TIE-POINTS, REGULAR CLOSED SETS, AND COPIES OF $\mathbb{N}^*$

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Abstract. We show that it is consistent to have a non-trivial embedding of $\mathbb{N}^*$ into itself even if all autohomeomorphisms of $\mathbb{N}^*$ are trivial.

1. Introduction

In this paper we are interested in the existence of non-trivial copies of $\mathbb{N}^*$ in $\beta\mathbb{N}$. A subspace $K \subset \mathbb{N}^*$ is said to be a trivial copy if there is an embedding of $\beta\mathbb{N}$ into $\beta\mathbb{N}$ which sends the remainder $\mathbb{N}^*$ onto K. A point $\mathcal{U}$ of $\mathbb{N}^*$ is said to be a tie-point if there are closed sets A, B covering $\mathbb{N}^*$ satisfying that $\overline{A \setminus B} \cap \overline{B \setminus A}$ is the single point $\mathcal{U}$. We shall use $A \underset{\mathcal{U}}{\bowtie} B$ to denote when this is the case, and say that A, B is a tie-point pair with base $\mathcal{U}$. In the exceptional case that there is an autohomeomorphism of $\mathbb{N}^*$ sending A onto B (and B onto A), we say in [2] that $\mathcal{U}$ is a symmetric tie-point. Obviously A and B are regular closed subsets of $\mathbb{N}^*$. It is of some interest to also be able to determine if either or both of A and B would be copies of $\mathbb{N}^*$; it is easily shown, in any case, that neither is a trivial copy. It is not known if $\mathbb{N}^*$ contains a non-trivial copy of $\mathbb{N}^*$, but Farah has shown that PFA implies that it has no regular closed non-trivial copies. We should mention the well-known fact that the continuum hypothesis implies that non-trivial copies of $\mathbb{N}^*$ abound.

A homomorphism ψ from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ into $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ is said to be trivial, if there is function h with domain and range subsets of $\mathbb{N}$ which induces ψ in the sense that $\psi(a) =^* h^{-1}[a]$ for all $a \subset \mathbb{N}$. Two ultrafilters on $\mathbb{N}$ are said to be RK-equivalent (Rudin-Keisler) if there is a trivial automorphism sending one to the other. We intend to deal with surjective homomorphisms only and so may assume that h

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is one-to-one. It will be more convenient then to work with the inverse map and to expand the domain of h to all of $\mathbb{N}$ by sending the additional points to 0; hence $\psi(a) =^* h[a] \setminus \{0\}$. Of course the kernel of ψ, $\ker(\psi)$, will consist of all sets which are almost contained in $h^{-1}(0)$. More generally, if $a \in \mathcal{P}(\mathbb{N})$ and $\psi \restriction \mathcal{P}(a)/\operatorname{fin}$ is trivial, then we let h_a denote any function into $\mathbb{N} \cup \{0\}$ with domain a which induces ψ as above. We let $\operatorname{triv}(\psi)$ denote the ideal of sets on which ψ is trivial; observe that $\ker(\psi) \subset \operatorname{triv}(\psi)$. If g is a homeomorphism from $\mathbb{N}^*$ onto $K \subset \beta\mathbb{N}$, then there is a corresponding homomorphism, ψ_g from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ onto $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ defined by $\psi_g(a)$ is the equivalence class of b if $b^* = g^{-1}(a^* \cap K)$. Of course g is said to be a trivial (auto)homeomorphism if ψ_g is a trivial automorphism. We will say that g is trivial at a point $x \in \mathbf{N}^*$ if some member of $\operatorname{triv}(\phi_g)$ is in the ultrafilter corresponding to x.

Velickovic introduced a poset, which we will denote $\mathbb{P}_2$, which introduces a symmetric tie-point (and a non-trivial automorphism). Several variations of $\mathbb{P}_2$ are possible and we continue the study of the properties of $\mathbb{N}^*$ that hold in the model(s) obtained when forcing with $\mathbb{P}_2$ (and its variants) over a model of PFA (see [14, 10, 12, 2]). It is known to follow from PFA that there are no tie-points (see remark [1, p.1662]) and we will prove a stronger statement in the second section.

Farah [5] defines the important notion of an ideal of $\mathcal{P}(\mathbb{N})$ being ccc over fin to mean that any uncountable almost disjoint family of subsets of $\mathbb{N}$ will intersect the ideal. By Stone duality, we define a closed subset K of $\mathbb{N}^*$ to be ccc over fin if there is no uncountable family of pairwise disjoint clopen subsets of $\mathbb{N}^*$ each meeting K in a non-empty set. Farah [5] proves that PFA implies that for each homomorphism ψ from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ onto $\mathcal{P}(\mathbb{N})/\operatorname{fin}$, $\operatorname{triv}(\psi)$ is ccc over fin. We show that this remains true in forcing extensions by the posets mentioned above.

maintheorems

2. Preliminaries

For ideals $\mathcal{I}, \mathcal{J}$ on $\mathcal{P}(\mathbb{N})$ we use $\mathcal{I} \perp \mathcal{J}$ to denote that $\mathcal{I} \cap \mathcal{J} = \operatorname{fin}$. Ideals $\mathcal{I}$ and $\mathcal{J}$ are separated if there is an $a \subset \mathbb{N}$ which mod finite contains every member of $\mathcal{I}$ and is mod finite disjoint from every member of $\mathcal{J}$. Let $\mathcal{I} \oplus \mathcal{J}$ denote the ideal generated by $\mathcal{I} \cup \mathcal{J}$. For $C \subset \mathbb{N}$, $\mathcal{I}_{\restriction C}$ is the ideal $\{a \cap C : a \in \mathcal{I}\} \oplus \operatorname{fin}$. The notation $\mathcal{I}^\perp$ denotes the ideal $\{b \subset \mathbb{N} : (\forall a \in \mathcal{I})\ b \cap a \in \operatorname{fin}\}$. Of course $\mathcal{I} \oplus \mathcal{I}^\perp$ is a dense ideal (every infinite set contains an infinite member). For a set $A \subset \mathbb{N}^*$, we let $\mathcal{I}_A$ denote the ideal $\{a \subset \mathbb{N} : a^* \subset A\}$. Observe that A being regular closed is equivalent to $\mathcal{I}_A^\perp$ being equal to $\{b \subset \mathbb{N} : b^* \cap A = \emptyset\} = \mathcal{I}_{\overline{\mathbb{N}^* \setminus A}}$. The boundary of a regular closed set A, denoted ∂A, is ccc over fin

precisely when $\mathcal{I}_A \oplus \mathcal{I}_A^\perp$ is ccc over fin. Indeed, an ultrafilter $\mathcal{U} \in \partial A$ if and only if $\mathcal{U} \cap (\mathcal{I}_A \oplus \mathcal{I}_A^\perp)$ is empty. Let us also recall that an ideal $\mathcal{I}$ is a P-ideal if every countable subset of $\mathcal{I}$ has a mod finite upper bound in $\mathcal{I}$, and similarly, $\mathcal{I}$ is a P_{ω_2}-ideal if every $<\omega_2$-sized subset of $\mathcal{I}$ has a mod finite upper bound in $\mathcal{I}$. In this notation we may define an (ω_2, ω_2)-gap to be an unseparated pair $\mathcal{I}, \mathcal{J}$ where each is an $\aleph_2$-generated P_{ω_2}-ideal. In a similar way, a pair $\mathcal{I}, \mathcal{J}$ is an (ω_1, ω_2)-gap if $\mathcal{I}$ is an $\aleph_1$-generated P-ideal and $\mathcal{J}$ is an $\aleph_2$-generated P_{ω_2}-ideal.

Following Farah ([5, p144-5]) (reporting on results of Todorcevic) we will make use of the notion of a *Luzin gap* which makes the unseparatedness of the orthogonal ideals upward absolute for ω_1-preserving extensions. Indeed, a pair $\mathcal{I}, \mathcal{J}$ is a Luzin gap if $\mathcal{I} \perp \mathcal{J}$ and there is a pair of maps $f_\mathcal{I}, f_\mathcal{J}$ sending ω_1 into $\mathcal{I}$ and $\mathcal{J}$ respectively, such that for all $\alpha, \beta \in \omega_1$ we have

$$f_\mathcal{I}(\alpha) \cap f_\mathcal{J}(\beta) \neq \emptyset \text{ if and only if } \alpha \neq \beta \ .$$

So long as ω_1 is preserved, $\mathcal{I}$ and $\mathcal{J}$ remain unseparated ([5, 5.2.3]). Let us note that we should actually be saying that $\mathcal{I}, \mathcal{J}$ *contains* a Luzin gap but the distinction does not seem important. It follows that neither $\mathcal{I}$ nor $\mathcal{J}$ can be a P_{ω_2}-ideal if $\mathcal{I}, \mathcal{J}$ is a Luzin gap.

Define $\mathcal{I}, \mathcal{J}$ to be countably separated if there is a family $\{C_n : n \in \omega\} \subset \mathcal{P}(\mathbb{N})$ such that for all $I \in \mathcal{I}$ and $J \in \mathcal{J}$, there is an n such that $I \subset^* C_n$ and $C_n \cap J =^* \emptyset$. A close reading of the proof of [5, 5.2.4] establishes the following (see also [8, Lemma 2] for a similar result that inspired this work).

Proposition 2.1. *If $\mathcal{I} \perp \mathcal{J}$ are not countably separated, then there is a proper poset which forces $\mathcal{I}, \mathcal{J}$ to be a Luzin gap.* luzin

It is immediate that if $\mathcal{I}$ and $\mathcal{J}$ are unseparated P-ideals then they are not countably separated. As we will need it later, this is a good place to record the following strengthening of Proposition 2.1 (using an idea from [5, 3.85]).

Proposition 2.2. *If $\mathcal{I} \perp \mathcal{J}$ are not countably separated, then there is a proper poset which introduces an uncountable almost disjoint family $\mathcal{C}$ of subsets of $\mathbb{N}$ such that for each $C \in \mathcal{C}$, $\mathcal{I}_{\restriction C}, \mathcal{J}_{\restriction C}$ contains a Luzin gap.* strongluzin

Proof. Start with the poset $2^{<\omega_1}$ which is countably closed and forces CH to hold. In the extension, let $\{y_\alpha : \alpha \in \omega_1\}$ be an enumeration of $\mathcal{P}(\mathbb{N})$ and inductively choose disjoint $a_\alpha \in \mathcal{I}$ and $b_\alpha \in \mathcal{J}$ so that for all $\beta < \alpha$, y_β does not mod finite separate $\{a_\alpha, b_\alpha\}$. Notice that we

now have that for any uncountable $\Lambda_0, \Lambda_1 \subset \omega_1$, the sets $\bigcup_{\alpha\in\Lambda_0} a_\alpha$ and $\bigcup_{\beta\in\Lambda_1} b_\beta$ have infinite intersection.

We define a poset Q where $q \in Q$ implies $q = \big(L_q, \psi_q, \langle C^q_\gamma : \gamma \in L_q\rangle\big)$ for some $L_q \in [\omega_1]^{<\omega}$, $\psi_q \in L_q^{L_q}$, and sequence, $\langle C^q_\gamma : \gamma \in L_q\rangle$ of finite sets of integers. We also require that for $\alpha, \beta \in L_q$ with $\gamma = \psi_q(\alpha) = \psi_q(\beta)$, then $a_\alpha \cap b_\beta$ must meet C^q_γ. We define $p < q$ if the obvious inclusions hold, and we also require that for $\gamma, \delta \in L_q$, we have $C^p_\gamma \cap C^p_\delta = C^q_\gamma \cap C^q_\delta$.

It is easy to see that if Q is ccc and if $G \subset Q$ is generic, then the family $\mathcal{C} = \{C_\gamma : \gamma \in \omega_1\}$ will form an almost disjoint family, where $C_\gamma = \bigcup\{C^p_\gamma : p \in G, \gamma \in L_p\}$. A simple density argument will also show that $\Gamma_\gamma = \{\alpha : (\exists p \in G)\ \alpha, \gamma \in L_p \text{ and } \psi_p(\alpha) = \gamma\}$ will be uncountable. In addition, the family $\{a_\alpha : \alpha \in \Gamma_\gamma\} \restriction C_\gamma$, $\{b_\alpha : \alpha \in \Gamma_\gamma\} \restriction C_\gamma$ will be a Luzin gap.

The proof is completed by showing that Q is ccc. Suppose that $\{q_\xi : \xi \in \omega_1\} \subset Q$. We may assume that there is a set $L \in [\omega_1]^{<\omega}$ and an integer m such that for $\xi \neq \zeta$, $L_{q_\xi} \cap L_{q_\zeta} = L$ and $|L_{q_\xi}| = |L_{q_\zeta}| = m$. Furthermore we may assume that there is some sequence $\langle C'_\gamma : \gamma \in L\rangle$ satisfying that for all ξ, $\langle C^{q_\xi}_\gamma : \gamma \in L\rangle = \langle C'_\gamma : \gamma \in L\rangle$.

The final reduction we make is to assume that the conditions are pairwise isomorphic. More specifically, for each ξ, let $\{\alpha^\xi_i : i < m\}$ be the order preserving enumeration of L_{q_ξ}. We assume that there is a $k \in \omega$ such that for each $\xi < \zeta$ and each $i, j < m$,

(1) $C^{q_\xi}_\gamma \subset k$ for each $\gamma \in L_{q_\xi}$,
(2) $\alpha^\xi_i \in L$ if and only if $\alpha^\zeta_i \in L$
(3) $a_{\alpha^\xi_i} \cap k = a_{\alpha^\zeta_i} \cap k$,
(4) $b_{\alpha^\xi_i} \cap k = b_{\alpha^\zeta_i} \cap k$,
(5) $\psi_{q_\xi}(\alpha^\xi_i) = \alpha^\xi_j$ if and only if $\psi_{q_\zeta}(\alpha^\zeta_i) = \alpha^\zeta_j$

Claim: if I, J are uncountable subsets of ω_1, $k' > k$ and $i < m$, then there are $n' > n > k'$ and uncountable $I' \subset I$, $J' \subset J$ such that $n \in a_{\alpha^\xi_i} \cap b_{\alpha^\zeta_i}$ and $n' \in b_{\alpha^\xi_i} \cap a_{\alpha^\zeta_i}$ for all $\xi \in I'$ and $\zeta \in J'$.

We first choose n in the infinite intersection of $\bigcup\{a_{\alpha^\xi_i} : \xi \in I\}$ with $\bigcup\{b_{\alpha^\zeta_i} : \zeta \in J\}$. By shrinking I and J we may now assume that $n \in a_{\alpha^\xi_i} \cap b_{\alpha^\zeta_i}$ for all $\xi \in I$ and $\zeta \in J$. Apply this idea again to choose $n' > n$ together with $I' \subset I$ and $J' \subset J$ so that $n' \in b_{\alpha^\xi_i} \cap a_{\alpha^\zeta_i}$ for all $\xi \in I'$ and $\zeta \in J'$. Now set $I_0 = J_0 = \omega_1$. We may recursively choose a sequence of pairs of uncountable sets I_i, J_i for $i \leq m$ and an increasing sequence, $\{n_i : i < 2m\} \subset \omega \setminus k$ by applying the above Claim so that $n_{2i} \in a_{\alpha^\xi_i} \cap b_{\alpha^\zeta_i}$ and $n_{2i+1} \in b_{\alpha^\xi_i} \cap a_{\alpha^\zeta_i}$ for all $\xi \in I_m$ and $\zeta \in J_m$.

Fix any $\xi \in I_m$ and $\zeta \in J_m$. We define a condition p which is below q_ξ and q_ζ. Let $L_p = L_{q_\xi} \cup L_{q_\zeta}$ and $\psi_p = \psi_{q_\xi} \cup \psi_{q_\zeta}$. For each $\gamma \in L_{q_\xi}$, we define C^p_γ to be $C^{q_\xi}_\gamma \cup C^{q_\zeta}_\gamma \cup \{n_{2i}, n_{2i+1} : i < m \text{ and } \psi_{q_\xi}(\alpha^\xi_i) = \gamma\}$. For each $\gamma \in L_{q_\zeta} \setminus L$, we define C^p_γ to be C^{q_ζ}.

It is routine to check that for $\gamma, \delta \in L_{q_\xi}$, $C^p_\gamma \cap C^p_\delta = C^{q_\xi}_\gamma \cap C^{q_\zeta}_\delta$. It is even easier to see this for $\gamma, \delta \in L_{q_\zeta}$. Finally, to show that p is a member of Q, we suppose that $\psi_p(\alpha) = \psi_p(\beta) = \gamma$ for distinct α, β and $\{\alpha, \beta, \gamma\} \subset L_p$. We must show that each of $a_\alpha \cap b_\beta \cap C^p_\gamma$ and $b_\alpha \cap a_\beta \cap C^p_\gamma$ are non-empty. Since each of q_ξ and q_ζ satisfy the conditions to be in Q, we may assume that $\alpha \in L_{q_\xi} \setminus L$ and $\beta \in L_{q_\zeta} \setminus L$. There is a pair of integers, $i, j < m$ such that $\alpha \in \{\alpha^\xi_i, \alpha^\zeta_i\}$ and $\beta \in \{\alpha^\zeta_j, \alpha^\zeta_j\}$. Again, if i is distinct from j, then by the isomorphism condition we already have that $a_\alpha \cap b_\beta$ and $b_\alpha \cap a_\beta$ will hit C'_γ. Of course if $i = j$, then the choices of n_{2i} and n_{2i+1} have been added to C^p_γ as required. □

Similar to Luzin gaps are uncountable pairwise incompatible families of partial functions on $\mathbb{N}$. Such a family will not have a common mod finite extension. The following result is a trivial consequence of a result of Todorcevic (see also [5, 2.2.1]).

Proposition 2.3. *If $\{h_\alpha : \alpha \in \omega_1\}$ is a family of partial functions on $\mathbb{N}$ with mod finite increasing domains, and if there is no common mod finite extension, then there is a proper poset which introduces an uncountable pairwise incompatible subfamily.* functiongaps

Now we define the partial order $\mathbb{P}_2$ [14]

Definition 2.4. The partial order $\mathbb{P}_2$ is defined to consist of all 1-to-1 functions f where poset

(1) $\text{dom}(f) = \text{range}(f) \subset \mathbb{N}$,
(2) for all $i \in \text{dom}(f)$ and $n \in \omega$, $f(i) \in [2^n, 2^{n+1})$ if and only if $i \in [2^n, 2^{n+1})$
(3) $\limsup_{n\to\omega} |[2^n, 2^{n+1}) \setminus \text{dom}(f)| = \omega$ growth
(4) for all $i \in \text{dom}(f)$, $i = f^2(i) \neq f(i)$.

The ordering on $\mathbb{P}_2$ is $\subseteq^*$.

Similar to $\mathbb{P}_2$, we define two additional posets (see [10, 2] for others) denoted $\mathbb{P}_0$ and $\mathbb{P}_1$. Let $\mathbb{P}_1$ denote the poset consisting of partial functions f with $\text{dom}(f) \subset \mathbb{N}$ satisfying condition (3) and having range $\{0, 1\} = 2$. The poset $\mathbb{P}_0$ consists of those $f \in \mathbb{P}_1$ satisfying the additional condition that for all $n \in \omega$, $f^{-1}(1) \cap [2^n, 2^{n+1})$ has size at most 1, and if it is non-empty, then $[2^n, 2^{n+1}) \subset \text{dom}(f)$. Each poset is ordered by $p < q$ if $p \supset^* q$.

Each of these posets introduces a new generic ultrafilter $\mathcal{U}$ which is a tie-point: $A \underset{\mathcal{U}}{\bowtie} B$. Let G denote a $\mathbb{P}$-generic filter. It is shown in [14], that the collection $\mathcal{U} = \{\mathbb{N} \setminus \mathrm{dom}(f) : f \in G\}$ is an ultrafilter. In the cases of $\mathbb{P}$ being one of $\mathbb{P}_0$ and $\mathbb{P}_1$, $\mathcal{I}_A$ would be $\{f^{-1}(1) : f \in G\}$, while, for $\mathbb{P}_2$, $\mathcal{I}_A = \{\{i \in \mathrm{dom}(f) : i < f(i)\} : f \in G\}$. This is discussed in [2]. One of our main motivations is to discover if A or B can be homeomorphic to $\mathbb{N}^*$ as this information can be quite useful in applications (again, see [2]).

If PFA holds, then each of $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$ is $\aleph_1$-closed and $\aleph_2$-distributive (see [12, p.4226]). In this paper we will restrict our study to forcing with these posets individually, but the reader is referred to [12] for the method to generalize to countable support infinite products. In particular, the result that $\mathrm{triv}(F)$ is ccc over fin for all homomorphisms F on $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ should hold in these more general models.

p0nstar **Theorem 2.5.** *If G is $\mathbb{P}_0$-generic and $A \underset{\mathcal{U}}{\bowtie} B$ are as above, then A is homeomorphic to $\mathbb{N}^*$.*

Proof. Let $\psi \in \mathbb{N}^{\mathbb{N}}$ be defined so that $\psi([2^n, 2^{n+1})) = \{n\}$ for all n, and let ψ^* denote the canonical extension with domain and range $\mathbb{N}^*$. In fact, for each free ultrafilter $\mathcal{W}$, the preimage of $\mathcal{W}$ under ψ^* is the set of ultrafilters extending $\{\psi^{-1}[W] : W \in \mathcal{W}\}$. Recall that A is the closure of the set $\bigcup\{(f^{-1}(1))^* : f \in G\}$. We will simply show that $\psi^* \restriction A$ is one-to-one. Let $\mathcal{V} = \{b \subset \mathbb{N} : \psi^{-1}(b) \in \mathcal{U}\}$. By the definition of $\mathbb{P}_0$, it follows that, for each $f \in G$, $\psi^* \restriction (f^{-1}(1))^*$ is one-to-one and that $\psi(f^{-1}(1)) \notin \mathcal{V}$. It follows easily that the preimage of any point of $\mathbb{N}^* \setminus \{\mathcal{V}\}$ contains a single point in A. Now suppose that $\mathcal{W} \neq \mathcal{U}$ is in the preimage of $\mathcal{V}$. Since $\mathcal{U}$ is generated by $\{\mathbb{N} \setminus \mathrm{dom}(f) : f \in G\}$, we may choose an $f \in G$ with $\mathrm{dom}(f) \in \mathcal{W}$. Since $\psi(f^{-1}(1)) \notin \mathcal{V}$, we have that $f^{-1}(0) \in \mathcal{W}$. But now, $f^{-1}(0)$ is disjoint from each member of $\mathcal{I}_A$, which shows that $\mathcal{W} \notin A$. □

The rest of the paper is devoted to proving the following theorems. We indicate where to find the proofs at the end of each statement.

cccfin **Theorem 2.6.** *In the extension obtained by forcing over a model of PFA by any of $\mathbb{P}_0$, $\mathbb{P}_1$, or $\mathbb{P}_2$, if Φ is a homomorphism from $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ onto $\mathcal{P}(\mathbb{N})/\mathrm{fin}$, then $\mathrm{triv}(\Phi)$ is a ccc over fin ideal. (see 4.14)*

P0stuff **Theorem 2.7.** *In the extension obtained by forcing over a model of PFA by $\mathbb{P}_0$, the following statements hold:*

1. *all automorphisms on $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ are trivial, (see 6.1)*
2. *there are non-trivial regular closed copies of $\mathbb{N}^*$, (see 2.5)*

P03 3. *all regular closed copies of $\mathbb{N}^*$ have finite boundaries, (see 6.2)*

P04

(4) *the intersection of two regular closed copies of* $\mathbb{N}^*$ *will also be regular closed (see 6.3)*
(5) *all tie-points are RK-equivalent, (follows from statement 1)* P05

Theorem 2.8. *In the extension obtained by forcing over a model of PFA by* $\mathbb{P}_1$*, the following statements hold:* P1stuff

(1) *there are non-trivial automorphisms and non-trivial regular closed copies of* $\mathbb{N}^*$*, (see 5.7)* P11
(2) *all regular closed copies of* $\mathbb{N}^*$ *have finite boundaries, (see 6.2)* P12
(3) *there are two regular closed copies of* $\mathbb{N}^*$ *that intersect in a single point, (see 5.7)* P13
(4) *each automorphism on* $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ *is trivial at the tie-point* $\mathcal{U}$*, (see 5.3)* P14
(5) *there is an automorphism on* $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ *which is not trivial at a tie-point, (see 5.7)* P15
(6) *automorphisms on* $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ *preserve RK-orbits. (see 2.9)* P16

The final result about the forcing $\mathbb{P}_1$ introduces the idea of something that might be called a nearly trivial automorphism. Statement 6 of Theorem 2.8 follows immediately from this theorem. The following theorem is proven in two parts: Theorem 6.1 and Proposition 5.3.

Theorem 2.9. *In the extension obtained by forcing over a model of PFA by* $\mathbb{P}_1$*, for each autohomeomorphism* ϕ *of* $\mathbf{N}^*$*, there is a trivial autohomeomorphism* $\bar{h}$ *of* $\mathbf{N}^*$ *and a regular closed set* $A \subset \mathbf{N}^*$*, such that* ϕ *and* $\bar{h}$ *agree on* A *and* ϕ *is trivial at every point not in* A*.* nearlytrivial

For the remainder of this section let $\mathbb{P}$ denote any one of the posets $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$. It is easily shown that $\mathbb{P}$ is σ-directed closed. The following partial order was introduced in [10] as a great tool to uncover the forcing preservation properties of $\mathbb{P}$, such as the fact that $\mathbb{P}$ is $\aleph_2$-distributive (and so introduces no new ω_1-sequences of subsets of $\mathbb{N}$).

Definition 2.10. Let $\mathfrak{F}$ denote any filter on $\mathbb{P}$. Define $\mathbb{P}(\mathfrak{F})$ to be the partial order consisting of all $q \in \mathbb{P}$ such that there is some $p \in \mathfrak{F}$ which is almost equal to it. The ordering on $\mathbb{P}(\mathfrak{F})$ is $p \leq q$ if $p \supseteq q$.

The forcing $\mathbb{P}(\mathfrak{F})$ introduces a new total function f which extends mod finite every member of $\mathfrak{F}$. Although f will not be a member of $\mathbb{P}$ it is only because its domain does not satisfy the growth condition (3) in the definition of $\mathbb{P}$. There is a simple σ-centered poset $\mathcal{S}$ which will force an appropriate set $I \subset \mathbb{N}$ which mod finite contains all the domains of members of $\mathfrak{F}$ and satisfies that $p = f \restriction I$ is a member of $\mathbb{P}$ which is below each member of $\mathfrak{F}$ (see [10, 2.1]).

A strategic choice of the filter $\mathfrak{F}$ will ensure that $\mathbb{P}(\mathfrak{F})$ is ccc, but remarkably even more is true. Again we are lifting results from [10, 2.6] and [12, proof of Thm. 3.1]. A poset is said to be ω^ω-bounding if every new function in ω^ω is bounded by some ground model function. Let $\kappa = 2^{|\mathbb{P}|}$, let $\kappa^{<\omega_1}$ denote the standard collapse which introduces a function from ω_1 onto κ, and let H be $\kappa^{<\omega_1}$-generic. In the extension $V[H]$, CH holds and no new countable sets have been added; $\mathbb{P}$ remains countably closed, and so there is a maximal filter $\mathfrak{F} \subset \mathbb{P}$ which is V-generic for $\mathbb{P}$ (in this extension $\mathfrak{F}$ only needs to meet $\aleph_1$ many dense sets). For the remainder of the paper $\mathfrak{F}$ refers to such a filter (or, when needed, a $\kappa^{<\omega_1}$-name of such a filter).

Lemma 2.11 ([12])**.** *In the forcing extension, $V[H]$, by $\kappa^{<\omega_1}$, there is a maximal filter $\mathfrak{F}$ on $\mathbb{P}$ which is $\mathbb{P}$-generic over V and for which $\mathbb{P}(\mathfrak{F})$ is ccc, ω^ω-bounding, and preserves that $\mathbb{R} \cap V$ is not meager.*

Almost all of the work we have to do is to establish additional preservation results for the poset(s) $\mathbb{P}(\mathfrak{F})$. Once these are established, we are able to apply the standard PFA type methodology as demonstrated in [10, 12].

We will also need several results from [2]. The following is a strengthening of [2, 2.5] in that we introduce gaps into the picture. For partial functions p and s with domains contained in $\mathbb{N}$, we let $s \sqcup p$ denote the function $s \cup (p \restriction \mathrm{dom}(p) \setminus \mathrm{dom}(s))$.

preserve1 **Lemma 2.12.** *Let $\dot{h}$ be a $\mathbb{P}(\mathfrak{F})$-name of a function in $\mathbb{N}^\mathbb{N}$. Also suppose that $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ and $\mathcal{I}^\perp$ are P-ideals. Then there are $I \in \mathcal{I}$, $C \in \mathcal{I}^\perp$, an increasing sequence $n_0 < n_1 < n_2 < \cdots$ of integers and a condition $p \in \mathfrak{F}$ such that, for each k, there is a single m_k with $[n_k, n_{k+1}) \setminus \mathrm{dom}(f) \subset [2^{m_k}, 2^{m_k+1})$ and such that either*

(1) *no extension of p forces a value on the function $\dot{h} \restriction I$ or*

two.preserve1 (2) *for each $i \in [n_k, n_{k+1}) \setminus C$ and each $q < p$ such that q forces a value on $\dot{h}(i)$, $p \cup (q \restriction [n_k, n_{k+1}))$ also forces a value on $\dot{h}(i)$.*

Proof. A fusion sequence for $\mathbb{P}(\mathfrak{F})$ is a descending sequence $\{p_k : k \in \omega\}$ of conditions together with an increasing sequence $\{n_k : k \in \omega\}$ of integers satisfying that for each k, there is an m_k such that $[2^{m_k}, 2^{m_k+1})$ contains more than k elements of $[n_k, n_{k+1}) \setminus \mathrm{dom}(p_k)$ and $p_{k+1} \restriction n_{k+1} \subset p_k$. Given such a sequence, it follow that the union, $\bigcup_k p_k$ is a condition in $\mathbb{P}(\mathfrak{F})$ (see [10, 2.4] or [12, 3.4]). Given $q, p \in \mathbb{P}(\mathfrak{F})$ and integer m, let $q <_m p$ denote the relation that $q \restriction m \subset p$ and $q \supset p$. For an integer i and $q \in \mathbb{P}(\mathfrak{F})$, let $q \| \dot{h}(i)$ abbreviate the statement that q forces a value on $\dot{h}(i)$.

Given any $p_0 \in \mathfrak{F}$, let M be a countable elementary submodel of a sufficiently large $H(\theta)$ with $\dot{h}, p_0, \mathbb{P}(\mathfrak{F})$ and $\mathcal{I}$ in M. Fix any $I \in \mathcal{I}$ such that each element of $\mathcal{I} \cap M$ is mod finite contained in I and $C \in \mathcal{I}^\perp$ such that each element of $\mathcal{I}^\perp \cap M$ is mod finite contained in C.

Construct a fusion sequence $p_k \in M$, n_k by first ensuring that there is an m_k with $n_k < 2^{m_k}$ and $[2^{m_k}, 2^{m_k+1}) \setminus \mathrm{dom}(p_k)$ has more than k elements. Let $\bar{n}_k = 2^{m_k+1}$ and, without loss of generality, assume that $\mathrm{dom}(p_k) \supset [n_k, 2^{m_k})$. We then choose $p_{k+1} <_{\bar{n}_k} p_k$ to satisfy several conditions which we accomplish through a finite recursion of choosing a descending sequence of $<_{\bar{n}_k}$-extensions of p_k. We consider each partial function s with $\mathrm{dom}(s) \subset \bar{n}_k$ such that $s \sqcup p_k$ is in $\mathbb{P}(\mathfrak{F})$. For each such s and each condition p, we define the set $a(s,k,p)$ to be the set $\{i : (\exists q <_{\bar{n}_k} s \sqcup p)\ (\forall q' <_{\bar{n}_k} q)\ q' \not\Vdash \dot{h}(i)\}$. We may arrange that either $a(s,k,p_{k+1})$ is $\mathcal{I}^\perp$ or for all $<_{\bar{n}_k}$-extension q of p_{k+1}, $a(s,k,q)$ is not in $\mathcal{I}^\perp$. Secondly, if $a(s,k,p_{k+1})$ is in $\mathcal{I}^\perp$, then the finite set $a(s,k,p_{k+1}) \setminus C$ is contained in n_{k+1}. Next, if $a(s,k,p_{k+1})$ is not in $\mathcal{I}^\perp$, then ensure that there is some $i \in [\bar{n}_k, n_{k+1}) \cap I$ such that no $<_{\bar{n}_k}$-extension of $s \sqcup p_{k+1}$ forces a value on $\dot{h}(i)$. Finally, arrange that for each $i < n_{k+1}$ and s as above, if $s \sqcup p_{k+1}$ has a $<_{n_{k+1}}$-extension forcing a value on $\dot{h}(i)$, then $s \sqcup p_{k+1}$ already does so.

Let $\bar{p} = \bigcup_k p_k$ be the condition that results. Assume that $q < \bar{p}$ and q forces a value on $\dot{h} \restriction I$. By extending q we may assume that there is an infinite set $K \subset \mathbb{N}$ such that $[n_k, n_{k+1}) \subset \mathrm{dom}(q)$ for all $k \in K$. Of course this means that $\mathbb{N} \setminus K$ is also infinite. We show that condition (2) holds by letting p be q. If k is any value less than or equal to the minimum of $\mathbb{N} \setminus K$ we already have that condition (2) holds for all $i \in [n_k, n_{k+1})$.

Fix any k' and $i \in [n_{k'}, n_{k'+1})$. If q' is any extension of q which forces a value on $\dot{h}(i)$, we may assume that $[n_{k'}, n_{k'+1}) \subset \mathrm{dom}(q')$ and then with the condition $(q' \restriction [n_{k'}, n_{k'+1})) \sqcup q$ we would have that $k' \in K$. Therefore to prove that condition (2) holds it suffices to prove that it holds for all $k' \in K$ for which there is some $k \notin K$ below k'. Let $k < k'$ be the largest value not in K strictly below k'. Set $s = q \restriction n_{k+1}$ and consider $a(s,k,p_{k+1})$. Since q is a $<_{\bar{n}_k}$ extension of $s \sqcup p_{k+1}$ which forces a value on each member of $\dot{h} \restriction [\bar{n}_k, n_{k+1}) \cap I$, we have that $a(s,k,p_{k+1})$ must be in $\mathcal{I}^\perp$. But then we have that $a(s,k,p_{k+1}) \setminus C \subset n_{k+1}$, hence our chosen i is not in $a(s,k,p_{k+1})$. This then means that there is a $<_{\bar{n}_k}$-extension q' of q which forces a value on $\dot{h}(i)$, so fix such a q'. But now, $q \restriction n_{k'+1}$ will equal $q' \restriction n_{k'+1}$ because the entire interval $[\bar{n}_k, n_{k'+1}]$ is contained in $\mathrm{dom}(q)$. Thus $q' <_{\bar{n}_{k'}} s \sqcup p_{k'+1}$ and so $s \sqcup p_{k'+1}$ is also

forcing a value on $\dot{h}(i)$. Since $q < s \sqcup p_{k'+1}$, we have shown that q is forcing a value on $\dot{h}(i)$.

Thus, we have proven that for each p_0, there is such an p as required below it; by the genericity, there is such an p in $\mathfrak{F}$. □

smallgaps **Lemma 2.13.** *In the model obtained by forcing with $\mathbb{P}$ over a model of PFA, there are no (ω_1, ω_2)-gaps.*

Proof. Assume otherwise and let $\mathcal{I}, \mathcal{J}$ be such a gap. Since $\mathbb{P}$ is $\aleph_2$-distributive, we may assume that $\mathcal{I}$ is in the ground model. The extension by $\mathbb{P}$ preserves $MA(\omega_1)$, hence $\mathcal{I}^\perp$ is a P-ideal. Fix a sequence of $\mathbb{P}$-names $\{\dot{b}_\gamma : \gamma \in \omega_2\}$ forced to be increasing mod finite and cofinal in $\mathcal{J}$. Let H be $\kappa^{<\omega_1}$-generic and let $\mathfrak{F} \subset \mathbb{P}$ be V-generic as discussed above. We now work in the forcing extension $V[H]$ and, since ω_2^V is collapsed, we avoid confusion by letting λ denote this ordinal in the extension. Let $\mathcal{B}$ be the P-ideal generated by $\{\mathrm{val}_{\mathfrak{F}}(\dot{b}_\gamma) : \gamma < \lambda\}$. Since $\mathfrak{F}$ is V-generic and $\kappa^{<\omega_1}$-closed, it follows that $\mathcal{I}, \mathcal{B}$ are unseparated and both are P-ideals.

We next prove that they remain unseparated after forcing with $\mathbb{P}(\mathfrak{F})$. Assume that $\dot{h}$ is the $\mathbb{P}(\mathfrak{F})$-name of a function which is mod finite equal to 0 on each member of $\mathcal{I}$. Apply Lemma 2.12 to select the condition p, $\{n_k : k \in \omega\}$, $I \in \mathcal{I}$, and $C \in \mathcal{I}^\perp$ as described. Since p does have an extension forcing value on $\dot{h} \restriction I$, we have that Case (2) holds. For each k, let I_k be the set of $i \in [n_k, n_{k+1}) \setminus C$ such that some $q < p$ forces that $\dot{h}(i) = 0$. By the assumption on $\dot{h}$, $\bigcup_k I_k$ must mod finite contains every member of $\mathcal{I}$. Therefore, there is some $b \in \mathcal{B}$ and an infinite set K such that $b \cap I_k$ is not empty for each $k \in K$. We may also assume that K has infinite complement. For each $k \in K$, choose any $i_k \in b \cap I_k$ and select $q_k < p$ such that $q_k \Vdash \dot{h}(i_k) = 0$. Then $p \cup \bigcup_{k \in K} q_k \restriction [n_k, n_{k+1})$ is a condition which forces that $\dot{h} \restriction b$ takes on value 0 infinitely often.

Now in the extension by forcing with $\mathbb{P}(\mathfrak{F})$ we know that $\mathcal{I}, \mathcal{B}$ form a gap which is not countably separated (because at least one is a P-ideal). Thus we can let Q denote the proper poset supplied by Lemma 2.1, and as before, let $\mathcal{S}$ denote the σ-centered poset which supplies a $\mathbb{P}$ lower bound for $\mathfrak{F}$. Returning to V, fix names $\{x_\alpha : \alpha \in \omega_1\} = X \subset 2^{\mathbb{N}}$, functions $p_{\mathcal{I}}, p_{\mathcal{J}}$. Meet ω_1-dense sets to get that $\mathcal{I}, \mathcal{J}$ contain a Luzin gap. □

3. Regular closed sets with small boundaries

Proposition 3.1. *PFA implies that if a regular closed subset of $\mathbb{N}^*$ has non-empty boundary, then this boundary is not ccc over fin.*

Proof. Let A be a regular closed set with non-empty boundary. Suppose first that $\mathcal{I}_A$ and $\mathcal{I}_A^\perp$ are P-ideals. Apply PFA to the proper poset provided by Proposition 2.2 to deduce that the boundary is not ccc over fin. Now suppose, by symmetry, that $\mathcal{I}_A$ is not a P-ideal and fix an increasing chain $\{a_n : n \in \omega\} \subset \mathcal{I}_A$ to witness. If we now assume that ∂A is ccc over fin and use that $\{a_n : n \in \omega\}^\perp$ is a P-ideal, we may actually assume that ∂A is contained in the boundary of $\bigcup_n a_n^*$. Therefore, for each $b \in \{a_n : n \in \omega\}^\perp$, there is a partition $b_0 \cup b_1$ of b such that $b_0 \in \mathcal{I}_A$ and $b_1 \in \mathcal{I}_A^\perp$. Because of the assumption on ∂A, the resulting families $\{b_0 : b \in \{a_n : n \in \omega\}^\perp\}$ and $\{b_1 : b \in \{a_n : n \in \omega\}^\perp\}$ form P-ideals; in fact they apparently form an (ω_2, ω_2)-gap which of course is inconsistent with PFA. Alternatively, just apply the same argument as above to these unseparated P-ideals to deduce that ∂A is not ccc over fin. □

Theorem 3.2. *In a model obtained by forcing with $\mathbb{P}$ over a model of PFA, if A is a regular closed subset whose boundary is ccc over fin, then each of $\mathcal{I}_A$ and $\mathcal{I}_A^\perp$ are P_{ω_2}-ideals.* smallboundary

Proof. Let A be a regular closed set with non-empty boundary. Again suppose first that $\mathcal{I}_A$ and $\mathcal{I}_A^\perp$ are P-ideals. Proceed as in Lemma 2.13.

Otherwise, if $\mathcal{I}_A$ is not a P-ideal, again fix $\{a_n : n \in \omega\} \subset \mathcal{I}_A$ to witness. There is some $b \in \{a_n : n \in \omega\}^\perp$ so that $\partial A \setminus b^*$ is contained in the boundary of $\bigcup_{n\in\omega} a_n^*$; so for simplicity we may assume that $b = \emptyset$. Let $\mathcal{I} = \mathcal{I}_A \cap \{a_n : n \in \omega\}^\perp$ and $\mathcal{J} = \mathcal{I}_A^\perp \cap \{a_n : n \in \omega\}^\perp$. Let H be κ^{ω_1}-generic as above, hence $\mathcal{I}$ and $\mathcal{J}$ remain as unseparated P-ideals such that, in addition, $\mathcal{I} \oplus \mathcal{J}$ is unseparated from $\{a_n : n \in \omega\}$. Let G be $\mathbb{P}(\mathfrak{F})$-generic and pass to the extension $V[H][G]$. Again proceed as in Lemma 2.13 to conclude that ∂A fails to be ccc over fin. □

4. σ-Borel liftings and ccc over fin four

A lifting of a map Φ from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ to itself is any function F from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ which satisfies that $F(a)/\operatorname{fin} = \Phi(a/\operatorname{fin})$ for all $a \in \mathcal{P}(\mathbb{N})$. For each $\ell \in \mathbb{N}$ and $s \subset \ell$, let $[s;\ell] = \{x \subset \mathbb{N} : x \cap \ell = s\}$. This defines the standard Polish topology on $\mathcal{P}(\mathbb{N})$. For a subset $a \subset \mathbb{N}$, let $[s;\ell;a] = [s;\ell] \cap \mathcal{P}(a)$.

We will need the following important and well-known theorem of [14] as presented in [12]

Proposition 4.1. *(Velickovic) If $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a lifting of a mod finite automorphism and there exist Borel functions $\{\psi_n : n \in \omega\}$ and a comeagre set $Z \subset \mathcal{P}(\mathbb{N})$ such that for every $a \in Z$ there is $n \in \omega$ such that $\psi_n(a) =^* F(a)$ then F is trivial.* velctble

In fact, we will need a strengthening to homomorphisms; which is probably folklore, but we do not have a reference. We should note however that Farah [5, p93] shows that this will not hold for homorphisms in general.

homomctble

Lemma 4.2. *If $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a lifting of a homomorphism from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ onto $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ and there exist Borel functions $\{\psi_n : n \in \omega\}$ and a comeagre set $Z \subset \mathcal{P}(\mathbb{N})$ such that for every $a \in Z$ there is $n \in \omega$ such that $\psi_n(a) =^* F(a)$ then F is trivial.*

Proof. Following [14], we first note that we may assume that each ψ_n is continous and show that $\operatorname{triv}(F)$ can not (under these assumptions) be a maximal ideal. For each $a \in \operatorname{triv}(F)$, fix a function h_a from a into $\{0\} \cup F(a)$ so that $h_a^{-1}(0) \in \ker(F)$, h_a is 1-to-1 on $\operatorname{dom}(h_a) \setminus h_a^{-1}(0)$, and $h_a(x) =^* F(x)$ for all $x \subset a$. For each $n, m \in \omega$, let $D^a_{n,m} = \{x \subset a : h_a(x) \setminus m = \psi_n(x) \setminus m\}$. By the Baire category theorem, there is basic clopen set $[s; \ell; a]$ such that $D^a_{n,m} \supset [s; \ell; a]$. Let $\mathcal{T}_{n,m,s,\ell} = \{a : D^a_{n,m} \supset [s; \ell; a] \neq \emptyset\}$. If some $\mathcal{T} = \mathcal{T}_{n,m,s,\ell}$ is $\subset^*$-cofinal in $\operatorname{triv}(F)$, then define $h = \bigcup_{a \in \mathcal{T}} h_a \restriction a \setminus \ell$. Since each h_a is 1-to-1 on $\operatorname{dom}(h_a) \setminus h_a^{-1}(0)$, it follows that $h_a \restriction a \cap b = h_b \restriction a \cap b$ for $a, b \in \mathcal{T}$ (just calculate $h_a(s \cup \{i\})$). Since $\mathcal{T}$ is cofinal in $\operatorname{triv}(F)$ and F is onto $\mathcal{P}(\mathbb{N})/\operatorname{fin}$, it follows that $\{j : |h^{-1}(j)| \neq 1\}$ is finite; and that $h^{-1}(\{j : |h^{-1}(j)| > 1\})$ is in the kernel of F. Redefine h so that all values in this member of $\ker(F)$ and all values of the finite set $\mathbb{N} \setminus \operatorname{dom}(h)$ are sent to 0. We check that h is a lifting of F (which is a contradiction). For any $x \subset \mathbb{N}$, a finite change to x will ensure that $x \in [s; \ell]$. We may assume that $x \in \operatorname{triv}(F)$ and so choose some $a \in \mathcal{T}$ so that $x \subset^* a$. Thus we have $F(x) =^* F(x \cap a) =^* h_a(x \cap a) =^* h(x \cap a) =^* h(x)$. Now we also have that $F(\mathbb{N} \setminus x) =^* h(\mathbb{N} \setminus x) =^* \{0\} \cup \mathbb{N} \setminus h(x)$.

Now we may select $\{a_n : n \in \omega\}$ disjoint from $\operatorname{triv}(F)$ so as to be a partition of $\mathbb{N}$. If Z is any dense G_δ subset of $\mathcal{P}(\mathbb{N})$, then adding a Cohen real will not introduce a Borel lifting for $F \restriction Z \cap V$. This is a simple Baire category argument using the fact that the Cohen poset is countable; we leave the details to the reader.

We will use the poset $\mathbb{C} = \{[s; \ell] : \ell \in \mathbb{N}; s \subset \ell\}$ ordered by $\subset$ as a device in this proof. For a (generic) filter $\mathcal{H}$ on $\mathbb{C}$ let $g_{\mathcal{H}}$ be the set $\bigcup\{s : (\exists \ell)[s; \ell] \in \mathcal{H}\}$. The continous function on $\mathcal{P}(a_0)$ defined by $\psi_0(x \cup g_{\mathcal{H}} \setminus a_0) \cap F(a_0)$ is therefore not a lifting of $F \restriction V \cap \mathcal{P}(a_0)$, and so we may choose $x_0 \subset a_0$ and some condition $[s_0; \ell_0] \in \mathbb{C}$ which forces that $\psi_0(x_0 \cup g_{\mathcal{H}} \setminus a_0) \cap F(a_0)$ is not mod finite equal to $F(x_0)$. There is a countable family $\mathcal{D}_0$ of dense open subsets of $\mathbb{C}$ so that any filter $\mathcal{H} \ni p_0$ on $2^{<\omega}$ meeting each of these will ensure that $\psi_0(x_0 \cup g_{\mathcal{H}} \setminus a_0) \cap F(a_0)$ is not mod finite equal to $F(x_0)$. In fact, by the continuity

assumption on ψ_0, we may let $\{D_0(m) : m \in \omega\}$ enumerate $\mathcal{D}_0$ in such a way that for $[s_0;\ell_0] \supset [s;\ell] \in D_0(m)$, if $y \in [s;\ell]$ is any point, then $\psi_0(x_0 \cup y \setminus a_0)\Delta F(x_0)$ has cardinality at least m. Moreover, notice that if $y \in [s \setminus a_0;\ell]$, then we still have that $\psi_0(x_0 \cup y \setminus a_0)\Delta F(x_0)$ has cardinality at least m, since there will be some $y' \in [s;\ell]$ with $y' \setminus a_0 = y \setminus a_0$. For each m, let $U_0(m) = \bigcup\{[s;\ell;\mathbb{N}\setminus a_0] : [s;\ell] \in D_0(m)\}$ (i.e. a dense open subset of $\mathcal{P}(\mathbb{N} \setminus a_0)$). So if we let $Z_1 = \bigcap_m U_0(m)$ be the dense G_δ of $\mathcal{P}(\mathbb{N} \setminus a_0)$ then $\psi_0(x_0 \cup y \setminus a_0)\Delta F(x_0)$ is infinite for all $y \in Z_1$. Also it follows easily that the generic $g_{\mathcal{H}}$ has the property that $Z_1(g_{\mathcal{H}}) = \{x \subset a_1 : x \cup (g_{\mathcal{H}} \setminus a_1) \in Z_1\}$ contains a dense G_δ in $\mathcal{P}(a_1)$.

Choose a condition $[s_1;\ell_1] < [s_0;\ell_0]$ which is in the first member of $\mathcal{D}_0(1)$ and repeating the previous step, for which there is an $x_1 \subset a_1$ such that $[s_1;\ell_1]$ forces that $x_1 \in Z_1(g_{\mathcal{H}})$ and $\psi_1(x_0 \cup x_1 \cup g_{\mathcal{H}} \setminus (a_0 \cup a_1)) \cap F(a_1)$ is not mod finite equal to $F(x_1)$. Let $\mathcal{D}_1$ be the countably many dense sets in $\mathbb{C}$ needed to force these properties of x_1 and analogously define the family $\{U_1(m) : m \in \omega\}$ of dense open subsets of $\mathcal{P}(\mathbb{N} \setminus (a_0 \cup a_1))$ and their intersection Z_2 a G_δ dense in $\mathcal{P}(\mathbb{N} \setminus (a_0 \cup a_1))$. In particular, we will have the enumeration $\{D_1(m) : m \in \omega\}$ of $\mathcal{D}_1$ so that for each $y \in [s;\ell] \in D_1(m)$, $x_1 \cup (y \setminus (a_0 \cup a_1)) \in U_0(m)$.

After so defining $x_n \subset a_n$ for each $n \in \omega$, we check that, for each $k \in \omega$, $F(\bigcup_n x_n) \cap F(a_k) \neq^* F(x_k)$. This of course contradicts that F is a homomorphism. To see this, one shows, by induction on $m > k$, that $\bigcup_{k<n} x_n$ is a member of $U_k(m)$. It then follows that $y_k = \bigcup_{k<n} x_n$ is in Z_k, which in turn ensures that $\psi_k(x_0 \cup \cdots x_k \cup y_k) \cap F(a_k)$ is not almost equal to $F(x_k)$. □

We continue the analysis of $\mathbb{P}$-names from V in the forcing extension $V[H]$ using a V-generic filter $\mathcal{F} \subset \mathbb{P}$. In particular, fix $\dot{\Phi}$ a $\mathbb{P}$-name which is forced by 1 to be a lifting of a homomorphism from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ onto $\mathcal{P}(\mathbb{N})/\operatorname{fin}$. Let F denote $\operatorname{val}_{\mathcal{F}}(\dot{\Phi})$. Of course it follows that F is a lifting of a homomorphism from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ onto $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.

For a set $\mathcal{C} \subset \mathcal{P}(\mathbb{N})$ and a function F on $\mathcal{P}(\mathbb{N})$, let us say that $F \restriction \mathcal{C}$ is σ-Borel if there is sequence $\{\psi_n : n \in \omega\}$ of Borel functions on $\mathcal{P}(\mathbb{N})$ such that for each $b \in \mathcal{C}$, there is an n such that $F(b) =^* \psi_n(b)$.

One of the main results which we can extract from [12] is the following.

Lemma 4.3. *$F \restriction (V \cap \mathcal{P}(\mathbb{N}))$ is σ-Borel in the extension obtained by forcing with $\mathbb{P}(\mathfrak{F})$.* sigmaBorel

Since it is not explicitly stated in [12], we will, for completeness, just sketch the main ideas from [12, Theorem 3.1] (see also [10, Theorem

2.2]). As usual, we pass to the extension $V[H]$ and then let G be $\mathbb{P}(\mathfrak{F})$-generic. Recall that G has introduced a total function f_G which mod finite extends every member of $\mathfrak{F}$. We let $\mathcal{I}$ denote the ideal generated by those sets $b \in V \cap \mathcal{P}(\mathbb{N})$ for which $F \restriction V \cap \mathcal{P}(b)$ is σ-Borel. If this ideal is not a dense ideal, then we could assume that it is the ideal fin. Following Shelah's original proof that it is consistent that all automorphisms are trivial, we inductively select an almost disjoint family $\{a_\alpha : \alpha \in \omega_1\}$ together with sets $x_\alpha \subset a_\alpha$ so as to build a ccc poset $Q = Q(\{x_\alpha; a_\alpha : \alpha \in \omega_1\})$ which introduces a set $X \subset \mathbb{N}$ satisfying that $X \cap a_\alpha =^* x_\alpha$ for all α, while at the same time ensuring that in the extension by Q, for each $Y \subset \mathbb{N}$, there is a β such that $Y \cap F(a_\alpha) \neq^* F(x_\alpha)$ for all $\alpha > \beta$. If we succeed, then we observe that $\{F(a_\alpha) \setminus F(x_\alpha) : \alpha \in \omega_1\}$ and $\{F(x_\alpha) : \alpha \in \omega_1\}$ are not countably separated and so we select the proper poset $\mathcal{R}$ guaranteed to exist by Lemma 2.1. Finally, we let $\mathcal{S}$ denote the σ-centered poset which will force a suitable I so that $f_G \restriction I$ will be a lower bound in $\mathbb{P}$ for each $f \in \mathfrak{F}$. Meeting ω_1-many dense subsets of $(\kappa^{<\omega_1}) * \mathbb{P}(\mathfrak{F}) * Q * \mathcal{R} * \mathcal{S}$ will produce a condition $p \in \mathbb{P}$ and a set X which satisfies that if we set $X \cap a_\alpha = x_\alpha$ for all $\alpha \in \omega_1$, then p forces that $\{F(a_\alpha) \setminus F(x_\alpha) : \alpha \in \omega_1\}$ and $\{F(x_\alpha) : \alpha \in \omega_1\}$ form a Luzin gap, and so there is no suitable value for $F(X)$.

The conclusion then is that the construction of Q must at some stage fail. At each stage of this construction, $Q(\{x_\beta; a_\beta : \beta < \alpha\})$ is a countable poset and under a suitable enumeration we consider a countable $\dot{Y}_\alpha$ - a "prediction" of a Q-name of a subset of $\mathbb{N}$. Fix any a_α which is almost disjoint from each a_β. An important technical detail arises here in that our choice of $x_\alpha \subset a_\alpha$ must be made so as to ensure a specific countable family of dense subsets of $Q(\{x_\beta; a_\beta : \beta < \alpha\})$ remain pre-dense in the partial order $Q(\{x_\beta; a_\beta : \beta \leq \alpha\})$. We omit this detail but point out that this is where it is used that $\mathbb{P}(\mathfrak{F})$ is ω^ω-bounding and preserves that the ground model reals are not meager. This ensures that there is a sufficiently rich supply of choices for x_α for this and the next requirement. The next step is to connect the expectation that, for each $\beta < \alpha$, the valuation of $\dot{Y}_\beta$ intersected with $F(a_\alpha)$ should not mod finite equal $F(x_\alpha)$. Each $q \in Q(\{x_\beta : a_\beta : \beta < \alpha\})$ defines a Borel map sending $x \in \mathcal{P}(a_\alpha) \cap V$ to the valuation of $\dot{Y}_\beta \cap F(a_\alpha)$ that "would result" if q were in the generic, and x was taken to be x_α. Thus, if x_α can not be chosen to continue the induction, it is because a_α is in $\mathcal{I}$. Since this induction must indeed stop, we of course have that $\mathcal{I}$ is dense. Moreover, $\mathcal{I}$ must be ccc over fin for the same reason (otherwise we could simply choose in advance the family

$\{a_\alpha : \alpha \in \omega_1\}$ from $\mathcal{I}^+$). It is shown in [12] that it must be a P-ideal. The final step to show that $\mathcal{I}$ is actually $\mathcal{P}(\mathbb{N})$ is similar. The case here is that $\{\psi_n : n \in \omega\}$ is a countable family of Borel functions such that for all $x \subset a \in \mathcal{I}$, there is an n such that $\psi_n(x) \cap F(a) =^* F(x)$. Using that $\mathcal{I}$ is a P-ideal we can find $b \in \mathcal{I}$ mod fin containing all a_β $(\beta < \alpha)$ and a partition $E_0 \cup E_1$ of $\mathbb{N} \setminus b$ so that any subset a_α of either E_0 or E_1 would fulfill the preserving countably many dense sets requirement. However, we are now committed to choosing $a_\alpha \in \mathcal{I}$ as well. Assume that $E_0 \notin \mathcal{I}$. Each ψ_n may again be assumed to be continuous and so we may select an increasing sequence $\{k_n : n \in \omega\}$ of integers such that for each $\ell < n$ and each $s \subset k_n$ and $t \subset [k_n, k_{n+1})$, $\psi(x) \restriction k_n = \psi(y) \restriction k_n$ for all $x, y \in [s \cup t; k_{n+1}]$. Since $\mathbb{P}(\mathfrak{F})$ is ω^ω-bounding this sequence may be selected to be a member of V. Expand the family of continuous functions to include $\psi^s_{n,m}(x) \equiv \psi(s \cup x \setminus k_m)$ for all n and $s \subset k_m$ and all m. Choose any $x \subset E_0$ so that for all such n, m, s, $\psi^s_{n,m}(x) \cap F(E_0) \neq^* F(x)$. By finding a subsequence of the k_n's we can assume that they also satisfy that for each n and $s \subset k_n$, and $y \in [s \cup (x \cap [k_n, k_{n+1})); k_{n+1}]$, $\psi_n(y) \cap F(E_0) \Delta F(x)$ meets $[k_n, k_{n+1})$. Since $\mathcal{I}$ is ccc over fin, there should be a $J \subset \mathbb{N}$ so that $x_\alpha = x \cap \bigcup_{n\in J}[k_n, k_{n+1})$ and $a_\alpha = E_0 \cap \bigcup_{n\in J}[k_n, k_{n+1})$ works. We need to also enlarge a_α so that $F(a_\alpha) \supset \bigcup_{n\in J}[k_n, k_{n+1})$ as well (which may require shrinking J so as to maintain that $a_\alpha \in \mathcal{I}$). Thus we now have that $\psi_n(x \cap \bigcup_{n\in J}[k_n, k_{n+1})) \cap F(a_\alpha) \Delta F(x) \supset (F(x) \Delta \psi_n(x)) \cap [k_m, k_{m+1})$ which is not empty for all $n < m \in J$.

Next we need to use a key Lemma from [2].

Lemma 4.4. *There is an increasing sequence $\{n_k : k \in \omega\} \subset \omega$ such that* $\mathrm{triv}(F)$ *contains all $a \subset \mathbb{N}$ for which there is an $r \in \mathfrak{F}$, such that $a \subset \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \mathrm{dom}(r)\}$.* mainlemma

We are now ready to complete the proof that each homomorphism from $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ onto $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ is trivial on every member of an ideal which is ccc over fin. We proceed by contradiction. We may fix an almost disjoint family $\{a_\alpha : \alpha \in \omega_1\} \subset [\mathbb{N}]^\omega$ which are not in the trivial ideal. Using that $\mathbb{P}$ is $\aleph_2$-distributive we may assume that we have a number of properties forced to hold for the function F. In particular we have that $a_\alpha \notin \mathrm{triv}(F)$ for each $\alpha \in \omega_1$. Since $\mathfrak{b} > \omega_1$ in the final model, we can assume that we have two sequences $\{A_\alpha : \alpha \in \omega_1\}$ and $\{B_\alpha : \alpha \in \omega_1\}$ satisfying that A_α mod finite separates the family $\{a_\beta : \beta < \alpha\}$ and $\{a_\beta : \beta \geq \alpha\}$, and $B_\alpha = F(A_\alpha)$ does the same for the families $\{F(a_\beta) : \beta < \alpha\}$ and $\{F(a_\beta) : \beta \geq \alpha\}$. Fix a family, $\{\mathcal{W}_\alpha : \alpha \in \omega_1\}$, of ultrafilters on $\mathbb{N}$ so that $a_\alpha \in \mathcal{W}_\alpha$ and F is not trivial

on any member of $\mathcal{W}_\alpha$ (dual to $\mathrm{triv}(F \restriction \mathcal{P}(a_\alpha))$). We next show that we can arrange it so that $a_\alpha \cup F(a_\alpha)$ is almost disjoint from $a_\beta \cup F(a_\beta)$ for $\alpha \neq \beta$. We assume without mention in the argument below that we are always choosing $a_\alpha \in \mathcal{W}_\alpha$.

First assume that for uncountably many α (and therefore for all) there is some $\gamma_\alpha < \omega_1$ such that $A_{\gamma_\alpha} \in F(\mathcal{W}_\alpha)$. We may choose γ_α to be minimal. If there are uncountably many α such that γ_α is some fixed γ, then we can pass to this subcollection and shrink a_α (for $\alpha > \gamma$) so that such that $F(a_\alpha)$ is almost contained in A_γ. If there is an uncountable set I such that the sequence $\{\gamma_\alpha : \alpha \in I\}$ is unbounded, then we can assume that for each α in I, there is a $\delta_\alpha < \gamma_\alpha$ such that $\gamma_\beta < \delta_\alpha$ for all $\beta \in I \cap \alpha$. In this case, we pass to the uncountable set I and we assume that $F(a_\alpha) \subset^* A_{\gamma_\alpha} \setminus A_{\delta_\alpha}$.

The final case is that each $F(\mathcal{W}_\alpha)$ is not in A_γ for each $\gamma < \omega_1$. We may assume then that $F(a_\alpha)$ is disjoint from A_α. If, for uncountably many α there is again a $\gamma_\alpha < \omega_1$ such that $B_{\gamma_\alpha} \in \mathcal{W}_\alpha$, then we proceed just as above (e.g. choose $a_\alpha \subset B_{\gamma_\alpha} \setminus B_{\delta_\alpha}$). So, instead, we must have that for each α and γ, B_γ is not in $\mathcal{W}_\alpha$. Shrink each a_α so that a_α is disjoint from B_α. Now if $\beta < \alpha$, then $a_\alpha \cap F(a_\beta)$ is finite since they are separated by B_α. Also $a_\beta \cap F(a_\alpha)$ is finite because they are separated by A_α.

For convenience let $F(a_\alpha) \cup \{0\}$ be denoted as b_α. Fix any $p \in \mathfrak{F}$ which forces that the final homomorphism has all of the above properties of F. Let $\{n_k : k \in \omega\}$ be the sequence guaranteed by Lemma 4.4 and let $r \in \mathfrak{F}$ be the condition constructed in that proof. Notice that for each $q \in \mathfrak{F}$, we have a function h_q with domain $a_q = \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \mathrm{dom}(q)\}$ which witnesses that $a_q \in \mathrm{triv}(F)$ (with h_q being one-to-one on $a_q \setminus h_q^{-1}(0)$). Therefore the family $\{h_q : q \in \mathfrak{F}\}$ is a σ-directed (mod finite) family of functions which has no extension. On the other hand, once we force with $\mathbb{P}(\mathfrak{F})$ (followed by any proper poset of our choice) there must be a further proper extension in which it does have an extension. By Proposition 2.3, we have that if $\mathbb{P}(\mathfrak{F})$ itself does not introduce a common extension, then there is a proper poset which will make the family indestructibly non-extendable. Thus we assume that $\dot{h}$ is a $\mathbb{P}(\mathfrak{F})$-name of a function on $\mathbb{N}$ which extends each such h_q.

Lemma 4.5. *The family* $\{\mathrm{dom}(h_q) : q \in \mathfrak{F}\}$ *generates a dense ideal in* $V[H]$.

Proof. The finite-to-one map sending $[n_k, n_{k+1})$ to k is easily seen to send the family $\{a_q : q \in \mathfrak{F}\}$ to a maximal ideal, and it is also the preimage of this maximal ideal. The forcing $\mathbb{P}(\mathfrak{F})$ is ω^ω-bounding and so does not diagonalize the dual ultrafilter. □

Since $\dot{h}$ is forced to mod finite extend each h_q, it follows easily from Lemma 2.12 that we may assume that it is also forced to have the property that $\dot{h}^{-1}(i)$ is finite for each $i \in \mathbb{N}$. Since $\mathbb{P}(\mathfrak{F})$ is ω^ω-bounding, we may therefore assume that the chosen sequence of n_k's has the property listed next. For later use, we refer to this as $\dot{h}$ being *locally decided*.

preserve2 **Lemma 4.6.** *The increasing sequence $n_0 < n_1 < \cdots$ of integers may also be assumed to satisfy that for each $k \in \mathbb{N}$ and for each $i \in [n_k, n_{k+1})$ and each $q < p$ such that q forces a value on $\dot{h}(i)$, $p \cup (q \restriction [n_k, n_{k+1}))$ also forces a value on $\dot{h}(i)$ and that value is in $\{0\} \cup [n_{k-1}, n_{k+2})$.*

Proof. Perform a standard fusion (see [10, 2.4] or [12, 3.4]) p_k, n_k by picking $L_k \subset [n_{k+1}, n_{k+2})$ (absorbed into $\mathrm{dom}(p_{k+1})$) so that for each partial function s on n_k which extends $p_k \restriction n_k$, if there is some integer $i \geq n_{k+1}$ for which no n_k-preserving extension of $s \sqcup p_k$ forces a value on $\dot{h}(i)$, then there is such an integer in L_k. Let $\bar{p}$ be the fusion and note that either $\bar{p}$ forces that $\dot{h} \restriction \mathrm{dom}(h_{\bar{p}})$ is not in V, or it forces that our sequence of n_k's does the job. □

Furthermore, we can suppose that there is a $q_0 \in \mathfrak{F}$ such that for each k, there is a single m_k such that $[2^{m_k}, 2^{m_k+1})$ properly contains $[n_k, n_{k+1}) \setminus \mathrm{dom}(q_0)$. By further grouping and by extending the condition q_0 we can assume that for all k and $j \in [n_k, n_{k+1})$, if q_0 does not force a value on $\dot{h}(j)$, then q_0 does force that $\dot{h}(j) \in \{0\} \cup [n_k, n_{k+1})$. For each k, let $H_k = [n_k, n_{k+1}) \setminus \mathrm{dom}(q_0) = [2^{m_k}, 2^{m_k+1}) \setminus \mathrm{dom}(q_0)$. Finally, let $\mathcal{H}_k = \mathcal{H}_k^{\mathbb{P}}$ denote the set of functions s with domain contained in H_k for which there is a $q \leq q_0$ with $s = q \restriction H_k$. Recall that for the posets $\mathbb{P}_0$ and $\mathbb{P}_1$ the conditions are functions into 2, while for the poset $\mathbb{P}_2$, the conditions q extending q_0 are permutations which send each H_k into itself. Therefore, with $\mathbb{P}$ being any of the three posets considered in this paper, $\mathcal{H}_k$ is a finite set of functions with domain and range contained in $2 \cup H_k$. For the remainder of the section we may assume that each condition we choose in $\mathbb{P}(\mathfrak{F})$ is below this q_0. For a $\dot{h}$ satisfying this lemma, we will say that it has the *selection property*.

grabY **Lemma 4.7.** *If $Y = \{y_k : k \in \omega\}$ is such that $y_k \in [n_k, n_{k+1})$ for each k, then for each $q \in \mathfrak{F}$, there is a $p < q$ such that p decides $\dot{h} \restriction Y$.*

Proof. Let K be the set of k such that q does not already force a value on $\dot{h}(y_k)$ and let $Y' = \{y_k : k \in K\}$. Now choose $q' < q$ in $\mathfrak{F}$ so that q' forces a value on $F(Y')$. For each $k \in K$, choose $j_k \in F(Y') \cap [n_k, n_{k+1})$ if it is non-empty, otherwise set $j_k = 0$. Assume that the set K', those $k \in K$ such that q' does not force $\dot{h}(y_k) = j_k$, is infinite. It then follows

from Lemma 4.6 that there is a condition $p < q'$ for which there are infinitely many $k \in K'$ such that $y_k \in \mathrm{dom}(h_p)$ and $p \Vdash \dot{h}(y_k) \neq j_k$. But now we have that p forces that $F(Y' \cap \mathrm{dom}(h_p)) =^* h_p(Y' \cap \mathrm{dom}(h_p))$ is not almost equal to $\dot{h}(Y' \cap \mathrm{dom}(h_p))$, since for each $k \in K'$ with $y_k \in \mathrm{dom}(h_p)$, $j_k \in F(Y') \cap F(\mathrm{dom}(h_p)) \setminus \dot{h}(Y' \cap \mathrm{dom}(h_p))$. □

Definition 4.8. For each condition $q \in \mathbb{P}(\mathfrak{F})$, and each $i \in \mathbb{N}$, let $\mathrm{Orb}_q(i) = \{j : (\exists p < q)\ p \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}(i) = j$"$\}$. Also let $S(k,q) = \{s \in \mathcal{H}_k : q \restriction H_k \subset s\}$.

bounding **Corollary 4.9.** *Our condition q_0 also satisfies that for each $i \in \mathbb{N}$ and $q < q_0$, if $\mathrm{Orb}_q(i)$ has more than one element, there is a k such that $\{i\} \cup \mathrm{Orb}_q(i) \subset \{0\} \cup [n_k, n_{k+1})$.*

Lemma 4.10. *For each $\alpha \in \omega_1$ and $q \in \mathbb{P}(\mathfrak{F})$, there are $r_\alpha < q$ in $\mathbb{P}(\mathfrak{F})$ and $W_\alpha \subset a_\alpha$ such that $r_\alpha \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$W_\alpha \in \mathcal{W}_\alpha$ and $\dot{h}[W_\alpha] \subset b_\alpha$"*

Proof. Otherwise we can choose a fusion sequence $\{r_k : k \in \omega\}$, an increasing sequence of integers ℓ_k and values $y_k \in a_\alpha \cap [n_{\ell_k}, n_{\ell_k+1})$, and conditions $s_k \in \mathcal{H}_k$ such that $s_k \sqcup r_k \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}(y_k) \notin b_\alpha$". There is an infinite set $L \subset \omega$ and an r such that $Y = \{y_k : k \in L\} \subset \mathrm{dom}(h_r)$ and r is below $s_k \sqcup r_k$ for all $k \in L$. Since r forces that $\dot{h}$ extends h_r, we have our contradiction since $h_r[Y] \subset^* \dot{F}(a_\alpha) = b_\alpha$ while $r \Vdash$ "$\dot{h}[Y] \cap b_\alpha$" is empty. □

aalpha **Lemma 4.11.** *For each integer L and each condition q, there is a condition $p < q$ and a set $I \in [\omega_1]^L$ such that $p < r_\alpha$ for each $\alpha \in I$.*

Proof. This is simply because $\mathbb{P}(\mathfrak{F})$ is ccc. □

balpha **Lemma 4.12.** *For each $\alpha \in \omega_1$, there is an integer ℓ_α such that for each k and each $s_k \in S(k, r_\alpha)$, if $|H_k \setminus \mathrm{dom}(s_k)| > \ell_\alpha$, then $s_k \sqcup r_\alpha$ does not decide $\dot{h} \restriction W_\alpha \cap [n_k, n_{k+1})$.*

Proof. If such an integer ℓ_α did not exist, then we could find an infinite $K \subset \omega$ and a sequence $\langle s_k : k \in \omega\rangle \in \Pi_{k\in\omega} S(k,r)$ with $\{|H_k \setminus \mathrm{dom}(s_k)| : k \in K\}$ diverging to infinity, and such that $s_k \sqcup r$ decides $\dot{h} \restriction W_\alpha \cap [n_k, n_{k+1})$ for each k. But then of course, $q = \bigcup_{k\in K} s_k \sqcup r_\alpha$ would force that $\dot{h} \restriction W_\alpha = h_\alpha$ for some $h_\alpha \in V$. It follows easily that there is some $q' < q$ and some infinite $W \subset W_\alpha$ such that $q' \Vdash_{\mathbb{P}}$ "$\dot{F}(W) \cap h_\alpha[W] = \emptyset$". By further extending q' we can assume that $W \subset \mathrm{dom}(h_{q'})$. This contradicts that $h_{q'}[W]$ is supposed to be forced by q' to be (mod finite) equal to both $\dot{h}[W]$ and $\dot{F}(W)$. □

By passing to an uncountable subcollection we may suppose that there is some $\bar{\ell}$ such that $\ell_\alpha = \bar{\ell}$ for all α. Now define $S'(k,q) = \{s \in S(k,q) : |H_k \setminus \mathrm{dom}(s)| > \bar{\ell}\}$.

Lemma 4.13. *There is a condition r and an infinite set K such that $\{|H_k \setminus \mathrm{dom}(r)| : k \in K\}$ diverges to infinity, and, for each $k \in K$, we can select $\{i_s : s \in S'(k,r)\} \subset [n_k, n_{k+1})$ such that for distinct s, s' in $S'(k,r)$, $s \sqcup r$ does not decide $\dot{h}(i_s)$, and $\mathrm{Orb}_r(i_s) \cap \mathrm{Orb}_r(i_{s'})$ is empty.* orbits

Proof. Fix any integer ℓ and let L be bigger than $(\ell+2)^{\ell+2}$. It is sufficient to find a single $k = k_\ell$ and a condition $r < q_0$ so that $|H_k \setminus \mathrm{dom}(r)| \geq \ell$ with the properties as required, since by the locally decided property of $\dot{h}$ we can then ensure that r has this property for all ℓ.

Apply Lemma 4.11 to find an r which is below r_α for each $\alpha \in I$ for some $I \subset \omega_1$ of cardinality at least L. For each $\alpha \in I$, we can assume that $\mathrm{dom}(h_r)$ contains $[n_k, n_{k+1})$ for each k such that r decides $\dot{h} \restriction W_\alpha \cap [n_k, n_{k+1})$. Since $a_\alpha \cup F(a_\alpha)$ is almost disjoint from $a_\beta \cup F(a_\beta)$ for $\alpha \neq \beta$, there is an m such that $[W_\alpha \cup \dot{F}(W_\alpha)] \cap [W_\beta \cup \dot{F}(W_\beta)] \subset m$ for each $\alpha \neq \beta \in I$. Let $K = \{k > m : |H_k \setminus \mathrm{dom}(r)| > \ell\}$. It follows from Lemma 4.12, that for each $\alpha \in I$, $k \in K$, and $s \in S'(k,r)$, there is an $i \in W_\alpha \cap [n_k, n_{k+1})$ for which $s \sqcup r$ does not decide $\dot{h}(i)$. Therefore, we can select any $k \in K$ and $s_k \in S(k,r)$ with $\ell \leq |H_k \setminus \mathrm{dom}(s_k)| < \ell + 2$ and fix any injection from $S'(k, s_k \cup r)$ into I (i.e. $\{\alpha_s : s \in S'(k, s_k \cup r)\}$). For each $s \in S'(k, s_k \sqcup r)$, there is an $i_s \in W_{\alpha_s} \cap [n_k, n_{k+1})$ such that $s \sqcup r$ does not decide $\dot{h}(i_s)$. Since r forces that $\{i_s, \dot{h}(i_s)\} \subset a_{\alpha_s} \cup b_{\alpha_s}$ and for $s' \neq s$, r forces that $i_{s'}, \dot{h}(i_{s'}) \notin a_{\alpha_s} \cup b_{\alpha_s}$, we have satisfied the requirement that $i_s \notin \mathrm{Orb}_r(i_{s'})$ (the hard part was making them distinct). □

Theorem 4.14. *The trivial ideal,* $\mathrm{triv}(F)$*, is ccc over fin.* cccproof

Proof. Let r and the sequence $X(r) = \{\{i_s : s \in S'(k,r)\} : k \in K\}$ be as constructed in Lemma 4.13. Since $\{|H_k \setminus \mathrm{dom}(r)| : k \in K\}$ diverges to infinity, we may assume that $\mathrm{dom}(r) \supset [n_k, n_{k+1})$ for each $k \notin K$. Choose $p < r$ so that p forces a value Y on $F(X(r))$. We reach a contradiction. Choose any infinite $K' \subset K$ such that each of $\{|H_k \setminus \mathrm{dom}(p)| : k \in K'\}$ and $\{|H_k \setminus \mathrm{dom}(p)| : k \in K \setminus K'\}$ are unbounded. For each $k \in K'$, let $s_k = p \restriction H_k$. By Lemma 4.13 and Lemma 4.6, $s_k \sqcup p$ does not decide $\dot{h}(i_{s_k})$. Choose s_k^0, s_k^1 extending s_k such that, for some $z_k^0 < z_k^1$, $s_k^0 \sqcup p \Vdash \dot{h}(i_{s_k}) = z_k^0$ and $s_k^1 \sqcup p \Vdash \dot{h}(i_{s_k}) = z_k^1$. Observe that $\bigcup_{k \in K'} s_k^1 \sqcup p$ forces that $\{z_k^1 : k \in K'\}$ is equal to $\dot{h}[\{i_{s_k} : k \in K'\}] \cap \bigcup_{k \in K'} [n_k, n_{k+1})$, and so, is almost contained in Y.

Let q be any condition extending $\bigcup_{k \in K'} s_k^0 \sqcup p$ such that $H_k \subset \mathrm{dom}(q)$ for all $k \in K'$. It follows that q forces that $Y \cap \bigcup_{k \in K'} [n_k, n_{k+1})$ is almost equal to $h_q[\{i_s : (\exists k \in K')\ s \in S'(k,r)\}]$. However, since $h_q(i_{s_k}) = z_k^0$ for each $k \in K'$ and, by the disjoint orbits assumption, we have that $\{z_k^1 : k \in K'\}$ must be almost disjoint from Y – a contradiction. □

five

5. Near liftings

In this section we examine some more combinatorics on $\mathbb{P}(\mathfrak{F})$ names of functions on $\mathbb{N}$. Assume that $\dot{f}$ is a $\mathbb{P}(\mathfrak{F})$-name of a function on $\mathbb{N}$ for which there is a condition q_0 forcing that $\dot{f}$ is locally decided and satisfies the selection property (i.e. $\dot{f}$ satisfies the conclusions of Lemma 4.6 and Lemma 4.7) with the same notation used above. Say that a condition q is standard, if for each $\ell > 0$, there are at most finitely many k such that $H_k \setminus \mathrm{dom}(q)$ has cardinality ℓ. The standard conditions are dense below q_0 in $\mathbb{P}$. For a standard condition q, let $K(q)$ denote those k such that $H_k \setminus \mathrm{dom}(q)$ is not empty. It follows then that $\{|H_k \setminus \mathrm{dom}(q)| : k \in K(q)\}$ diverges to infinity.

p0trivial

Proposition 5.1. *If $\mathbb{P}$ is $\mathbb{P}_0$ and p_0 is a condition such that no extension of p_0 decides $\dot{f}(t)$ for all values of t, then there is an extension $p < p_0$ such that for all $i \in \mathbb{N} \setminus \mathrm{dom}(p)$, there is a value t_i so that for some distinct pair u_i, v_i, $p \cup \{(i,0)\} \Vdash \dot{f}(t_i) = u_i$ and $p \cup \{(i,1)\} \Vdash \dot{f}(t_i) = v_i$.*

Proof. We proceed by a simple recursion. By induction on ℓ, suppose we have chosen p_ℓ together with a family $\{i(k,j) : j < \ell\} \subset H_k \setminus \mathrm{dom}(p_\ell)$ for all $k \in K(p_\ell)$. We assume that for each $j < \ell$ and $k \in K(p_\ell)$, there is a value $t_{k,j}$ so that $p_{j+1} \cup \{(i(k,j'),0) : j' \leq j\}$ and $p_{j+1} \cup \{(i(k,j'),0) : j' < j\} \cup \{(i(k,j),1)\}$ force distinct values, $u_{k,j}, v_{k,j}$, on $\dot{f}(t_{k,j})$. As usual in such a fusion, we assume that $p_{j+1} \restriction n_{m_j} \subset p_j$ so that we will have that $\bigcup_\ell p_\ell$ is a condition. Now we may choose a sequence $\langle t_{k,\ell} : k \in K\rangle$ (for some infinite $K \subset K(p_\ell)$) such that, for each $k \in K$, $t_{k,\ell} \in [n_k, n_{k+1})$ and $p_\ell \cup \{(i(k,j),0) : j < \ell\}$ does not force a value on $\dot{f}(t_{k,\ell})$. For each $k \in K$, there are two values $\bar{\imath}_0^k, \bar{\imath}_1^k$ from $H_k \setminus (\mathrm{dom}(p_\ell) \cup \{i(k,j) : j < \ell\})$, such that $p_\ell \cup \{(\bar{\imath}_0^k, 1)\}$ and $p_\ell \cup \{(\bar{\imath}_1^k, 1)\}$ force distinct values, v_0^k, v_1^k, on $\dot{f}(t_{k,\ell})$. Choose $\bar{p}_{\ell+1} < p_\ell$ such that for all $k \in K(\bar{p}_{\ell+1}) \subset K$, $\{i(k,j) : j < \ell\}$ is disjoint from $\mathrm{dom}(\bar{p}_{\ell+1})$ and $\bar{p}_{\ell+1} \cup \{(i(k,j),0) : j < \ell\}$ forces a value, $u_{k,\ell}$, on $\dot{f}(t_{k,\ell})$. Suppose, without loss of generality, that $v_1^k \neq u_{k,\ell}$ and let $i_{k,\ell} = \bar{\imath}_1^k$. It follows that $i_{k,\ell} \in \mathrm{dom}(\bar{p}_{\ell+1})$ and so define $p_{\ell+1}$ to be the condition we get by removing $i_{k,\ell}$ from the domain of $\bar{p}_{\ell+1}$ for all $k \in K = K(p_{\ell+1})$.

When the recursion is finished, we choose any increasing sequence $\{k_\ell : \ell \in \omega\}$ so that $k_\ell \in K(p_{\ell+1})$, and $p < p_0$ any condition so that $K(p) = \{k_\ell : \ell \in \omega\}$, and $S_{k_\ell} \setminus \mathrm{dom}(p) = \{i(k_\ell, j) : j < \ell\}$. Of course this implies that p is constantly 0 on each $S_{k_\ell} \cap \mathrm{dom}(p)$. For each $k = k_\ell$ and $j < \ell$, we have that $p \cup \{(i(k,j),1)\}$ forces the value $v_{k,j}$ on $\dot{f}(t_{k,j})$ because of Lemma 4.6. And similarly, since $p_{\ell+1} \restriction H_k \subset p$, we have

that $p \cup \{(i(k,j'),0) : j' \leq j\}$ forces that $\dot{f}(t_{k,j}) = u_{k,j}$. Because of this, we have that if $q < p$ is such that $k \in K(q)$ and q forces a value on $\dot{f}(t_{k,j})$, then this value has to be $u_{k,j}$. We finish the construction by another more routine recursion. For each $k \in K(p)$, let $j_{k,0}$ denote the largest value so that $i(k, j_{k,0}) \notin \mathrm{dom}(p)$. By Lemma 4.7, there is a condition $p_1 < p$ so that p_1 forces a value on $\dot{f} \restriction \{t_{k,j_{k,0}} : k \in K(p)\}$. Again, as discussed above, we have that p_1 forces that $\dot{f}(t_{k,j_{k,0}}) = u_{k,j_{k,0}}$ for each $k \in K(p_1)$. There is an infinite set $K_1 \subset K(p_1)$ such that there is a largest $j_{k,1} < j_{k,0}$ such that $i(k, j_{k,1}) \notin \mathrm{dom}(p_1)$. Find a condition $p_2 < p_1$ which forces a value on $\dot{f} \restriction \{t_{k,j_{k,1}} : k \in K_1$. Continue this induction. Again there is a sequence $\{k_\ell : \ell \in \omega\}$ such that $j_{k,\ell}$ was successfully chosen for $k = k_{\ell+1}$. We extend p to a condition p' so that $K(p') = \{k_\ell : \ell \in \omega\}$ and $S_{k_\ell} \setminus \mathrm{dom}(p')$ is equal to $\{i(k_\ell, j_{k_\ell,m}) : m < \ell\}$. We still have that $p' \cup \{(i,1)\} \Vdash \dot{f}(t_{k,i}) = v_{k,i}$ for each $i \in H_k \setminus \mathrm{dom}(p')$, but we can now show that $p' \cup \{(i,0)\}$ forces that $\dot{f}(t_{k,i}) = u_{k,i}$. The simplest way to do this is to consider any $i' \in H_k \setminus \mathrm{dom}(p')$ with $i' \neq i$. If $i' < i$, then the condition $p' \cup \{(i',1)\}$ is compatible with $p_{m+1} \restriction H_k$ where m is chosen so that $i = i(k, j_{k,m})$. On the other hand if $i' > i$, then $p' \cup \{(i',1)\}$ is compatible with $p \cup \{(i(k,j),0) : j \leq j_{k,m}\}$. Since each of these force that $\dot{f}(t_{k,i}) = u_{k,i}$, we have that $p' \cup \{(i,0)\}$ forces that $\dot{f}(t_{k,i}) = u_{k,i}$. □

For the rest of this section we work with the case that $\mathbb{P} = \mathbb{P}_1$.

Definition 5.2. Say that $\bar{p}$ forces that $\bar{f}$ is a **near lifting** of $\dot{f}$ if there is some $\bar{m}$ such that for all $k \in K(\bar{p})$ and $t \in [n_k, n_{k+1})$, if $q < \bar{p}$ is such that $q \Vdash \dot{f}(t) \neq \bar{f}(t)$, then $|H_k \setminus \mathrm{dom}(q)|$ has cardinality less than $\bar{m}$. nearlifting

Proposition 5.3. *If $\mathbb{P}$ is $\mathbb{P}_1$, $\dot{h}$ is a lifting of an autohomeomorphism φ on $\mathbb{N}^*$ and if $\bar{p}$ forces that $\dot{h}$ has a near lifting $\bar{f}$, then $\bar{p}$ forces (over $\mathbb{P}_1$) that $\bar{h}$ witnesses that φ is nearly trivial as stated in Theorem 2.9. In addition, some member of the special ultrafilter $\mathcal{U}$ is in* $\mathrm{triv}(\phi)$. really29

Proof. Let G be a generic filter for $\mathbb{P}_1$ and suppose that φ is an autohomeomorphism on $\mathbb{N}^*$. Let F be a lifting for φ, and let $\dot{h}$ be the $\mathbb{P}_1(\mathfrak{F})$-name of the lifting for F as per the results in section 4. Now suppose that $\bar{p}, \bar{m}, \bar{h}$ be as in the definition 5.2 with $\bar{p} \in G$. We simply define the ideal $\mathcal{I}_A$ to be the family $\{\{t : h_p(t) = \bar{h}(t)\} : p \in G\}$. Then A is the closure of the open set $\bigcup\{I^* : I \in \mathcal{I}_A\}$. Certainly A is regular closed, and, since each $p \in G$ forces that h_p is a lifting of F on $\mathrm{dom}(h_p)$, it follows that $F \restriction A$ is equal to $(\bar{h})^* \restriction A$ where $(\bar{h})^*$ is the trivial automorphism induced by $\bar{h}$. If $Y \subset \mathbb{N}$ is forced by some $p \in G$ to not be in $\mathrm{triv}(F)$, then it follows immediately from Lemma 4.7 and

the definition of $\bar{h}$ being a near lifting, that Y has an infinite subset in $\mathcal{I}_A$.

Now we produce a condition $p \in G$ which forces that $\dot{h}$ is trivial on the set $\mathbb{N} \setminus \mathrm{dom}(p)$. For a condition p and $k \in K(p)$, let $\{i(p,k,j) : j < \ell_{p,k}\}$ denote the increasing enumeration of $H_k \setminus \mathrm{dom}(p)$. For each ℓ and $\rho \in 2^\ell$, let p^ρ denote the condition extending p defined by assigning $p^\rho(i(p,k,j)) = \rho(j)$ for each $k \in K(p)$ and $j < \min\{\ell, \ell_k\}$. Let $p_0 = \bar{p}$ and apply Lemma 4.7, to select a condition $p_1 < p_0$ so that for all $k \in K(p_0)$, $i(p_1,k,0) = i(p_0,k,0)$ and for each $\rho \in 2^1$ (i.e. the two functions $\{(0,0)\}$ and $\{(0,1)\}$) p_1^ρ forces a value on $\dot{h}(i(p_0,k,0))$ for all $k \in K(p_0)$. It follows that for all but finitely many $k \in K(p_1)$, p_1 forces that $\dot{h}(i(p_1,k,0)) = \bar{h}(i(p_1,k,0)$. By extending p_1, we may assume this is the case for all $k \in K(p_1)$. Choose an integer $\bar{n}_1$ large enough so that for all k in $K(p_1) \cap \bar{n}_1$ is not empty, and that $\ell_{p_1,k}$ is larger than 1 for all $\bar{n}_1 < k \in K(p_1)$ is non-empty and by extending, assume that $\ell_{p_1,k} = 1$ for exactly one $k \in K(p_1)$ with $n_{k+1} < \bar{n}_1$. Choose an extension p_2, which agrees with p_1 up $\bar{n}_1$ and satisfies that for all $k \in K(p_1) \setminus \bar{n}_1$, $i(p_2,k,1) = i(p_1,k,1)$ and for all $\rho \in 2^2$, p_2^ρ forces a value on $\dot{h}(i(p_1,k,1))$ for all $k \in K(p_1)$. Again there is a value $\bar{n}_2$ large enough so that for all $k \in K(p_2)$ above $\bar{n}_2$, $\ell_{p_2,k} > 2$ and p_2 forces that $\dot{h}(i(p_2,k,1)) = \bar{h}(i(p_2,k,1))$. Additionally we may assume that there is just one element k in $K(p_2) \cap \bar{n}_2 \setminus \bar{n}_1$ and that $\ell_{p_2,k} = 2$ and $n_{k+1} < \bar{n}_2$. The condition p that is constructed by this infinite recursion will force that $\dot{h}(i(p,k,\ell)) = \bar{h}(i(p,k,\ell))$ for all $k \in K(p)$ and $\ell < \ell_{p,k}$. □

The next lemma might be thought of as producing a sequence of values at which the condition will *flip* the value of $\dot{f}$ while preserving the growth requirement on $|H_k \setminus \mathrm{dom}(p)|$. It is almost exactly a reformulation of $\dot{f}$ not having a near lifting so we omit the proof.

nonear **Lemma 5.4.** *If p forces that $\dot{f}$ does not have a near lifting, and if $\mathbb{P}$ is $\mathbb{P}_0$ or $\mathbb{P}_1$, then there is a condition $\bar{p} < p$ and a sequence $\{i_k, t_k : k \in K(\bar{p})\}$ such that, for each $k \in K(\bar{p})$, $i_k \in H_k \setminus \mathrm{dom}(\bar{p})$, $\bar{p} \cup \{(i_k,0)\}$ and $\bar{p} \cup \{(i_k,1)\}$ force distinct values on $\dot{f}(t_k)$. We may further assume that $H_k \cap i_k \subset \mathrm{dom}(\bar{p})$ for all $k \in K(\bar{p})$.*

Then this leads to the first major step towards the main result of this section.

allflipping **Lemma 5.5.** *If p forces that $\dot{f}$ does not have a near lifting, then there is a condition $\bar{p} < p$ and a sequence $\langle T(i) : i \notin \mathrm{dom}(\bar{p})\rangle$ of pairwise disjoint finite sets such that for $k \in K(\mathrm{dom}(\bar{p}))$ and each $i \in H_k \setminus$*

$\mathrm{dom}(\bar{p})$ *and each function* ρ *from* $H_k \cap i$ *into* 2 *such that* $\rho \cup \bar{p}$ *is a condition,* $\rho \cup \bar{p} \cup \{(i,0)\}$ *and* $\rho \cup \bar{p} \cup \{(i,1)\}$ *force distinct values on* $\dot{f} \restriction T(i)$.

Proof. We proceed by a sequence of Claims. We continue with the notation from Proposition 5.3.

Claim 5.5.1. For each p, ℓ and $\psi \in 2^\ell$, there is a condition $\bar{p} < p$ satisfying c1

(1) for each $k \in K(\bar{p})$, $\{i(p,k,j) : j < \ell\} = \{i(\bar{p},k,j) : j < \ell\}$,
(2) for each $k \in K(\bar{p})$ and $j \in [\ell, \ell_{\bar{p},k})$ and for each $\psi \subset \rho \in 2^j$ such that $\rho \restriction [\ell, j)$ is constantly 0, there is a t such that $\bar{p}^\rho \cup \{(i(\bar{p},k,j),0)\}$ and $\bar{p}^\rho \cup \{(i(\bar{p},k,j),1)\}$ force distinct values on $\dot{f}(t)$, c1.2
(3) for each $k \in K(\bar{p})$ and $j \in [\ell, \ell_{\bar{p},k})$ and some t specified as in condition 2, we also have that for all $\rho \in 2^{j+1}$, $\bar{p}^\rho$ forces a value on $\dot{f}(t)$.

The proof of Claim 5.5.1 is by a simple fusion with repeated applications of the assumption that no condition forces that $\dot{f}$ is nearly trivial and Lemma 5.4. The third condition is made to hold at each step by repeated applications of Lemma 4.7.

Next, by a similar recursion we can consider each $\psi \in 2^\ell$ in turn so as to prove the next claim.

Claim 5.5.2. For each p, ℓ there is a condition $\bar{p} < p$ satisfying c2

(1) for each $k \in K(\bar{p})$, $\{i(p,k,j) : j < \ell\} = \{i(\bar{p},k,j) : j < \ell\}$,
(2) for each $k \in K(\bar{p})$ and $j \in [\ell, \ell_{\bar{p},k})$ and for each $\rho \in 2^j$ such that $\rho \restriction [\ell, j)$ is constantly 0, there is a t such that $\bar{p}^\rho \cup \{(i(\bar{p},k,j),0)\}$ and $\bar{p}^\rho \cup \{(i(\bar{p},k,j),1)\}$ force different values on $\dot{f}(t)$.
(3) for each $k \in K(\bar{p})$ and $j \in [\ell, \ell_{\bar{p},k})$ and some t specified as in condition 2, we also have that for all $\rho \in 2^{j+1}$, $\bar{p}^\rho$ forces a value on $\dot{f}(t)$.

Proof of Claim 5.5.2. This involves a doubly indexed induction. Let $\{\rho_m : m < 2^\ell\}$ be any enumeration of 2^ℓ. We assume, by induction on m, that for any condition $p < p_{m-1}$, there is a condition $p_m < p$ so that for all $\rho \in \{\rho_0, \ldots, \rho_{m-1}\}$ and all $\ell < \ell_{p_m,k}$ $(k \in K(p_m))$, there is a t so that $p_m^\rho \cup \{(i(p_m,k,j),0) : j \leq \ell\}$ and $p_m^\rho \cup \{(i(p_m,k,j),0) : j < \ell\} \cup \{(i(p_m,k,\ell),1)\}$ force distinct values on $\dot{f}(t)$. Then, to produce p_{m+1}, we assume, by induction on ℓ, that for each $p < p_m$ we can have, in addition, that for each $j < \ell < \ell_{p_m,k}$ $(k \in K(p_m))$, there is some t such that for all $\psi \in 2^{j+1}$, p_m^ψ forces a value on $\dot{f}(t)$, and, for $\rho = \rho_m$, the two conditions $p_{m+1}^\rho \cup \{(i(p_m,k,j),0) : j \leq \ell\}$ and

$p^{\rho}_{m+1} \cup \{(i(p_m,k,j),0) : j<\ell\} \cup \{(i(p_m,k,\ell),1)\}$ force distinct values on $\dot f(t)$. Assume that p_m is such a condition. Apply Lemma 5.4 to the condition $p' < p^{\rho_m}_m$ where $p'(i(p_m,k,j)) = 0$ for all $j<\ell$ and $k \in K(p_m)$, to select an infinite $K \subset K(p')$ and a sequence $\{i_k : k \in K\}$ disjoint from $\mathrm{dom}(p')$ so that, for each $k \in K$, there is a t_k such that $p' \cup \{(i_k,0)\}$ and $p' \cup \{(i_k,1)\}$ force distinct values on $\dot f(t_k)$. Let $\tilde p$ denote the condition so that,

(1) for all $k \in K = K(\tilde p)$, $\{i(\tilde p,k,j) : j<\ell\} = \{i(p_m,k,j) : j<\ell\}$,
(2) for all $k \in K(p')$, $i(\tilde p,k,\ell) = i_k$
(3) for all $k \in K(p')$, $\tilde p \restriction (\{i(p_m,k,j) : \ell < j < \ell_{p_m,k}\} \cap i_k)$ is constantly 0, and $\tilde p \restriction H_k \setminus i_k \subset p'$.

Extend $\tilde p$ (by a length $2^{\ell+1}$ recursion using Lemma 4.7) to a condition still satisfying the above three criteria, but which in addition satisfies that for each $\psi \in 2^{\ell+1}$ and each $k \in K(\tilde p)$, $(\tilde p)^{\psi}$ forces a value on $\dot f(t_k)$. It is routine to verify that this $\tilde p$ satisfies that for each $j \le \ell$, and each $\rho \in \{\rho_0,\dots,\rho_m\}$, there is a t such that $(\tilde p)^{\rho} \cup \{(i(\tilde p,k,j'),0) : j' \le j\}$ and $(\tilde p)^{\rho} \cup \{(i(\tilde p,k,j'),0) : j'<j\} \cup \{(i(\tilde p,k,\ell),1)\}$ force distinct values on $\dot f(t)$, while at the same time, for each $\psi \in 2^{\ell+1}$, $(\tilde p)^{\psi}$ forces a value on this $\dot f(t)$. By the inductive hypothesis there is an extension q (with $\{i(q,k,j) : j \le \ell\} = \{i(\tilde p,k,j) : j \le \ell\}$ for all $k \in K(q)$) which also satisfies that for each $k \in K(q)$, $\rho \in \{\rho_0,\dots,\rho_{m-1}\}$, and $\bar\ell < \ell_{q,k}$, there is a t so that, q^{ψ} forces a value on $\dot f(t)$ for all $\psi \in 2^{\bar\ell+1}$ and $q^{\rho} \cup \{(i(q,k,j),0) : j \le \bar\ell\}$ and $q^{\rho} \cup \{(i(q,k,j),1) : j<\bar\ell\} \cup \{(i(q,k,\bar\ell),1)\}$ force distinct values on $\dot f(t)$. Now, with a simple fusion, it should be clear that we can produce p_{m+1} as required. □

Finally it should be clear that repeated applications of Claim 5.5.2 and a fusion argument will produce a condition as in the next Claim.

c3 *Claim* 5.5.3. There is a condition $\bar p$ such that for each $k \in K(\bar p)$, each $\ell < \ell_{\bar p,k}$, and each $\rho \in 2^{\ell}$, there is a t such that each of $\bar p^{\rho} \cup \{(i(\bar p,k,\ell),0)\}$ and $\bar p^{\rho} \cup \{(i(\bar p,k,\ell),1)\}$ force distinct values on $\dot f(t)$, and such that for each $\psi \in 2^{\ell+1}$, $\bar p^{\psi}$ forces a value on $\dot f(t)$.

Now with $\bar p$ chosen as in Claim 5.5.3, the selection of the sequence $\langle T(i) : i \notin \mathrm{dom}(\bar p)\rangle$ is straightforward. For each $k \in K(\bar p)$ and $\ell < \ell_{\bar p,k}$, choose a $t(\bar p,k,\rho)$ to be any t fulfilling the conclusion of the claim. Set $T(i(\bar p,k,\ell))$ to be the set $\{t(\bar p,k,\rho) : \rho \in 2^{\ell}\}$. It remains only to show that if $i \ne j$ are both in $\omega \setminus \mathrm{dom}(\bar p)$, then $T(i)$ and $T(j)$ are disjoint. Since we are assuming that Lemma 4.6 holds for $\dot f$, we may fix a $k \in K(\bar p)$ and a pair $\ell < \ell'$ such that $i = i(\bar p,k,\ell)$ and $j = i(\bar p,k,\ell')$. Fix any $t \in T(j)$ and choose $\rho \in 2^{\ell'}$ such that $t = t(\bar p,k,\rho)$. It follows

then that $\psi = \rho \restriction \ell + 1$ is such that $\bar{p}^\psi$ does not force a value on $\dot{f}(t)$ which shows that t can not be in $T(i)$. □

Define $\mathrm{gnd}(\dot{f})$ to be the ideal of sets Y such that $\dot{f} \restriction Y$ is from V. For the next, one of the main results of the section, we will need to assume that $\mathrm{gnd}(\dot{f})$ is ccc over fin.

Lemma 5.6. *If $\bar{p}$ and the sequence $\langle T(i) : i \notin \mathrm{dom}(\bar{p})\rangle$ are as in Lemma 5.5, and if $\bar{p}$ forces that $\mathrm{gnd}(\dot{f})$ is ccc over fin, then there is a condition $p < \bar{p}$ and a sequence $\langle t_i : i \notin \mathrm{dom}(p)\rangle$ such that for each $i \notin \mathrm{dom}(p)$, $p \cup \{(i,1)\}$ and $p \cup \{(i,1)\}$ force distinct values on $\dot{f}(t_i)$.* nearmain

Proof. The proof of Lemma 5.5 was oriented with an induction through increasing values of $i \notin \mathrm{dom}(\bar{p})$. This proof will now need to be in reverse. To facilitate this notationally, we will now let, for a condition $q \leq \bar{p}$, $\{\bar{\imath}(q,k,\ell) : \ell < \ell_{q,k}\}$ be a descending enumeration of $H_k \setminus \mathrm{dom}(q)$ for $k \in K(q)$. Similarly, for each ℓ and $\rho \in 2^\ell$, we let $q^{\downarrow\rho}$ denote the condition $q \cup \bigcup_{k\in K(q)}\{(\bar{\imath}(q,k,j),\rho(j)) : j < \min(\ell,\ell_{q,k})\}$.

Claim 5.6.1. If $p \leq \bar{p}$ and e is 0 or 1, then there exists a $q < p$ and a sequence $\langle i_k : k \in K(q)\rangle$ such that, $i_k \in H_k \setminus \mathrm{dom}(p)$, $q(i_k) = e$, q forces a value on $\dot{f} \restriction \bigcup_{k\in K(q)} T(i_k)$, and the sequence $\{|H_k \cap i_k \setminus \mathrm{dom}(\bar{p})| : k \in K(q)\}$ diverges to infinity. usesccccoverfin

Proof of Claim 5.6.1. If $\mathbb{P}$ is $\mathbb{P}_0$ and e is 1, then there is nothing to prove since the assumption that $q(i_k) = 1$ automatically ensures that q forces a value on $\dot{f} \restriction T(i_k)$. Otherwise, we may first extend p to the condition p' obtained by setting $p'(\bar{\imath}(p,k,j)) = e$ for all $j < \ell_{p,k}/2$. Now we use the fact that p forces that $\mathrm{gnd}(\dot{f})$ is ccc over fin to assert that there is a selection $\langle i_k : k \in K(p')\rangle$ with $i_k \in \{\bar{\imath}(p,k,j) : j < \ell_{p,k}/2\}$ and an extension $q < p'$ satisfying that q forces a value on $\dot{f} \restriction \bigcup\{T(i_k) : k \in K(p')\}$. This is simply because there is an uncountable almost disjoint family of such selections and the family of $T(i)$ are pairwise disjoint and finite. □

By a standard fusion argument using Claim 5.6.1, we may assume first that we have constructed a condition p_0 satisfying that for each ℓ and the constantly 0 element ρ of 2^ℓ, $p_0^{\downarrow\rho}$ forces a value on $\dot{f} \restriction T(\bar{\imath}(p_0,k,\ell))$ for each $k \in K(p_0)$ with $\ell < \ell_{p_0,k}$. Then repeat the construction to find $p_1 < p_0$ so that this holds for ρ being the constantly 1 function in 2^ℓ for all ℓ.

Now we show that we can choose the sequence $\langle t_i : i \notin \mathrm{dom}(p)\rangle$ as required. Fix any $k \in K(p)$ and $i \in H_k \setminus \mathrm{dom}(p)$. There are ℓ, ℓ_0 and ℓ_1 such that $i = i(\bar{p},k,\ell) = \bar{\imath}(p_0,k,\ell_0) = \bar{\imath}(p_1,k,\ell_1)$. Choose any $\rho \in 2^\ell$

such that $p_1 \restriction H_k$ is compatible with $\bar{p}^\rho$. By construction, there is a $t_i \in T(i)$ and distinct u_i, v_i such that $\bar{p}^\rho \cup \{(i,0)\} \Vdash \dot{f}(t_i) = u_i$ and $\bar{p}^\rho \cup \{(i,1)\} \Vdash \dot{f}(t_i) = v_i$. We show that $p_0 \cup \{(i,0)\} \Vdash \dot{f}(t_i) = u_i$ and $p_1 \cup \{(i,1)\} \Vdash \dot{f}(t_i) = v_i$. From this it follows of course that $p_1 \cup \{(i,0)\}$ also forces that $\dot{f}(t_i) = u_i$ which finishes the proof. We show this by showing that if ψ is any other member of 2^ℓ such that $\bar{p}^\psi$ is compatible with $p_1 \restriction H_k$, then $\bar{p}^\psi$ must also force that $\dot{f}(t_i) = u_i$, since we then have that all extensions extend to a condition that forces that $\dot{f}(t_i) = u_i$. Let $\chi \in 2^{\ell_0}$ be the constantly 0 function, and recall that $p_0^{\chi_0}$ forces a value on $\dot{f} \restriction T(i)$. Now each of $\bar{p}^\psi \restriction H_k$ and $\bar{p}^\rho \restriction H_k$ are compatible with the p_0^χ and each force a value on $\dot{f}(t_i)$. Therefore they must force the same value. The argument for $p_1 \cup \{(i,1)\}$ is the same. □

The final result in this section is the proof of statements 1, 3, and 5 of Theorem 2.8.

ramsey **Theorem 5.7.** *In the model obtained by forcing with the poset $\mathbb{P}_1$ over a model of PFA, there is a non-trivial autohomeomorphism φ of $\mathbb{N}^*$ and two regular closed copies A, B of $\mathbb{N}^*$ and a tie-point $\mathcal{W}$ such that*

1. *$\varphi[A] = B$ and $\varphi[B] = A$, and $A \cap B = \{\mathcal{W}\}$,*
2. *$\mathcal{W}$ is the only point on the boundary of each of A and B,*
3. *φ is the identity on $\mathbb{N}^* \setminus (A \cup B)$,*
4. *the identity function is a near lifting of φ in the sense developed above.*

Proof. We will define a strange sequence, $\{\dot{t}_m : m \in \omega\}$, of $\mathbb{P}_1$-names of pairs. These will code liftings of the maps between A and B (each will "pick" a point from the pair) and the mappings of each onto $\mathbb{N}^*$ (each member from the m-th pair being sent to m). When these are viewed as $\mathbb{P}_1(\mathfrak{F})$-names as above, they will have the empty function as a near lifting. The difficult part of the construction is to ensure that A and B meet in a single ultrafilter.

For each $m \in \omega$ and each function $\sigma \in 2^{[2^m, 2^{m+1})}$, we will choose a pair $a_\sigma \subset [2^m, 2^{m+1})$. The definition of $\dot{t}_m$ will simply be that a condition $p \in \mathbb{P}_1$ such that $[2^m, 2^{m+1}) \subset \mathrm{dom}(p)$, will force that $\dot{t}_m$ is equal to $a_{p\restriction[2^m,2^{m+1})}$. Analogous to the definition of $K(p)$ above, let $M(p)$ denote the set $\{m \in \omega : [2^m, 2^{m+1}) \not\subset \mathrm{dom}(p)\}$ for each $p \in \mathbb{P}_1$. Without mention, we will assume that we work with the dense set of conditions which satisfy that $\{|[2^m, 2^{m+1}) \setminus \mathrm{dom}(p)| : m \in M(p)\}$ diverges to infinity.

For each $p \in \mathbb{P}_1$, let $T(p) = \{t_m : m \notin M(p) \text{ and } p \Vdash t_m = \dot{t}_m\}$, $A(p) = \{\min(t_m) : t_m \in T(p)\}$ and $B(p) = \{\max(t_m) : t_m \in T(p)\}$. If

G is a generic filter on $\mathbb{P}_1$, then we will set A to be the closure of the open set $\bigcup\{A(p)^* : p \in G\}$ and B to be the closure of the open set $\bigcup\{B(p)^* : p \in G\}$.

For each $p \in \mathbb{P}_1$, let $W(p)$ equal $\bigcup_{m\in M(p)}\{a_\sigma : \sigma \cup p \in \mathbb{P}_1\}$. Now define the filter $\mathcal{W}$ to be the filter generated by the family $\{W(p) : p \in G\}$. If we define these names so that $\mathcal{W}$ is forced to be an ultrafilter, then it is quite routine to check that $\mathcal{W}$ is the only boundary point of each of A and B and that the map sending $\mathbb{N}^*$ onto $\mathbb{N}^*$ obtained by extending the map sending each interval $[2^m, 2^{m+1})$ to $\{m\}$ will restrict to a homeomorphism on each of A and B. It may help to recall that $\mathbb{P}_1$ does not add any new countable sets.

Now we set about showing that there is such a sequence of names. We will define, for $m \in \omega$ and $\sigma \in 2^{[2^m,2^{m+1})}$, the value of a_σ based only on the cardinality of $\sigma^{-1}(1)$. In fact some Ramsey theory says this will effectively be the case anyway. To make these choices we now intoduce the idea of an ℓ-structure for $\ell \in \omega$.

The 0-structure will be the empty set. We let $L_0 = 2$ and $n_0 = 6 = L_0 + L_0^{L_0}$. We next define a family of pairs $\{a_{\langle i\rangle} : i < n_0\}$:

$$a_{\langle 0\rangle} = \{0, 1\}, \quad a_{\langle 1\rangle} = \{2, 3\}, \quad a_{\langle 2\rangle} = \{0, 2\},$$
$$a_{\langle 3\rangle} = \{0, 3\}, \quad a_{\langle 4\rangle} = \{1, 2\}, \quad a_{\langle 5\rangle} = \{1, 3\}$$

This assignment satisfies that for each set Y such that $Y \cap a_{\langle i\rangle}$ is not empty for each $i < L_0$, there is a $j < n_0$ such that $Y \supset a_{\langle j\rangle}$. This is the process by which we will ensure that the above defined $\mathcal{W}$ is an ultrafilter.

By recursion on ℓ, we define $L_\ell = 2\, n_0\, n_1 \cdots n_{\ell-1}$, $n_\ell = L_\ell + L_\ell^{L_\ell}$, and our ℓ-structure based on the cartesian product $\mathcal{N}_\ell = n_\ell \times n_{\ell-1} \times \cdots \times n_1 \times n_0$. It is awkward, but ultimately more convenient, to have this product in descending order. For each $j < \ell$, also let $\mathcal{N}_{\ell,j} = n_\ell \times \cdots \times n_j$.

An ℓ-structure is a family $\langle\{a_x : x \in \mathcal{N}_\ell\}, \{Y_\rho : \rho \in \bigcup_{j<\ell}\mathcal{N}_{\ell,j}\}\rangle$ satisfying

(1) for each $x \in \mathcal{N}_\ell$, a_x is a pair of integers
(2) for each $\rho \in \bigcup_{j<\ell}\mathcal{N}_{\ell,j}$, Y_ρ is the union of all a_x with $x \in \mathcal{N}_\ell$ and $\rho \subset x$,
(3) for each $j < \ell - 1$ and $\rho \in \mathcal{N}_{\ell,j}$, the family $\langle\{a_x : \rho \subset x \in \mathcal{N}_\ell\}, \{Y_\psi : \rho \subset \psi \in \bigcup_{i<\ell}\mathcal{N}_{\ell,i}\}\rangle$ is an $(\ell - j)$-structure (with a confusing prefix of ρ on each index),
(4) the family $\{Y_{\langle m\rangle} : m < L_\ell\}$ are pairwise disjoint and $Y_\emptyset$ is the union,
(5) for each $Y \subset Y_\emptyset$ such that $Y \cap Y_{\langle m\rangle} \neq \emptyset$ for each $m < L_\ell$, there is a $k < n_\ell$ such that $Y \supset Y_{\langle k\rangle}$.

The construction is quite straightforward. Let $\{\bar{a}_x : x \in \mathcal{N}_{\ell-1}\}$ be the pairs from an $\ell-1$-structure. The definition of L_ℓ ensures that it exceeds the cardinality of, $\bar{Y}_\emptyset$, the union of these pairs. Let $\{Y_{\langle m\rangle} : m < L_\ell\}$ be a pairwise disjoint family of sets of integers each of the same cardinality as $\bar{Y}_\emptyset$. Similarly, let $\{Y_{\langle k\rangle} : m \leq k < n_\ell\}$ be a family of sets, each of cardinality $|\bar{Y}_\emptyset|$, so that the last inductive assumption is satisfied. For each $k < n_\ell$, fix a bijection, f_k, between $\bar{Y}_\emptyset$ and $Y_{\langle k\rangle}$ and define $a_{k^\frown\bar{x}} = f_k[\bar{a}_{\bar{x}}]$ for each $\bar{x} \in \mathcal{N}_{\ell-1}$.

For each ℓ, let c_ℓ denote the order-preserving mapping from $\mathcal{N}_\ell$ with the natural lexicographic ordering into an initial segment of $[0, L_{\ell+1})$. It is important to observe that for each $\rho \in \bigcup_{j<\ell}\mathcal{N}_{\ell,j}$, the set $[\rho] = \{x \in \mathcal{N}_\ell : \rho \subset x\}$ is an interval in the lexicographic ordering, and so, $c_\ell([\rho])$ is an interval in $[0, L_{\ell+1})$. We may also assume that we have, for each ℓ, a fixed ℓ-structure so that the set $Y^\ell_\emptyset = Y_\emptyset$ from this structure is an initial segment of integers.

For each integer m, choose $\ell = \ell_m$ maximal so that $L_{\ell_m+1} \leq 2^m$ (hence the interval $[2^m, 2^{m+1})$ will support an ℓ-structure). For each $\sigma \in 2^{[2^m,2^{m+1})}$ such that there is an $x \in \mathcal{N}_\ell$ with $c_\ell(x) = |\sigma^{-1}(1)|$, define a_σ to be the pair obtained by adding 2^m to each member of a_x. It follows that $a_\sigma \subset [2^m, 2^{m+1})$. If there is no such $x \in \mathcal{N}_\ell$, let $a_\sigma = \{2^m, 2^m{+}1\}$.

Define the condition $p_0 \in \mathbb{P}_1$ by the prescription that for all m and $\ell = \ell_m$, $p_0 \restriction [2^m, 2^{m+1})$ is the partial function which is 0 on the segment $[2^m + |Y^\ell_\emptyset|, 2^{m+1})$. This ensures that for all $\sigma \in 2^{[2^m,2^{m+1})}$ which extend $p_0 \restriction [2^m, 2^{m+1})$, there will be an $x \in \mathcal{N}_\ell$ such that $c_\ell(x) = |\sigma^{-1}(1)|$.

To finish the proof, we prove that p_0 forces that $\mathcal{W}$ is an ultrafilter. That is, if $q < p_0$ and $Y \subset \mathbb{N}$, then there is a $p < q$ such that Y either contains, or is disjoint from, $W(p)$.

We may assume that the sequence $\{k_m = |[2^m, 2^{m+1}) \setminus \mathrm{dom}(q)| : m \in M(q)\}$ diverges to infinity. For each $m \in M(q)$, let $q_m = q \restriction [2^m, 2^{m+1})$. Also, for $m \in M(q)$, let i_m be the largest integer so that $n_{i_m} < k_m/3$. By thinning out further (using any extension of q), we can also assume that $\{i_m : m \in M(q)\}$ diverges to infinity. It then follows that for each $m \in M(q)$ there is a $\rho_m \in \mathcal{N}_{\ell_m,i_m}$ such that the interval $c_{\ell_m}[\rho_m]$ is contained in $[|q_m^{-1}(1)|, |q_m^{-1}(1)| + k_m)$. By inductive hypotheses (3) and (5) in the definition of an ℓ_m-structure, there is an extension ψ_m of ρ_m with $\psi_m \in \mathcal{N}_{\ell_m,i_m+1}$ such that Y either contains, or is disjoint from, $W_m = 2^m + Y^{\ell_m}_{\psi_m}$ (i.e. $W_m = \bigcup\{a_\sigma : \sigma \in 2^{[2^m,2^{m+1})}$ and $|\sigma^{-1}(1)| \in c_{\ell_m}[\psi_m]\}$). By symmetry, we may assume that there is an infinite $K \subset M(q)$ such that Y contains W_m for all $m \in K$. Let $p < q$ be any condition such that $p^{-1}(0) = q^{-1}(0)$ (no more 0's are added) and for

each $m \in K$, the minimum element of $c_{\ell_m}[\psi_m]$ is the number of values in $[2^m, 2^{m+1})$ which are sent to 1 by p. Notice that for $m \in K$, $|[2^m, 2^{m+1}) \setminus \mathrm{dom}(p)| > n_{i_m+2}$ and so we may assume that $M(p) = K$. In other words, p will satisfy that $W(p) = \bigcup_{m\in M(p)} W_m$. This shows that p forces that Y contains a member of $\mathcal{W}$. □

6. ALL HOMEOMORPHISMS CAN BE TRIVIAL

In this section we now assume that $\mathbb{P}$ is either $\mathbb{P}_0$ or $\mathbb{P}_1$. In each result, when we are discussing any $\mathbb{P}$-name of an automorphism on $\mathcal{P}(\mathbb{N})/\mathrm{fin}$, we will assume without mention that the properties of F, $\dot{h}$ and the sequences $\{n_k, m_k : k \in \omega\}$ established in sections 4 and 5 are valid below some condition q_0. In particular, that $[2^{m_k}, 2^{m_k+1})$ contains $H_k = [n_k, n_{k+1}) \setminus \mathrm{dom}(q_0)$ for all $k \in K(q_0)$.

Now we are ready to prove the main theorems from Section 2. We start with restating and proving item 1 of Theorem 2.7 and Theorem 2.9.

Theorem 6.1. *In the extension obtained by forcing over a model of PFA by $\mathbb{P}_0$ all automorphisms on $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ are trivial. In the extension obtained by forcing over a model of PFA by $\mathbb{P}_1$, all automorphisms on $\mathcal{P}(\mathbb{N})$ are nearly trivial.* repeatnoauto

Proof. In this result, F is a lifting of an automorphism, hence we may assume that $\dot{h}$ is forced to be 1-to-1. In the case that $\mathbb{P}$ is $\mathbb{P}_0$ we refer to Lemma 5.1. In the case that $\mathbb{P}$ is $\mathbb{P}_1$ we refer to Lemma 5.6. Thus we have a condition p satisfying that for each $i \notin \mathrm{dom}(p)$ there is a t_i such that there are distinct values u_i, v_i that $p \cup \{(i,0)\}$ and $p \cup \{(i,1)\}$, respectively, force on $\dot{h}(t_i)$. Choose a condition $q < p$ and a set Y so that q forces that $F(Y) = \{u_i : i \notin \mathrm{dom}(p)\}$. Let $L_0 = \{i \notin \mathrm{dom}(q) : t_i \notin Y\}$. If L_0 is infinite, then we have a contradiction by choosing any infinite subset L' of L_0 so that $L' \cap H_k$ has at most one element for each k, and considering the condition $q' = q \cup \{(i,0) : i \in L'\}$. We now have that q' forces that $F(\{t_i : i \in L'\}) =^* h_{q'}(\{t_i : i \in L'\})$ will be almost contained in $F(Y)$ while $\{t_i : i \in L'\}$ is disjoint from Y.

Now suppose that L_0 is finite. If $\{v_i : i \notin \mathrm{dom}(q)\} \setminus F(Y)$ is infinite, then we choose an infinite L' so that $L' \cap H_k$ is empty for infinitely many $k \in K(q)$ and so that $\{v_i : i \in L'\}$ is disjoint from $F(Y)$. Again the extension $q' = q \cup \{(i,1) : i \in L'\}$ will force that $F(\{t_i : i \in L'\}) =^* h_{q'}(\{t_i : i \in L'\})$, but this contradicts that it is supposed to be mod finite contained in $F(Y)$.

The final case then is that there is an infinite sequence $L' \subset K(q)$ such that $K(q) \setminus L'$ is still infinite and there is a sequence of pairs

$\{i_k, i'_k : k \in L'\}$ such that i_k, i_k are distinct members of $H_k \setminus \mathrm{dom}(q)$ and $v_{i_k} = u_{i'_k}$ for each $k \in L'$. Now we have that the extension $q' = q \cup \{(i_k, 1), (i'_k, 0) : k \in L'\}$ will force that $\dot{h}$ is not 1-to-1. □

This next result gives the proof of items 3 and 2 of Theorem 2.7 and 2.8.

boundaries

Theorem 6.2. *In the forcing extensions obtained by forcing over a model of PFA with either $\mathbb{P}_0$ or $\mathbb{P}_1$, all regular closed copies of $\mathbb{N}^*$ have finite boundaries.*

Proof. Let A be a regular closed copy of $\mathbb{N}^*$ and let F denote the homomorphism onto $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ witnessing that A is homeomorphic to $\mathbb{N}^*$. Let us note that for any set $I \in \mathrm{triv}(F)$, I^* does not meet the boundary of A. By the results of section 4, there is a $\mathbb{P}(\mathfrak{F})$-name $\dot{h}$, and a sequence $\{n_k : k \in \omega\}$, and some condition p which forces that $\dot{h}$ is a lifting of F. Also, we may assume that p forces that $\dot{h}$ is forced to be 1-to-1 among values that are sent to a positive integer. If $\mathbb{P} = \mathbb{P}_0$ we apply Proposition 5.1, and if $\mathbb{P} = \mathbb{P}_1$ and no extension of p forces that $\dot{h}$ has a near lifting (Definition 5.2), we apply the sequence of results culminating in Lemma 5.6 so as to find a condition $p_0 < p$ so that for all $i \notin \mathrm{dom}(p_0)$ there is a t_i such that $p_0 \cup \{(i, 0)\}$ and $p_0 \cup \{(i, 1)\}$ forces $\dot{h}(t_i)$ is $u_i \neq v_i$ respectively. If $i \neq j$, then $p_0 \cup \{(i, 0), (j, 0)\}$ and $p_0 \cup \{(i, 0), (j, 1)\}$ force the same value on $\dot{h}(t_i)$ which shows that $t_i \neq t_j$. A simple argument shows that if $u_i > 0$, we can also choose a t^0_i and (if necessary extend p_0) so that $p_0 \cup \{(i, 1)\}$ forces that $\dot{h}(t^0_i) = u_i$ and $p_0 \cup \{(i, 0)\}$ forces a value on $\dot{h}(t^0_i)$. By reversing t_i and t^0_i we may now assume that, for all $i \notin \mathrm{dom}(p_0)$, $v_i > 0$ and, if $u_i > 0$, then $p_0 \cup \{(i, 1)\}$ forces that $\dot{h}(t^0_i) = u_i$. We also note that for distinct i, j not in $\mathrm{dom}(p_0)$, $p_0 \cup \{(i, 1), (j, 0)\}$ forces that $\dot{h}(t_i) = v_i$ and $\dot{h}(t_j) = u_j$ which shows that $v_i \neq u_j$ by the 1-to-1 condition on $\dot{h}$.

We will next argue that for all $k \in K(p_0)$, we may arrange that the family $\{v_i : i \in H_k \setminus \mathrm{dom}(p_0)\}$ is a singleton which we denote v^0_k. It is interesting to note then that this is impossible if $\mathbb{P}$ is $\mathbb{P}_1$, because there will be extensions of p_0 which force a violation to the 1-to-1 condition on $\dot{h}$ since we can arrange more than one t_i is sent to the same v^0_k. Thus, if $\mathbb{P} = \mathbb{P}_1$, we only have the case when $\dot{h}$ is forced to have a near lifting. In the case that $\mathbb{P}$ is $\mathbb{P}_0$, we note a critical property for $p_0, \{v^0_k : k \in K(p_0)\}$ is that for each $q < p_0$ and each $k \in K(q)$, there is no t such that $q \Vdash \dot{h}(t) = v^0_k$ (since some extension can otherwise force a violation of the 1-to-1 property of $\dot{h}$).

By a simple thinning out process, we can arrange that if we cannot arrange that each $\{v_i : i \in H_k \setminus \mathrm{dom}(p_0)\}$ is a singleton, then we can assume that, for each $k \in K(p_0)$, the elements in $\{v_i : i \in H_k \setminus \mathrm{dom}(p_0)\}$ are pairwise distinct. But now choose a condition $q_0 < p_0$ which forces values on $F(\{t_i : i \notin \mathrm{dom}(p_0)\})$. Fix any infinite set $K' \subset K(q_0)$ such that $K(q_0) \setminus K'$ is also infinite. Since $q_0 \cup \{(i_k, 1) : k \in K'\}$ forces that $\{v_{i_k} : k \in K'\} \subset^* F(\{t_i : i \notin \mathrm{dom}(p_0)\})$, it follows that q_0 forces that $\{v_{i_k} : k \in K'\}$ is almost contained in $F(\{t_i : i \notin \mathrm{dom}(p_0)\}$. However, if we choose any other $\bar{\imath}_k \in H_k \setminus \mathrm{dom}(q_0)$ for each $k \in K'$, then the condition $q' < q_0$ which sends, for each $k \in K'$, $\bar{\imath}_k$ to 1 and the remainder of $H_k \setminus \mathrm{dom}(q_0)$ to 0, then q' forces that $F(\{t_i : i \in \bigcup_{k\in K'} H_k \setminus \mathrm{dom}(p_0)\})$ is almost equal to $\bigcup_{k\in K'} (\{v_{\bar{\imath}_k}\} \cup \{u_i : \bar{\imath}_k \neq i \in H_k \setminus \mathrm{dom}(p_0)\})$. Of course we also have that p_0 forces that $F(\{t_i : i \in \bigcup_{k\in K'} H_k \setminus \mathrm{dom}(p_0)\})$ is almost equal to $F(\{t_i : i \notin \mathrm{dom}(p_0)\}) \cap \bigcup_{k\in K'}[n_k, n_{k+1})$ because of the properties assumed to hold for $\dot{h}$. This means then that $\{v_{i_k} : k \in K'\}$ is supposed to be mod finite contained in $\bigcup_{k\in K'} (\{v_{\bar{\imath}_k}\} \cup \{u_i : \bar{\imath}_k \neq i \in H_k \setminus \mathrm{dom}(p_0)\})$. We know that it is actually disjoint because v_{i_k} is not equal to any u_j.

Now that we have the sequence $\{v^0_k : k \in K(p_0)\}$ as above, we choose a condition $p_1 < p_0$ and a set Y_0 such that p_1 forces that $F(Y_0)$ is almost equal $\{v^0_k : k \in K(p_0)\}$. We will show below that Y_0 is not in $\mathrm{triv}(F)$ and hits the boundary in exactly one point. However, if p_1 does not force that $\mathbb{N} \setminus Y_0$ is in $\mathrm{triv}(F)$, then, working with $\dot{h} \restriction \mathbb{N} \setminus Y_0$, we may assume that there is a sequence of $\{t^1_i : i \notin \mathrm{dom}(p_1)\} \subset \mathbb{N} \setminus Y_0$ and non-zero values $\{v^1_k : k \in K(p_1)\}$ so that for each $k \in K(p_1)$, and $i \in H_k \setminus \mathrm{dom}(p_1)$, $p_1 \cup \{(i, 1)\} \Vdash \dot{h}(t^1_i) = v^1_k$.

Choose $p_2 < p_1$ and $Y_1 \subset \mathbb{N} \setminus Y_0$, so that p_2 forces that $F(Y_1)$ is almost equal $\{v^1_k : k \in K(p_1)\}$, and again ask if p_2 forces that the complement of $Y_0 \cup Y_1$ is in $\mathrm{triv}(F)$. We will argue that this process must stop at some finite stage ℓ and at each stage we "captured" exactly one point on the boundary. If the process continues to infinity, then after a simple fusion, we have a condition p and an increasing sequence $\{\ell_k : k \in K(p)\}$ so that for each $k \in K(p)$ and $\ell < \ell_k$ we have the pairwise distinct elements $\{v^\ell_k : \ell < \ell_k\}$ so that for each $q < p$ each $k \in K(q)$ and each $\ell < \ell_k$, there is no value t such that $q \Vdash \dot{h}(t) = v^\ell_k$. From this we can choose an almost disjoint family $\{Y_\alpha : \alpha \in \omega_1\}$ so that, for some condition $q < p$, q forces that $F(Y_\alpha)$ will contain some $v_{k,\ell}$ for infinitely many $k \in K(p)$. This is a contradiction if we show that no such Y_α is in $\mathrm{triv}(F)$. Assume that Y is any set such that there is a condition $q < p$ such that q forces the value $F(Y)$ satisfying that for each $k \in K(q)$, $F(Y) \cap \{v^\ell_k : \ell < \ell_k\}$ is a singleton. We can

additionally assume that each such intersection is actually a singleton, v_k. We finish this portion of the proof by proving that Y is not in triv(F) and that Y^* meets the boundary of A in a singleton (because then the induction stopped at some finite stage, and at each stage we captured just one point of the boundary). To do so, we show that if we choose any set $Y' \subset Y$, then exactly one of Y' and $Y \setminus Y'$ meets the boundary of A (and the other is in triv(F)).

Choose a condition $q' < q$ which forces the values $\{v_k : k \in L_0\}$ and $\{v_k : k \in L_1\}$ on $F(Y')$ and $F(Y \setminus Y')$ respectively. It follows that $L_0 \cup L_1 = K(q)$ and $L_0 \cap L_1$ is empty. By further extending q' we may assume that $L_1 \cap K(q')$ is empty. It follows easily that q' forces that $F((Y \setminus Y') \setminus \mathrm{dom}(h_{q'}))$ is finite, and that $(Y \setminus Y') \cap \mathrm{dom}(h_{q'})$ is in triv(F).

Since q' forces that $F(Y')$ is infinite, it follows that $Y^* \cap A$ is not empty. Since q' already forces the values on $\dot{h} \restriction (Y' \cap \mathrm{dom}(h_{q'}))$ we know that q' forces that $F(Y' \setminus \mathrm{dom}(h_{q'}))$ is almost equal to $\{v_k : k \in K(q')\}$. Let K' be the infinite set of $k \in K(q')$ such that we can choose $y_k \in Y' \cap [n_k, n_{k+1})$. By extending q' we may assume that $K = K(q')$. Apply Lemma 4.7, to choose $\tilde{q} < q'$ so that $\tilde{q}$ forces a value u_k on $\dot{h}(y_k)$ for each $k \in K$. Since $\dot{h}$ is forced to be a lifting, it follows that for almost all $k \in K$, this value $u_k \in \{0, v_k\}$. Thus for all $k \in K(\tilde{q})$, we have that $u_k = 0$, and we have shown that $\{y_k : k \in K(\tilde{q})\}$ is forced to be in the kernel of F, and so its closure misses A.

Now we work the case when $\mathbb{P}$ is $\mathbb{P}_1$ and some condition p_0 forces that $\bar{h}$ is a near lifting of $\dot{h}$. Let $\bar{m}$ be the integer which witnesses the property that p_0 forces that $\bar{h}$ is a near lifting of $\dot{h}$. Let $T_0 = \{t \in \mathbb{N} : \bar{h}(t) = 0\}$ and suppose that $p_1 < p_0$ forces a value J on $F(T_0)$, and let $T_1 = \mathbb{N} \setminus T_0$. If J is finite, then $\bar{h}(T_1) =^* \mathbb{N}$. For each $k \in K(p_1)$, let $T_1(k) = \{t \in [n_k, n_{k+1}) : p \not\Vdash \dot{h}(t) = \bar{h}(t)\}$. Recall that our assumptions on $\dot{h}$ guarantee that p_1 forces that $\dot{h}[T_1(k)] \cap \dot{h}[T_1(k')] \subset \{0\}$ for $k \neq k'$. Also we have that $\bar{h}$ is 1-to-1 and so p_1 forces that, as sets, $\dot{h}[T_1(k)] = \bar{h}[T_1(k)]$ for almost all k. Thus, p_1 forces that for almost all k, $0 \notin \dot{h}[T_1(k)]$. Of course, this shows that A is simply the set T_1^*, and so has empty boundary.

Now we assume that J is infinite. By choosing an extension of p_1 we may suppose that $J_k = J \cap [n_k, n_{k+1})$ is not empty for all $k \in K(p_1)$. Assume that Y is any infinite subset of J which meets J_k for each $k \in K(p_1)$. Choose $q < p_1$ and $T \subset T_0$ such that $q \Vdash F(T) =^* Y$. Fix any $t \in T$ such that there is a $k \in K(q)$ with $|H_k \setminus \mathrm{dom}(q)| > \bar{m}$ and $t \in [n_k, n_{k+1})$. It follows that there is some extension of q which forces that $\bar{h}(t) = \dot{h}(t) = 0$. For this reason, no extension of q forces that T^*

is contained in A. On the other hand, since $F(T) =^* Y$, it follows that T^* does meet A. In fact, this shows that T is not in triv(F). Therefore the sequence $\{J_k : k \in K(p_1)\}$ must have a finite bound on the sizes, since otherwise there would be an uncountable almost disjoint family of sets none of which are in triv(F). To finish the proof, we must show that the set T selected above is such that T^* meets the boundary in exactly one point (and so there is a finite cover of the boundary by clopen sets, each meeting the boundary in a singleton). This proceeds exactly as in the previous case and so the details can be skipped. □

Next we restate and prove item 4 of Theorem 2.7

Theorem 6.3. *In the extension obtained by forcing over a model of PFA by $\mathbb{P}_0$, if A and B are regular closed copies of $\mathbb{N}^*$, then $A \cap B$ is regular closed.* P04proof

Proof. If $A \cap B$ is not regular closed, then there is clopen set W of $\mathbb{N}^*$ which meets $A \cap B$ in a non-empty nowhere dense set. Since $A \cap W$ and $B \cap W$ will also be copies of $\mathbb{N}^*$, it suffices to show that if A and B meet, then $A \cap B$ has interior. By Theorem 6.2, we may assume that A and B each have exactly one point, $\mathcal{W}$, on their boundary (which they must share).

Let F_0 and F_1 be the liftings of the homomorphisms onto $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ corresponding to the closed sets A and B respectively. Let $\dot{h}_0$ and $\dot{h}_1$ be the $\mathbb{P}(\mathfrak{F})$-names of the liftings for F_0 and F_1 respectively.

As in Theorem 6.2, there is a condition p_0 and a sequence $\{v^0_k : k \in K(p_0)\}$ satisfying that for each $k \in K(p_0)$ and $i \in H_k \setminus \operatorname{dom}(p_0)$, there is a t_i such that $p_0 \cup \{(i,0)\} \Vdash \dot{h}_0(t_i) = 0$ and $p_0 \cup \{(i,1)\} \Vdash \dot{h}_0(t_i) = v^0_k$.

Now we notice that p_0 forces that $F(\{t_i : i \notin \operatorname{dom}(p_0)\}) =^* \{v^0_k : k \in K(p_0)\}$ since p_0 forces that $\dot{h}_0$ is a lifting of F.

It then further follows that for each $p_1 < p_0$, $\{t_i : i \notin \operatorname{dom}(p_1)\}$ is not in triv(F) and so is a member of the ultrafilter $\mathcal{W}$.

Since the same holds for $\dot{h}_1$, we may assume that there is a $p_1 < p_0$ and a sequence $\{v^1_k : k \in K(p_1)\}$ so that again that for each $k \in K(p_1)$ and $i \in H_k \setminus \operatorname{dom}(p_1)$, there is a $t^1_i \in \{t_j : j \notin \operatorname{dom}(p_0)\}$ such that $p_1 \cup \{(i,0)\} \Vdash \dot{h}_1(t^1_i) = 0$ and $p_1 \cup \{(i,1)\} \Vdash \dot{h}_1(t^1_i) = v^1_k$.

Notice that $\{t^1_i : i \notin \operatorname{dom}(q)\} \cap \{t_i : i \notin \operatorname{dom}(q)\}$ is infinite for all $q < p_1$ since q forces the intersection is in the ultrafilter $\mathcal{W}$. It follows easily from this fact that there is an infinite set of $i \in \operatorname{dom}(p_1)$ such that $t^1_i = t_i$. Now we can extend p_1 to a condition q which forces that $\dot{h}_0(t_i) > 0$ and $\dot{h}_1(t_i) > 0$ for all i in an infinite set I. This then shows that $\{t_i : i \in I\}^* \subset A \cap B$ as required. □

Next we prove item 5 of Theorem 2.7.

Theorem 6.4. *In the extension obtained by forcing over a model of PFA by $\mathbb{P}_0$, all tie-points are RK-equivalent to the new point $\mathcal{U}$.*

Proof. Assume now that $C \underset{\mathcal{W}}{\bowtie} D$ is a tie-point and that Let $\{c_\alpha : \alpha \in \omega_2\}$ and $\{d_\alpha : \alpha \in \omega_2\}$ be the $\mathbb{P}_0$-names for the generators (increasing mod finite) for $\mathcal{I}_C$ and $\mathcal{I}_D$ respectively as per Lemma 3.2. Let H be $\kappa^{<\omega_1}$-generic and let $\mathfrak{F} \subset \mathbb{P}$ be as discussed earlier. To avoid confusion, again let λ denote the ordinal in the extension $V[H]$ corresponding to ω_2^V. We again have that the interpretations by the filter $\mathfrak{F}$ of the names C, D etc. form a tie-point in $V[H]$. Following the standard approach it suffices to show that no $\mathbb{P}(\mathfrak{F})$-name $\dot{f} \in 2^{\mathbb{N}}$ will satisfy that $c_\alpha \subset^* \dot{f}^{-1}(0)$ and $d_\alpha \subset^* \dot{f}^{-1}(1)$ for all $\alpha \in \lambda$. Otherwise we can use Lemma 2.1 to show that would have to $\mathcal{I}_C$ and $\mathcal{I}_D$ contain a Luzin gap.

Well now, then it is easy to see that there is a condition p and sequence $\{n_k : k \in \omega\}$ so that $\dot{f}$ satisfies each of the conditions 4.6, 4.7, and that $\mathrm{gnd}(\dot{f})$ is ccc over fin (since every set not in $\mathrm{triv}(\dot{f})$ is a member of $\mathcal{W}$. Apply Lemma 5.1 to obtain $p_0 < p$ and the sequence $\{t_i : i \notin \mathrm{dom}(p_0)\}$ so that for each $i \notin \mathrm{dom}(p_0)$, there is a pair $u_i \neq v_i$ such that $p_0 \cup \{(i,0)\} \Vdash \dot{f}(t_i) = u_i$ and $p_0 \cup \{(i,1)\} \Vdash \dot{f}(t_i) = v_i$.

It follows easily now that the mapping sending each $i \notin \mathrm{dom}(p_0)$ to t_i will send the ultrafilter $\mathcal{U}$ to the ultrafilter $\mathcal{W}$. □

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