Resource-Bounded Balanced Genericity, Stochasticity and Weak Randomness*

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Abstract. We introduce balanced $t(n)$-genericity which is a refinement of the genericity concept of Ambos-Spies, Fleischhack and Huwig [2] and which in addition controls the frequency with which a condition is met. We show that this concept coincides with the resource-bounded version of Church's stochasticity [6]. By uniformly describing these concepts and weaker notions of stochasticity introduced by Wilber [19] and Ko [11] in terms of prediction functions, we clarify the relations among these resource-bounded stochasticity concepts. Moreover, we give descriptions of these concepts in the framework of Lutz's resource-bounded measure theory [13] based on martingales: We show that $t(n)$-stochasticity coincides with a weak notion of $t(n)$-randomness based on so-called simple martingales but that it is strictly weaker than $t(n)$-randomness in the sense of Lutz.

1 Introduction

Over the last years resource-bounded versions of Baire category and Lebesgue measure have been introduced in complexity theory. These concepts allow a quantitative analysis of the structural properties of complexity classes. In most cases the concepts were introduced for deterministic time classes, where in general the $t(n)$-time bounded concepts correspond to the class $DTIME(t(2^n))$. In particular, polynomial time bounded versions of these concepts have been used to analyse the structure of the class E of the deterministic exponential time sets.

Many applications of category and measure can be reduced to questions about the typical sets for these concepts, i.e., the generic sets in case of Baire category and the random sets in the case of Lebesgue measure. These typical sets have all properties which are shared by a large class of sets, i.e., by a comeager respectively measure-1 class (in the corresponding resource-bounded sense).

Resource-bounded genericity concepts have been introduced by Ambos-Spies, Fleischhack and Huwig [2], Lutz [12], Fenner [8, 9], Ambos-Spies [1], and others.

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Resource-bounded randomness concepts can be found e.g. in Schnorr [15], Wilber [19], Ko [11] and Lutz [13]. While an attempt to clarify the relations among the various genericity notions has been recently made by Ambos-Spies in [1], it seems that the relations among the different resource-bounded randomness notions have not yet been explored systematically, though some isolated results have been obtained.

As also shown in [1], as in the classical case, most of the genericity concepts are incompatible with the randomness concepts in the resource-bounded case too. There is a notable exception, however: the genericity concept of Ambos-Spies, Fleischhack and Huwig [2] is compatible with Lutz's randomness [13]. In fact, Ambos-Spies, Neis and Terwijn [3] have used this type of genericity to get new measure results and simpler proofs of certain older measure results. As observed by them too, however, genericity cannot control the density of a set whereas random sets are exponentially dense. More generally, in case of genericity, certain events which may happen infinitely often are forced to actually happen infinitely often. Beyond this, however, genericity cannot determine the relative frequency with which the events will happen, i.e., which fractions of the chances are actually realized. In contrast, for a random set, the distribution of the events (captured by the measure concept) will be determined. In particular, all $t(n)$-random sets satisfy the law of large numbers, whereas this law fails for some $t(n)$-generic sets.

These observations have motivated our investigations here. We address the question, whether the gap between $t(n)$-genericity (in the sense of [2]) and $t(n)$-randomness (in the sense of [13]) can be bridged by considering an extension of the former which in addition controls the frequency. The answer we obtain is negative, but we hope that our investigations help to better understand the relations among some of the resource-bounded randomness notions in the literature. We show that our new balanced $t(n)$-genericity concept coincides with the resource-bounded version of some other, weaker, randomness concept, namely that of Church [6]. According to the classification of randomness concepts by Kolmogorov (see [16]), Church's randomness, which is based on the distribution of the 0s and 1s in effectively chosen subsequences, is a stochasticity notion, while Lutz's concept, a resource-bounded version of Schnorr's randomness concept based on martingales [15], is a notion of typicalness. Our equivalence proof is based on the characterization of genericity and stochasticity in terms of prediction functions. By giving characterizations of other, weaker, resource-bounded stochasticity notions in terms of prediction functions too, we clarify the relations among these concepts, which originally were defined in quite different terms, and we prove some new separation results for these notions.

Finally we compare stochasticity and typicalness. We show that the expressive power of prediction functions is that of so-called simple martingales. So, $t(n+1)$-stochasticity, hence balanced $t(n)$-genericity, coincides with weak $t(n+1)$-randomness, where weak $t(n)$-randomness is defined by simple martingales. We also show, however, that general martingales are more powerful, i.e., that there are weakly $t(n)$-random sets which are not $t(n)$-random.

The outline of the paper is as follows. In Section 2 we review the $t(n)$-

genericity concept of Ambos-Spies et al. [2] and introduce its balanced counterpart. Section 3 is devoted to the equivalence of balanced $t(n)$-genericity and $t(n+1)$-stochasticity and a classification of various stochasticity concepts. Finally, in Section 4, we derive from Lutz's $t(n)$-randomness the new, weaker concept of weak $t(n)$-randomness, and show the equivalence of this concept with stochasticity.

We close this section by introducing some notation. Let ω be the set of natural numbers and let $\{0,1\}^*$ be the set of (finite) binary strings. For a string x, $x(m)$ denotes the $(m+1)$th bit in x, i.e., $x = x(0)...x(n-1)$, where $n = |x|$ is the length of x. λ is the empty string. We identify strings with numbers by letting n be the $(n+1)$th string under the canonical ordering. Note that $|n| \approx \log(n)$. Lower case letters $\cdots, k, l, m, n, \cdots, x, y, z$ from the middle and the end of the alphabet will denote numbers and strings. The letters i and j are reserved for elements of $\{0,1\}$, and lower case Greek letters denote nonnegative real numbers.

A set of strings is called a problem or shortly a set, while sets of sets are called classes. Capital letters denote sets, $||A||$ denotes the cardinality of A. We identify a set with its infinite characteristic string, i.e., $n \in A$ iff $A(n) = 1$ and $n \notin A$ iff $A(n) = 0$, so that $\{0,1\}^\omega$, the set of infinite binary sequences, is identified with the power class of $\{0,1\}^*$. We let $A \upharpoonright n$ denote the initial segment $A(0)...A(n-1) \in \{0,1\}^*$ of A of length n. I.e., interpreted as a set, $A \upharpoonright n = \{x : x < n \;\&\; x \in A\}$.

We will use strings in two different meanings: as elements of sets and as finite initial segments of sets. In an attempt to avoid confusion, usually we will write $X \upharpoonright x$ for strings intended to denote initial segments. Then $X \upharpoonright x$ denotes a string of length x and, for $y < x$, $X(y)$ or $(X \upharpoonright x)(y)$ will denote the $(y+1)$th bit of $X \upharpoonright x$. Also note the difference in the length of an initial segment $A \upharpoonright x$ and the length of its bound x : $2^{|x|} - 1 \leq |A \upharpoonright x| \leq 2^{|x|+1} - 1$. Since, as mentioned before, many of the genericity and randomness concepts discussed in this paper are based on functions defined on initial segments, this will be responsible for the fact that the $DTIME(t(O(n)))$ bounded concepts will correspond to the class $DTIME(t(O(2^n)))$.

Throughout this paper, $t(n)$ is a time constructible recursive function such that $t(n)$ is nondecreasing and $t(n) \geq n$ for all n. In addition, if $t(n)$ is not the identity function, we assume that $DTIME(t(n)) = DTIME(O(t(n)))$. Finally, for a partial function f, we let $f(x)\downarrow$ ($f(x)\uparrow$) denote that $f(x)$ is (un)defined.

2 Genericity and Balanced Genericity

Ambos-Spies, Fleischhack and Huwig [2] introduced resource-bounded genericity notions corresponding to finite-extension diagonalization arguments in which every single diagonalization step requires only a one-bit extension. The properties enforced by the single diagonalization steps are formalized by *conditions* in these concepts.

Definition 1. A *condition* C is a set $C \subseteq \{0,1\}^*$. A $t(n)$-*condition* is a condition $C \in DTIME(t(n))$. A condition C is *dense along* a set A if there are infinitely many x such that $(A\restriction x)i \in C$ for some $i \in \{0, 1\}$. A set A *meets* a condition C if $A\restriction x \in C$ for some x. A is $t(n)$-*generic* if A meets every $t(n)$-condition which is dense along A.

Intuitively, a condition C is dense along a set A if in the inductive definition of A there are infinitely many chances to extend $A\restriction x$ to $A\restriction(x+1)$ in such a way that $A\restriction(x+1)$ will force the property encoded by C for A. So a $t(n)$-generic set A has all properties which can be encoded by $t(n)$-conditions which can be forced infinitely often along the construction of A. In the following lemma we give an example.

Lemma 2. *(Ambos-Spies et al. [2]) Let A be $t(n)$-generic. Then A is P-bi-immune.*

Proof. By symmetry, it suffices to show that A is P-immune. So let $B \in P$ be infinite. Define a condition C by $C = \{(X\restriction x)0 \ : \ x \in B\}$. Then, by $B \in P$, C is an n- (hence $t(n)$-) condition and, by infinity of B, C is dense along all sets. Moreover, by definition, no superset of B meets C. So, by $t(n)$-genericity, A meets C whence B is not contained in A. □

As observed already in [2], a $t(n)$-generic set A meets a condition C which is dense along A not just once but infinitely often.

Lemma 3. *(Ambos-Spies et al. [2]) Let A be $t(n)$-generic and let C be a $t(n)$-condition which is dense along A. There are infinitely many strings x such that $A\restriction x \in C$.*

For analyzing the frequency with which a set meets conditions which are dense along it, it is convenient to consider only proper conditions.

Definition 4. A condition C is *proper* if for every string x, $x0 \notin C$ or $x1 \notin C$. A set A *meets (avoids)* a proper condition C *at* x if $(A\restriction x)A(x) \in C$ $((A\restriction x)(1 - A(x)) \in C)$.

As the following observation shows, in the definition of $t(n)$-genericity, it suffices to consider proper conditions.

Lemma 5. *For any set A, the following are equivalent.*

1. *A is $t(n)$-generic.*
2. *A meets every proper $t(n)$-condition which is dense along A.*
3. *A infinitely often meets every proper $t(n)$-condition which is dense along A.*

Moreover, we can replace "meets" by "avoids" in *3* of Lemma 5. So a $t(n)$-generic set A infinitely often meets and infinitely often avoids every proper $t(n)$-condition which is dense along A. As the following theorem shows, however, we cannot say anything about the relative frequency of these events.

Theorem 6. *(Ambos-Spies et al. [3]) Let C be a proper $t(n)$-condition and let f be an unbounded, nondecreasing recursive function. There is a $t(n)$-generic set A such that, for all n, $\|\{y < n \ : \ A \text{ meets } C \text{ at } y\}\| \leq f(n)$. In particular, there is a sparse $t(n)$-generic set.*

To overcome this shortcoming we introduce the following strengthening of $t(n)$-genericity.

Definition 7. A set A *meets* a condition C *balancedly* if

$$\lim_n \frac{\|\{y < n \ : \ A\restriction(y+1) \in C\}\|}{\|\{y < n \ : \ \exists i \ ((A\restriction y)i \in C)\}\|} = \frac{1}{2}. \tag{1}$$

A set A is *balancedly $t(n)$-generic* if A meets balancedly every proper $t(n)$-condition which is dense along A.

Note that A meets a proper $t(n)$-condition C balancedly iff, in the limit, the frequency of A meeting and avoiding C is the same. The following theorem demonstrates the additional power of balanced genericity compared with genericity.

Theorem 8. *Let A be balancedly $t(n)$-generic.*

1. *A satisfies the law of large numbers, i.e.,*

$$\lim_n \|\{y < n : A(y) = 1\}\| / \|\{y < n : A(y) = 0\}\| = 1. \tag{2}$$

2. *A is exponentially dense.*

Proof. *1.* The proper n-condition $C = \{x1 \ : \ x \in \{0,1\}^*\}$ is dense along all sets. So A meets C balancedly, which immediately implies (2).

2. By *1*, $\|A^{\leq n}\| \geq 2^{n-1}$ for almost all n. □

Corollary 9. *There is a $t(n)$-generic set which is not balancedly n-generic, hence not balancedly $t(n)$-generic.*

3 Resource-Bounded Stochasticity

The first notion of randomness was proposed by von Mises [18]. He called a sequence random if every subsequence obtained by an admissible selection rule satisfies the law of large numbers. A formalization of this notion, based on formal computability was given by Church [6] in 1940. Following Kolmogorov (see [16]) we call randomness in the sense of von Mises and Church stochasticity.

For a formal definition of Church's stochasticity concept, we first formalize the notion of a selection rule.

Definition 10. A *selection function* f is a total recursive function $f : \{0,1\}^* \to \{0,1\}$. A selection function f is *dense along A* if $f(A\restriction x) = 1$ for infinitely many x.

By interpreting A as the infinite 0-1-sequence $A(0)A(1)A(2)\cdots$, a selection function f selects the subsequence $A(x_0)A(x_1)A(x_2)\cdots$ of A where $x_0 < x_1 < x_2 < \cdots$ are the strings x such that $f(A\restriction x) = 1$. In particular, f selects an infinite subsequence S of A iff f is dense along A. So Church's stochasticity concept can be defined as follows.

Definition 11. (Church [6]) A set A is *stochastic* if, for every selection function f which is dense along A and for $i \in \{0,1\}$,

$$\lim_n \frac{\|\{y < n \ : \ f(A\restriction y) = 1 \ \& \ A(y) = i\}\|}{\|\{y < n \ : \ f(A\restriction y) = 1\}\|} = \frac{1}{2}. \tag{3}$$

Di Paola [7] studied subrecursive versions of Church stochasticity corresponding to the Ritchie and Grzegorczyk hierarchies. Here we will consider $t(n)$-time bounded Church stochasticity corresponding to $DTIME(t(2^n))$.

Definition 12. A $t(n)$-*selection function* is a selection function f such that $f \in DTIME(t(n))$. A set A is $t(n)$-*stochastic* if, for every $t(n)$-selection function f which is dense along A and for $i \in \{0,1\}$, (3) holds.

To show that stochasticity and balanced genericity coincide, we characterize these concepts in terms of prediction functions. A prediction function f is a procedure which, given a finite initial segment of a 0-1-sequence, predicts the value of the next member of the sequence. We will show that a sequence A is stochastic (balancedly generic) iff, for every partial prediction function which makes infinitely many predictions along A, the number of the correct and incorrect predictions is asymptotically the same.

Definition 13. A *prediction function* f is a partial function $f : \{0,1\}^* \to \{0,1\}$. A $t(n)$-*prediction function* f is a prediction function f such that $f \in DTIME(t(n))$ and $domain(f) \in DTIME(t(n))$. A prediction function f is *dense along* A if $f(A\restriction x)$ is defined for infinitely many x. A *meets (avoids)* f *at* x if $f(A\restriction x)$ is defined and $f(A\restriction x) = A(x)$ ($f(A\restriction x) = 1 - A(x)$). A *meets* f *balancedly* if

$$\lim_n \frac{\|\{y < n \ : \ f(A\restriction y) = A(y)\}\|}{\|\{y < n \ : \ f(A\restriction y)\downarrow\}\|} = \frac{1}{2}. \tag{4}$$

Note that (4) can be rephrased by

$$\lim_n \frac{\|\{y < n \ : \ A \text{ meets } f \text{ at } y\}\|}{\|\{y < n \ : \ A \text{ avoids } f \text{ at } y\}\|} = 1. \tag{5}$$

Theorem 14. *For any set A, the following are equivalent.*

1. *A is balancedly $t(n)$-generic.*
2. *A is $t(n+1)$-stochastic.*
3. *A meets balancedly every $t(n+1)$-prediction function which is dense along A.*

Proof. (Idea). We prove the implications *1* $\Rightarrow$ *3* $\Rightarrow$ *2* $\Rightarrow$ *1*.

1 $\Rightarrow$ *3*. For any $t(n+1)$-prediction function f, define the proper $t(n)$-condition C_f by $C_f = \{(X\restriction x)f(X\restriction x) : f(X\restriction x)\downarrow\}$. Then, for any set A, C_f is dense along A iff f is dense along A, and A meets C_f balancedly iff A meets f balancedly.

3 $\Rightarrow$ *2*. For any $t(n+1)$-selection function f, define a $t(n+1)$-prediction function f' by $f'(X\restriction x) = 0$ if $f(X\restriction x) = 1$ and by letting $f'(X\restriction x)$ be undefined otherwise. Then, for any set A, f' is dense along A iff f is dense along A, and A meets f' balancedly iff (3) holds for f.

2 $\Rightarrow$ *1*. Let A be any set and let C be any proper $t(n)$-condition which is dense along A. In fact, for simplicity, assume that, for $i \in \{0,1\}$, there are infinitely many x such that $(A\restriction x)i \in C$. Define $t(n+1)$-selection functions f_i, $i = 0, 1$, by letting $f_i(X\restriction x) = 1$ for $(X\restriction x)i \in C$ and letting $f_i(X\restriction x) = 0$ otherwise. Then f_0 and f_1 are dense along A, and, if (3) holds for f_0 and f_1 in place of f then A meets C balancedly. □

In the remainder of this section we shortly discuss some other, weaker resource-bounded stochasticity concepts. We will characterize these concepts by different types of prediction functions thereby clarifying the relations among these notions. The first concept, we will consider, was introduced by Ko in [11] and was defined in terms of prediction functions already.

Definition 15. (Ko [11]) A set A is Ko-$t(n)$-*stochastic* if, for every total $t(n)$-prediction function f, A meets f balancedly, i.e.,

$$\lim_n \frac{\|\{y < n : f(A\restriction y) = A(y)\}\|}{n} = \frac{1}{2}. \tag{6}$$

Lemma 16. *Every $t(n)$-stochastic set is Ko-$t(n)$-stochastic.*

A still older notion of stochasticity can be found in [19].

Definition 17. (Wilber [19]) A set A is Wilber-$t(n)$-*stochastic* if, for every set $B \in DTIME(t(n))$,

$$\lim_n \frac{\|\{y < n : A(y) = B(y)\}\|}{n} = \frac{1}{2}. \tag{7}$$

To relate Wilber's notion to the other stochasticity concepts, we use the following property of prediction functions.

Definition 18. A prediction function f is *oblivious* if, for all strings x and y with $|x| = |y|$, $f(x)$ is defined if and only if $f(y)$ is defined, and, if $f(x)$ is defined, then $f(x) = f(y)$.

Intuitively, for an oblivious prediction function f, the predicted value $A(y)$ of a set A at y does not depend on the previously seen values $A\restriction y$ of A, so that f makes the same predictions at y for all sets.

For the comparison of $t(n)$-stochasticity and Wilber-$t(n)$-stochasticity, we have to observe that the former is measured in the length of the initial segment, whereas the latter is measured in the input length. As remarked in the introduction, there is roughly an exponential relation between these parameters, but this correspondence is not exact. To overcome this problem, we use the following notation: a set A is *$t(O(t'(n)))$-stochastic* if and only if A is $t(c \cdot t'(n))$-stochastic for all $c \geq 0$.

Lemma 19. *For any set A, the following are equivalent.*

1. *A is Wilber-$t(O(2^n))$-stochastic.*
2. *A meets balancedly every total oblivious $t(O(n))$-prediction function.*

Definition 20. A set A is *weakly $t(n)$-stochastic* if, for every infinite set $B \in DTIME(t(n))$,

$$\lim_n \frac{||A \cap B \restriction n||}{||B \restriction n||} = \frac{1}{2}. \tag{8}$$

Note that weak $t(n)$-stochasticity may be viewed as balanced $DTIME(t(n))$-bi-immunity: If A is weakly $t(n)$-stochastic and $B \in DTIME(t(n))$ is infinite then $A \cap B$ and $\overline{A} \cap B$ are infinite and, moreover, $||\{y < n : y \in A \cap B\}||$ and $||\{y < n : y \in \overline{A} \cap B\}||$ grow at the same rate.

Lemma 21. *For any set A, the following are equivalent.*

1. *A is weakly $t(O(2^n))$-stochastic.*
2. *A meets balancedly every oblivious $t(O(n))$-prediction function which is dense along A.*

The relation between stochasticity and weak stochasticity can be further illustrated by the following characterization of $t(n)$-stochasticity in the style of Definition 20.

Lemma 22. *A set A is $t(O(n))$-stochastic if and only if, for every infinite set $B \in DTIME^{<A}(t(O(2^n)))$, (8) holds. Here $B \in DTIME^{<A}(t'(n))$ means that there is a $t'(n)$-time bounded deterministic oracle Turing machine M such that $B(x) = M(A \restriction x; x)$ for all x.*

The above characterizations of the stochasticity concepts in terms of prediction functions imply the following relations:

$$\begin{array}{ccccc} & & \text{Ko-}t(O(n))\text{-stochastic} & & \\ & \nearrow & & \searrow & \\ t(O(n))\text{-stochastic} & & & & \text{Wilber-}t(O(2^n))\text{-stochastic} \\ & \searrow & & \nearrow & \\ & & \text{weakly } t(O(2^n))\text{-stochastic} & & \end{array} \tag{9}$$

By analyzing closure properties of these concepts, we can show that no other implications hold.

We should remark that many results in this section have some parallels in the theory of genericity. There prediction functions are usually viewed as extension functions. Corresponding to Theorem 14, Ambos-Spies [1] has shown that a set A is $t(n)$-generic iff A meets every $t(n+1)$-prediction function which is dense along A. As shown in [1] too, total $t(n)$-prediction functions yield an almost trivial genericity concept (cf. Lemma 6.6 of [1]), while in [2], it was implicitly shown that genericity for oblivious $t(O(n))$-prediction functions coincides with $DTIME(t(O(2^n)))$-bi-immunity. Finally, a characterization of $t(n)$-genericity corresponding to Lemma 22 was given by Balcazar and Mayordomo in [5].

4 Randomness and Weak Randomness

Ville [17] used betting strategies, called martingales, to point out some limitations of von Mises' stochasticity concept.

Definition 23. A *martingale* is a function $d : \{0,1\}^* \rightarrow R_+$, where R_+ is the set of nonnegative reals, such that

$$\forall x \in \{0,1\}^* \ (d(x0) + d(x1) = 2d(x)). \tag{10}$$

A martingale d *succeeds* on a set A if $\limsup_n d(A \restriction n) = \infty$. $S^\infty[d]$ denotes the class of the sets on which the martingale d succeeds.

The classical measure can be defined in terms of martingales. In particular, a class has measure 0 iff there is a martingale which succeeds on all sets in the class. Schnorr [15] defined randomness concepts based on recursively approximable martingales and on recursive, rational-valued martingales, respectively, and he showed that these concepts are equivalent. Corresponding approaches have been taken to define resource-bounded randomness. While Schnorr [15] and, later, Ambos-Spies et al. [4] defined that a $t(n)$-martingale d is a rational-valued martingale $d \in DTIME(t(n))$, Lutz [13] defined that a $t(n)$-martingale is a martingale with approximations in $DTIME(t(n))$. Again, as independently shown in [4], [10] and [14], these concepts are more or less equivalent, though the correspondence does not preserve the exact time bounds.

Here we will take still another approach which was introduced by Ambos-Spies and Mainhardt (unpublished). This approach, which is also more or less equivalent to the previous ones (see Lemma 26 below), will yield sharp bounds when comparing prediction functions and martingales.

Definition 24. The *betting strategy* or *stake function* $s_d : \{0,1\}^* \rightarrow [0,2]$ of a martingale d is given by $s_d(x) = d(x0)/d(x)$ if $d(x) \neq 0$ and $s_d(x) = 1$ otherwise.

Note that any martingale d can be inductively computed from its betting strategy s_d and its initial value $d(\lambda)$. Here we will measure the complexity of a martingale by the complexity of its stake function.

Definition 25. (Ambos-Spies and Mainhardt) A martingale $d : \{0,1\}^* \rightarrow Q_+$ is a $t(n)$-*martingale* if its stake function s_d is in $DTIME(t(n))$. A set A is $t(n)$-*random* if no $t(n)$-martingale succeeds on A.

As observed by Schnorr [15], in the general recursive case, this concept coincides with the previous ones. In the subrecursive case we obtain the following relations.

Lemma 26. *For any $t(n)$-martingale d, $d \in DTIME(n \cdot t(n))$. Conversely if $d : \{0,1\} \to Q_+$ is a martingale such that $d \in DTIME(t(n))$, then d is a $t(n+1)$-martingale.*

The following definition will allow us to give a martingale characterization of stochasticity, hence balanced genericity.

Definition 27. A martingale d is *simple* if there is a rational number $\alpha \in [0,1]$ such that

$$range(s_d) \subseteq \{1, 1+\alpha, 1-\alpha\}. \tag{11}$$

A set A is *weakly $t(n)$-random* if there is no simple $t(n)$-martingale which succeeds on A.

We now state our main theorem.

Theorem 28. *For any set A, the following are equivalent.*

1. *A is balancedly $t(n)$-generic.*
2. *A is $t(n+1)$-stochastic.*
3. *A is weakly $t(n+1)$-random.*

By Theorem 14, for a proof of Theorem 28 it suffices to show the equivalence of the prediction function and simple martingale concepts. Since the weak stochasticity notions in Section 3 could be characterized by special types of prediction functions, we first define corresponding restrictions for martingales. Then the equivalence proof will also yield martingale characterizations of these stochasticity concepts.

Definition 29. A martingale d is *strict*, if, for all $x \in \{0,1\}^*$, $s_d(x) \neq 1$. A martingale d is *oblivious* if, for all strings x and y with $|x| = |y|$, $s_d(x) = s_d(y)$.

Note that $s_d(A \upharpoonright x) = 1$ expresses that the strategy s_d does not bet on $A(x)$. For prediction functions, this corresponds to making no prediction for $A(x)$. So strictness of a martingale corresponds to totality of a prediction function.

Lemma 30. *For any set A, the following are equivalent.*

1. *A meets balancedly every (total, oblivious, total and oblivious) $t(n)$-prediction function which is dense along A.*
2. *No simple (strict, oblivious, strict and oblivious) $t(n)$-martingale succeeds on A.*

Proof. (Idea). *1* ⇒*2.* For any simple $t(n)$-martingale d, define a $t(n)$-prediction function f by

$$f(X\upharpoonright x) = \begin{cases} 1 & s_d(X\upharpoonright x) < 1 \\ 0 & s_d(X\upharpoonright x) > 1 \\ \uparrow & s_d(X\upharpoonright x) = 1 \end{cases}$$

Then f is total and oblivious if d is strict and oblivious, respectively. Moreover, for any set A, if A meets f balancedly, then d does not succeed on A.

2 ⇒*1.* For any $t(n)$-prediction function f, define $t(n)$-martingales d^i_α ($i \in \{0,1\}, \alpha \in Q \cap (0,1)$) by letting $d^i_\alpha(\lambda) = 1$ and

$$s_{d^i_\alpha}(x) = \begin{cases} 1-\alpha & \text{if } f(x) = i \\ 1+\alpha & \text{if } f(x) = 1-i \\ 1 & \text{if } f(x)\uparrow \end{cases}$$

Then, for any set A such that f is dense along A but A does not meet f balancedly, for some i and α, d^i_α succeeds A. □

Now Theorem 28 is immediate by Lemma 30 and Theorem 14. Moreover, we obtain the following martingale characterizations of the weak stochasticity notions.

Theorem 31. *1. A set A is Ko-$t(n)$-stochastic if and only if no strict simple $t(n)$-martingale succeeds on A.*
2. A set A is weakly $t(O(2^n))$-stochastic if and only if no oblivious simple $t(O(n))$-martingale succeeds on A.
3. A set A is Wilber-$t(O(2^n))$-stochastic if and only if no simple $t(O(n))$-martingale which is strict and oblivious succeeds on A.

In the definition of weak $t(n)$-randomness, we can replace simple martingales by almost simple martingales, where a martingale d is *almost simple* if there is a finite set $F = \{\alpha_0, \cdots, \alpha_m\}$ of rational numbers $\alpha_k \in [0,1]$ such that

$$range(s_d) \subseteq \cup_{k \leq m}\{1, 1+\alpha_k, 1-\alpha_k\}.$$

As our final result shows, however, we can not use arbitrary martingales.

Theorem 32. *There is a weakly $t(n)$-random set which is not $t(n)$-random.*

Proof. (Idea). In [17] Ville constructed a stochastic set A such that for all n

$$0 \leq \frac{1}{2} - \frac{1}{n}||A\upharpoonright n|| \leq \frac{f(n)}{n} \tag{12}$$

where $f(n)$ is any given unbounded nondecreasing time constructible function. By Theorem 28, A is weakly $t(n)$-random. On the other hand, however, we can show that no $t(n)$-random set A satisfies (12) for $f(n) = \log\log\log n$. □

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