Homogeneous Faults, Colored Edge Graphs, and Cover Free Families

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Abstract. In this paper, we use the concept of colored edge graphs to model homogeneous faults in networks. We then use this model to study the minimum connectivity (and design) requirements of networks for being robust against homogeneous faults within certain thresholds. In particular, necessary and sufficient conditions for most interesting cases are obtained. For example, we will study the following cases: (1) the number of colors (or the number of non-homogeneous network device types) is one more than the homogeneous fault threshold; (2) there is only one homogeneous fault (i.e., only one color could fail); and (3) the number of non-homogeneous network device types is less than five.

1 Background and colored edge graph

In network communications, the communication could fail if some nodes or some edges are broken. Though the failure of a modem could be considered the failure of a node, we can model this scenario also as the failure of the communication link (the edge) attached to this modem. Thus it is sufficient to consider edge failures in communication networks. It is also important to note that several nodes (or edges) in a network could fail at the same time. For example, all brand X routers in a network could fail at the same time due to a platform dependent computer worm (virus) attack. In order to design survivable communication networks, it is essential to consider this kind of homogeneous faults for networks. Existing works on network quality of services have not addressed this issue in detail and there is no existing model to study network reliability in this aspect. In this paper, we use the colored edge graphs which could be used to model homogeneous faults in networks. The model is then used to optimize the design of survivable networks and to study the minimum connectivity (and design) requirements of networks for being robust against homogeneous faults within certain thresholds.

Definition 1. *A colored edge graph is a tuple $G(V, E, C, f)$, with V the node set, E the edge set, C the color set, and f a map from E onto C. The structure*

$$\mathcal{Z}_{C,t} = \{Z : Z \subseteq E \text{ and } |f(Z)| \leq t\}.$$

is called a t-color adversary structure. Let $A, B \in V$ be distinct nodes of G. A, B are called $(t+1)$-color connected *for $t \geq 1$ if for any color set $C_t \subseteq C$ of size t,* *there is a path p from A to B in G such that the edges on p do not contain any color in C_t. A colored edge graph G is* $(t+1)$-color connected *if and only if for any two nodes A and B in G, they are $(t+1)$-color connected.*

The interpretation of the above definition is as follows. In a network, if two edges have the same color, then they could fail at the same time. This may happen when the two edges are designed with same technologies (e.g., with same operating systems, with same application software, with same hardware, or with same hardware and software). If a colored edge network is $(t+1)$-color connected, then the network communication is robust against the failure of edges of any t colors (that is, the adversary may tear down any t types of devices).

In practice, one communication link may be attached to different brands of network devices (e.g., routers, modems) on both sides. For this case, the edge can have two different colors. If any of these colors is broken, the edge is broken. Thus from a reliability viewpoint, if one designs networks with two colors on the same edge, the same reliability/security can be obtained by having only one color on each edge. In the following discussion, we will only consider the case with one color on each edge. Meanwhile, multiple edges between two nodes are not allowed either.

We are interested in the following practical questions. For a given number n of nodes in V (i.e., the number of network nodes), a given number m of the colors (e.g., the number of network device types), and a given number t, how can we design a $(t+1)$-color connected colored edge graph $G(V, E)$ with minimum number λ of edges? In another word, how can we use minimum resources (e.g., communication links) to design a network that will keep working even if t types of devices in the network fail?

For practical network designs, one needs first to have an estimate on the number of homogeneous faults. For example, the number t of brands of routers that could fail at the same time. Then it is sufficient to design a $(t+1)$-color connected network with $m = t+1$ colors (e.g., with $t+1$ different brands of routers). Necessary and sufficient conditions for this kind of network design will be obtained in this paper.

Another important issue that should be taken into consideration in practical network designs is that the number m of colors (e.g., the number of brands for routers) is quite small. For example, m is normally less than five. Necessary and sufficient conditions for network designs with $m \leq 5$ and with optimized resources will be obtained in this paper. Note that for cases with small m, we may have $m > t+1$.

The outline of the paper is as follows. Section 3 describes the necessary and sufficient conditions for the case of $m = t+1$ without optimizing the number of edges in the networks. Section 4 gives a necessary condition for colored edge networks in terms of optimized number of edges. Section 5 shows that the necessary conditions in Section 4 are also sufficient for the most important three cases: (1) $m = t+1$; (2) $t = 1$; and (3) $m \leq 5$. Section 6 shows that it is **coNP**-hard to determine whether a given colored edge graph is $(t+1)$-connected.

2 Related works

Though colored-edge graph is a new concept which we used to model network survivability issues, there are related research topics in this field. For example, edge-disjoint (colorful) spanning trees have been extensively studied in the literature (see, e.g., [1]). These results are mainly related to our discussion in the next section for the case of $m = t + 1$. A colored edge graph G is *proper* if whenever two edges share an end point they carry different colors. A spanning tree for a colored edge graph is called colorful if no two of its edges have the same color. Two spanning trees of a graph are edge disjoint if they do not share common edges. For a non-negative integer s, let K_s denote the complete graph on s vertices. A classical result from Euler (see [1]) shows that the edges of K_{2n} can be partitioned into n isomorphic spanning trees (paths, for example) and each of these spanning trees can easily be made colorful, but the resulting edge colored graph usually fails to be proper.

Though it is important to design colored edge graphs with required security parameters, for several scenarios it is also important to calculate the robustness of a given colored edge graphs. Roskind and Tarjan [7] designed a greedy algorithm to find $(t+1)$-edge disjoint spanning trees in a given graph. This is related to the questions $(t+1)$-color connectivity for the case of $m = t+1$. We are not aware of any approximate algorithms for deciding $(t+1)$-color connectivity of a given colored edge graph. Indeed, we will show that this problem is **coNP**-hard.

3 Necessary and sufficient conditions for special cases

In this section, we show necessary and sufficient conditions for some special cases.

Lemma 1. *A colored edge graph $G(V, E, C, f)$ is $(t+1)$-color connected if and only if, for all $i_1, i_2, \ldots, i_{m-t} \leq m$, $(V, E_{i_1} \cup E_{i_2} \cup \cdots \cup E_{i_{m-t}})$ is a connected graph, where $E_1, E_2, \ldots, E_m$ is a partition of E under the m different colors.*

As we have mentioned in the previous section, the classical result from Euler shows that K_{2n} can be partitioned into n spanning trees. Thus, by Lemma 1, we have the following theorem.

Theorem 1. *(Euler) For $n = 2m$, there is a coloration $G(V, E, C, f)$ of K_n such that G is $(m-1)$-color connected.*

In the following, we extend Theorem 1 to the general case of $n \geq 2m$.

Lemma 2. *For $n \geq 2m$ and $m \geq 2$, there exists a graph $G(V, E)$ with $|V| = n, |E| = m(n-1)$, and $E = E_1 \cup E_2 \cup \cdots \cup E_m$ such that the following conditions are satisfied:*

1. *$G(V, E_i)$ is a connected graph for all $0 < i \leq m$;*
2. *$E_i \cap E_j = \emptyset$ for all $i, j \leq m$.*

Proof. We prove the Lemma by induction on n and m. For $n = 2$ and $m = 1$, the Lemma holds obviously. Assume that the Lemma holds for $n_0 = 2m_0$.

In the following, we show that the Lemma holds for $n = n_0 + 1, m = m_0$ and for $n = n_0 + 2, m = m_0 + 1$. Let $G(V_0, E_0)$ be the graph with $|V_0| = n_0, |E_0| = m_0(n_0 - 1)$, and $E_0 = E_1^0 \cup E_2^0 \cup \cdots \cup E_{m_0}^0$ such that the conditions in the Lemma are satisfied:

For the case of $n = n_0 + 1$ and $m = m_0$, let $V = V_0 \cup \{u\}$ where u is a new node that is not in V_0, and let $E_1 = E_1^0 \cup \{(u, u_1)\}$, $E_2 = E_2^0 \cup \{(u, u_2)\}$, $\ldots$, $E_{m_0} = E_{m_0}^0 \cup \{(u, u_{m_0})\}$ where $u_1, u_2, \ldots, u_{m_0}$ are distinct nodes from V_0. It is straightforward to show that $|V| = n, |E| = m(n-1)$, $G(V, E_i)$ is a connected graph, and $E_i \cap E_j = \emptyset$ for all $i, j \leq m$. Thus the Lemma holds for this case.

For the case of $n = n_0 + 2$ and $m = m_0 + 1$, let $V = V_0 \cup \{u, v\}$ where u, v are new nodes that are not in V_0, and define $E_1, \ldots, E_m$ as follows.

1. Set $E_m = \emptyset$ and $U = \emptyset$, where U is a temporary variable.
2. Define E_1:
   (a) Select an edge $(v_1, v_2) \in E_1^0$.
   (b) Let $E_1 = \left(E_1^0 \setminus \{(v_1, v_2)\}\right) \bigcup \{(v_1, u), (u, v), (v, v_2)\}$.
   (c) Let $E_m = E_m \cup \{(v, v_1), (v_1, v_2), (v_2, u)\}$ and $U = U \cup \{v_1, v_2\}$.
3. Define E_i for $2 \leq i \leq m_0$:
   (a) Select $v_{2i-1}, v_{2i} \notin U$.
   (b) Let $E_i = E_i^0 \cup \{(u, v_{2i-1}), (v, v_{2i})\}$.
   (c) Let $E_m = E_m \cup \{(v, v_{2i-1}), (u, v_{2i})\}$ and $U = U \cup \{v_{2i-1}, v_{2i}\}$.

It is straightforward to show that $|V| = n, |E_i| = (n-1)$ (thus $|E| = m(n-1)$), $G(V, E_i)$ is a connected graph, and $E_i \cap E_j = \emptyset$ for all $i, j \leq m$. This completes the proof of the Lemma. Q.E.D.

Theorem 2. *Given n, m, t with $m = t+1$, there exists a $(t+1)$-color connected colored edge graph $G(V, E, C, f)$ with $|V| = n$ and $|C| = m$ if and only if $n \geq 2m$.*

Proof. By Lemma 1, a $(t+1)$-color connected colored edge graph $G(V, E, C, f)$ with $|V| = n$ and $|C| = m = t + 1$ contains at least $m(n-1)$ edges. Meanwhile, $G(V, E, C, f)$ contains at most $n(n-1)/2$ edges. Thus for $n < 2m$, we have $n(n-1)/2 < m(n-1)$. In another word, for $n < 2m$, there is no $(t+1)$-color connected colored edge graph $G(V, E, C, f)$ with $|V| = n$ and $|C| = m = t + 1$. Now the theorem follows from Lemmas 1 and 2. Q.E.D

4 Necessary conditions for general cases

First we note that for a colored edge graph G to be $(t+1)$-color connected, each node must have a degree of at least $t + 1$. Thus the total degree of an n-node graph should be at least $n(t + 1)$. This implies the following lemma.

Lemma 3. *For $m \geq t + 1 > 1$, and a $(t+1)$-color connected colored edge graph $G(V, E, C, f)$ with $|V| = n$, $|E| = \lambda$, and $|C| = m$, we have $2\lambda \geq (t + 1)n$.*

In the following, we use cover free family concepts to study the necessary conditions for colored edge graphs connectivity.

Definition 2. *Let X be a finite set with $|X| = \lambda$ and $\mathcal{F}$ be a set of mutually disjoint subsets of X with $|\mathcal{F}| = m$. Then $(X, \mathcal{F})$ is called a (λ, m)-partition of X if $X = \bigcup_{P \in \mathcal{F}} P$. Let n, t be positive integers. An (λ, m)-partition $(X, \mathcal{F})$ is called a $(t; n-1)$-cover free family (or $(t; n-1)$-CFF(λ, m)) if, for any t elements $B_1, \ldots, B_t \in \mathcal{F}$, we have that*

$$\left| X \setminus \left(\bigcup_{i=1}^{t} B_i \right) \right| \geq n-1 \qquad \left(\text{or} \left| \bigcap_{i=1}^{t} (X \setminus B_i) \right| \geq n-1 \right)$$

It should be noted that our above definition of cover-free family is different from the generalized cover-free family definition for set systems in the literature. In [8], a set system $(X, \mathcal{F})$ is called a $(w, t; n-1)$-cover free family if for any w blocks $A_1, \ldots, A_w \in \mathcal{F}$ and any t blocks $B_1, \ldots, B_t \in \mathcal{F}$, one has $\left| \left(\cap_{j=1}^{w} A_j \right) \setminus \left(\cup_{i=1}^{t} B_i \right) \right| \geq n-1$. Specifically, there are two major differences between our (λ, m)-partition system and the set systems in the literature[3].

1. For a set system $(X, \mathcal{F})$, $\mathcal{F}$ may contain repeated elements.
2. For a set system $(X, \mathcal{F})$, the elements in $\mathcal{F}$ are not necessarily mutually disjoint.

It is straightforward to show that a colored edge graph G is $(t+1)$-color connected if and only if for any color set $C_t \subseteq C$ of size t, after the removal of edges in G with colors in C_t, G remains connected. Assume that G contains n nodes. Then a necessary condition for connectivity is that G contains at least $n-1$ edges. From this discussion, we get the following lemma.

Lemma 4. *For a colored edge graph $G(V, E, C, f)$, with $|V| = n$, $|E| = \lambda$, $|C| = m$, a necessary condition for $G(V, E, C, f)$ to be $(t+1)$-color connected is that the (λ, m)-partition $(X, \mathcal{F})$ is a $(t; n-1)$-CFF(λ, m) with $X = E$ and $\mathcal{F} = \{E_c : c \in C\}$ where $E_c = \{e : f(e) = c, e \in E\}$.*

In the following, we analyze lower bounds for the number λ of edges for the existence of a $(t; n-1)$-CFF(λ, m). For a set partition $(X, \mathcal{F})$ and a positive integer t, let

$$\mu(X, \mathcal{F}; t) = \min \left\{ \left| X \setminus \left(\bigcup_{i=1}^{t} B_i \right) \right| : B_1, \ldots, B_t \in \mathcal{F} \right\}$$

It is straightforward to see that a (λ, m)-partition $(X, \mathcal{F})$ is a $(t; n-1)$-CFF(λ, m) if and only if $\mu(X, \mathcal{F}; t) \geq n-1$.

Given positive integers λ, m, t, let

$$\mu(\lambda, m; t) = \max \left\{ \mu(X, \mathcal{F}; t) : (X, \mathcal{F}) \text{ is a } (\lambda, m)\text{-partition} \right\}$$

From the above discussion and Lemma 3, we have the following theorem.

[3] The first author of this paper would like to thank Prof. Doug Stinson for pointing this out to the author.

Theorem 3. *Let λ, m, t be given positive integers. $\mu(\lambda, m; t) \geq n-1$ and $2\lambda \geq (t+1)n$ are necessary conditions for the existence of a $(t+1)$-color connected colored edge graph $G(V, E, C, f)$, with $|V| = n$, $|E| = \lambda$, $|C| = m$.*

Theorem 4. *Let λ, m, t be given positive integers. Then we have*

$$\mu(\lambda, m; t) = \begin{cases} (m-t) \cdot \lfloor \frac{\lambda}{m} \rfloor & \text{if } t \geq \lambda - \lfloor \frac{\lambda}{m} \rfloor \cdot m \\ (m-t) \cdot \lfloor \frac{\lambda}{m} \rfloor + \left(\lambda - \lfloor \frac{\lambda}{m} \rfloor \cdot m - t\right) & \text{otherwise} \end{cases}$$

Proof. For a given (λ, m)-partition $(X, \mathcal{F})$, let $B_1, \ldots, B_m$ be an enumeration of elements in $\mathcal{F}$ such that $|B_i| \leq |B_{i+1}|$ for all $i < m$. It is straightforward to show that $\mu(X, \mathcal{F}; t) = \sum_{i=1}^{m-t} |B_i|$. Thus $\mu(\lambda, m; t)$ takes the maximum value if $\sum_{i=1}^{m-t} |B_i|$ is maximized. It is straightforward to show that this value is maximized when the (λ, m)-partition $(X, \mathcal{F})$ satisfies the following conditions:

1. $|B_i| = \lfloor \frac{\lambda}{m} \rfloor$ for $i \leq m - \left(\lambda - \lfloor \frac{\lambda}{m} \rfloor \cdot m\right)$, and
2. $|B_i| = \lfloor \frac{\lambda}{m} \rfloor + 1$ for $m \geq i > m - \left(\lambda - \lfloor \frac{\lambda}{m} \rfloor \cdot m\right)$.

The theorem follows from the above discussion. Q.E.D.

Example 1. For $n = 7, \lambda = 10, m = 5$, and $t = 2$, we have $\mu(10, 5; 2) = 6 = n-1$. However, $2\lambda = 20 < (t+1)n = 21$. This shows that the condition $2\lambda \geq (t+1)n$ in Theorem 3 is not redundant.

Example 2. There are no $(t+1)$-color connected colored edge graph $G(V, E, C, f)$ for the following special cases:

1. $m = 2, t = 1, n = 3$.
2. $m = 4, t = 2, n = 4$.
3. $m = 3, t = 2, n \leq 5$.

Proof. Before we consider the specific cases, we observe that, when m and t are fixed, the function μ is nondecreasing when λ increases.

1. In this case, the maximum value that λ could take is 3. Thus $\mu(3, 2; 1) = 1 < n-1 = 2$. That is, there is no $(1; 2)$-CFF$(3, 2)$, which implies the claim. Note that this result also follows from Theorem 2.

2. In this case, the maximum value that λ could take is 6. Thus $\mu(6, 4; 2) = 2 < n-1 = 3$.

3. We only show this for the case $m = 3, t = 2, n = 5$. In this case, the maximum value that λ could take is 10. Thus $\mu(10, 3; 2) = 3 < n-1 = 4$. Note that this result also follows from Theorem 2. Q.E.D

The following theorem is a variant of Theorem 3.

Theorem 5. *For $m-1 > t > 0$, a necessary condition for the existence of a $(t+1)$-color connected colored edge graph $G(V, E, C, f)$ with $|V| = n$, $|E| = \lambda$, and $|C| = m$ is that $2\lambda \geq (t+1)n$ and the following conditions are satisfied:*

– *If $n = (m-t)k$ for some integer $k > 0$, then $\lambda \geq mk - 1$.*

- *If $n = (m-t)k + 1$ for some integer $k > 0$, then $\lambda \geq mk$.*
- *If $n = (m-t)k + 2$ for some integer $k > 0$, then $\lambda \geq mk + t + 1$.*
- ......
- *If $n = (m-t)k + m - t - 1$ for some integer $k > 0$, then $\lambda \geq mk + m - 2$.*

Proof. For $m > t + 1$, by Theorem 4, we have

$$\mu(\lambda, m; t) = \begin{cases} (m-t)k' & \text{if } \lambda = mk' + i \text{ for } \ 0 \leq i \leq t \\ (m-t)k' + 1 & \text{if } \lambda = mk' + t + 1 \\ \cdots\cdots & \\ (m-t)k' + m - t - 1 & \text{if } \lambda = mk' + m - 1 \end{cases}$$

Thus the necessary condition $\mu(\lambda, m; t) \geq n - 1$ in Theorem 3 can be interpreted as the following conditions:

$$k' \geq \begin{cases} \frac{n-1}{m-t} & \text{if } \lambda = mk' + i \text{ for } \ 0 \leq i \leq t \\ \frac{n-2}{m-t} & \text{if } \lambda = mk' + t + 1 \\ \cdots\cdots & \\ \frac{n-m+t}{m-t} & \text{if } \lambda = mk' + m - 1 \end{cases}$$

In aother word, for a $(t+1)$-color connected colored edge graph $G(V, E, C, f)$, the following $m - t$ conditions (the disjunction not conjunction) are satisfied:

- $|V| = n, |E| \geq m \left\lceil \frac{n-1}{m-t} \right\rceil$, and $|C| = m$.
- $|V| = n, |E| \geq m \left\lceil \frac{n-2}{m-t} \right\rceil + t + 1$, and $|C| = m$.
- ......
- $|V| = n, |E| \geq m \left\lceil \frac{n-m+t}{m-t} \right\rceil + m - 1$, and $|C| = m$.

By distinguishing the cases for $n = (m-t)k$, $n = (m-t)k + 1$, $\cdots$, and $n = (m-t)k + m - t - 1$, and by reorganizing above lines, these necessary conditions can be interpreted as the following $m - t$ conditions:

- $n = (m-t)k$ and $\lambda \geq mk - 1$ for some $k > 0$. Note that this follows from the last line of the above conditions (one can surely take other lines, but then the value of λ would be larger). This comment applies to following cases also.
- $n = (m-t)k + 1$ and $\lambda \geq mk$ for some $k > 0$.
- $n = (m-t)k + 2$ and $\lambda \geq mk + t + 1$ for some $k > 0$.
- ......
- $n = (m-t)k + m - t - 1$ and $\lambda \geq mk + m - 2$ for some $k > 0$.

Q.E.D.

5 Necessary and sufficient conditions for practical cases (with small m and t)

Generally we are interested in the question whether the necessary condition in Theorems 3 and 5 are also sufficient. In the following, we show that this is true for several important practical cases.

Theorem 6. *The necessary condition in Theorem 3 is sufficient for the case of* $m = t + 1$.

Proof. Since $\lambda - \lfloor\frac{\lambda}{m}\rfloor \cdot m$ is the remainder of λ divided by m, we trivially have $t = m - 1 \geq \lambda - \lfloor\frac{\lambda}{m}\rfloor \cdot m$. Now assume that $m > \frac{n}{2}$. By Theorem 4, we have $\mu(\lambda, m; t) = \lfloor\frac{\lambda}{m}\rfloor \leq \lfloor\frac{n(n-1)}{2m}\rfloor < n - 1$. The rest follows from Theorem 2. Q.E.D.

Before we show that the necessary conditions in Theorems 3 and 5 are sufficient for the case of $t = 1$, we first present two lemmas whose proofs are straightforward.

Lemma 5. *For* $n = m = \lambda \geq 3$ *and* $t = 1$*, the following* m*-node circle graph is* $(1 + 1)$*-color connected:*

$$\{(v_1, v_2), (v_2, v_3), \ldots, (v_m, v_1)\}$$

with $f(v_i, v_{i+1}) = c_i$ *for* $i < m$ *and* $f(v_m, v_1) = c_m$.

Lemma 6. *For* $t = 1$, $m \geq 3$*, and* $m < n \leq 2m - 2$*, the* $(1 + 1)$*-color connected graph in Figure 1 has the edges:*

$$\{(v_1, v_2), (v_2, v_3), \ldots, (v_m, v_1)\} \cup \{(v_m, v_{m+1}), (v_{m+1}, v_{m+2}), \ldots, (v_n, v_1)\}$$

and colors defined by the following map:

$$\begin{aligned} f(v_i, v_{i+1}) &= c_i && \text{for } 1 \leq i \leq m - 1 \\ f(v_m, v_1) &= c_m \\ f(v_{m+i-1}, v_{m+i}) &= c_i && \text{for } 1 \leq i \leq n - m \\ f((v_n, v_1)) &= c_{n-m+1} \end{aligned}$$

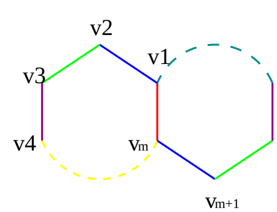


Fig. 1: For Lemma 6

Before we show that the necessary condition in Theorem 3 is also sufficient for the case of $t = 1$, we first prove this for $m = 3$.

Theorem 7. *The necessary condition in Theorem 3 is sufficient for the case of $m = 3$ and $t = 1$.*

Proof. For $m = 3$ and $t = 1$, we have

$$\mu(\lambda, m; t) = \begin{cases} 2k' & \text{if } \lambda = 3k' \text{ or } \lambda = 3k' + 1 \\ 2k' + 1 & \text{if } \lambda = 3k' + 2 \end{cases}$$

By the condition $\mu(\lambda, m; t) \geq n - 1$, the necessary condition is converted to the following conditions:

$$k' \geq \begin{cases} \frac{n-1}{2} & \text{if } \lambda = 3k' \text{ or } \lambda = 3k' + 1 \\ \frac{n-2}{2} & \text{if } \lambda = 3k' + 2 \end{cases}$$

Thus in order to prove the theorem, it is sufficient to construct $(1+1)$-color connected colored edge graph $G(V, E, C, f)$ for each of the following two conditions:

– $|V| = n, |E| = 3\left\lceil \frac{n-1}{2} \right\rceil$, and $|C| = 3$.
– $|V| = n, |E| = 3\left\lceil \frac{n-2}{2} \right\rceil + 2$, and $|C| = 3$.

By distinguishing the cases for $n = 2k$ and $n = 2k+1$, it is sufficient to construct the required colored edge graph for each of the following two conditions:

– $n = 2k$, $\lambda = 3k - 1$, and $m = 3$.
– $n = 2k + 1$, $\lambda = 3k$, and $m = 3$.

For the case of $n = 2k$, let

$$\begin{aligned} V &= \{v_1, \cdots, v_{2k}\}, \\ E_1 &= \{(v_1, v_{2i}) : 1 \leq i < k\} \\ E_2 &= \{(v_1, v_{2i+1}) : 1 \leq i < k\} \cup \{(v_1, v_{2k})\} \\ E_3 &= \{(v_{2i}, v_{2i+1}) : 1 \leq i < k)\} \cup \{(v_2, v_{2k})\} \\ E &= E_1 \cup E_2 \cup E_3 \end{aligned}$$

For each $e \in E_i$ $(i \leq 3)$, let $f(e) = c_i$. Then it is straightforward to check that the colored edge graph $G(V, E, C, f)$ is $(1 + 1)$-color connected, $|V| = n$, and $|E| = 3k - 1$.

For the case of $n = 2k + 1$, let

$$\begin{aligned} V &= \{v_1, \cdots, v_{2k+1}\}, \\ E_1 &= \{(v_1, v_{2i}) : 1 \leq i \leq k\} \\ E_2 &= \{(v_1, v_{2i+1}) : 1 \leq i \leq k\} \\ E_3 &= \{(v_{2i}, v_{2i+1}) : 1 \leq i \leq k)\} \\ E &= E_1 \cup E_2 \cup E_3 \end{aligned}$$

For each $e \in E_i$ $(i \leq 3)$, let $f(e) = c_i$. Then it is straightforward to check that the colored edge graph $G(V, E, C, f)$ is $(1 + 1)$-color connected, $|V| = n$, and $|E| = 3k - 1$, Q.E.D.

Corollary 1. *For $m = 3$, $t = 1$, and $n, \lambda > 0$, there exists an $(1+1)$-color connected colored edge graph $G(V,E,C,f)$ with $|V| = n$ and $|E| = \lambda$ if and only if $\lambda \geq \min\left\{3\left\lceil\frac{n-1}{2}\right\rceil, 3\left\lceil\frac{n-2}{2}\right\rceil + 2\right\}$.*

Now let us prove the theorem for the general case of $t = 1$.

Theorem 8. *The necessary conditions in Theorems 3 and 5 are sufficient for the case of $t = 1$.*

Proof. For the case of $m = 2$ and $t = 1$, it follows from Theorem 6. Now assume that $m > 2$ and $t = 1$. In this special case, the necessary conditions in Theorem 5 is as follows:

- $n = (m-1)k$ and $\lambda \geq mk - 1$ for some $k > 0$.
- $n = (m-1)k + 1$ and $\lambda \geq mk$ for some $k > 0$.
- $n = (m-1)k + 2$ and $\lambda \geq mk + 2$ for some $k > 0$.
- ......
- $n = (m-1)k + m - 2$ and $\lambda \geq mk + m - 2$ for some $k > 0$.

In the following we first show that the condition "$n = (m-1)k+1$ and $\lambda \geq km$" is sufficient. Let the graph in Figure 2a be defined as follows:

$$\begin{aligned}
V &= \{v_0, v_1 \cdots, v_{(m-1)k}\}, \\
E_1 &= \{(v_0, v_{(m-1)i+1}) : 0 \leq i \leq k-1\} \\
E_j &= \{(v_{(m-1)i+j-1}, v_{(m-1)i+j}) : 0 \leq i \leq k-1\} \text{ for } 2 \leq j \leq m-1 \\
E_m &= \{(v_{(m-1)i}, v_0) : 1 \leq i \leq k\} \\
E &= E_1 \cup E_2 \cup \cdots \cup E_m
\end{aligned}$$

For each $e \in E_j$ with $i \leq m$, let $f(e) = c_j$. Then it is straightforward to check that the colored edge graph $G(V,E,C,f)$ is $(1+1)$-color connected, $|V| = (m-1)k + 1$, and $|E| = mk$.

Now we show that the condition "$n = (m-1)k + j$ and $\lambda \geq km + j$ for $2 \leq j \leq m-1$" is sufficient. Let $G(V,E,C,f)$ be the colored edge graph that we have just constructed with $|V| = (m-1)k+1$, and $|E| = mk$.

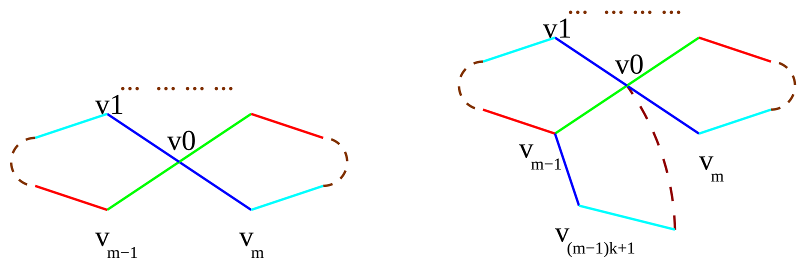
Let $V' = V \cup \{v_{(m-1)k+1}, \ldots, v_{(m-1)k+j-1}\}$. Define a new colored edge graph $G(V', E', C, f')$ (see Figure 2b) by attaching the following edges to the m-node circle $\{(v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_0)\}$:

$$\{(v_{m-1}, v_{(m-1)k+1}), (v_{(m-1)k+1}, v_{(m-1)k+2}), \ldots, (v_{(m-1)k+j-1}, v_0)\}$$

The colors for the new edges are defined by letting $f'(v_{(m-1)k+i}, v_{(m-1)k+i+1}) = c_{i+1}$ for $0 \leq i \leq j-2$ and $f'(v_{(m-1)k+j-1}, v_0) = c_j$. It is straightforward to check that $G(V', E', C, f')$ is $(1+1)$-color connected, $|V| = (m-1)k + j$, and $|E| = mk + j$. Q.E.D.

Corollary 2. *For $t = 1$ and m, $n, \lambda > 1$, there exists an $(1+1)$-color connected colored edge graph $G(V,E,C,f)$ with $|V| = n$ and $|E| = \lambda$ if and only if*

$$\lambda \geq \min\left\{m\left\lceil\frac{n-1}{m-1}\right\rceil, m\left\lceil\frac{n-2}{m-1}\right\rceil + 2, \ldots, m\left\lceil\frac{n-m+1}{m-1}\right\rceil + m - 1\right\}.$$



(a) $n = (m-1)k+1$, $\lambda \geq km$ (b) $n = (m-1)k+j$, $\lambda \geq km+j$

Fig. 2: Figures for Theorem 8

Proof. It follows from the proof of Theorem 8. Q.E.D

Theorem 9. *The conditions in Theorems 3 and 5 are sufficient for the case of* $m = 4, t = 2$.

Proof. It is sufficient to show that both of the conditions "$n = (m-t)k+1$ and $\lambda \geq km$" and "$n = (m-t)k+2$ and $\lambda \geq mk+t+1$" are sufficient (note that $m = 4$ and $t = 2$). In the following we first show that the condition "$n = (m-t)k+1$ and $\lambda \geq km$" is sufficient by induction on k.

For the case of $k = 2$, we have $n = 5, \lambda = 8, m = 4$, and $t = 2$. Let the graph G_1 in Figure 3a be defined as

$$G_1 = \{(v_1, v_2)_1, (v_2, v_3)_2, (v_3, v_4)_1, (v_4, v_5)_3, (v_5, v_1)_2, (v_1, v_3)_3, (v_1, v_4)_4, (v_2, v_5)_4\}$$

where $(v, v')_i$ means that the edge (v, v') takes color c_i. It is straightforward to check that G_1 is $(2+1)$-color connected.

For the case of $k = 3$, we have $n = 7, \lambda = 12, m = 4$, and $t = 2$. Let the graph G_2 in Figure 3b be defined as

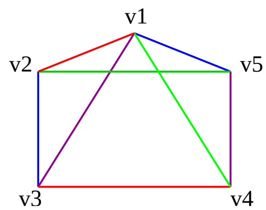
$$\{(v_1, v_2)_1, (v_2, v_3)_2, (v_4, v_5)_3, (v_5, v_1)_2, (v_1, v_3)_3, (v_1, v_4)_4, \\ (v_2, v_5)_4, (v_3, v_6)_1, (v_6, v_7)_3, (v_7, v_4)_1, (v_4, v_6)_4, (v_3, v_7)_2\}$$

where $(v, v')_i$ means that the edge (v, v') takes color c_i. It is straightforward to check that G_2 is $(2+1)$-color connected.

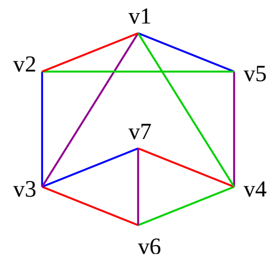
Now for $k = 2r$ ($r \geq 2$), we have $n = (m-t)k+1 = 4r+1$ and $\lambda = km = 8r$. If we glue the v_1 node of r copies of G_1, we get a $(t+1)$-color connected colored graph G with $n = 4r+1$ and $\lambda = 8r$. Thus the condition for the case of $k = 2r$ holds.

For $k = 2r+1$ ($r \geq 2$), we have $n = (m-t)k+1 = 4r+3$ and $\lambda = km = 8r+4$. If we glue the v_1 node of $r-1$ copies of G_1 and one copy of G_2, we get a $(t+1)$-color connected colored graph G with $n = 4(r-1)+1+6 = 4r+3$ and $\lambda = 8(r-1)+12 = 8r+4$. Thus the condition for the case of $k = 2r+1$ holds. This completes the induction.

For the condition "$n = (m-t)k+2$ and $\lambda \geq mk+t+1$", one can add one node to the graph for the case "$n = (m-t)k+1$ and $\lambda \geq km$" with 3 edges (with



(a) $n = 5, m = 4, t = 2$



(b) $n = 7, m = 4, t = 2$

Fig. 3: Figures for Theorem 9

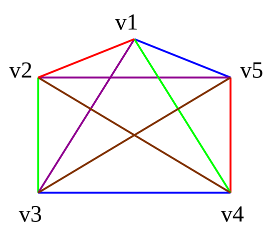
distinct colors) to any three nodes. The resulting graph meets the requirements. Q.E.D.

Theorem 9 could be extended to the case of $m = 5$ and $t = 3$.

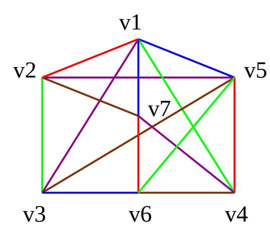
Theorem 10. *The conditions in Theorems 3 and 5 are sufficient for the case of $m = 5$ and $t = 3$.*

Proof. It is sufficient to show that both of the conditions "$n = (m-t)k+1$ and $\lambda \geq km$" and "$n = (m-t)k+2$ and $\lambda \geq mk+t+1$" are sufficient (note that $m-t = 2$). In the following we first show that the condition "$n = 2k+1$ and $\lambda \geq km$" is sufficient by induction on k and m.

For $m = 5$ and $k = 2$, we have $n = 5, \lambda = 10$. The graph in Figure 4a shows that the condition is sufficient also. For the case of $k = 3$, we have $n = 7, \lambda = 15$. The graph in Figure 4b shows that the condition is sufficient also.



(a) $n = 5, m = 5, t = 3$



(b) $n = 7, m = 5, t = 3$

Fig. 4: Figures for Theorem 10

For $k = 2r$ ($r \geq 2$), the condition becomes $n = (m-t)k+1 = 4r+1$ and $\lambda = km = 10r$. If we glue the v_1 node of r copies of $G_{5,1}$, we get a $(t+1)$-color connected colored graph G with $n = 4r+1$ and $\lambda = 10r$. Thus the condition for the case of $k = 2r$ holds.

For $k = 2r+1$ ($r \geq 2$), the condition becomes $n = (m-t)k+1 = 4r+3$ and $\lambda = km = 10r+5$. If we glue the v_1 node of $r-1$ copies of $G_{5,1}$ and one copy of

$G_{5,2}$, we get a $(t+1)$-color connected colored graph G with $n = 4(r-1)+1+6 = 4r+3$ and $\lambda = 10(r-1)+15 = 10r+5$. Thus the condition for the case of $k = 2r+1$ holds. This completes the induction.

For the condition "$n = (m-t)k+2$ and $\lambda \geq mk+t+1$", we have $n = 2k+2$ and $\lambda \geq 5k+4$. We can add one node to the graph for the case "$n = (m-t)k+1$ and $\lambda \geq km$" with 4 edges (with distinct colors) to any four nodes. The resulting graph meets the requirements. Q.E.D.

Open Questions: We showed in this section that the conditions in Theorems 3 and 5 are sufficient for practical cases. It would be interesting to show that these conditions are also sufficient for general cases. We leave this as an open question.

6 Hardness results

We have given necessary and sufficient conditions for $(t+1)$-color connected colored edge graphs. Sometimes, it is also important to determine whether a given graph is $(t+1)$-color connected. Unfortunately, the following Theorem shows that the problem ceConnect is **coNP**-complete. The ceConnect problem is defined as follows.

INSTANCE: A colored edge graph $G = G(V, E, C, f)$, two nodes $A, B \in V$, and a positive integer $t \leq |C|$.
QUESTION: Are A and B t-color connected?

Before we prove the hardness result, we first introduce the concept of color separator. For a colored edge graph $G = G(V, E, C, f)$, a color separator for two nodes A and B of the graph G is a color set $C' \subseteq C$ such that the removal of all edges with colors in C' from the graph G will disconnect A and B. It is straightforward to observe that A and B are $(t+1)$-color connected if and only there is no t-size color separator for A and B.

Theorem 11. *The problem ceConnect is* **coNP***-complete.*

Proof. It is straightforward to show that the problem is in **coNP**. Thus it is sufficient to show that it is **coNP**-hard. The reduction is from the Vertex Cover problem. The VC problem is as follows (definition taken from [6]):

INSTANCE: A graph $G = (V, E)$ and a positive integer $t \leq |V|$.
QUESTION: Is there a vertex cover of size t or less for G, that is, a subset $V' \subseteq V$ such that $|V'| \leq t$ and, for each edge $(u, v) \in E$, at least one of u and v belongs to V'?

For a given instance $G = (V, E)$ of VC, we construct a colored edge graph $G_c = (V_c, E_c, f, C)$ as follows. First assume that the vertex set V is ordered as in $V = \{v_1, \ldots, v_n\}$. Let

$$\begin{aligned}
V_c &= \{A, B\} \bigcup \left\{e_{(v_i,v_j)} : (v_i, v_j) \in E \text{ and } i < j\right\} \\
E_c &= \left\{(A, e_{(v_i,v_j)}), (e_{(v_i,v_j)}, B) : (v_i, v_j) \in E\right\} \\
C &= \{c_v : v \in V\} \\
f &= \left\{f(A, e_{(v_i,v_j)}) = c_{v_i}, f(e_{(v_i,v_j)}, B) = c_{v_j} : (v_i, v_j) \in E, i < j\right\}
\end{aligned}$$

In the following, we show that there is a vertex cover of size t in G if and only if there is a t-color edge separator for G_c.

Without loss of generality, assume that $V' = \{v'_1, \ldots, v'_k\}$ is a vertex cover for G. Then it is straightforward to show that $C' = \{c_{v'_i} : v'_i \in V'\}$ is a color separator for G_c since each incoming path for B in G_c contains two colors corresponding to one edge (v_i, v_j) in G.

For the other direction, assume that $C' = \{c_{v'_i} : i = 1, \ldots, t\}$ is a t-color separator for G_c. Let $V' = \{v'_i : c_{v'_i} \in C'\}$. By the fact that C' is a color separator for G_c, for each edge $(v_i, v_j) \in E$ in G, the path $(A, e_{(v_i,v_j)}, B)$ in G_c contains at least one color from C'. Since this path contains only two colors c_{v_i} and c_{v_j}, we know that v_i or v_j or both belong to V'. In another word, V' is a t-size vertex cover for G. This completes the proof of the Theorem. Q.E.D.

7 Disjunct systems

We conclude our paper with some observations on the relationship between disjunct system and cover free families.

Incidence matrix is usually used to describe set systems. Let $(X, \mathcal{F})$ be a (λ, m)-partition of X with $X = \{x_1, \ldots, x_\lambda\}$ and $\mathcal{F} = \{B_1, \ldots, B_m\}$. Then the incidence matrix of $(X, \mathcal{F})$ is the $\lambda \times m$ matrix $(a_{i,j})$ where $a_{i,j} = 1$ if $x_i \in B_j$ and $a_{i,j} = 0$ otherwise. If A is an incidence matrix of a set system, then A^T (the transpose of A) is an extended incidence matrix of a disjunct system. Note that by extended incidence matrix, we mean that, after consolidating repeated columns of the matrix we get the incident matrix of a disjunct system.

Definition 3. *Let Y be a set of m elements, and $\mathcal{B}$ be a set of λ subsets of Y. Then the set system $(Y, \mathcal{B})$ is called a $(t; n-1)$-disjunct system (or $(t; n-1)$-DS(m, λ)) if for any $P \subseteq Y$ such that $|P| \le t$, there exist at least $n-1$ blocks $B \in \mathcal{B}$ such that $P \cap B = \emptyset$.*

Theorem 12.
1. *If there exists a $(t; n-1)$-CFF(λ, m) then there exists a $(t; n'-1)$-DS(m, λ') for some $1 < n' \le n$ and $\lambda' \le \lambda$.*
2. *If there exists a $(t; n-1)$-DS(m, λ), then there exists a $(t; n-1)$-CFF(λ', m) for some $0 < \lambda' \le \lambda$.*

Proof. Assume that $(X, \mathcal{F})$ is a $(t; n-1)$-CFF(λ, m) with incidence matrix A. Let $Y = \mathcal{F}$ and $\mathcal{B} = \{[x] : x \in X\}$ where $[x] = \{P : x \in P \text{ and } P \in \mathcal{F}\}$. In the following, we show that $(Y, \mathcal{B})$ is a $(t; n'-1)$-DS(m, λ') with extended incidence matrix A^T for some $1 < n' \le n$ and $\lambda' \le \lambda$. By the fact that $(X, \mathcal{F})$ is a $(t; n-1)$-CFF(λ, m), for any $P = \{B_1, \ldots, B_t\} \subseteq Y$, there exist distinct $x_1, \ldots, x_{n-1} \in X \setminus (\cup_{i=1}^t B_i)$. That is, for any $i \le n-1$ and $j \le t$, we have $x_i \notin B_j$ which means $B_j \notin [x_i]$. Thus $P \cap [x_i] = \emptyset$ for all $i \le n-1$. Note that for $i \ne j$, we may have $[x_i] = [x_j]$. Thus the above arguments only guarantee that there exists $n' > 1$ such that $(Y, \mathcal{B})$ is a $(t; n'-1)$-DS(m, λ').

For the other direction, assume that $(Y, \mathcal{B})$ is a $(t; n-1)$-DS(m, λ) with incidence matrix A. Let $X = \mathcal{B}$ and $\mathcal{F} = \{[y] : y \in Y\}$ where $[y] = \{P :$

$y \in P$ and $P \in \mathcal{B}\}$. In the following, we show that $(X, \mathcal{F})$ is a $(t, n-1)$-CFF(λ, m) with incidence matrix A^T. For any t blocks $[y_1], \ldots, [y_t] \in \mathcal{F}$, let $P = \{y_1, \ldots, y_t\}$. By the fact that $(Y, \mathcal{B})$ is a $(t, n-1)$-DS(m, λ), there exist distinct blocks $B_1, \ldots, B_{n-1} \in \mathcal{B}$ such that $P \cap B_i = \emptyset$. That is, for each $i \leq t$ and $j \leq n-1$, we have $y_i \notin B_j$ which means $B_j \notin [y_i]$. Thus $\{B_1, \ldots, B_{n-1}\} \in X \setminus (\cup_{i=1}^{t} [y_t])$. It follows that $(X, \mathcal{F})$ is a $(t, n-1)$-CFF(λ, m). Q.E.D.

Acknowledgement

The first author of this paper would like to thank Prof. Doug Stinson and Prof. Ruizhong Wei for some discussions on generalized cover-free families.

References

1. G. Constantine. Colorful isomorphic spanning trees in complete graphs. *Annals of Combinatorics* **9**:163–167, 2005.
2. Y. Desmedt and Y. Wang. Perfectly Secure Message Transmission Revisited. In *Proc. EuroCrypt'02*, pages 502-517. LNCS 2332, Springer-Verlag
3. Y. Desmedt, Y. Wang, and M. Burmester. A complete characterization of tolerable adversary structures for secure point-to-point transmissions without feedback. In *Proc. ISAAC 2005*, pages 277–287. Lecture Notes in Computer Science 3827. Springer Verlag 2005.
4. Y. Desmedt, Y. Wang, R. Safavi-Naini, and H. Wang. Radio networks with reliable communications. In *Proc. COCOON*, pages 156-166, LNCS 3595, 2005.
5. K. Engel. Interval packing and covering in the boolean lattice. *Combin. Probab. Comput.* **5**:373–384, 1995.
6. M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of* **NP**-*Completeness*. W. H. Freeman and Company, San Francisco, 1979.
7. J. Roskind and R. Tarjan, A note on finding minimum-cost edge-disjoint spanning trees. *Math. Oper. Res.* **10**:701–708, November 1985.
8. D. Stinson and R. Wei. Generalized cover-free families. *Discrete Math.* **279**:463–477, 2004.
9. Y. Wang and Y. Desmedt. Secure communication in multicast channels. *Journal of Cryptology* **14**(2):121–135, 2001.