_0+i))^{5kk_0} \\ &\geq (n_0+i)^{5kk_0+5kk_0(k^f_{n_0+i}-1)} \\ &= (n+1)^{5kk_0k^f_{n+1}} \\ &\geq (n+1)^{k^g_{n+1}} \\ &\geq g(n+1).\end{aligned}$$

Claim 5. *For any $f \in PTGF$, $grow_{C_f}(n+1) \leq (grow_A(n+1))^{3k^f_{n+1}k_0}$.*

Proof. If $grow_{C_f}(n+1) = (grow_A(n+1))^{k^f_{n+1}}$, then the claim holds obviously.

If $grow_{C_f}(n+1) = grow_{C_f}(n)+1$, then let n_0 and i be defined as in the proof of Claim 3.. By Claim 2.,

$$\begin{aligned}grow_{C_f}(n+1) &= (grow_A(n_0))^{k^f_{n_0}}+i \\ &= 2^{\log((grow_A(n_0))^{k^f_{n_0}}+i)} \\ &\leq 2^{k^f_{n_0}\log(grow_A(n_0))+\log(2i)} \\ &\leq 2^{(k^f_{n_0+i}+1)\frac{\log(n_0+i)}{\log n_0}\log(grow_A(n_0))+\log(2i)} \\ &\leq 2^{(k^f_{n_0+i}+1)\frac{\log(n_0+i)}{\log n_0}k_0\frac{\log n_0}{\log(n_0+i)}\log(grow_A(n_0+i))+\log(2i)} \\ &= 2^{(k^f_{n_0+i}+1)k_0\log(grow_A(n_0+i))+\log(2i)} \\ &\leq 2^{\log((grow_A(n_0+i))^{k_0k^f_{n_0+i}+k_0+1})} \\ &\leq (grow_A(n+1))^{3k_0k^f_{n+1}}.\end{aligned}$$

Claim 6. *For all $f, g \in PTGF$, $k^{f\cup g}_n = \min\{k^f_n, k^g_n\}$ and $k^{f\cap g}_n = \max\{k^f_n, k^g_n\}$.*

Proof. Straightforward.

Claim 7. *For all $f, g \in PTGF$, $\mathbf{deg}_1(C_f \oplus \emptyset) \cup \mathbf{deg}_1(C_g \oplus \emptyset) = \mathbf{deg}_1(C_{f\cup g} \oplus \emptyset)$.*

Proof. Let $C_1 \oplus \emptyset \in \mathbf{deg}_1(C_f \oplus \emptyset) \cup \mathbf{deg}_1(C_g \oplus \emptyset)$ be defined in such a way that for all n,

$$grow_{C_1}(n) = \min\{grow_{C_f}(n), grow_{C_g}(n)\}.$$

It suffices to show that $C_1 \equiv^{\#} C_{f\cup g}$. By Claim 6.,

$$\begin{aligned} &grow_{C_{f\cup g}}(n+1) \\ &\quad = \max\{grow_{C_{f\cup g}}(n)+1, (grow_A(n+1))^{k_{n+1}^{f\cup g}}\} \\ &\quad = \max\{grow_{C_{f\cup g}}(n)+1, \min\{(grow_A(n+1))^{k_{n+1}^{f}}, (grow_A(n+1))^{k_{n+1}^{g}}\}\}. \end{aligned}$$

So, by Claim 5.,

$$\begin{aligned} &(grow_{C_{f\cup g}}(n+1))^{3k_0} \\ &\qquad = \max\{(grow_{C_{f\cup g}}(n)+1)^{3k_0}, \min\{(grow_A(n+1))^{k_{n+1}^{f}3k_0}, \\ &\qquad (grow_A(n+1))^{k_{n+1}^{g}3k_0}\}\} \\ &\qquad \geq \max\{(grow_{C_{f\cup g}}(n)+1)^{3k_0}, \min\{grow_{C_f}(n+1), grow_{C_g}(n+1)\}\} \\ &\qquad \geq \min\{grow_{C_f}(n+1), grow_{C_g}(n+1)\} \\ &\qquad = grow_{C_1}(n+1). \end{aligned}$$

Hence, $C_{f\cup g} \leq^{\#} C_1$.

It is straightforward to check that $C_1 \leq^{\#} C_{f\cup g}$. This completes the proof of the claim.

Claim 8. *For all* $f, g \in PTGF$, $\mathbf{deg}_1(C_f \oplus \emptyset) \cap \mathbf{deg}_1(C_g \oplus \emptyset) = \mathbf{deg}_1(C_{f\cap g} \oplus \emptyset)$.

Proof. Let $C_2 \oplus \emptyset \in \mathbf{deg}_1(C_f \oplus \emptyset) \cap \mathbf{deg}_1(C_g \oplus \emptyset)$ be defined in such a way that for all n,

$$grow_{C_2}(n) = \max\{grow_{C_f}(n), grow_{C_g}(n)\}.$$

It suffices to show that $C_2 = C_{f\cap g}$. In order to prove this, we prove by induction on n that

$$grow_{C_{f\cap g}}(n) = grow_{C_2}(n).$$

By Claim 6.,

$$grow_{C_{f\cap g}}(0) = 0 = grow_{C_2}(0),$$

$$\begin{aligned} &grow_{C_{f\cap g}}(n+1) \\ &\quad = \max\{grow_{C_{f\cap g}}(n)+1, (grow_A(n+1))^{k_{n+1}^{f\cap g}}\} \\ &\quad = \max\{grow_{C_{f\cap g}}(n)+1, \max\{grow_A(n+1))^{k_{n+1}^{f}}, grow_A(n+1))^{k_{n+1}^{g}}\}\} \\ &\quad = \max\{\max\{grow_{C_f}(n), grow_{C_g}(n)\}+1, \max\{grow_A(n+1))^{k_{n+1}^{f}}, \\ &\qquad grow_A(n+1))^{k_{n+1}^{g}}\}\} \\ &\quad = \max\{grow_{C_f}(n+1), grow_{C_g}(n+1)\}. \end{aligned}$$

This completes the proof of the claim.

Claim 9. *If there exists a number series $\{a_n\}$ as in the hypothesis, then, for all $C \in \hat{D}$, $C \leq_1^p A$ if and only if there exist $C_A \oplus \emptyset \in \hat{D}$ and $f \in PTGF$ such that* $\mathbf{deg}_1(C_A \oplus \emptyset) = \mathbf{deg}_1(C)$ *and* $\mathcal{G}_A(f) = C_A \oplus \emptyset$.

Proof. We define the set $C_A \oplus \emptyset \in \hat{D}$ and the census function f at the same time by induction on n.

Assume that $f(n)$ and $grow_{C_A}(n)$ have been defined. First let

$$k'_{n+1} = \mu k\ ((grow_A(n+1))^k > grow_C(n+1)),$$
$$k''_{n+1} = \mu k\ ((n+1)^k \geq f(n)+1).$$

We distinguish the following two cases:

Case 1. $grow_{C_A}(n) + 1 \geq \max\{(grow_A(n+1))^{k''_{n+1}}, grow_C(n+1)\}$.

Let

$$\begin{aligned} k_{n+1} &= k''_{n+1}, \\ f(n+1) &= f(n)+1, \\ grow_{C_A}(n+1) &= grow_{C_A}(n)+1. \end{aligned}$$

Case 2. $grow_{C_A}(n) + 1 < \max\{(grow_A(n+1))^{k''_{n+1}}, grow_C(n+1)\}$.

Let

$$\begin{aligned} k_{n+1} &= \max\{k'_{n+1}, k''_{n+1}\}, \\ f(n+1) &= \max\{(n+1)^{k_{n+1}-1}+1, f(n)+1\}, \\ grow_{C_A}(n+1) &= (grow_A(n+1))^{k_{n+1}}. \end{aligned}$$

It is easily checked that $k_n = \mu k\ (n^k \geq f(n))$, $(grow_A(n))^{k_n} \leq grow_{C_A}(n)$ and $\mathcal{G}_A(f) = C_A \oplus \emptyset$, that is, $C_A = C_f$.

Obviously, $grow_{C_A}(n) \geq grow_C(n)$ for all n, so $C_A \leq^\# C$.

Now we prove $C \leq^\# C_A$ by finding a $k \in \omega$ such that $(grow_C(n))^k \geq grow_{C_A}(n)$ for all n.

Let $s \geq 2$ satisfy $grow_A(n) \leq (grow_C(n))^{s/2}$. We will prove by induction on n that

$$grow_{C_A}(n) \leq (grow_C(n))^{s\Pi_{i=0}^n(1+a_i)}.$$

Obviously,

$$grow_{C_A}(0) = 0 \leq 0 = (grow_C(0))^{s(1+a_0)}.$$

For the induction step, we distinguish the following two cases.

Case 1. $grow_{C_A}(n+1) = grow_{C_A}(n) + 1$.

$$\begin{aligned} (grow_C(n+1))^{s\Pi_{i=0}^{n+1}(1+a_i)} &\geq (grow_C(n)+1)^{s\Pi_{i=0}^{n+1}(1+a_i)} \\ &\geq (grow_C(n))^{s\Pi_{i=0}^{n}(1+a_i)} + 1 \\ &\geq grow_{C_A}(n) + 1 = grow_{C_A}(n+1). \end{aligned}$$

Case 2. $grow_{C_A}(n+1) > grow_{C_A}(n) + 1$.

$$grow_{C_A}(n+1) = (grow_A(n+1))^{k_{n+1}}.$$

If $k_{n+1} = k'_{n+1}$, then

$$\begin{aligned} grow_{C_A}(n+1) &= (grow_A(n+1))^{k_{n+1}} \\ &= grow_A(n+1)(grow_A(n+1))^{k_{n+1}-1} \\ &\leq grow_A(n+1)grow_C(n+1) \\ &\leq (grow_C(n+1))^{1+s/2} \\ &\leq (grow_C(n+1))^s. \end{aligned}$$

If $k_{n+1} = k''_{n+1}$, then, by

$$\begin{aligned} (n+1)^{k_n} &\geq n^{k_n} + 1 \\ &\geq f(n) + 1 \end{aligned}$$

and the definition of k''_{n+1}, $k_n \geq k''_{n+1} = k_{n+1}$ and

$$\begin{aligned} grow_{C_A}(n+1) &= (grow_A(n+1))^{k_{n+1}} \\ &\leq (grow_A(n+1))^{k_n} \\ &\leq (grow_A(n))^{k_n(1+a_{n+1})} \\ &\leq (grow_{C_A}(n))^{1+a_{n+1}} \\ &\leq (grow_C(n))^{s\Pi_{i=0}^{n}(1+a_i)(1+a_{n+1})} \\ &= (grow_C(n))^{s\Pi_{i=0}^{n+1}(1+a_i)} \\ &\leq (grow_C(n+1))^{s\Pi_{i=0}^{n+1}(1+a_i)}. \end{aligned}$$

This completes the induction.

Now let $a = \Pi_{i=0}^{\infty}(1+a_i)$ and $k = sa$. Then, for all n,

$$\begin{aligned} grow_{C_A}(n) &\leq (grow_C(n))^{s\Pi_{i=0}^{n}(1+a_i)} \\ &\leq (grow_C(n))^k. \end{aligned}$$ □

Corollary 8. *If a set $A \in \hat{D}$ satisfies any one of the following conditions, then $\langle \hat{\mathbf{D}}; \leq, \cap, \cup \rangle$ can be embedded into $\langle (\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}}; \leq, \cap, \cup \rangle$.*

1. *$grow_A(n) = a^n$ for some $a \geq 2$.*
2. *$grow_A(n) = n^n$.*
3. *$grow_A(1) = 1$ and $grow_A(n+1) = 2^{grow_A(n)}$.*

Proof. It is straightforward to check that for these A, the requirement of the Theorem 14 is satisfied. □

Corollary 9. *Let $A \in \hat{D}$ be defined by $grow_A(n) = n^{(\log n)^m}$ for some $m \in \omega$. Then*

$$\langle \hat{\mathbf{D}}; \leq, \cap, \cup \rangle \cong \langle (\cdot, \mathbf{deg}_1(A)] \cap \hat{\mathbf{D}}; \leq, \cap, \cup \rangle \quad \text{via } \mathcal{G}_A.$$

Proof.

$$\begin{aligned} \frac{grow_A(n+1)}{grow_A(n)} &= 2^{(\log(n+1))^{m+1} - (\log n)^{m+1}} \\ &= (2^{(\log n)^{m+1}})^{\frac{(\log(n+1))^{m+1} - (\log n)^{m+1}}{(\log n)^{m+1}}} \\ &= (grow_A(n))^{\frac{(\log(n+1))^{m+1} - (\log n)^{m+1}}{(\log n)^{m+1}}}. \end{aligned}$$

Let $a_{n+1} = \frac{(\log(n+1))^{m+1} - (\log n)^{m+1}}{(\log n)^{m+1}}$. Then it is easily checked that $\Sigma_{n=1}^{\infty} a_n$ converges. Now the corollary follows from Theorem 14. □

Theorem 15. *There exist infinitely many nontrivial automorphisms for* $\langle \hat{\mathbf{D}}; \leq, \cup, \cap \rangle$.

Proof. Let A_m be defined by $grow_{A_m}(n) = n^{(\log n)^m}$ and $\mathcal{F}_m$, $\mathcal{G}_m$ be the isomorphisms in Corollary 7 and Corollary 9 corresponding to A_m. Then, for all m, $\mathcal{I}_m = \mathcal{G}_m^{-1}\mathcal{F}_m$ is an automorphism for $\langle \hat{\mathbf{D}}; \leq, \cup, \cap \rangle$. We have to prove that there exists f, such that $\mathbf{f} \neq \mathcal{I}_1(\mathbf{f}) \neq \mathcal{I}_2(\mathbf{f}) \neq \ldots$.

Let f be defined by $f(n) = n^n$. Then

$$grow_{\mathcal{F}_m(f)}(n) = (n^n)^{(\log(n^n))^m} = n^{n^{m+1}(\log n)^m}$$

and

$$grow_{\mathcal{G}_m^{-1}(\mathcal{F}(f)\oplus\emptyset)}(n) = n^{n^{m+1}}.$$

That is, $\mathcal{I}_m(\mathbf{f}) = \mathbf{g}$, where $g(n) = n^{n^{m+1}}$. Hence $\mathbf{f} \neq \mathcal{I}_1(\mathbf{f}) \neq \mathcal{I}_2(\mathbf{f}) \neq \ldots$. □

4. The class S

In this section, we study the p-isomorphic type structure of S, which is the class of scatted sets (i.e., neither dense nor co-dense). It is shown that $\langle \mathbf{deg}_1(S); \leq \rangle$ is not distributive, but any interval in $\langle \mathbf{deg}_1(S); \leq \rangle$ is a countable distributive lattice. The cap, cup and lattice embedding properties of $\langle \mathbf{deg}_1(S); \leq \rangle$ are discussed also.

4.1. Basic facts

For a study of **S**, we first introduce a concept: census function pair. Let $PTCFP_1 = \{(u, v) : u, v \in PTCF\}$ and $PTCFP_2 = \{(u, v) : u, v \in PTCF$ and $u(n) + v(n) = n$ for all $n\}$. Obviously $PTCFP_2 \subset PTCFP_1$.

Definition 17. *Let* $(u_1, v_1), (u_2, v_2) \in PTCFP_i$ $(i = 1, 2)$. $(u_1, v_1) \leq^{\#} (u_2, v_2)$ *if and only if* $u_1 \leq^{\#} u_2$ *and* $v_1 \leq^{\#} v_2$. $(u_1, v_1) \equiv^{\#} (u_2, v_2)$ *if and only if* $(u_1, v_1) \leq^{\#} (u_2, v_2)$ *and* $(u_2, v_2) \leq^{\#} (u_1, v_1)$.

Later we will use the following notation: For $(u_1, v_1), (u_2, v_2) \in PTCFP_1$, $(u_1, v_1) \cup (u_2, v_2)$ denotes $(u_1 \cup u_2, v_1 \cup v_2)$ and $(u_1, v_1) \cap (u_2, v_2)$ denotes $(u_1 \cap u_2, v_1 \cap v_2)$. $\mathbf{PTCFP}_i$ $(i = 1, 2)$ denotes the quotient structure of $PTCFP_i$ under $\equiv^{\#}$.

Theorem 16. *1.* $\langle \mathbf{PTCFP}_1; \leq, \cup, \cap \rangle$ *is a countable distributive lattice.*
2. $\langle \mathbf{PTCFP}_2; \leq, \cup \rangle$ *can be embedded into* $\langle \mathbf{PTCFP}_1; \leq, \cup \rangle$ *as an upper semi-lattice.*
3. $\langle \mathbf{PTCFP}_2; \leq \rangle \cong \langle \mathbf{deg}_1(\mathbf{P}_T) - \mathbf{F} - \hat{\mathbf{F}}; \leq \rangle$.

Proof. The proof of the first item is the same as the proof of Theorem 1 and the other two items are straightforward. □

In the following, we will use the census function pairs to give some properties of the operators $\cup$ and $\cap$ for $\mathbf{deg}_1(\mathbf{P}_T)$.

Definition 18. *For* $(u, v) \in PTCFP_i$ $(i = 1, 2)$*, let*

$$(u, v)_2 = \begin{cases} (u_1, v_1) & \text{if for some } (u_1, v_1) \in PTCFP_2 \text{ and } (u_1, v_1) \equiv^{\#} (u, v) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Lemma 18. *Let* $(u, v) \in PTCFP_1$*. Then* $(u, v)_2$ *exists if and only if there exists* $k \in \omega$ *such that* $n \leq u(n^k) + v(n^k)$ *for all* n*.*

Proof. Assume that $(u_1, v_1) \in PTCFP_2$ and $(u_1, v_1) \equiv^{\#} (u, v)$ via n^k. Then $n = u_1(n) + v_1(n) \leq u(n^k) + v(n^k)$ for all n.

For the other direction, assume that $n \leq u(n^k) + v(n^k)$ for all n. Let

$$T = \{(u_1, v_1) : u(n/2) \leq u_1(n) \leq u(n^k), v(n/2) \leq v_1(n) \leq v(n^k), \\ u_1(n) + v_1(n) = n\}.$$

Then $T \subseteq PTCFP_2$ and, by the hypothesis, T is not empty. Obviously, for every element (u_1, v_1) of T, $(u_1, v_1) \equiv^{\#} (u, v)$. □

Corollary 10. *Let* $(u_1, v_1), (u_2, v_2) \in PTCFP_2$*. Then* $(u_1 \cap u_2, v_1 \cap v_2)_2$ *exists if and only if there exists* k *such that, for all* n*,*

$$n \leq \min\{u_1(n^k), u_2(n^k)\} + \min\{v_1(n^k), v_2(n^k)\}.$$

Corollary 11. *For* $A, B \in \mathbf{P}_T - F - \hat{F}$*,* $\mathbf{deg}_1(A) \cap \mathbf{deg}_1(B)$ *exists if and only if there exists* k *such that for all* n*,*

$$n \leq \min\{cens_A(n^k), cens_B(n^k)\} + \min\{cens_{\bar{A}}(n^k), cens_{\bar{B}}(n^k)\},$$

and then $\mathbf{deg}_1(A) \cap \mathbf{deg}_1(B) = \mathbf{deg}_1(C)$*, where* $C \in \mathbf{P}_T$ *satisfies*

$$(cens_C, cens_{\bar{C}}) = (\min\{cens_A, cens_B\}, \min\{cens_{\bar{A}}, cens_{\bar{B}}\})_2.$$

Corollary 12. *For* $A, B \in \mathbf{P}_T - F - \hat{F}$*,* $\mathbf{deg}_1(A) \cap \mathbf{deg}_1(B)$ *exists if and only if there exists* $C \in \mathbf{P}_T$ *such that* $C \leq_1^p A$ *and* $C \leq_1^p B$*.*

We close this section with two observations that, for some $\mathbf{a} \in \mathbf{S}$, $\langle(\cdot, \mathbf{a}]; \leq, \cup\rangle$ is not a lattice and, for some $\mathbf{b} \in \mathbf{S}$, $\langle(\cdot, \mathbf{b}]; \leq, \cup\rangle$ is a lattice,

Theorem 17. *There exists* $\mathbf{a} \in \mathbf{S}$ *such that* $\langle(\cdot, \mathbf{a}]; \leq, \cup\rangle$ *is not a lattice.*

Proof. Let $f(0) = 0$, $f(n + 1) = 2^{f(n)}$, and

$$A = \{0^n : f(3i) \leq n < f(3i + 1)\} \cup \{0^n : f(3i + 1) \leq n < f(3i + 2) \text{ and } n \text{ is even}\}$$
$$B = \{0^n : f(3i) \leq n < \tfrac{f(3i+1)+f(3i+2)}{2}\}$$
$$C = \{0^n : f(3i) \leq n < f(3i + 1)\} \cup \{0^n : \tfrac{f(3i+1)+f(3i+2)}{2} \leq n < f(3i + 2)\}.$$

Then it can be easily checked that $A \in S$, $\mathbf{deg}_1(B) < \mathbf{deg}_1(A)$ and $\mathbf{deg}_1(C) < \mathbf{deg}_1(A)$, but $\mathbf{deg}_1(B) \cap \mathbf{deg}_1(C)$ does not exist. □

Theorem 18. *Let* $f(0) = 0$*,* $f(n + 1) = 2^{f(n)}$ *and* $A = \{0^n : f(2i) \leq n < f(2i + 1)\}$*. Then* $A \in S$ *and* $\langle(\cdot, \mathbf{deg}_1(A)]; \leq, \cup, \cap\rangle$ *is a lattice.*

Proof. It suffices to show that, for all $B, C \leq_1^p A$, $\mathbf{deg}_1(B) \cap \mathbf{deg}_1(C)$ exists.

Let $p(n)$ be a polynomial such that, for all n

$$\begin{aligned} cens_B(n) &\leq cens_A(p(n)),\\ cens_C(n) &\leq cens_A(p(n)),\\ cens_{\bar{B}}(n) &\leq cens_{\bar{A}}(p(n)),\\ cens_{\bar{C}}(n) &\leq cens_{\bar{A}}(p(n)). \end{aligned}$$

Let n_0 be fixed such that, for all i with $f(i) > n_0$, $p(2f(i)) < f(i+1)$. For $n > n_0$, let $i_n = \mu i\ (n \in [f(i), f(i+1)])$. We distinguish the following two cases.

Case 1. $p(2n) < f(i_n + 1)$.

W.l.o.g, assume that i_n is even. Then

$cens_{\bar{B}}(2n) \leq cens_{\bar{A}}(p(2n)) = cens_{\bar{A}}(2n) \leq f(i_n)$.

$cens_{\bar{C}}(2n) \leq cens_{\bar{A}}(p(2n)) = cens_{\bar{A}}(2n) \leq f(i_n)$.

So

$$\begin{aligned} &\min\{cens_B(2n), cens_C(2n)\} + \min\{cens_{\bar{B}}(2n), cens_{\bar{C}}(2n)\}\\ &\geq \min\{cens_B(2n), cens_C(2n)\}\\ &= \min\{2n - cens_{\bar{B}}(2n), 2n - cens_{\bar{C}}(2n)\}\\ &\geq \min\{2n - f(i_n), 2n - f(i_n)\}\\ &= 2n - f(i_n)\\ &\geq n. \end{aligned}$$

Case 2. $p(2n) \geq f(i_n + 1)$.

W.l.o.g., assume that $i_n + 1$ is even. Then

$cens_{\bar{B}}(2f(i_n + 1)) \leq cens_{\bar{A}}(p(2f(i_n + 1))) \leq f(i_n + 1)$.

$cens_{\bar{C}}(2f(i_n + 1)) \leq cens_{\bar{A}}(p(2f(i_n + 1))) \leq f(i_n + 1)$.

So

$$\begin{aligned} &\min\{cens_B(2p(2n)), cens_C(2p(2n))\} + \min\{cens_{\bar{B}}(2p(2n)), cens_{\bar{C}}(2p(2n))\}\\ &\geq \min\{cens_B(2f(i_n + 1)), cens_C(2f(i_n + 1))\}\\ &= \min\{2f(i_n + 1) - cens_{\bar{B}}(2f(i_n + 1)), 2f(i_n + 1) - cens_{\bar{C}}(2f(i_n + 1))\}\\ &\geq \min\{2f(i_n + 1) - f(i_n + 1), 2f(i_n + 1) - f(i_n + 1)\}\\ &= f(i_n + 1)\\ &\geq n. \end{aligned}$$

Hence, by Corollary 11, $\mathbf{deg}_1(B) \cap \mathbf{deg}_1(C)$ exists. □

4.2. *Distributivity*

In this section, we discuss the distributivity of $\langle \mathbf{S}; \leq \rangle$.

Definition 19. *An upper semilattice $L = \langle A; \leq \rangle$ is distributive if for all $a, b, c \in A$*

$$a \leq b \cup c \to \exists d \leq b\, \exists e \leq c\ (a = d \cup e).$$

Theorem 19. *$\langle \mathbf{S}; \leq \rangle$ is not distributive.*

Proof. Let $\mathbf{a} \in \mathbf{S}$. Then $\mathbf{a} \leq \mathbf{1} = \mathbf{a} \cup \bar{\mathbf{a}}$. By Corollary 11, it can been easily checked that $\mathbf{a} \cap \bar{\mathbf{a}}$ does not exist. Hence there is no $\mathbf{d} \leq \mathbf{a}, \mathbf{e} \leq \bar{\mathbf{a}}$ such that $\mathbf{a} = \mathbf{d} \cup \mathbf{e}$. Otherwise, $\mathbf{e} \leq \bar{\mathbf{a}}, \mathbf{a}$, which implies that $\mathbf{a} \cap \bar{\mathbf{a}}$ exists. □

In the following, we will show some distributive properties inside $\langle \mathbf{S}; \leq \rangle$.

Lemma 19. *For* $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in PTCFP_2$ *with* $(c_1, c_2) \leq^{\#} (a_1, a_2) \cup (b_1, b_2)$*, if both* $((a_1, a_2) \cap (c_1, c_2))_2$ *and* $((b_1, b_2) \cap (c_1, c_2))_2$ *exist, then* $(c_1, c_2) \equiv^{\#} (((a_1, a_2) \cap (c_1, c_2))_2 \cup ((b_1, b_2) \cap (c_1, c_2))_2)_2$.

Proof.

$$\begin{aligned}
&(((a_1, a_2) \cap (c_1, c_2))_2 \cup ((b_1, b_2) \cap (c_1, c_2))_2)_2 \\
&\quad \equiv^{\#} (a_1 \cap c_1, a_2 \cap c_2)_2 \cup (b_1 \cap c_1, b_2 \cap c_2)_2 \\
&\quad \equiv^{\#} ((a_1 \cap c_1) \cup (b_1 \cap c_1), (a_2 \cap c_2) \cup (b_2 \cap c_2))_2 \\
&\quad \equiv^{\#} (c_1 \cap (a_1 \cup b_1), c_2 \cap (a_2 \cup b_2))_2 \\
&\quad \equiv^{\#} (c_1, c_2)_2 \\
&\quad \equiv^{\#} (c_1, c_2).
\end{aligned}$$
□

Theorem 20. *For* $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbf{PTCFP}_2$ *with* $\mathbf{h} \leq \mathbf{f} \cup \mathbf{g}$*, if both* $\mathbf{h} \cap \mathbf{f}$ *and* $\mathbf{h} \cap \mathbf{g}$ *exist, then* $\mathbf{f} = (\mathbf{h} \cap \mathbf{f}) \cup (\mathbf{h} \cap \mathbf{g})$.

Proof. Follows from Lemma 19. □

Corollary 13. *For* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{S}$ *with* $\mathbf{a} < \mathbf{b}$*, the following properties hold.*

1. $\langle [\mathbf{a}, \mathbf{b}]; \leq, \cup, \cap \rangle$ *is a distributive lattice.*
2. $\langle (\cdot, \mathbf{c}]; \leq \rangle$ *is a lattice if and only if* $\langle (\cdot, \mathbf{c}]; \leq \rangle$ *is distributive.*

4.3. Density

In this section, we show the dense property of $\langle \mathbf{S}; \leq \rangle$. We start with a lemma.

Lemma 20. *Given* $A \in S$*, there exists a set* $B \in S$ *such that* $B \leq_1^p A$ *and, for all* $k, n \in \omega$*, there exist* $n_1, n_2 > n$ *such that* $(\{0\}^*)^{[n_1, n_1^k]} \subseteq B$ *and* $(\{0\}^*)^{[n_2, n_2^k]} \subseteq \bar{B}$.

Proof. Define a set B by letting

$$cens_B(n+1) = cens_B(n) + 1 \iff cens_A(2n+2) \geq n+1.$$

We show that B satisfies the requirements by establishing a series of claims.

Claim 1. *For all n,* $cens_{\bar{B}}(n) \leq cens_{\bar{A}}(2n)$ *and* $cens_B(n) \leq cens_A(2n)$*. That is,* $B \leq_1^p A$.

Proof. Straightforward.

Claim 2. $B \in S$.

Proof. Follows from Claim 1. and Lemma 7.

Claim 3. *For all $n, k \in \omega$, there exist $n_1, n_2 > n$ such that $(\{0\}^*)^{[n_1, n_1^k]} \subseteq B$ and $(\{0\}^*)^{[n_2, n_2^k]} \subseteq \bar{B}$.*

Proof. Because $A \in S$, we know that, for $n \in \omega$, there exists $n_0 > n$ such that $cens_{\bar{A}}(2^{k+1}n_0^k) < n_0$. By the construction, there exists $n_1 < 2n_0$ such that $cens_B(n_1) \geq cens_A(n_1)$. Again, by the construction and the property $cens_{\bar{A}}(2^{k+1}n_0^k) < n_0$, $(\{0\}^*)^{[n_1, 2^k n_0^k]} \subseteq B$. Hence $(\{0\}^*)^{[n_1, n_1^k]} \subseteq B$. In the same way, we can show that there exists n_2 such that $(\{0\}^*)^{[n_2, n_2^k]} \subseteq \bar{B}$.

All these claims complete the proof of Lemma 20. □

Lemma 21. *Let $A \in S$ such that, for all polynomial p and $n \in \omega$, there exist $n_1, n_2 > n$ satisfying $(\{0\}^*)^{[n_1, p(n_1)]} \subseteq A$ and $(\{0\}^*)^{[n_2, p(n_2)]} \subseteq \bar{A}$. Then there exists $B \in S$ such that $B <_1^p A$.*

Proof. Let $B = \{x : x0 \in A\}$. Obviously $B \leq_1^p A$. W.l.o.g., we may assume that $0 \in A$. In order to show $A \not\leq_1^p B$, we have to show that, for all $k, n \in \omega$, there exists $n_0 > n$ such that $cens_A(n_0) > cens_B(n_0^k)$. By the hypothesis, for $p(n) = n^k + 1$ and for all n, there exists $n_0 > n$ such that $(\{0\}^*)^{[n_0, n_0^k + 1]} \subseteq \bar{A}$. Hence $cens_A(n_0) = 1 + cens_B(n_0) = 1 + cens_B(n_0^k)$, that is, $cens_A(n_0) > cens_B(n_0^k)$. □

Theorem 21. *Given $A \in S$, there exists $B \in S$ such that $B <_1^p A$.*

Proof. This follows from Lemma 20 and Lemma 21. □

Theorem 22. *For $A, B \in S$ with $B <_1^p A$, there exists a countable set of independent degrees in the interval $[\mathbf{deg}_1(B), \mathbf{deg}_1(A)]$.*

Proof. W.l.o.g., we may assume that $cens_A \not\leq^\# cens_B$. Then there exists a countable set $\{\mathbf{u}_n : n \in \omega\}$ of independent degrees of **PTCF** in the interval $[\mathbf{cens}_B, \mathbf{cens}_A]$. Obviously, for each n, $(u_n, cens_{\bar{B}})_2$ exists. Let C_n be defined in such a way that $(cens_{C_n}, cens_{\bar{C}_n}) = (u_n, cens_{\bar{B}})_2$. Then, for all n, $B <_1^p C_n <_1^p A$. Hence $\{\mathbf{deg}_1(C_n) : n \in \omega\}$ is a countable set of independent degrees in the interval $[\mathbf{deg}_1(B), \mathbf{deg}_1(A)]$. □

Corollary 14. *For $A, B \in S$ such that $B <_1^p A$, any countable distributive lattice can be embedded into $\langle [\mathbf{deg}_1(B), \mathbf{deg}_1(A)]; \leq \rangle$.*

4.4. Cap, cup, and lattices embedding

In this section, we briefly mention some cap, cup properties and we show that the diamond can be embedded into any intervals of $\langle \mathbf{S}; \leq \rangle$ preserving both the least and the greatest elements.

Theorem 23. *There exist $A, B \in S$ with $B <_1^p A$ such that $\mathbf{deg}_1(B)$ is not cuppable to $\mathbf{deg}_1(A)$ and $\mathbf{deg}_1(A)$ is not cappable to $\mathbf{deg}_1(B)$.*

Proof. The proof of the theorem is an extension of the proofs of Theorems 2 and 4. The details are omitted. □

Theorem 24. *Let $A, B \in S$ such that $B <_1^p A$. Then the diamond can be embedded into $\langle[\mathbf{deg}_1(B), \mathbf{deg}_1(A)];\ \leq, \cup, \cap\rangle$ preserving both the least and the greatest elements.*

Proof. We distinguish the following three cases.

Case 1. $cens_{\bar{A}} \equiv^{\#} cens_{\bar{B}}$. First it is straightforward to check that $\mathbf{cens}_B <_w \mathbf{cens}_A$. By Corollary 3, there exist $\mathbf{f}, \mathbf{g} \in (\mathbf{cens}_B, \mathbf{cens}_A)$ such that $\mathbf{f} \cup \mathbf{g} = \mathbf{cens}_A$ and $\mathbf{f} \cap \mathbf{g} = \mathbf{cens}_B$. Let $C, D \in S$ be defined by

$$(cens_C, cens_{\bar{C}}) = (f, cens_{\bar{A}})_2$$
$$(cens_D, cens_{\bar{D}}) = (g, cens_{\bar{A}})_2$$

Then $\mathbf{deg}_1(C), \mathbf{deg}_1(D) \in (\mathbf{deg}_1(B), \mathbf{deg}_1(A))$, $\mathbf{deg}_1(C) \cup \mathbf{deg}_1(D) = \mathbf{deg}_1(A)$ and $\mathbf{deg}_1(C) \cap \mathbf{deg}_1(D) = \mathbf{deg}_1(B)$.

Case 2. $cens_A \equiv^{\#} cens_B$. The same as the proof of Case 1.

Case 3. $cens_A \not\equiv^{\#} cens_B$ and $cens_{\bar{A}} \not\equiv^{\#} cens_{\bar{B}}$.

Let $C, D \in S$ be defined by

$$(cens_C, cens_{\bar{C}}) = (cens_B, cens_{\bar{A}})_2$$
$$(cens_D, cens_{\bar{D}}) = (cens_A, cens_{\bar{B}})_2$$

Then it is easily checked that $\mathbf{deg}_1(C), \mathbf{deg}_1(D) \in (\mathbf{deg}_1(B), \mathbf{deg}_1(A))$, $\mathbf{deg}_1(C) \cup \mathbf{deg}_1(D) = \mathbf{deg}_1(A)$ and $\mathbf{deg}_1(C) \cap \mathbf{deg}_1(D) = \mathbf{deg}_1(B)$. □

Corollary 15. *For $A \in S$ and $B \in \hat{D}$, $Th(\langle(\cdot, \mathbf{deg}_1(A)]; \leq, \cup, \cap\rangle) \neq Th(\langle(\cdot, \mathbf{deg}_1(B)]; \leq, \cup, \cap\rangle)$.*

We close this section with an observation that there are infinitely many isomorphic intervals in $\langle \mathbf{S}; \leq\rangle$.

Theorem 25. *For $A \in S$, there exist infinitely many different degrees: $\mathbf{deg}_1(B_1), \mathbf{deg}_1(B_2), \ldots < \mathbf{deg}_1(A)$ such that, for all i and j, $I_i = \langle(\cdot, \mathbf{deg}_1(B_i)]; \leq, \cup, \cap\rangle \cong \langle(\cdot, \mathbf{deg}_1(B_j)]; \leq, \cup, \cap\rangle = I_j$.*

Proof. Let B be the set constructed in Lemma 20, and $cens_{B_i}(n) = cens_B(n+i)-i$. Obviously, for any i and j, $B_i <_1^p B_j$ if and only if $i > j$. We have to give an isomorphism between I_i and I_j $(i < j)$. For any $X \leq_1^p B_i$, let $cens_Y(n) = cens_X(n + j - i) - (j - i)$ and let $\mathcal{F}(\mathbf{deg}_1(X)) = \mathbf{deg}_1(Y)$. Then it is easily checked that $\mathcal{F}$ is an isomorphism between I_i and I_j. □

We conclude this section with several straightforward observations (without proofs) on the relationships between $\hat{\mathbf{D}}$ and $\mathbf{S}$.

Theorem 26. *1. For $A \in S$ and $B \in \mathbf{P}_T$, if $B \leq_1^p A$ then $B \in S$.*

2. *For $A \in \hat{D}$, there exists $B \in S$ such that $B \leq_1^p A$ if and only if $\mathbf{deg}_1(A)$ is cuppable to $\mathbf{1}$.*
3. *For $A, B \in S$ such that $\mathbf{deg}_1(B) < \mathbf{deg}_1(A)$. There exist $f, g \in PTGF$ such that $\langle[\mathbf{f}, \mathbf{g}];\ \leq, \cup, \cap\rangle$ can be embedded into $\langle[\mathbf{deg}_1(B), \mathbf{deg}_1(A)]; \leq, \cup, \cap\rangle$ preserving the least or the greatest element. Moreover, if $A \equiv^{\#} B$ or $\bar{A} \equiv^{\#} \bar{B}$, then $\langle[\mathbf{deg}_1(B), \mathbf{deg}_1(A)]; \leq, \cup, \cap\rangle \cong \langle[\mathbf{f}, \mathbf{g}]; \leq, \cup, \cap\rangle$.*

In this paper we have discussed the isomorphic type structure of $\mathbf{P}_T$. However, we were unable to get any (un)decidability results on the first-order or second-order theory of the structure $\langle \mathbf{deg}_1(\mathbf{P}_T); <, \cup, \cap\rangle$. It would be interesting to characterize this structure completely or to prove a negative result: the first-order theory of this structure is undecidable.

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