SIAM J. COMPUT.
Vol. 28, No. 2, pp. 394–408

GENERICITY, RANDOMNESS, AND POLYNOMIAL-TIME APPROXIMATIONS*

YONGGE WANG†

Abstract. Polynomial-time safe and unsafe approximations for intractable sets were introduced by Meyer and Paterson [Technical Report TM-126, Laboratory for Computer Science, MIT, Cambridge, MA, 1979] and Yesha [*SIAM J. Comput.*, 12 (1983), pp. 411–425], respectively. The question of which sets have optimal safe and unsafe approximations has been investigated extensively. Duris and Rolim [*Lecture Notes in Comput. Sci.* 841, Springer-Verlag, Berlin, New York, 1994, pp. 38–51] and Ambos-Spies [*Proc. 22nd ICALP*, Springer-Verlag, Berlin, New York, 1995, pp. 384–392] showed that the existence of optimal polynomial-time approximations for the safe and unsafe cases is independent. Using the law of the iterated logarithm for p-random sequences (which has been recently proven in [*Proc. 11th Conf. Computational Complexity*, IEEE Computer Society Press, Piscataway, NJ, 1996, pp. 180–189]), we extend this observation by showing that both the class of polynomial-time Δ-levelable sets and the class of sets which have optimal polynomial-time unsafe approximations have p-measure 0. Hence typical sets in **E** (in the sense of p-measure) do not have optimal polynomial-time unsafe approximations. We will also establish the relationship between resource bounded genericity concepts and the polynomial-time safe and unsafe approximation concepts.

Key words. computational complexity, resource bounded genericity, resource bounded randomness, approximation

AMS subject classifications. 68Q05, 68Q25, 68Q30, 03D15, 60F99

PII. S009753979630235X

1. Introduction.

The notion of polynomial-time safe approximations was introduced by Meyer and Paterson in [13] (see also [8]). A safe approximation algorithm for a set A is a polynomial-time algorithm M that on each input x outputs either 1 (accept), 0 (reject), or ? (I do not know) such that all inputs accepted by M are members of A and no member of A is rejected by M. An approximation algorithm is optimal if no other polynomial-time algorithm correctly decides infinitely many more inputs, that is to say, outputs infinitely many more correct 1s or 0s. In Orponen, Russo, and Schöning [14], the existence of optimal approximations was phrased in terms of **P**-levelability: a recursive set A is **P**-levelable if for any deterministic Turing machine M accepting A and for any polynomial p there is another machine M' accepting A and a polynomial p' such that for infinitely many elements x of A, M does not accept x within $p(|x|)$ steps while M' accepts x within $p'(|x|)$ steps. It is easy to show that A has an optimal polynomial-time safe approximation if and only if neither A nor $\bar{A}$ is **P**-levelable.

The notion of polynomial-time unsafe approximations was introduced by Yesha in [19]: an unsafe approximation algorithm for a set A is just a standard polynomial-time bounded deterministic Turing machine M with outputs 1 and 0. Note that, different from the polynomial-time safe approximations, here we are allowed to make errors, and we study the amount of inputs on which M are correct. Duris and Rolim [6] further investigated unsafe approximations and introduced a levelability concept, Δ-levelability, which implies the nonexistence of optimal polynomial-time unsafe approximations. They showed that complete sets for **E** are Δ-levelable and there exists

* Received by the editors April 22, 1996; accepted for publication (in revised form) March 4, 1997; published electronically July 7, 1998.
http://www.siam.org/journals/sicomp/28-2/30235.html

† Department of Computer Science, The University of Auckland, Private Bag 92019, Auckland, New Zealand (wang@cs.auckland.ac.nz).

an intractable set in **E** which has an optimal safe approximation but no optimal unsafe approximation. But they did not succeed in producing an intractable set with optimal unsafe approximations. Ambos-Spies [1] defined a concept of weak Δ-levelability and showed that there exists an intractable set in **E** which is not weakly Δ-levelable (hence it has an optimal unsafe approximation).

Like resource-bounded randomness concepts, different kinds of resource-bounded genericity concepts were introduced by Ambos-Spies [2], Ambos-Spies, Fleischhack, and Huwig [3], Fenner [7], and Lutz [9]. It has been proved that resource-bounded generic sets are useful in providing a coherent picture of complexity classes. These sets embody the method of diagonalization construction; that is, requirements which can always be satisfied by finite extensions are automatically satisfied by generic sets.

It was shown in Ambos-Spies, Neis, and Terwijn [4] that the generic sets of Ambos-Spies are **P**-immune, and that the class of sets which have optimal safe approximations is large in the sense of resource-bounded Ambos-Spies category. Mayordomo [11] has shown that the class of **P**-immune sets is neither meager nor comeager both in the sense of resource-bounded Lutz category and in the sense of resource-bounded Fenner category. We extend this result by showing that the class of sets which have optimal safe approximations is neither meager nor comeager both in the sense of resource-bounded Lutz category and in the sense of resource-bounded Fenner category. Moreover, we will show the following relations between unsafe approximations and resource-bounded categories.

1. The class of weakly Δ-levelable sets is neither meager nor comeager in the sense of resource-bounded Ambos-Spies category [4].
2. The class of weakly Δ-levelable sets is comeager (is therefore large) in the sense of resource-bounded general Ambos-Spies [2], Fenner [7], and Lutz [9] categories.
3. The class of Δ-levelable sets is neither meager nor comeager in the sense of resource-bounded general Ambos-Spies [2], Fenner [7], and Lutz [9] categories.

In the last section, we will show the relationship between polynomial-time approximations and p-measure. Mayordomo [12] has shown that the class of **P**-bi-immune sets has p-measure 1. It follows that the class of sets which have optimal polynomial-time safe approximations has p-measure 1. Using the law of the iterated logarithm for p-random sequences which we have proved in Wang [16, 17], we will show that the following hold.

1. The class of Δ-levelable sets has p-measure 0.
2. The class of sets which have optimal polynomial-time unsafe approximations have p-measure 0. That is, the class of weakly Δ-levelable sets has p-measure 1.
3. p-Random sets are weakly Δ-levelable but not Δ-levelable.

Hence typical sets in the sense of resource-bounded measure do not have optimal polynomial-time unsafe approximations.

It should be noted that the above results show that the class of weakly Δ-levelable sets is large both in the sense of the different notions of resource-bounded category and in the sense of resource-bounded measure. That is to say, typical sets in $\mathbf{E}_2$ (in the sense of resource-bounded category or in the sense of resource-bounded measure) are weakly Δ-levelable.

In contrast to the results in this paper, we have recently shown (in [18]) the following results.

1. There is a p-stochastic set $A \in \mathbf{E}_2$ which is Δ-levelable.

2. There is a p-stochastic set $A \in \mathbf{E}_2$ which has an optimal unsafe approximation.

2. Definitions. N and $Q(Q^+)$ are the set of natural numbers and the set of (nonnegative) rational numbers, respectively. $\Sigma = \{0, 1\}$ is the binary alphabet, Σ^* is the set of (finite) binary strings, Σ^n is the set of binary strings of length n, and Σ^∞ is the set of infinite binary sequences. The length of a string x is denoted by $|x|$. $<$ is the length-lexicographical ordering on Σ^*, and z_n $(n \geq 0)$ is the nth string under this ordering. λ is the empty string. For strings $x, y \in \Sigma^*$, xy is the concatenation of x and y, $x \sqsubseteq y$ denotes that x is an initial segment of y. For a sequence $x \in \Sigma^* \cup \Sigma^\infty$ and an integer number $n \geq -1$, $x[0..n]$ denotes the initial segment of length $n+1$ of x ($x[0..n] = x$ if $|x| \leq n+1$) and $x[i]$ denotes the ith bit of x, i.e., $x[0..n] = x[0] \cdots x[n]$. Lowercase letters $\ldots, k, l, m, n, \ldots, x, y, z$ from the middle and the end of the alphabet will denote numbers and strings, respectively. The letter b is reserved for elements of Σ, and lowercase Greek letters $\xi, \eta, \ldots$ denote infinite sequences from Σ^∞.

A subset of Σ^* is called a language, a problem, or simply a set. Capital letters are used to denote subsets of Σ^* and boldface capital letters are used to denote subsets of Σ^∞. The cardinality of a language A is denoted by $\|A\|$. We identify a language A with its characteristic function, i.e., $x \in A$ if and only if $A(x) = 1$. The characteristic sequence of a language A is the infinite sequence $A(z_0)A(z_1)A(z_2)\cdots$. We freely identify a language with its characteristic sequence and the class of all languages with the set Σ^∞. For a language $A \subseteq \Sigma^*$ and a string $z_n \in \Sigma^*$, $A\restriction z_n = A(z_0) \cdots A(z_{n-1}) \in \Sigma^*$. For languages A and B, $\bar{A} = \Sigma^* - A$ is the complement of A, $A \Delta B = (A - B) \cup (B - A)$ is the symmetric difference of A and B; $A \subseteq B$ (resp., $A \subset B$) denotes that A is a subset of B (resp., $A \subseteq B$ and $B \not\subseteq A$). For a number n, $A^{=n} = \{x \in A : |x| = n\}$ and $A^{\leq n} = \{x \in A : |x| \leq n\}$.

We fix a standard polynomial-time computable and invertible pairing function $\lambda x, y\langle x, y\rangle$ on Σ^* such that, for every string x, there is a real $\alpha(x) > 0$ satisfying

$$\|\Sigma^{[x]} \cap \Sigma^n\| \geq \alpha(x) \cdot 2^n \text{ for almost all } n,$$

where $\Sigma^{[x]} = \{\langle x, y\rangle : y \in \Sigma^*\}$ and $\Sigma^{[\leq x]} = \{\langle x', y\rangle : x' \leq x \ \&\ y \in \Sigma^*\}$. We will use $\mathbf{P}$, $\mathbf{E}$, and $\mathbf{E}_2$ to denote the complexity classes $DTIME(poly)$, $DTIME(2^{linear})$, and $DTIME(2^{poly})$, respectively. Finally, we fix a recursive enumeration $\{P_e : e \geq 0\}$ of $\mathbf{P}$ such that $P_e(x)$ can be computed in $O(2^{|x|+e})$ steps (uniformly in e and x).

We define a *finite function* to be a partial function from Σ^* to Σ whose domain is finite. For a finite function σ and a string $x \in \Sigma^*$, we write $\sigma(x)\downarrow$ if $x \in dom(\sigma)$, and $\sigma(x)\uparrow$ otherwise. For two finite functions σ, τ, we say σ and τ are compatible if $\sigma(x) = \tau(x)$ for all $x \in dom(\sigma) \cap dom(\tau)$. The concatenation $\sigma\tau$ of two finite functions σ and τ is defined as $\sigma\tau = \sigma \cup \{(z_{n_\sigma+i+1}, b) : z_i \in dom(\tau) \ \&\ \tau(z_i) = b\}$, where $n_\sigma = \max\{n : z_n \in dom(\sigma)\}$ and $n_\sigma = -1$ for $\sigma = \lambda$. For a set A and a string x, we identify the characteristic string $A\restriction x$ with the finite function $\{(y, A(y)) : y < x\}$. For a finite function σ and a set A, σ is extended by A if for all $x \in dom(\sigma)$, $\sigma(x) = A(x)$.

3. Genericity versus polynomial-time safe approximations. In this section, we summarize some known results on the relationship between the different notions of resource-bounded genericity and the notion of polynomial-time safe approximations.

We first introduce some concepts of resource-bounded genericity.

DEFINITION 3.1. *A partial function f from Σ^* to $\{\sigma : \sigma$ is a finite function $\}$ is* dense along *a set A if there are infinitely many strings x such that $f(A\restriction x)$ is defined.*

A set A meets *f if, for some x, the finite function $(A\upharpoonright x)f(A\upharpoonright x)$ is extended by A. Otherwise, A* avoids *f.*

Definition 3.2. *A class $\mathbf{C}$ of sets is* nowhere dense *via f if f is dense along all sets in $\mathbf{C}$ and for every set $A \in \mathbf{C}$, A avoids f.*

Definition 3.3. *Let $\mathbf{F}$ be a class of (partial) functions from Σ^* to $\{\sigma : \sigma$ is a finite function$\}$. A class $\mathbf{C}$ of sets is* $\mathbf{F}$-meager *if there exists a function $f \in \mathbf{F}$ such that $\mathbf{C} = \cup_{i\in N}\mathbf{C}_i$ and $\mathbf{C}_i$ is nowhere dense via $f_i(x) = f(\langle i, x\rangle)$. A class $\mathbf{C}$ of sets is* $\mathbf{F}$-comeager *if $\bar{\mathbf{C}}$ is $\mathbf{F}$-meager.*

Definition 3.4. *A set G is* $\mathbf{F}$-generic *if G is an element of all $\mathbf{F}$-comeager classes.*

Lemma 3.5 (see [2, 7, 9]). *A set G is $\mathbf{F}$-generic if and only if G meets all functions $f \in \mathbf{F}$ which are dense along G.*

For a class $\mathbf{F}$ of functions, each function $f \in \mathbf{F}$ can be considered as a finitary property $\mathcal{P}$ of sets. If $f(A\upharpoonright x)$ is defined, then all sets extending $(A\upharpoonright x)f(A\upharpoonright x)$ have the property $\mathcal{P}$. So a set A has the property $\mathcal{P}$ if and only if A meets f. f is dense along A if and only if in a construction of A along the ordering $<$, where at stage s of the construction we decide whether or not the string z_s belongs to A, there are infinitely many stages s such that by appropriately defining $A(z_s)$ we can ensure that A has the property $\mathcal{P}$ (that is to say, for some string x, $(A\upharpoonright x)f(A\upharpoonright x)$ is extended by A).

For different function classes $\mathbf{F}$, we have different notions of $\mathbf{F}$-genericity. In this paper, we will concentrate on the following four kinds of function classes which have been investigated by Ambos-Spies [2], Amos-Spies, Neis, and Terwijn [4], Fenner [7], and Lutz [9], respectively. $\mathbf{F}_1$ is the class of polynomial-time computable partial functions from Σ^* to Σ; $\mathbf{F}_2$ is the class of polynomial-time computable partial functions from Σ^* to $\{\sigma : \sigma$ is a finite function$\}$; $\mathbf{F}_3$ is the class of polynomial-time computable total functions from Σ^* to $\{\sigma : \sigma$ is a finite function$\}$; and $\mathbf{F}_4$ is the class of polynomial-time computable total functions from Σ^* to Σ^*.

Definition 3.6.

1. (*See Ambos-Spies, Neis, and Terwijn* [4].) *A set G is* A-generic *if G is $\mathbf{F}_1$-generic.*
2. (*See Ambos-Spies* [2].) *A set G is* general A-generic *if G is $\mathbf{F}_2$-generic.*
3. (*See Fenner* [7].) *A set G is* F-generic *if G is $\mathbf{F}_3$-generic.*
4. (*See Lutz* [9].) *A set G is* L-generic *if G is $\mathbf{F}_4$-generic.*

Obviously, we have the following implications.

Theorem 3.7.

1. *If a set G is general A-generic, then G is A-generic, F-generic, and L-generic.*
2. *If a set G is F-generic, then G is L-generic.*

Proof. The proof is straightforward. □

In this paper, we will also study the following n^k-time ($k > 1$) bounded genericity concepts. A set G is Ambos-Spies n^k-generic (resp., general Ambos-Spies n^k-generic, Fenner n^k-generic, Lutz n^k-generic) if and only if G meets all n^k-time computable functions $f \in \mathbf{F}_1$ (resp., $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$) which are dense along G.

Theorem 3.8 (see Ambos-Spies [2]). *A class $\mathbf{C}$ of sets is meager in the sense of Ambos-Spies category (resp., general Ambos-Spies category, Fenner category, Lutz Category) if and only if there exists a number $k \in N$ such that there is no Ambos-Spies n^k-generic (resp., general Ambos-Spies n^k-generic, Lutz n^k-generic, Fenner n^k-generic) set in $\mathbf{C}$.*

As an example, we show that Ambos-Spies n-generic sets are $\mathbf{P}$-immune.

THEOREM 3.9 (see Ambos-Spies, Neis, Terwijn [4]). *Let G be an Ambos-Spies n-generic set. Then G is* **P***-immune.*

Proof. For a contradiction assume that $A \in \mathbf{P}$ is an infinite subset of G. Then the function $f : \Sigma^* \to \Sigma$ defined by

$$f(x) = \begin{cases} 0 & z_{|x|} \in A, \\ \uparrow & z_{|x|} \notin A \end{cases}$$

is computable in time n and is dense along G. So, by the Ambos-Spies n-genericity of G, G meets f. By the definition of f, this implies that there exists some string $z_i \in A$ such that $z_i \notin G$, a contradiction. □

It has been shown (see Mayordomo [12]) that neither F-genericity nor L-genericity implies **P**-immunity or non-**P**-immunity.

A *partial* set A is defined by a partial characteristic function $f : \Sigma^* \to \Sigma$. A partial set A is polynomial-time computable if $dom(A) \in \mathbf{P}$ and its partial characteristic function is computable in polynomial time.

DEFINITION 3.10 (see Meyer and Paterson [13]). *A* polynomial-time safe approximation *of a set A is a polynomial-time computable partial set Q which is consistent with A, that is to say, for every string $x \in dom(Q)$, $A(x) = Q(x)$. The approximation Q is* optimal *if, for every polynomial-time safe approximation Q' of A, $dom(Q') - dom(Q)$ is finite.*

DEFINITION 3.11 (see Orponen, Russo, and Schöning [14]). *A set A is* **P**-levelable *if, for any subset $B \in \mathbf{P}$ of A, there is another subset $B' \in \mathbf{P}$ of A such that $\|B' - B\| = \infty$.*

LEMMA 3.12 (see Orponen, Russo, and Schöning [14]). *A set A possesses an optimal polynomial-time safe approximation if and only if neither A nor $\bar{A}$ is* **P***-levelable.*

Proof. The proof is straightforward. □

LEMMA 3.13. *If a set A is* **P***-immune, then A is not* **P***-levelable.*

Proof. The proof is straightforward. □

THEOREM 3.14 (see Ambos-Spies [2]). *Let G be an Ambos-Spies n-generic set. Then neither G nor $\bar{G}$ is* **P***-levelable. That is to say, G has an optimal polynomial-time safe approximation.*

Proof. This follows from Theorem 3.9. □

Theorem 3.14 shows that the class of **P**-levelable sets is "small" in the sense of resource-bounded (general) Ambos-Spies category.

COROLLARY 3.15. *The class of* **P***-levelable sets is meager in the sense of resource-bounded (general) Ambos-Spies category.*

Now we show that the class of **P**-levelable sets is neither meager nor comeager in the sense of resource-bounded Fenner category and Lutz category.

THEOREM 3.16.

1. *There exists a set G in $\mathbf{E}_2$ which is both F-generic and* **P***-levelable.*
2. *There exists a set G in $\mathbf{E}_2$ which is F-generic but not* **P***-levelable.*

Proof. 1. Let $\delta(0) = 0, \delta(n+1) = 2^{2^{\delta(n)}}$, $I_1 = \{x : \delta(2n) \leq |x| < \delta(2n+1), n \in N\}$, $I_2 = \Sigma^* - I_1$, and $\{f_i : i \in N\}$ be an enumeration of $\mathbf{F}_3$ such that $f_i(x)$ can be computed uniformly in time $2^{\log^k(|x|+i)}$ for some $k \in N$.

In the following, we construct a set G in stages which is both F-generic and **P**-levelable. In the construction we will ensure that

$$G \cap \Sigma^{[e]} \cap I_1 =^* \Sigma^{[e]} \cap I_1$$

for $e \geq 0$. Hence $G \cap \Sigma^{[e]} \cap I_1 \in \mathbf{P}$ for $e \geq 0$. In order to ensure that G is $\mathbf{P}$-levelable, it suffices to satisfy for all $e \geq 0$ the following requirements:

$L_e : P_e \subseteq G \cap I_1 \Rightarrow P_e \subseteq^* \Sigma^{[\leq e]} \cap I_1$.

To show that the requirements $L_e (e \geq 0)$ ensure that G is $\mathbf{P}$-levelable (fix a subset $C \in \mathbf{P}$ of G) we have to define a subset $C' \in \mathbf{P}$ of G such that $C' - C$ is infinite. Fix e such that $P_e = C \cap I_1$. Then, by the requirement L_e, $C \cap I_1 \subseteq^* \Sigma^{[\leq e]} \cap I_1$. So, for $C' = G \cap \Sigma^{[e+1]} \cap I_1$, $C' \in \mathbf{P}$ and C' is infinite. Since $C' \cap C = \emptyset$, C' has the required property.

The strategy for meeting a requirement L_e is as follows: if there is a string $x \in (I_1 \cap P_e) - \Sigma^{[\leq e]}$, then we let $G(x) = 0$ to refute the hypothesis of the requirement L_e (so L_e is trivially met). To ensure that G is F-generic, it suffices to meet for all $e \geq 0$ the following requirements:

G_e: There exists a string x such that G extends $(G{\upharpoonright}x) f_e(G{\upharpoonright}x)$.

Because the set I_1 is used to satisfy L_e, we will use I_2 to satisfy G_e. The strategy for meeting a requirement G_e is as follows: for some string $x \in I_2$, let G extend $(G{\upharpoonright}x) f_e(G{\upharpoonright}x)$.

Define a priority ordering of the requirements by letting $R_{2n} = G_n$ and $R_{2n+1} = L_n$. Now we give the construction of G formally.

Stage s.

If $G(z_s)$ has been defined before stage s, then go to stage $s + 1$.

A requirement L_e *requires* attention if

1. $e < s$.
2. $z_s \in P_e \cap \Sigma^{[>e]} \cap I_1$.
3. For all $y < z_s$, if $y \in P_e$ then $y \in G \cap I_1$.

A requirement G_e *requires* attention if $e < s$, G_e has not received attention yet, and $x \in I_2$ for all $z_s \leq x \leq z_t$ where z_t is the greatest element in $dom((G{\upharpoonright}z_s) f_e(G{\upharpoonright}z_s))$.

Fix the minimal n such that R_n requires attention. If there is no such n, then let $G(z_s) = 1$. Otherwise, we say that R_n *receives* attention. Moreover, if $R_n = L_e$ then let $G(z_s) = 0$. If $R_n = G_e$ then let $G{\upharpoonright}z_{t+1} = fill_1((G{\upharpoonright}z_s) f_e(G{\upharpoonright}z_s), t)$, where z_t is the greatest element in $dom((G{\upharpoonright}z_s) f_e(G{\upharpoonright}z_s))$ and for a finite function σ and a number k, $fill_1(\sigma, k) = \sigma \cup \{(x, 1) : x \leq z_k \ \&\ x \notin dom(\sigma)\}$.

This completes the construction of G.

It is easy to verify that the set G constructed above is both $\mathbf{P}$-levelable and F-generic; the details are omitted here.

2. For a general A-generic set G, by Theorem 3.9, G is $\mathbf{P}$-immune. By Theorem 3.7, G is F-generic. Hence, G is F-generic but not $\mathbf{P}$-levelable. □

COROLLARY 3.17. *The class of* $\mathbf{P}$*-levelable sets is neither meager nor comeager in the sense of resource-bounded Fenner category and Lutz category.*

Proof. This follows from Theorem 3.16. □

4. Genericity versus polynomial-time unsafe approximations.

DEFINITION 4.1 (see Duris and Rolim [6] and Yesha [19]). *A* polynomial-time unsafe approximation *of a set A is a set* $B \in \mathbf{P}$. *The set* $A \Delta B$ *is called the* error *set of the approximation. Let f be an unbounded function on the natural numbers. A set A is* Δ-levelable with density f *if, for any set* $B \in \mathbf{P}$, *there is another set* $B' \in \mathbf{P}$ *such that*

$$\|(A \Delta B){\upharpoonright}z_n\| - \|(A \Delta B'){\upharpoonright}z_n\| \geq f(n)$$

for almost all $n \in N$*. A set* A *is* Δ-levelable *if* A *is* Δ*-levelable with density* f *such that* $\lim_{n\to\infty} f(n) = \infty$*.*

Note that, in Definition 4.1, the density function f is independent of the choice of $B \in \mathbf{P}$.

DEFINITION 4.2 (see Ambos-Spies [1]). *A polynomial-time unsafe approximation* B *of a set* A *is* optimal *if, for any approximation* $B' \in \mathbf{P}$ *of* A*,*

$$\exists k \in N\ \forall n \in N\ (\|(A\Delta B)\upharpoonright z_n\| < \|(A\Delta B')\upharpoonright z_n\| + k).$$

A set A *is* weakly Δ-levelable *if, for any polynomial-time unsafe approximation* B *of* A*, there is another polynomial-time unsafe approximation* B' *of* A *such that*

$$\forall k \in N\ \exists n \in N\ (\|(A\Delta B)\upharpoonright z_n\| > \|(A\Delta B')\upharpoonright z_n\| + k).$$

It should be noted that our above definitions are a little different from the original definitions of Ambos-Spies [1], Duris and Rolim [6], and Yesha [19]. In the original definitions, they considered the errors on strings up to certain length (i.e., $\|(A\Delta B)^{\leq n}\|$) instead of errors on strings up to z_n (i.e., $\|(A\Delta B)\upharpoonright z_n\|$). But it is easy to check that all our results except Theorem 5.14 in this paper hold for the original definitions also.

LEMMA 4.3 (see Ambos-Spies [1]).

1. *A set* A *is weakly* Δ*-levelable if and only if* A *does not have an optimal polynomial time unsafe approximation.*
2. *If a set* A *is* Δ*-levelable then it is weakly* Δ*-levelable.*

LEMMA 4.4. *Let* A, B *be two sets such that* A *is* Δ*-levelable with linear density and* $A\Delta B$ *is sparse. Then* B *is* Δ*-levelable with linear density.*

Proof. Let p be the polynomial such that, for all n, $\|(A\Delta B)^{\leq n}\| \leq p(n)$, and assume that A is Δ-levelable with density αn ($\alpha > 0$). Then there is a real number $\beta > 0$ such that, for large enough n, $\alpha n - 2p(1 + [\log n]) > \beta n$. We will show that B is Δ-levelable with density βn.

Now, given any set $C \in \mathbf{P}$, by Δ-levelability of A, choose $D \in \mathbf{P}$ such that

$$\|(A\Delta C)\upharpoonright z_n\| > \|(A\Delta D)\upharpoonright z_n\| + \alpha n$$

for almost all n. Then

$$\begin{aligned} \|(B\Delta C)\upharpoonright z_n\| &\geq \|(A\Delta C)\upharpoonright z_n\| - p(1 + [\log n]) \\ &> \|(A\Delta D)\upharpoonright z_n\| + \alpha n - p(1 + [\log n]) \\ &\geq \|(B\Delta D)\upharpoonright z_n\| + \alpha n - 2p(1 + [\log n]) \\ &> \|(B\Delta D)\upharpoonright z_n\| + \beta n \end{aligned}$$

for almost all n. Hence, B is Δ-levelable with density βn. □

THEOREM 4.5.

1. *There exists a set* G *in* $\mathbf{E}_2$ *which is both A-generic and* Δ*-levelable.*
2. *There exists a set* G *in* $\mathbf{E}_2$ *which is A-generic but not weakly* Δ*-levelable.*

Proof. 1. Duris and Rolim [6] constructed a set A in $\mathbf{E}$ which is Δ-levelable with linear density and, in [4], Ambos-Spies, Neis, and Terwijn showed that, for any set $B \in \mathbf{E}$, there is an A-generic set B' in $\mathbf{E}_2$ such that $B\Delta B'$ is sparse. So, for any set A which is Δ-levelable with linear density, there is an A-generic set G in $\mathbf{E}_2$ such that $A\Delta G$ is sparse. It follows from Lemma 4.4 that G is Δ-levelable with linear density.

2. Ambos-Spies [1, Theorem 3.3] constructed a **P**-bi-immune set in **E** which is not weakly Δ-levelable. In his proof, he used the requirements

$BI_{2e} : P_e \subseteq G \Rightarrow P_e$ is finite,

$BI_{2e+1} : P_e \subseteq \bar{G} \Rightarrow P_e$ is finite,

to ensure that the constructed set G is **P**-bi-immune. In order to guarantee that G is not weakly Δ-levelable, he used the requirements

$R : \forall e \in N\ \forall n \in N\ (\|(G\Delta B)\restriction z_n\| \leq \|(G\Delta P_e)\restriction z_n\| + e + 1)$

to ensure that $B = \cup_{i\geq 0}\Sigma^{[2i]}$ will be an optimal unsafe approximation of G. If we change the requirements BI_{2e} and BI_{2e+1} to the requirements

R_e : if $f_e \in \mathbf{F}_1$ is dense along G, then G meets f_e,

then a routine modification of the finite injury argument in the proof of Ambos-Spies [1, Theorem 3.3] can be used to construct an A-generic set G in $\mathbf{E}_2$ which is not weakly Δ-levelable. The details are omitted here. □

COROLLARY 4.6. *The class of (weakly) Δ-levelable sets is neither meager nor comeager in the sense of resource-bounded Ambos-Spies category.*

Corollary 4.6 shows that the class of weakly Δ-levelable sets is neither large nor small in the sense of resource-bounded Ambos-Spies category. However, as we will show next, it is large in the sense of resource-bounded general Ambos-Spies category, resource-bounded Fenner category, and resource-bounded Lutz category.

THEOREM 4.7. *Let G be a Lutz n^3-generic set. Then G is weakly Δ-levelable.*

Proof. Let $B \in \mathbf{P}$. We show that $\bar{B}$ witnesses that the unsafe approximation B of G is not optimal. For any string x, define $f(x) = y$, where $|y| = |x|^2$ and $y[j] = 0$ if and only if $z_{|x|+j} \in B$. Obviously, f is computable in time n^3. Since G is Lutz n^3-generic, G meets f infinitely often. Hence, for any k and n_0, there exists $n > n_0$ such that $n^2 - 2n > k$ and, for all strings x with $z_n \leq x < z_{n^2}$, $x \in G$ if and only if $x \in \bar{B}$. Hence

$$\begin{aligned} \|(G\Delta B)\restriction z_{n^2}\| &\geq n^2 - n \\ &> n + k \\ &\geq \|(G\Delta \bar{B})\restriction z_{n^2}\| + k, \end{aligned}$$

which implies that G is weakly Δ-levelable. □

COROLLARY 4.8. *The class of weakly Δ-levelable sets is comeager in the sense of resource-bounded Lutz, Fenner, and general Ambos-Spies categories.*

Proof. This follows from Theorems 3.7, 3.8, and 4.7. □

Now we show that the class of Δ-levelable sets is neither meager nor comeager in the sense of all these resource-bounded categories we have discussed above.

THEOREM 4.9. *There exists a set G in $\mathbf{E}_2$ which is both general A-generic and Δ-levelable.*

Proof. Let $\delta(0) = 0, \delta(n+1) = 2^{2^{\delta(n)}}$. For each set $P_e \in \mathbf{P}$, let $P_{g(e)}$ be defined in such a way that

$$P_{g(e)}(x) = \begin{cases} 1 - P_e(x) & \text{if } x = 0^{\delta(<e,n>)} \text{ for some } n \in N, \\ P_e(x) & \text{otherwise.} \end{cases}$$

In the following we construct a general A-generic set G which is Δ-levelable by keeping $P_{g(e)}$ to witness that the unsafe approximation P_e of G is not optimal. Let

$\{f_i : i \in N\}$ be an enumeration of all functions in $\mathbf{F}_2$ such that $f_i(x)$ can be computed uniformly in time $2^{\log^k(|x|+i)}$ for some $k \in N$.

The set G is constructed in stages. To ensure that G is general A-generic, it suffices to meet for all $e \in N$ the following requirements:

G_e : if f_e is dense along G, then G meets f_e.

To ensure that G is Δ-levelable, it suffices to meet for all $e, k \in N$ the following requirements, as shown at the end of the proof:

$L_{\langle e,k\rangle} : \exists n_1 \in N\ \forall n > n_1\ (\|(G\Delta P_e)\upharpoonright z_n\| > \|(G\Delta P_{g(e)})\upharpoonright z_n\| + k)$.

The strategy for meeting a requirement G_e is as follows: at stage s, if G_e has not been satisfied yet and $f_e(G\upharpoonright z_s)$ is defined, then let G extend $(G\upharpoonright z_s)f_e(G\upharpoonright z_s)$. But this action may injure the satisfaction of some requirements $L_{\langle i,k\rangle}$ and G_m. The conflict is solved by delaying the action until it will not injure the satisfaction of the requirements $L_{\langle i,k\rangle}$ and G_m which have higher priority than G_e.

The strategy for meeting a requirement $L_{\langle e,k\rangle}$ is as follows: at stage s, if $L_{\langle e,k\rangle}$ has not been satisfied yet and $P_e(z_s) \neq P_{g(e)}(z_s)$, then let $G(z_s) = P_{g(e)}(z_s)$. When a requirement G_e becomes satisfied at some stage, it is satisfied forever, so $L_{\langle e,k\rangle}$ can only be injured finitely often and then it will have a chance to become satisfied forever.

Stage s.

In this stage, we define the value of $G(z_s)$.

A requirement G_n *requires* attention if

1. $n < s$.
2. G_n has not been satisfied yet.
3. There exists $t \leq s$ such that
   - A. $f_n(G\upharpoonright z_t)$ is defined.
   - B. $G\upharpoonright z_s$ is consistent with $(G\upharpoonright z_t)f_n(G\upharpoonright z_t)$.
   - C. For all $e, k \in N$ such that $\langle e,k\rangle < n$, there is at most one $\langle e,m\rangle \in N$ such that $0^{\delta(\langle e,m\rangle)} \in dom((G\upharpoonright z_t)f_n(G\upharpoonright z_t))$.
   - D. For all $e, k \in N$ such that $\langle e,k\rangle < n$,

$$\|(G\Delta P_e)\upharpoonright z_s\| - \|(G\Delta P_{g(e)})\upharpoonright z_s\| > k + n. \tag{1}$$

Fix the minimal m such that G_m requires attention, and fix the minimal t in the above item 3 corresponding to the requirement G_m. If there is no such m, then let $G(z_s) = 1 - P_e(z_s)$ if $z_s = 0^{\delta(\langle e,n\rangle)}$ for some $e, n \in N$, and let $G(z_s) = 0$ otherwise. Otherwise we say that G_m *receives* attention. Moreover, let

$$G(z_s) = \begin{cases} ((G\upharpoonright z_t)f_m(G\upharpoonright z_t))(z_s) & \text{if } z_s \in dom((G\upharpoonright z_t)f_m(G\upharpoonright z_t)), \\ 1 - P_e(z_s) & \text{if } z_s \notin dom((G\upharpoonright z_t)f_m(G\upharpoonright z_t))\ \&\ z_s = 0^{\delta(\langle e,n\rangle)} \\ & \text{for some } e, n, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the construction.

We show that all requirements are met by proving a sequence of claims.

Claim 1. *Every requirement G_n requires attention at most finitely often.*

Proof. The proof is by induction. Fix n and assume that the claim is correct for all numbers less than n. Then there is a stage s_0 such that no requirement G_m with $m < n$ requires attention after stage s_0. So G_n receives attention at any stage $s > s_0$

at which it requires attention. Hence it is immediate from the construction that G_n requires attention at most finitely often. □

CLAIM 2. *Given $n_0 \in N$, if no requirement $G_n (n < n_0)$ requires attention after stage s_0 and G_{n_0} requires attention at stage s_0, then for all $\langle e,k \rangle < n_0$ and $s > s_0$,*

$$\|(G\Delta P_e)\restriction z_s\| - \|(G\Delta P_{g(e)})\restriction z_s\| > k + n_0 - 1.$$

Proof. The proof is straightforward from the construction. □

CLAIM 3. *Every requirement G_n is met.*

Proof. For a contradiction, fix the minimal n such that G_n is not met. Then f_n is dense along G. We have to show that G_n requires attention infinitely often which is contrary to Claim 1. Since $\|P_e \Delta P_{g(e)}\| = \infty$ for all $e \in N$, by the construction and Claim 2, there will be a stage s_0 such that at all stages $s > s_0$, (1) holds for all $e, k \in N$ such that $\langle e,k \rangle < n$. Hence G_n requires attention at each stage $s > s_0$ at which $f_n(G\restriction z_s)$ is defined. □

CLAIM 4. *Every requirement $L_{\langle e,k \rangle}$ is met.*

Proof. This follows from Claims 2 and 3. □

Now we show that G is both A-generic and Δ-levelable. G is A-generic since all requirements G_n are met. For $\langle e,k \rangle \in N$, let $n_{\langle e,k \rangle}$ be the least number s_0 such that for all $s > s_0$,

$$\|(G\Delta P_e)\restriction z_s\| > \|(G\Delta P_{g(e)})\restriction z_s\| + k$$

and let $f(n)$ be the biggest k such that

$$\forall e \le k \ (n \ge n_{\langle e,k \rangle}).$$

Then $\lim_{n\to\infty} f(n) = \infty$ and, for all $e \in N$,

$$\|(G\Delta P_e)\restriction z_n\| \ge \|(G\Delta P_{g(e)})\restriction z_n\| + f(n) \ \text{ a.e.}$$

That is to say, G is Δ-levelable with density f. □

THEOREM 4.10. *There exists a set G in $\mathbf{E}_2$ which is general A-generic but not Δ-levelable.*

Proof. As in the previous proof, a set G is constructed in stages. To ensure that G is general A-generic, it suffices to meet for all $e \in N$ the following requirements:

G_e : if f_e is dense along G, then G meets f_e.

Fix a set $B \in \mathbf{P}$. Then the requirements

$NL_{\langle e,k \rangle} : P_e \Delta B$ infinite $\Rightarrow \exists n\ (\|(G\Delta P_e)\restriction z_n\| - \|(G\Delta B)\restriction z_n\| \ge k)$

will ensure that B witnesses the failure of Δ-levelability of G.

To meet the requirements G_e, we use the strategy in Theorem 4.9. The strategy for meeting a requirement $NL_{\langle e,k \rangle}$ is as follows: at stage s such that $P_e(z_s) \ne B(z_s)$ and $\|(G\Delta P_e)\restriction z_n\| - \|(G\Delta B)\restriction z_n\| < k$ for all $n < s$, let $G(z_s) = B(z_s)$. If $P_e \ne^* B$, this action can be repeated over and over again. Hence $\|G\Delta P_e\|$ is growing more quickly than $\|G\Delta B\|$, and eventually the requirement $NL_{\langle e,k \rangle}$ is met at some sufficiently large stage.

Define a priority ordering of the requirements by letting $R_{2n} = G_n$ and $R_{2\langle e,k \rangle+1} = NL_{\langle e,k \rangle}$. We now describe the construction of G formally.

Stage s.

In this stage, we define the value of $G(z_s)$.

A requirement $NL_{\langle e,k \rangle}$ *requires* attention if $\langle e,k \rangle < s$ and

1. $P_e(z_s) \neq B(z_s)$.
2. $\|(G\Delta P_e)\restriction z_n\| - \|(G\Delta B)\restriction z_n\| < k$ for all $n < s$.

A requirement G_n *requires* attention if

1. $n < s$.
2. G_n has not been satisfied yet.
3. There exists $t \leq s$ such that
   A. $f_n(G\restriction z_t)$ is defined.
   B. $G\restriction z_s$ is consistent with $(G\restriction z_t)f_n(G\restriction z_t)$.
   C. There is no $e, k \in N$ such that
      (1). $\langle e, k\rangle < n$.
      (2). $\forall u < s\ (\|(G\Delta P_e)\restriction z_u\| - \|(G\Delta B)\restriction z_u\| < k)$.
      (3). There exists $y \in dom((G\restriction z_t)f_n(G\restriction z_t)) - dom(G\restriction z_s)$ such that $P_e(y) \neq B(y)$.

Fix the minimal m such that R_m requires attention. If there is no such m, let $G(z_s) = B(z_s)$. Otherwise we say that R_m *receives* attention. Moreover, if $R_m = NL_{\langle e,k\rangle}$ then let $G(z_s) = B(z_s)$. If $R_m = G_n$ then fix the least t in the above item 3 corresponding to the requirement G_m. Let $G(z_s) = ((G\restriction z_t)f_m(G\restriction z_t))(z_s)$ if $z_s \in dom((G\restriction z_t)f_m(G\restriction z_t))$ and let $G(z_s) = B(z_s)$ otherwise.

This completes the construction of G.

It suffices to show that all requirements are met. Note that, by definition of requiring attention, R_m is met if and only if R_m requires attention at most finitely often. So, for a contradiction, fix the minimal m such that R_m requires attention infinitely often. By minimality of m, fix a stage s_0 such that no requirement $R_{m'}$ with $m' < m$ requires attention after stage s_0. Then R_m receives attention at any stage $s > s_0$ at which R_m requires attention. Now, we first assume that $R_m = G_n$. Then at some stage $s > s_0$, G_n receives attention and becomes satisfied forever. Finally assume that $R_m = NL_{\langle e,k\rangle}$. Then $B\Delta P_e$ is infinite and, at all stages $s > s_0$ such that $B(z_s) \neq P_e(z_s)$, the requirement $NL_{\langle e,k\rangle}$ receives attention; hence $G(z_s) = B(z_s)$. Since, for all other stages s with $s > s_0$, $B(z_s) = P_e(z_s)$, $G\Delta P_e$ grows more rapidly than $G\Delta B$; hence

$$\lim_n(\|(G\Delta P_e)\restriction z_n\| - \|(G\Delta B)\restriction z_n\|) = \infty$$

and $NL_{\langle e,k\rangle}$ is met contrary to assumption. □

COROLLARY 4.11. *The class of Δ-levelable sets is neither meager nor comeager in the sense of resource-bounded (general) Ambos-Spies, Lutz, and Fenner categories.*

Proof. The proof follows from Theorems 3.7, 4.9, and 4.10. □

5. Resource-bounded randomness versus polynomial-time approximations. We first introduce a fragment of Lutz's effective measure theory which will be sufficient for our investigation.

DEFINITION 5.1. *A* martingale *is a function $F : \Sigma^* \to R^+$ such that, for all $x \in \Sigma^*$,*

$$F(x) = \frac{F(x1) + F(x0)}{2}.$$

A martingale F succeeds *on a sequence $\xi \in \Sigma^\infty$ if $\limsup_n F(\xi[0..n-1]) = \infty$. $S^\infty[F]$ denotes the set of sequences on which the martingale F succeeds.*

DEFINITION 5.2 (see Lutz [10]). *A set $\mathbf{C}$ of infinite sequences has* p-measure 0 *($\mu_p(\mathbf{C}) = 0$) if there is a polynomial-time computable martingale $F : \Sigma^* \to Q^+$ which* *succeeds on every sequence in* $\mathbf{C}$*. The set* $\mathbf{C}$ *has* p-measure 1 ($\mu_p(\mathbf{C}) = 1$) *if* $\mu_p(\bar{\mathbf{C}}) = 0$ *for the complement* $\bar{\mathbf{C}} = \{\xi \in \Sigma^\infty : \xi \notin \mathbf{C}\}$ *of* $\mathbf{C}$*.*

Definition 5.3 (see Lutz [10]). *A sequence* ξ *is* n^k-random *if, for every* n^k*-time computable martingale* F*,* $\limsup_n F(\xi[0..n-1]) < \infty$*; that is to say,* F *does not succeed on* ξ*. A sequence* ξ *is* p-random *if* ξ *is* n^k*-random for all* $k \in N$*.*

The following theorem is straightforward from the definition.

Theorem 5.4. *A set* $\mathbf{C}$ *of infinite sequences has* p-measure 0 *if and only if there exists a number* $k \in N$ *such that there is no* n^k*-random sequences in* $\mathbf{C}$*.*

Proof. See, e.g., [16]. ∎

The relation between p-measure and the class of $\mathbf{P}$-levelable sets is characterized by the following theorem.

Theorem 5.5 (see Mayordomo [11]). *The class of* $\mathbf{P}$*-bi-immune sets has p-measure* 1*.*

Corollary 5.6. *The class of* $\mathbf{P}$*-levelable sets has p-measure* 0*.*

Corollary 5.7. *The class of sets which possesses optimal polynomial-time safe approximations has p-measure* 1*.*

Corollary 5.8. *For each p-random set A, A has an optimal polynomial-time safe approximation.*

Now we turn our attention to the relations between the p-randomness concept and the concept of polynomial-time unsafe approximations. In our following proof, we will use the law of the iterated logarithm for p-random sequences.

Definition 5.9. *A sequence* $\xi \in \Sigma^\infty$ *satisfies the law of the iterated logarithm if*

$$\limsup_{n\to\infty} \frac{2\sum_{i=0}^{n-1}\xi[i] - n}{\sqrt{2n\ln\ln n}} = 1$$

and

$$\liminf_{n\to\infty} \frac{2\sum_{i=0}^{n-1}\xi[i] - n}{\sqrt{2n\ln\ln n}} = -1.$$

Theorem 5.10 (see Wang [17]). *There exists a number* $k \in N$ *such that every* n^k*-random sequence satisfies the law of the iterated logarithm.*

For the sake of convenience, we will identify a set with its characteristic sequence. The symmetric difference of two sets can be characterized by the parity function on sequences.

Definition 5.11.

1. *The parity function* $\oplus : \Sigma \times \Sigma \to \Sigma$ *on bits is defined by*

$$b_1 \oplus b_2 = \begin{cases} 0 & \textit{if } b_1 = b_2, \\ 1 & \textit{otherwise}, \end{cases}$$

   where $b_1, b_2 \in \Sigma$*.*
2. *The parity function* $\oplus : \Sigma^\infty \times \Sigma^\infty \to \Sigma^\infty$ *on sequences is defined by* $(\xi \oplus \eta)[n] = \xi[n] \oplus \eta[n]$*.*
3. *The parity function* $\oplus : \Sigma^* \times \{f : f$ *is a partial function from* Σ^* *to* $\Sigma\} \to \Sigma^*$ *on strings and functions is defined by* $x \oplus f = b_0 \cdots b_{|x|-1}$*, where* $b_i = x[i] \oplus f(x[0..i-1])$ *if* $f(x[0..i-1])$ *is defined and* $b_i = \lambda$ *otherwise.*
4. *The parity function* $\oplus : \Sigma^\infty \times \{f : f$ *is a partial function from* Σ^* *to* $\Sigma\} \to \Sigma^* \cup \Sigma^\infty$ *on sequences and functions is defined by* $\xi \oplus f = b_0 b_1 \cdots$ *where* $b_i = \xi[i] \oplus f(\xi[0..i-1])$ *if* $f(\xi[0..i-1])$ *is defined and* $b_i = \lambda$ *otherwise.*

The intuitive meaning of $\xi \oplus f$ is as follows: Given a sequence ξ and a number $n \in N$ such that $f(\xi[0..n-1])$ is defined, we use f to predict the value of $\xi[n]$ from the first n bits $\xi[0..n-1]$. If the prediction is successful, then output 0, else output 1. And $\xi \oplus f$ is the output sequence.

We first explain a useful technique which is similar to the invariance property of p-random sequences.

LEMMA 5.12. *Let $\xi \in \Sigma^\infty$ be n^k-random and $f : \Sigma^* \to \Sigma$ be a partial function computable in time n^k such that $\xi \oplus f$ is an infinite sequence. Then $\xi \oplus f$ is n^{k-1}-random.*

Proof. For a contradiction assume that $\xi \oplus f$ is not n^{k-1}-random and let $F : \Sigma^* \to Q^+$ be an n^{k-1}-martingale that succeeds on $\xi \oplus f$. Define $F' : \Sigma^* \to Q^+$ by letting $F'(x) = F(x \oplus f)$ for all $x \in \Sigma^*$. It is a routine to check that F' is an n^k-martingale. Moreover, since F succeeds on $\xi \oplus f$, F' succeeds on ξ, which is a contradiction with the hypothesis that ξ is n^k-random. □

LEMMA 5.13. *Let k be the number in Theorem* 5.10*, and let $A, B, C \subseteq \Sigma^*$ be three sets such that the following conditions hold.*

1. *$B, C \in \mathbf{P}$.*
2. *$\|B\Delta C\| = \infty$.*
3. *There exists $c \in N$ such that, for almost all n,*

$$\|(A\Delta C)\restriction z_n\| - \|(A\Delta B)\restriction z_n\| \geq -c. \tag{2}$$

Then A is not n^{k+1}-random.

Proof. Let α, β, and γ be the characteristic sequences of A, B, and C, respectively.

By Lemma 5.12, it suffices to define an n^2-time computable partial function $f : \Sigma^* \to \Sigma$ such that $\alpha \oplus f$ is an infinite sequence which is not n^k-random. Define the function f by

$$f(x) = \begin{cases} \beta[|x|] & \text{if } \beta[|x|] \neq \gamma[|x|], \\ \text{undefined} & \text{if } \beta[|x|] = \gamma[|x|]. \end{cases}$$

Then f is n^2-time computable and, since $\|B\Delta C\| = \infty$, $\alpha \oplus f$ is an infinite sequence. In order to show that $\alpha \oplus f$ is not n^k-random, we show that $\alpha \oplus f$ does not satisfy the law of the iterated logarithm.

We first show that, for all $n \in N^+$, the following equation holds:

$$\sum_{i=0}^{n-1}(\alpha \oplus \gamma)[i] - \sum_{i=0}^{n-1}(\alpha \oplus \beta)[i] = l_n - 2\sum_{i=0}^{l_n-1}(\alpha \oplus f)[i], \tag{3}$$

where $l_n = |\alpha[0..n-1] \oplus f|$.

Let

$$a(n) = \|\{i < n : \alpha[i] \neq \gamma[i] = \beta[i]\}\|,$$

$$b(n) = \|\{i < n : \alpha[i] \neq \gamma[i] \neq \beta[i]\}\|,$$

$$c(n) = \|\{i < n : \alpha[i] = \gamma[i] \neq \beta[i]\}\|,$$

$$d(n) = \|\{i < n : \alpha[i] = \gamma[i] = \beta[i]\}\|.$$

Then

$$\begin{aligned} \textstyle\sum_{i=0}^{n-1}(\alpha \oplus \gamma)[i] &= a(n) + b(n), \\ \textstyle\sum_{i=0}^{n-1}(\alpha \oplus \beta)[i] &= a(n) + c(n), \\ l_n &= b(n) + c(n), \\ \textstyle\sum_{i=0}^{l_n-1}(\alpha \oplus f)[i] &= c(n). \end{aligned}$$

Obviously, this implies (3).

The condition (2) is equivalent to

$$\sum_{i=0}^{n-1}(\alpha \oplus \gamma)[i] - \sum_{i=0}^{n-1}(\alpha \oplus \beta)[i] \geq -c.$$

So, by (3),

$$l_n - 2\sum_{i=0}^{l_n-1}(\alpha \oplus f)[i] \geq -c \tag{4}$$

for almost all n, where $l_n = |\alpha[0..n-1] \oplus f|$. By (4),

$$\liminf_{n\to\infty} \frac{n - 2\sum_{i=0}^{n-1}(\alpha \oplus f)[i]}{\sqrt{2n \ln \ln n}} \geq 0.$$

Hence, by Theorem 5.10, $\alpha \oplus f$ is not n^k-random. This completes the proof. □

Now we are ready to prove our main theorems of this section.

THEOREM 5.14. *The class of* Δ*-levelable sets has p-measure* 0.

Proof. Let A be a Δ-levelable set. Then there is a function $f(n) \geq 0$ satisfying $\lim_{n\to\infty} f(n) = \infty$ and polynomial-time computable sets B, C such that for all n,

$$\|(A\Delta C)\upharpoonright z_n\| - \|(A\Delta B)\upharpoonright z_n\| \geq f(n).$$

By Lemma 5.13, A is not n^{k+1}-random, where k is the number in Theorem 5.10. So the theorem follows from Theorem 5.4. □

THEOREM 5.15. *The class of sets which have optimal polynomial-time unsafe approximations has p-measure* 0.

Proof. If A has an optimal polynomial-time unsafe approximation, then there is a polynomial-time computable set B and a number $c \in N$ such that, for all n,

$$\|(A\Delta B)\upharpoonright z_n\| - \|(A\Delta \bar{B})\upharpoonright z_n\| < c;$$

i.e.,

$$\|(A\Delta \bar{B})\upharpoonright z_n\| - \|(A\Delta B)\upharpoonright z_n\| > -c.$$

By Lemma 5.13, A is not n^{k+1}-random, where k is the number in Theorem 5.10. So the theorem follows from Theorem 5.4. □

COROLLARY 5.16. *The class of sets which are weakly* Δ*-levelable but not* Δ*-levelable has p-measure* 1.

COROLLARY 5.17. *Every p-random set is weakly* Δ*-levelable but not* Δ*-levelable.*

Acknowledgments. I would like to thank Professor Ambos-Spies for many comments on an early version of this paper, and I would like to thank two anonymous referees for their valuable comments on this paper.

REFERENCES

[1] K. Ambos-Spies, *On optimal polynomial time approximations: **P**-levelability vs. Δ-levelability*, in Proc. 22nd ICALP, Springer-Verlag, Berlin, New York, 1995, pp. 384–392.

[2] K. Ambos-Spies, *Resource-bounded genericity*, in Proc. 10th Conf. on Structure in Complexity Theory, IEEE Computer Society Press, Piscataway, NJ, 1995, pp. 162–181.

[3] K. Ambos-Spies, H. Fleischhack, and H. Huwig, *Diagonalizations over polynomial time computable sets*, Theoret. Comput. Sci., 51 (1987), pp. 177–204.

[4] K. Ambos-Spies, H.-C. Neis, and S. A. Terwijn, *Genericity and measure for exponential time*, Theoret. Comput. Sci., 168 (1996), pp. 3–19.

[5] L. Berman, *On the structure of complete sets*, in Proc. 17th Annual Symposium on Foundations of Computer Science, IEEE Computer Society Press, Piscataway, NJ, 1976, pp. 76–80.

[6] P. Duris and J. D. P. Rolim, ***E**-complete sets do not have optimal polynomial time approximations*, Lecture Notes in Comput. Sci. 841, Springer-Verlag, New York, 1994, pp. 38–51.

[7] S. Fenner, *Notions of resource-bounded category and genericity*, in Proc. 6th Conf. on Structure in Complexity Theory, IEEE Computer Society Press, Piscataway, NJ, 1991, pp. 196–212.

[8] K. Ko and D. Moore, *Completeness, approximation and density*, SIAM J. Comput., 10 (1981), pp. 787–796.

[9] J. H. Lutz, *Category and measure in complexity classes*, SIAM J. Comput., 19 (1990), pp. 1100–1131.

[10] J. H. Lutz, *Almost everywhere high nonuniform complexity*, J. Comput. System Sci., 44 (1992), pp. 220–258.

[11] E. Mayordomo, *Almost every set in exponential time is **P**-bi-immune*, Theoret. Comput. Sci., 136 (1994), pp. 487–506.

[12] E. Mayordomo, *Contributions to the Study of Resource-Bounded Measure*, Ph.D. thesis, Universidad Polytecnica de Catalunya, Barcelona, 1994.

[13] A. R. Meyer and M. S. Paterson, *With what frequency are apparently intractable problems difficult?* Technical Report TM-126, Laboratory for Computer Science, MIT, Cambridge, MA, 1979.

[14] P. Orponen, A. Russo, and U. Schöning, *Optimal approximations and polynomially levelable sets*, SIAM J. Comput., 15 (1986), pp. 399–408.

[15] D. A. Russo, *Optimal approximation of complete sets*, Lecture Notes in Comput. Sci. 223, Springer-Verlag, New York, 1986, pp. 311–324.

[16] Y. Wang, *Randomness and Complexity*. Ph.D. thesis, Heidelberg, 1996.

[17] Y. Wang, *The law of the iterated logarithm for p-random sequences*, in Proc. 11th Conf. Computational Complexity (formerly Conf. on Structure in Complexity Theory), IEEE Computer Society Press, Piscataway, NJ, 1996, pp. 180–189.

[18] Y. Wang, *Randomness, stochasticity, and approximations*, Lecture Notes in Comput. Sci. 1269, Springer-Verlag, New York, 1997, pp. 213–225.

[19] Y. Yesha, *On certain polynomial-time truth-table reducibilities of complete sets to sparse sets*, SIAM J. Comput., 12 (1983), pp. 411–425.