COMPACT C-CLOSED SPACES NEED NOT BE SEQUENTIAL

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Abstract. We obtain an independence result connected to the classic Moore-Mrowka problem. A property known to be intermediate between sequential and countable tightness in the class of compact spaces is the notion of a space being C-closed. A space is C-closed if every countably compact subset is closed. We prove it is consistent to have a compact C-closed space that is not sequential. Our example also answers a question of Arhangelskii by producing a compactification of the countable discrete space which is not itself sequential and yet it has a Fréchet-Urysohn remainder. Ismail and Nyikos showed that compact C-closed spaces are sequential if $2^{\mathfrak{t}} > 2^{\omega}$. We prove that compact C-closed spaces are sequential also holds in the standard Cohen model.

1. Introduction

In this note we consider three problems from the open problems book [9, p578]:

(1) [Arhangelskii, 1973] Are σ-sequential compact spaces sequential?
(2) [Ismail and Nyikos, 1980 [2]] If all countably compact subsets of a compact space are closed, is the space sequential (or, equivalently, sequentially compact)?
(3) [Rančin, 1973 [7]] A reformulation of Arhangelskii's question: Let $b\mathbb{N}$ be a compact extension of the countable discrete space $\mathbb{N}$ so that $b\mathbb{N} \setminus \mathbb{N}$ is sequential. Must then $b\mathbb{N}$ also be sequential, or equivalently, is there a sequence from $\mathbb{N}$ converging to a point in $b\mathbb{N} \setminus \mathbb{N}$?

The background to these problems is the challenge of distinguishing between three convergence properties in the class of compact spaces. In this paper all spaces are regular and Hausdorff. A space X is said to be Fréchet-Urysohn if for any subset A of X, each limit point of A

2010 *Mathematics Subject Classification.* 54E65, 54D55, 03E75, 54A35.
Key words and phrases. sequential space, tightness, compactness.

is the limit of a converging sequence of points from A. A subset A of a space X is said to be sequentially closed if every converging sequence of points from A converges to a point of A. A space is sequential if every sequentially closed subset is actually closed. Finally, a space X has countable tightness if the closure of a set is equal to the union of the closures of its countable subsets. Naturally a space is said to be σ-sequential (respectively, σ-Fréchet-Urysohn) if it can be written as a countable union of such spaces.

The notion of C-closed is much more recent and asserts that every countably compact subset is actually closed. It is easily seen that a sequential space (in fact σ-sequential space [2, 1.10]) is C-closed and that [2] countably compact C-closed spaces have countable tightness. Notice also that every non-isolated point in a compact C-closed space is the limit of a converging sequence.

The celebrated Moore-Mrowka problem asks if every compact space of countable tightness is sequential. Naturally each of Problems (1)-(3) have a positive answer in any model in which the Moore-Mrowka problem has a positive answer. Moreover, the very method of constructing the models relies on the fact that C-closed spaces were already known to be sequential in such models. While it is consistent that the Moore-Mrowka problem also has a negative answer, the known examples are not C-closed.

The main result of the paper is a forcing construction in Section 3 establishing the consistency of a negative answer to problems (1)-(3). In section 2, we first show that many of the standard iterated ccc forcing models (specifically iterating a single ccc forcing notion) will be models in which C-closed compact spaces are sequential and, even in many countable support proper forcing extensions problems (1) and (3) have positive answer in these models.

2. COMPACT C-CLOSED SPACES IN THE COHEN MODEL

A space is said to be sequentially compact if every infinite sequence has a converging subsequence. Evidently, a sequentially compact C-closed space is sequential. Therefore we are interested in the existence and structure of compact spaces with a "completely divergent copy of ω". The well-known cardinal $\mathfrak{t} \leq \mathfrak{c}$ is the minimal cardinal such that there is a maximal mod finite descending chain of infinite subsets of ω with that cardinality.

Proposition 2.1 ([2]). *If a compact space X contains a completely divergent copy of ω, then the closure of ω has cardinality at least $2^{\mathfrak{t}}$.*

Proof. Recursively construct subsets $\{a_r : r \in 2^{<\mathfrak{t}}\} \subset [\omega]^{\aleph_0}$ so that

(1) $r \subset s$ implies $a_s \subset^* a_r$
(2) a_{r0} and a_{r1} have disjoint closures.

Conclusion: $\bigcap\{\overline{a_t} : t \subset f \in 2^{\mathfrak{t}}\}$ is disjoint from $\bigcap\{\overline{a_t} : t \subset g \in 2^{\mathfrak{t}}\}$ when $f \neq g$. □

Let us remark that an easy improvement of the previous lemma is that the cardinality of a compact space containing a completely divergent sequence is at least $\mathfrak{n}$ (namely, the minimum cardinality of a cover of $\mathbb{N}^*$ by nowhere dense sets). The following proposition is an easy consequence of [4, Lemma 5].

Proposition 2.2. *If $K \subset [0,1]^\kappa$ is compact with countable tightness, then in a Cohen forcing extension, $\overline{K}$ has a dense set of points of countable character.*

Proof. Put $K \in M \prec H(\theta)$ with M countable. It will suffice to simply prove that the set of points in $\overline{K}$ with countable character is not empty, and so we may also assume that K has no isolated points.

Since K has countable tightness, every point of K has countable π-character. It follows then that $RO(K) \cap M$ is a local π-base for each point of $K \cap M$. In addition, it $RO(K) \cap M$ is a local π-base for each limit point of $K \cap M$ (even in the forcing extension).

Notice that Cohen forcing will add a generic g for the countable poset $RO(K) \cap M$. Since g is a neighborhood base for the closed set $\bigcap g = \bigcap\{\overline{W} : W \in g\}$, we will prove that this set is a singleton.

Suppose that z_1, z_2 are distinct points of K. Choose disjoint regular open sets U_1, U_2 containing z_1, z_2 respectively. Let U_3 be a third regular open set containing neither of z_1, z_2 in its closure and chosen so that $\{U_1, U_2, U_3\}$ is a cover of K.

Let D denote the set of $W \in RO(K) \cap M$ satisfying that there is an $1 \leq i \leq 3$ so that $W \subset U_i$. We check that D is a dense subset of the poset $RO(K) \cap M$. Choose any $W \in RO(K) \cap M$ and any $x \in W \cap M$. Select $1 \leq i \leq 3$ so that $x \in U_i$. Since $RO(K) \cap M$ is a local π-base for x, there is a $W' \in RO(K) \cap M$ such that $W' \subset W \cap U_i$. Therefore W' is an element of D that is below W and this proves that D is dense. Choose $W \in D \cap g$. It follows that at least one of z_1, z_2 is not in W. This completes the proof that $\bigcap g$ is a singleton. □

Our interest in this model comes from the connection mentioned above between C-closed spaces and sequentially compact spaces.

Corollary 2.3. If $K \subset [0,1]^I$ (for any index set I) is compact and has countable tightness, then in the extension by adding more than $|I|$ many Cohen reals, the closure $\overline{K}$, in $[0,1]^I$, is sequentially compact.

Proof. Suppose that $\kappa > |I|$ and $\{\dot{x}_n : n \in \omega\}$ is a sequence of $Fn(\kappa, 2)$-names of points of $\overline{K}$. We may assume that each $\dot{x}_n$ is a canonical or nice name and thereby assume that there is a set $J \subset \kappa$ of cardinality $|I|$ so that each $\dot{x}_n$ is a $Fn(J, 2)$-name. Let $G \subset Fn(\kappa, 2)$ and define $G_J = G \cap Fn(J, 2)$, which is a generic filter for $Fn(J, 2)$. If we prove that, in $V[G_J]$, $\overline{K}$ has countable tightness, then we can invoke Proposition 2.2 to conclude that, in $V[G]$, $\{\mathrm{val}_{G_J}(\dot{x}_n) : n \in \omega\}$ has a converging sequence in $\overline{K}$.

Assume that $\{\dot{y}_\alpha : \alpha \in \omega_1\}$ are $Fn(J, 2)$-names of members of $\overline{K}$ and assume that $p \in Fn(J, 2)$ forces that $\{\dot{y}_\alpha : \alpha \in \omega_1\}$ is a free sequence. For each $\alpha \in \omega_1$, choose a $p_\alpha < p$ (in $Fn(J, 2)$) and a pair, $\{W_\alpha, U_\alpha\}$, of basic open subsets of $[0, 1]^I$ satisfying that $\overline{W_\alpha} \subset U_\alpha$, and that p_α forces that $\{\dot{y}_\beta : \beta < \alpha\} \subset W_\alpha$ and, for each $\gamma \geq \alpha$, $\dot{y}_\gamma \notin U_\alpha$. Choose an uncountable $S \subset \omega_1$ so that $\{p_\alpha : \alpha \in S\}$ is centered. For each $\alpha \in S$, the family $\mathcal{F}_\alpha = \{K \cap W_\gamma : \alpha \leq \gamma \in S\} \cup \{K \setminus U_\beta : \beta \in S \cap \alpha\}$ has the finite intersection property. It is easily checked that if x_α is chosen from $\bigcap \mathcal{F}_\alpha$ for each $\alpha \in S$, then the sequence $\{x_\alpha : \alpha \in S\}$ is an uncountable free sequence from K. □

We do not know if Corollary 2.3 can be strengthened to holding if we only assume that uncountably many Cohen reals are added. See Question 3 at the end.

Theorem 2.4 (CH)**.** *For any index set I and forcing extension by the standard poset* $\mathrm{Fn}(I, 2)$*, compact C-closed spaces are sequentially compact, hence sequential.*

Proof. Since the conclusion of the Theorem holds in any model of CH, we may assume that $|I| > \aleph_1$. Let $G \subset \mathrm{Fn}(I, 2)$ be a generic filter and let λ be any infinite cardinal. Also let $\dot{K}$ and $\{\dot{a}_n : n \in \omega\}$ be $\mathrm{Fn}(I, 2)$-names such that some condition $p \in G$ forces that $\dot{K} \subset [0, 1]^\lambda$ is a compact subset with countable tightness and that $\{\dot{a}_n : n \in \omega\}$ is a completely divergent subset of $\dot{K}$. We will prove that $\dot{K}$ is forced to not be C-closed.

Let θ be a large enough regular cardinal so that $\dot{K}, 2^{|I|}$ and λ are in $H(\theta)$. Choose $M \prec H(\theta)$ so that $\{\dot{a}_n : n \in \omega\} \in M$ with $M^\omega \subset M$ and $|M| = \omega_1$. Let $J = M \cap I$ and let π_M denote the projection map, in $V[G]$, from $[0, 1]^\lambda$ onto $[0, 1]^{M \cap \lambda}$.

By basic elementarity we have that, for each n, $\dot{x}_n = \dot{a}_n \cap M$ is a $\mathrm{Fn}(J, 2)$-name of a member of $[0, 1]^{M \cap \lambda}$. Furthermore, since $M^\omega \subset M$, $\{\dot{x}_n : n \in \omega\}$ is forced, with respect to the poset $\mathrm{Fn}(J, 2)$, to be completely divergent.

Let $G_M = G \cap M$ and let us note that $V[G]$ models that $\pi_M(\mathrm{val}_G(\dot{a}_n))$ is equal to $x_n = \mathrm{val}_{G_M}(\dot{x}_n)$. Moreover, in $V[G]$, we have from Corollary 2.3 that the sequential closure, Y, of $\{x_n : n \in \omega\}$ is countably compact. It follows then that $\mathrm{val}_G(K) \cap \pi^{-1}[Y]$ is countably compact. To finish the proof, we show that Y itself is not compact.

Working in $V[G_M]$, we have by Proposition 2.1, that the closure of $\{x_n : n \in \omega\}$ has cardinality greater than $\aleph_1$. Let L_M denote the points in $V[G_M]$ that are limit points of $\{x_n : n \in \omega\}$. We also note that $V[G_{M\cup J_1}]$ is a model of CH. If we let Y_{J_1} denote the sequential closure of $\{x_n : n \in \omega\}$ in the model $V[G_{M\cup J_1}]$, then Y_{J_1} has cardinality (at most) $\aleph_1$. It follows that $L_M \setminus Y_{J_1}$ is not empty and that, by compactness of K in $V[G]$, the closure of Y is compact and maps onto $\overline{L_M}$. We will show that, in $V[G]$, $Y \cap L_M \subset Y_{J_1}$.

As is standard, in $V[G]$, Y is equal to the union of the increasing chain $\{Y^{(\alpha)} : \alpha \in \omega_1\}$ where, $Y^{(0)} = \{x_n : n \in \omega\}$, and for each $\alpha < \omega_1$, $Y^{(\alpha)} = \bigcup\{Y^{(\beta)} : \beta < \alpha\}$ if α is a limit and $Y^{(\alpha)} = Y^{(\beta)} \cup \{y : (\exists\{y_n : n \in \omega\} \subset Y^{(\beta)})\ \ \langle y_n\rangle_n \to y\}$ if $\alpha = \beta + 1$. In fact, if $z \in Y \cap L_M$ then there is a well-founded tree $T \subset \omega^{<\omega}$ and a function $\rho_T : T \to Y$ such that

(1) for each maximal $t \in T$, $\rho_T(t) \in \{x_n : n \in \omega\}$,
(2) for each non-maximal $t \in T$, $t^\frown n \in T$ for all $n \in \omega$,
(3) for each non-maximal $t \in T$, the sequence $\{\rho_T(t^\frown n) : n \in \omega\}$ converges to $\rho_T(t)$,
(4) $\rho_T(\emptyset) = z$.

Now choose a pair of countable $\mathrm{Fn}(I \setminus M, 2)$-names $\dot{T}, \dot{\rho}$ coding T and ρ_T as follows. Simply $\dot{T}$ is a name for T, and $\dot{\rho}$ is a name for a function from $\omega^{<\omega}$ into ω so that it is forced that for each maximal element t of $\dot{T}$, $\dot{\rho}(t) = n$ if $\rho_T(t) = x_n$. Fix any condition $q \in \mathrm{Fn}(I \setminus M, 2)$ that forces that $\dot{\rho}(\emptyset) = z$.

Let ψ be any injection of J_1 into $I \setminus M$ chosen so that each of $\dot{T}$ and $\dot{\rho}$ are $\mathrm{Fn}(\psi[\omega], 2)$-names and $q \in \mathrm{Fn}(\psi[\omega], 2)$. This function ψ will lift to a bijection between $\mathrm{Fn}(J_1, 2)$ and $\mathrm{Fn}(\psi[\omega], 2)$ and further, to an isomorphism of names. Thus, there are isomorphic $\mathrm{Fn}(J_1, 2)$-names, $\dot{T}_\psi$ and $\dot{\rho}_\psi$ which are members of $V[G_M]$. These names naturally induce $\mathrm{Fn}(J_1, 2)$-names $\{\dot{y}^\psi_t : t \in \dot{T}_\psi\}$ defined canonically as follows. If some condition r forces that t is a maximal element of $\dot{T}_\psi$ and forces that $\dot{\rho}_\psi(t) = n$, then r forces that $\dot{y}^\psi_t$ is equal to x_n. If r forces that t is a non-maximal member of T_ψ, then r forces that $\dot{y}^\psi_t$ is the limit (if it is forced to exist) of the sequence $\{\dot{y}^\psi_{t^\frown n} : n \in \omega\}$. By induction on countable ordinals β, if r forces that $\{s \in \dot{T}_\psi : t \subseteq s\}$ has rank β, the limit $\dot{y}_t^\psi$ is forced to exist. Notice that this is simply a restatement of the fact that, for each $\gamma \in \lambda \cap M$, the sequence $\{\dot{y}^\psi_{t^\frown n}(\gamma) : n \in \omega\}$ is forced to be Cauchy. This is going to hold because of the assumption that $\{\dot\rho_T(t^\frown n)(\gamma) : n \in \omega\}$ is forced to be Cauchy. In addition, since q forces $\dot\rho_T(\emptyset) = z \in V[G_M]$, the condition q_ψ also forces that the value of $\dot{y}^\psi_\emptyset$ is equal to z. This proves that $z \in Y_{J_1}$ as required. □

Remark 2.5. This result generalizes immediately to show that compact C-closed spaces will be sequential in many of the standard finite support iterated ccc forcing models. The key property needed, besides adding Cohen reals, is that there are, up to isomorphism, only $\mathfrak{c}$ many nice names of reals. We do not know if it is necessary to assume that Cohen reals are added (i.e. finite support iteration) to make this claim. However, affirmative answers to questions (1) and (3) of the introduction do hold in such forcing models over a model of CH. To sketch the proof of (3), let us note that if we have a $\mathbb{P}$-name of a compact extension $b\mathbb{N} \subset [0,1]^{\omega_2}$ of an embedded completely divergent copy of $\mathbb{N}$ as in (3) for such a poset $\mathbb{P}$, then reflecting to a model M as in Theorem 2.4 will yield a countable subset of the image Y of $M[G] \cap (b\mathbb{N} \setminus \mathbb{N})$ in $[0,1]^{M\cap\omega_2}$ that has closure of cardinality greater than $\aleph_1$. The reason is simply that the image of the copy of $\mathbb{N}$ is completely divergent and, in $V[G_M]$, its remainder will have cardinality greater than $\aleph_1$. This remainder has countable tightness and Y has cardinality $\aleph_1$, so some countable subset of Y has large closure. Since $b\mathbb{N} \setminus \mathbb{N}$ is assumed to be sequential, the proof proceeds as in Theorem 2.4.

3. A T-algebra counterexample

In this section we prove that it is consistent to have a non-sequential compactification $X = \omega \cup K$ of ω such that K is Fréchet-Urysohn.

Theorem 3.1. *There is a ccc FS-iteration $\mathbb{P}_{\omega_2}$ forcing that there is a compact Fréchet-Urysohn K as a remainder of ω so that ω is completely divergent. Thus $X = \omega \cup K$ is C-closed but not sequential.*

Before starting the proof we review Koszmider's notion of a T-algebra [5] (see also [1]). T-algebras are special kinds of the minimally generated Boolean algebras first studied by Koppelberg [3]. They have a generating family indexed by a binary tree and satisfying corresponding combinatorial properties. We construct a T-algebra with indexing tree $2^{<\omega_1}$ and we make a small generalization by stating the properties in terms of mod finite containment as subsets of ω.

An indexed list $\{a_\beta : \beta < \delta\}$ ($\delta \le \omega_1$) is an mgen-list (minimally generating list) if, for each $\xi < \beta < \delta$, a_β is not in $I(\{a_\xi : \xi < \beta\})$, the ideal generated by $[\omega]^{<\aleph_0} \cup \{a_\xi : \xi < \beta\}$, while, for such $\xi < \beta$, one of $a_\xi \cap a_\beta$ or $a_\xi \setminus a_\beta$ is in $I(\{a_\zeta : \zeta < \xi\})$. Let us note that, by induction on $\beta > \xi$, it follows that if $a_\xi \cap a_\beta$ is not in the ideal $I(\{a_\zeta : \zeta < \xi\})$, then $a_\xi \setminus a_\beta$ is not only in the ideal, it is also in the Boolean algebra generated by $[\omega]^{<\aleph_0} \cup \{a_\zeta : \zeta < \xi\}$. Similarly, if $a_\xi \cap a_\beta$ is in the ideal $I(\{a_\zeta : \zeta < \xi\})$, then it is in the Boolean algebra generated by $[\omega]^{<\aleph_0} \cup \{a_\zeta : \zeta < \xi\}$.

It will be convenient to have our mgen-lists enumerated only by successor ordinals. An mgen-list $\mathcal{A} = \{a_{\beta+1} : \beta < \delta\}$ contained in $[\omega]^{\aleph_0}$ can be used to define a locally compact scattered topology on the ordinal $\delta + 1$. Given such an mgen-list we define $a[\mathcal{A}, \beta] = \{\xi \leq \beta : a_{\xi+1} \cap a_{\beta+1} \notin I(\{a_{\zeta+1} : \zeta < \xi\})\}$. This is explained in [3]. With these definitions in hand, we can better describe the required properties for the generator of a T-algebra.

Definition 3.2. The family $\mathcal{A} = \{a_t : t \in 2^{<\omega_1}\} \subset \mathcal{P}(\omega)$ is a T-algebra generating family providing:

(1) for $o(t)$ not a successor a_t is empty,
(2) for any t, $a_{t0} \ \dot\cup \ a_{t1} = \omega$,
(3) for any t, the family $\{a_{t\restriction\xi+1} : \xi + 1 \in \mathrm{dom}(t)\}$ is an mgen-list.

When we have such a list $\mathcal{A} = \{a_t : t \in 2^{<\omega_1}\}$ (even a partial downwards closed list) then for any $\rho \in 2^{\leq\omega_1}$ we will let $\mathcal{A}_\rho = \{a_{\rho\restriction\xi+1} : \xi + 1 \in \mathrm{dom}(\rho)\}$, and I_ρ will denote $I(\{a_{\rho\restriction\xi+1} : \xi + 1 \in \mathrm{dom}(\rho)\})$. Similarly, $a[\rho, \alpha]$ will be alternate notation for $a[\mathcal{A}_\rho, \alpha]$. As mentioned above (also see [3]), there is a unique locally compact scattered topology on the set $\{\rho \restriction \alpha : \alpha \in \omega_1\}$ generated by declaring to be clopen each member of the family $\{a[\mathcal{A}_\rho, \alpha] : \alpha \in \omega_1\}$. We let τ_ρ denote this topology on the predecessors of ρ, and let τ_ρ^+ denote the one-point compactification topology with ρ denoting the added point.

Proposition 3.3. *Let $\mathcal{A} = \{a_t : t \in 2^{<\omega_1}\}$ be a T-algebra generating family and let $\mathcal{B}(\mathcal{A})$ denote the boolean subalgebra of $\mathcal{P}(\omega)$ generated by $[\omega]^{<\aleph_0} \cup \mathcal{A}$. The Stone space, or space of ultrafilters, on $\mathcal{B}(\mathcal{A})$ can described as $\{\mathcal{U}_n : n \in \omega\} \cup \{\mathcal{U}_\rho : \rho \in 2^{\omega_1}\}$ where $\mathcal{U}_n$ is the fixed ultrafilter generated by the singleton n, and $\mathcal{U}_\rho$ is the filter generated by the family of all complements from the set $[\omega]^{<\aleph_0} \cup \{a_{\rho\restriction\xi+1} : \xi \in \omega_1\}$.*

Proof. We first prove that each such $\mathcal{U}_\rho$ is an ultrafilter, which is the same thing as proving that I_ρ is a maximal ideal. We must prove that for each $t \in 2^{<\omega_1}$, one of a_{t0} or a_{t1} is in I_ρ. Of course if $t \subset \rho$, then this is immediate. Let $\xi \in \omega_1$ be chosen minimal so that $\rho \restriction \xi + 1$ is not below t. By the definition of a T-algebra generating family, we have that one of $a_{t\restriction\xi+1} \cap a_{t0}$ or $a_{t\restriction\xi+1} \cap a_{t1}$ is in the ideal $I_{t\restriction\xi}$. Suppose, by symmetry, that $a_{t\restriction\xi+1} \cap a_{t0} \in I_{t\restriction\xi}$. Since $a_{\rho\restriction\xi+1} = \omega \setminus a_{t\restriction\xi+1}$, this is the same as $a_{t0} \setminus a_{\rho\restriction\xi+1}$ and is in $I_{t\restriction\xi}$. Since $I_{t\restriction\xi} \cup \{a_{\rho\restriction\xi+1}\} \subset I_\rho$, this shows that $a_{t0} \in I_\rho$.

Now suppose that $\mathcal{U}$ is a free ultrafilter on $\mathcal{B}(\mathcal{A})$. It will be easier to work with the dual non-principle maximal ideal $\mathcal{I}$. By recursion on $\xi \in \omega_1$ we identify $t_\xi \in 2^{\xi+1}$ so that $I_{t_\xi} \subset \mathcal{I}$. Since $\mathcal{U}$ is an ultrafilter, there is a $t_0 \in 2^1$ so that $a_{t_0} \in \mathcal{I}$. Assume that $\langle t_\eta : \eta < \xi\rangle$ have been chosen, and by induction, assume that this is an increasing sequence. If ξ is a successor, $\xi = \beta + 1$, then let $t = t_\beta$, otherwise, let $t = \bigcup\{t_\beta : \beta < \xi\}$. Again, just as in the base case, there is a unique immediate successor t_ξ of t such that $a_{t_\xi} \in \mathcal{I}$. Clearly the completed induction shows that $\mathcal{U}$ is equal to $\mathcal{U}_\rho$ where $\rho = \bigcup\{t_\xi : \xi \in \omega_1\}$. □

By identifying each $\mathcal{U}_\rho$ with ρ itself, for $\rho \in 2^{\omega_1}$, we have that our T-algebra generating family indexed by $2^{<\omega_1}$ induces a topology, denoted $\tau_{\mathcal{A}}$, on 2^{ω_1}. The next result explains the connection between the topology $\tau_{\mathcal{A}}$ on 2^{ω_1} and the family of topologies $\{\tau_\rho^+ : \rho \in 2^{\omega_1}\}$ described above. Given distinct $\rho, \psi \in 2^{\omega_1}$, let $\rho \wedge \psi$ denote the largest common predecessor and let $o(\rho \wedge \psi)$ denote the order type of its domain.

Proposition 3.4. *If $\{\rho\} \cup \{\rho_n : n \in \omega\}$ is a subset of 2^{ω_1}, then $\rho \in \overline{\{\rho_n\}_n}$ iff $\omega_1 \in cl_{\tau_\rho^+}(\{o(\rho_n \wedge \rho) : n \in \omega\})$. More generally, if $Y \subset 2^{\omega_1} \setminus \{\rho\}$, then $\rho \in \overline{Y}$, if and only if, $\omega_1 \in cl_{\tau_\rho^+}(\{o(y \wedge \rho) : y \in Y\})$.*

Proof. Let $\rho \in 2^{\omega_1}$ and let $Y \subset 2^{\omega_1} \setminus \{\rho\}$. It will be simpler to prove the contrapositive implications. First assume that $\rho \notin \overline{Y}$. Since we are working in the Stone space, there is a finite $H \subset \omega_1$ such that the element $\bigcup\{a_{\rho\restriction\beta+1} : \beta \in H\}$ is a member of the ultrafilter $\mathcal{U}_y$ for all $y \in Y$. Fix $y \in Y$ and choose minimal $\beta_y = \beta \in H$ so that $a_{\rho\restriction\beta+1} \in \mathcal{U}_y$. Now let ξ_y be the domain of $y \wedge \rho$ and note that $\xi_y \leq \beta_y$ since, otherwise $a_{\rho\restriction\beta_y+1} \in \mathcal{I}_y$. Furthermore, $a_{\rho\restriction\xi_y+1} \in \mathcal{U}_y$ since its complement is in $\mathcal{I}_y$. Since $a_{\rho\restriction\xi_y+1} \cap a_{\rho\restriction\beta_y+1}$ is in $\mathcal{U}_y$, it follows that it is not in $I_{\rho\restriction\xi_y}$. This is the same as saying that $\xi_y \in a[\rho, \beta_y]$. Now it follows that $\{\xi_y : y \in Y\}$ is contained in $\bigcup\{a[\rho, \beta] : \beta \in H\}$. This completes the verification that ω_1 is not in the closure of $\{o(y \wedge \rho) : y \in Y\}$.

The converse proceeds completely analogously and we refer the reader to [5] for the verification. □

Corollary 3.5. The topology induced on 2^{ω_1} is Fréchet-Urysohn if and only if τ_ρ^+ is Fréchet-Urysohn for each $\rho \in 2^{\omega_1}$.

Finally, we identify the condition that guarantees that the set ω of fixed ultrafilters of $S(\mathcal{B}(\mathcal{A}))$ is a completely divergent sequence. The proof is just a restatement of the fact that a set $Y \subset \omega$ will converge to

ρ in the Stone space providing Y is almost disjoint from each member of I_ρ.

Proposition 3.6. *If $\mathcal{A} = \{a_t : t \in 2^{<\omega_1}\} \subset [\omega]^{\aleph_0}$ is a T-algebra generating sequence, then the set of fixed ultrafilters is completely divergent in the Stone space providing, for each infinite $Y \subset \omega$ and each $\rho \in 2^{\omega_1}$, there is an $\alpha \in \omega_1$ such that $Y \cap a_{\rho\restriction\alpha+1}$ is infinite.*

Definition 3.7. For an indexed list $\mathcal{A} = \{a_\beta : \beta < \delta\}$, of infinite subsets of ω, we define a poset $Q[\mathcal{A}]$ according to the rule that $q = (b_q, c_q) \in Q[\mathcal{A}]$ if and only if there is a finite $H_q \subset \delta$ such that

1. b_q, c_q are disjoint subsets of ω,
2. $b_q \cup c_q$ is almost contained in $\bigcup\{a_\xi : \xi \in H_q\}$, and
3. for each $\beta \in H_q$, $a_\beta \setminus \bigcup\{a_\xi : \xi \in H_q \cap \beta\}$ is mod finite contained in one of b_q or c_q.

We order $Q[\mathcal{A}]$ as a forcing poset by $(b_r, c_r) < (b_q, c_q)$ providing $b_q \subset b_r$ and $c_q \subset c_r$.

Note that if, in Definition 3.7, $\mathcal{A}$ is an mgen-list and if $\delta \in \omega_1$, then $Q[\mathcal{A}]$ is actually countable. This is because, for $q = (b_q, c_q) \in Q[\mathcal{A}]$, each of b_q and c_q are members of the Boolean algebra generated by $[\omega]^{<\aleph_0} \cup \{a_\beta : \beta \in H_q\}$.

Definition 3.8. We define the FS-iteration $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\rangle$, together with $\mathbb{P}_{\alpha+1}$-names $\dot{t}_\alpha, \dot{\rho}_\alpha, \{\dot{a}_{t_\beta 0}, \dot{a}_{t_\beta 1} : \beta < \alpha\}$ and $\dot{Y}_\alpha$ according to the following rules:

1. if α is a successor, then a $\mathbb{P}_\alpha$-name $\dot{t}_\alpha$ has been selected with $1 \Vdash \dot{t}_\alpha$ is minimal in $2^{<\omega_1} \setminus \{\dot{t}_\beta : \beta < \alpha\}$, and $\dot{Q}_\alpha = \dot{Q}[\{\dot{a}_{\dot{t}_\alpha\restriction\xi+1} : \xi + 1 \in \mathrm{dom}(\dot{t}_\alpha)\}]$. Each of $\dot{a}_{t_\alpha 0}$ and $\dot{a}_{t_\alpha 1}$ are added canonically by the generic for $\dot{Q}_\alpha$, in the sense that $(b, c) \Vdash b \subset \dot{a}_{t_\alpha 0} \subset \omega \setminus c$
2. if α is a limit and $\alpha = \mu + \delta$ for some $\delta \in \omega_1$ and $cf(\mu) = \omega_1$, then $\dot{\rho}_\alpha = \dot{\rho}_\mu \in 2^{\omega_1}$ is a branch of $\{\dot{t}_\beta : \beta < \mu\}$, and $\dot{Q}_\alpha = [\delta]^{<\omega} \times [\omega_1]^{<\omega}$ ordered so that the generic filter canonically defines a subset $\dot{Y}_\alpha$ of δ such that $(h, H) \Vdash \dot{Y}_\alpha \cap a[\mathcal{A}_\alpha, \xi] \subset h$ for all $\xi \in H$, where $\mathcal{A}_\alpha$ is the mgen-list $\{\dot{a}_{\rho_\alpha\restriction\beta+1} : \beta \in \omega_1\}$. For convenience, we also use $a[\rho_\alpha, \xi]$ to denote $a[\mathcal{A}_\alpha, \xi]$.

We use the representation of FS-iterations that members of $\mathbb{P}_\alpha$ are functions with domain a finite subset of α. In this way, $\{\mathbb{P}_\alpha : \alpha \leq \omega_2\}$ is a continuous increasing chain of ccc posets. We arrange that for each $\mathbb{P}_{\omega_2}$-name $\dot{\rho}$ of a member of 2^{ω_1}, it is forced that there are cofinally many $\mu \in \omega_2$ such that $\dot{\rho}_\mu$ is equal to $\dot{\rho}$. Similarly, we may assume that it is forced that $\{\dot{t}_\alpha : \alpha \in \omega_2\}$ is equal to $2^{<\omega_1}$.

We begin with a very routine lemma.

Lemma 3.9. *Let $\alpha \in \omega_2$ be a successor, let $\dot{Y}$ be a $\mathbb{P}_\alpha$-name. Let $\mathcal{A}_\alpha$ denote the mgen-list $\{\dot{a}_{\dot{t}_\alpha\restriction\xi+1} : \xi+1 \in \mathrm{dom}(\dot{t}_\alpha)\}$, and let $\mathcal{A}_{\alpha,0}$ denote the mgen-list $\mathcal{A}_\alpha$ extended by $\dot{a}_{\dot{t}_\alpha 0}$, and similarly for $\mathcal{A}_{\alpha,1}$. Finally suppose that $p \in \mathbb{P}_\alpha$ forces that $\dot{Y} \subset \mathrm{dom}(\dot{t}_\alpha)$ and is not in the ideal generated by the family $\{a[\mathcal{A}_\alpha, \beta] : \beta < \mathrm{dom}(\dot{t}_\alpha)\}$.*

Then p also forces that $\dot{Y}$ is not in either of the ideals similarly generated by $\mathcal{A}_{\alpha,0}$ and $\mathcal{A}_{\alpha,1}$.

Proof. Let $p \in G_\alpha$ for any generic filter $G_\alpha \subset \mathbb{P}_\alpha$. Let $Y = \mathrm{val}_{G_\alpha}(\dot{Y})$ and let (b,c) be any member of $Q_\alpha = Q[\mathcal{A}_\alpha]$. Let $\delta = \mathrm{dom}(t_\alpha)$. Recall that there is a finite $H \subset \delta$ such that $b \cup c$ is almost equal to $\bigcup\{a_{t_\alpha\restriction\xi+1} : \xi \in H\}$. By extending (b,c) we can assume that H contains any potential witness to contradicting the conclusion of the lemma.

Evidently $Y \setminus \bigcup\{a[\mathcal{A}_\alpha, \xi] : \xi \in H\}$ is infinite. Choose any pair y_0, y_1 from $Y \setminus \bigcup\{a[\mathcal{A}_\alpha, \xi] : \xi \in H\}$ with $y_0 < y_1$. Since $a[\mathcal{A}_\alpha, y_0] \subset [0, y_0]$, we have that $y_1 \notin a[\mathcal{A}_\alpha, y_0]$. Set $b_1 = b \cup (a_{t_\alpha\restriction y_0+1} \setminus c)$ and $c_1 = c \cup (a_{t_\alpha\restriction y_1+1} \setminus b_1)$. It is routine decoding to show that (b_1, c_1) forces that y_0 is not in $a[\mathcal{A}_{\alpha,1}, \delta] \cup \bigcup\{a[\mathcal{A}_{\alpha,1}, \xi] : \xi \in H\}$ and that y_1 is not in $a[\mathcal{A}_{\alpha,0}, \delta] \cup \bigcup\{a[\mathcal{A}_{\alpha,1}, \xi] : \xi \in H\}$. This proves that p does not force that H is a witness to the failure of the lemma. Since p and H were arbitrary, the lemma is proven. □

Although the forcing iteration is not the same, the first and main part of this next result is essentially proven in [5].

Lemma 3.10. *Let $G \subset \mathbb{P}_{\omega_2}$ be a generic filter, and for each $\lambda < \omega_2$, let $G_\lambda = G \cap \mathbb{P}_\lambda$. For each $\rho \in 2^{\omega_1}$, there is minimal λ, that will have uncountable cofinality, so that $\tau_\rho^+ \in V[G_\lambda]$. For this λ, $V[G_\lambda]$ models that τ_ρ^+ has countable tightness.*

Proof. We begin by choosing, in V, a $\mathbb{P}_{\omega_2}$-name, $\dot{\rho}$, of a member of 2^{ω_1}. Let $p_0 \in G$ be an arbitrary member of $\mathbb{P}_{\omega_2}$. Choose minimal $\mu_0 < \omega_2$ so that there is a $p_1 < p_0$, also in G, forcing that $\dot{\rho}$ is equal to some some $\mathbb{P}_{\mu_0}$-name.

Let us briefly work in the forcing extension $V[G]$. For each $\gamma \in \omega_1$, let α_γ be identified so that (the valuation of) $\dot{t}_{\alpha_\gamma}$ is equal to (the valuation of) $\dot{\rho} \restriction \gamma$. Fix names $\{\dot{\alpha}_\gamma : \gamma \in \omega_1\}$ for this strictly increasing sequence of ordinals. Choose an extension $p_2 \in G$ of p_1 so as to force a value λ on the supremum of this sequence.

Now let $\dot{Y}$ be the $\mathbb{P}_\lambda$-name of an uncountable subset of ω_1. Choose any countable elementary submodel $M \prec H(\omega_3)$ such that p_2, the recursive construction of $\mathbb{P}_{\omega_2}$ and the set of name $\{\dot{Y}, \dot{\rho}\} \cup \{\dot{\alpha}_\gamma : \gamma \in \omega_1\}$

are all in M. Let $\delta = M \cap \omega_1$ and let $\mu = \sup(M \cap \lambda)$. Fix any $q < p_2 \restriction \lambda$ ($q \in \mathbb{P}_\lambda$) and finite $H \in [\omega_1]^{<\omega}$ such that q forces a value on $\dot{\alpha}_\xi$ for each $\xi \in H$. We prove that there is an $r < q$ and a $y \in \delta$ such that $r \Vdash y \in \dot{Y}$ and $r \Vdash y \notin a[\dot{\rho}, \xi]$ for each $\xi \in H$. In fact, this follows from applying Lemma 3.9 finitely many times to the set $\dot{Y} \cap \delta$. This proves that τ_ρ^+ has countable tightness. □

Since our poset is a finite support iteration of σ-centered posets, the iteration will preserve countable tightness of compact scattered spaces.

Corollary 3.11. If G is $\mathbb{P}_{\omega_2}$-generic, then $V[G]$ models that τ_ρ^+ has countable tightness for all $\rho \in 2^{\omega_1}$.

Proof. Choose $\lambda < \omega_2$ so that τ_ρ^+ is in the model $V[G_\lambda]$ and has countable tightness. The quotient forcing $\mathbb{Q} = \mathbb{P}_{\omega_2}/G_\lambda$ has property K. Suppose that $\{\dot{\gamma}_\alpha : \alpha \in \omega_1\}$ is a sequence of $\mathbb{Q}$-names of distinct ordinals in ω_1. Let $q \in \mathbb{Q}$ be arbitrary. We show by contradiction that $q \in \mathbb{Q}$ forces that some initial segment of $\{\rho \restriction (\dot{\gamma}_\alpha) : \alpha \in \omega_1\}$ has ρ as an accumulation point. Otherwise, we may choose, for each $\alpha \in \omega_1$, a $q > q_\alpha \in \mathbb{Q}$ and a finite $H_\alpha \subset \omega_1$ such that q_α forces a value γ_α on $\dot{\gamma}_\alpha$, and also forces that $\rho \restriction \dot{\gamma}_\beta$ is in $\bigcup\{a[\rho, \xi] : \xi \in H_\alpha\}$ for all $\beta < \alpha$. But now, apply that $\mathbb{Q}$ has property K, there is an uncountable $L \subset \omega_1$ such that q_β is compatible with q_α for all $\beta, \alpha \in L$. It follows that, for all $\alpha \in L$, $\{\rho \restriction \gamma_\beta : \beta \in L \cap \alpha\}$ is contained in $\bigcup\{a[\rho, \xi] : \xi \in H_\alpha\}$. This contradicts that τ_ρ^+ has countable tightness before forcing with $\mathbb{Q}$. □

When we combine countable tightness of τ_ρ^+ with the forcings utilized at limit steps, we obtain that τ_ρ^+ is Fréchet-Urysohn.

Lemma 3.12. *If G is $\mathbb{P}_{\omega_2}$-generic, then $V[G]$ models that τ_ρ^+ is Fréchet-Urysohn for all $\rho \in 2^{\omega_1}$.*

Proof. If $\dot{Y}$ is a $\mathbb{P}_{\omega_2}$-name of a countable subset of ω_1, there is a $\mu < \omega_2$ with cofinality ω_1 such that $\dot{Y}$ is equal to a $\mathbb{P}_\mu$-name and such that $\dot{\rho}_\mu$ is forced to equal $\dot{Y}$. Working in $V[G_\mu]$, there is a limit $\delta \in \omega_1$ such that $Y \subset \delta$. It should be apparent that if Y is not in the ideal generated by $\{a[\rho, \xi] : \xi \in \omega_1\}$, then the poset $\dot{Q}_{\mu+\delta}$ will force an infinite subset of Y which is almost disjoint from each member of $\{a[\rho, \xi] : \xi \in \omega_1\}$. It follows that Y contains a sequence converging to ω_1. □

Now we introduce notions to facilitate the proof that ω will be completely divergent in the Stone space.

Definition 3.13. For each successor $\alpha \in \omega_2$ and finite $H \subset \omega_1$, let $\dot{m}(\alpha, H)$ be a $\mathbb{P}_\alpha$-name such that 1 forces that $\dot{m}(\alpha, H)$ is the minimal integer m such that every finite set in the Boolean algebra generated by $\{\dot a_{\dot t_\alpha\restriction\xi+1} : \xi\in H\cap\mathrm{dom}(\dot t_\alpha)\}$ is contained in m.

Definition 3.14. A condition $p\in\mathbb{P}_\lambda$ is determined if for each $\alpha\in\mathrm{dom}(p)$ there are finite sets $h^\alpha_p, H^\alpha_p\subset\omega_1$ and integer m^α_p such that $p\restriction\alpha$ forces as follows:

(1) if α is a successor and $p(\alpha)=(\dot b^\alpha_p,\dot c^\alpha_p)$, then
  (a) the value of $\mathrm{dom}(\dot t_\alpha)$ is decided and $h^\alpha_p\subset H^\alpha_p\subset\mathrm{dom}(\dot t_\alpha)$,
  (b) $\dot m(\alpha,H^\alpha_p)$ is forced to be less or equal to m^α_p,
  (c) for each $\xi\in H^\alpha_p$, there is a $\beta\in\mathrm{dom}(p)\cap\alpha$ such that $\dot t_\beta=\dot t_\alpha\restriction\xi$ and $\dot t_\alpha(\xi)$ is decided,
  (d) $\dot b^\alpha_p\cup\dot c^\alpha_p$ is equal to $m^\alpha_p\cup\bigcup\{\dot a_{\dot t_\alpha\restriction\xi+1} : \xi\in H^\alpha_p\}$,
  (e) values are forced on $\dot b^\alpha_p\cap m^\alpha_p$ and $\dot c^\alpha_p\cap m^\alpha_p$,
  (f) for each $\xi\in h^\alpha_p$, $\dot a_{\dot t_\alpha\restriction\xi+1}\setminus(m^p_\alpha\cup\bigcup\{\dot a_{\dot t_\alpha\restriction\beta+1} : \beta\in H^p_\alpha\cap\xi\})$ is contained in $\dot b^\alpha_p$,
  (g) for each $\xi\in H^\alpha_p\setminus h^\alpha_p$, $\dot a_{\dot t_\alpha\restriction\xi+1}\setminus(m^p_\alpha\cup\bigcup\{\dot a_{\dot t_\alpha\restriction\beta+1} : \beta\in H^p_\alpha\cap\xi\})$ is contained in $\dot c^\alpha_p$,

(2) if α is a limit, then $p(\alpha)=(h^\alpha_p,H^\alpha_p)$, and, for each $\xi\in h^\alpha_p\cup H^\alpha_p$, there is a $\beta\in\mathrm{dom}(p)$ such that $\dot\rho_\alpha\restriction\xi=\dot t_\beta$, and the value of $\dot\rho_\alpha(\xi)$ is decided.

For successor α we may think of $\dot b^p_\alpha$ and $\dot c^p_\alpha$ as being determined by h^α_p, H^α_p and the pair $\dot b^p_\alpha\cap m^\alpha_p, \dot c^p_\alpha\cap m^\alpha_p$ partitioning m^p_α.

The following lemma is a standard fact about finite support iterations and we feel the proof can be omitted.

Lemma 3.15. *For each $\lambda\le\omega_2$, the set of determined conditions is dense in $\mathbb{P}_\lambda$.*

Definition 3.16. Let $\alpha\in\omega_2$ be a successor, and let $p\in\mathbb{P}_{\alpha+1}$ be determined. Also let $p\Vdash\mathrm{dom}(\dot t_\alpha)=\delta$. Then for any $h\subset H\in[\delta\setminus H^\alpha_p]^{<\aleph_0}$ and $k\in\omega$, let $q(\alpha,H,h,\dot b^\alpha_p,\dot c^\alpha_p,k)$ denote the pair $(\dot b,\dot c)$ where $\dot b$ is the $\mathbb{P}_\alpha$-name for

$$\dot b^\alpha_p\cup\left(\{k\}\cup\bigcup_{\xi\in h}\dot a_{\dot t_\alpha\restriction\xi+1}\setminus\bigcup_{\eta\in H\cap\xi}\dot a_{\dot t_\alpha\restriction\eta+1}\right)\setminus\dot c^\alpha_p$$

and $\dot c$ is the $\mathbb{P}_\alpha$-name for

$$\dot c^\alpha_p\cup\left(\bigcup_{\xi\in H\setminus h}\dot a_{\dot t_\alpha\restriction\xi+1}\setminus\bigcup_{\eta\in H\cap\xi}\{k\}\cup\dot a_{\dot t_\alpha\restriction\eta+1}\right)\setminus\dot b^\alpha_p\quad .$$

Similarly, let $q(\alpha, H, h, \dot{b}^\alpha_p, \dot{c}^\alpha_p)$ be defined by omitting any mention of k.

Lemma 3.17. *With p and H, h, k as in Definition 3.16, $p \restriction \alpha$ forces that $q(\alpha, H, h, \dot{b}^\alpha_p, \dot{c}^\alpha_p)$ and $q(\alpha, H, h, \dot{b}^\alpha_p, \dot{c}^\alpha_p, k)$ are members of $\dot{Q}_\alpha$, and p forces that each of them are extensions of $p(\alpha)$.*

Lemma 3.18. *Suppose that $p \in \mathbb{P}_\lambda$ is a determined condition and that α is a limit in $\mathrm{dom}(p)$, $\zeta \in H^\alpha_p$, and $\beta \in \mathrm{dom}(p)$ is such that $p \Vdash \dot{t}_\beta = \dot{\rho}_\alpha \restriction \zeta$. Then p forces that $\xi \notin a[\rho_\alpha, \zeta]$ if and only if one of the following:*

(1) *$\xi \in h^\beta_p$ and $p \Vdash \rho(\zeta) = 1$, or*
(2) *$\xi \in H^\beta_p \setminus h^\beta_p$ and $p \Vdash \rho(\zeta) = 0$.*

Proof. If $\xi \notin H^\beta_p$, then use $q(\beta, \{\xi\}, \{\xi\}, \dot{b}^p_\beta, \dot{c}^p_\beta)$ or $q(\beta, \{\xi\}, \emptyset, \dot{b}^p_\beta, \dot{c}^p_\beta)$ to force the *wrong* decision on $\xi \in a[\dot{\rho}_\alpha, \zeta]$. □

Lemma 3.19. *If $\dot{Y}$ is a $\mathbb{P}_{\omega_2}$-name of an infinite subset of ω, and if $\dot{\rho}$ is a $\mathbb{P}_{\omega_2}$-name of an element of 2^{ω_1}, then for each $p \in \mathbb{P}_{\omega_2}$, there is a $q < p$ and a $\delta \in \omega_1$ such that $q \Vdash \dot{Y} \cap \dot{a}_t$ is infinite, where $t = \dot{\rho} \restriction \delta + 1$.*

Proof. We may assume that p is determined, $0 \in \mathrm{dom}(p)$, and that p forces that $\dot{\rho}$ equals $\dot{\rho}_\lambda$ for some limit $\lambda \in \omega_2$. We also assume that p forces that $\dot{Y}$ is almost disjoint from $\dot{a}_t$ for all $t \in \{\dot{\rho} \restriction \xi + 1 : \xi \in \omega_1\}$. We work towards a contradiction.

Choose any countable elementary submodel M with each of the above in M. Let $\delta = M \cap \omega_1$ and, by strengthening p, assume that p forces that $\dot{t}_{\gamma_\delta} = \dot{\rho} \restriction \delta$ (with γ_δ a successor in $\mathrm{dom}(p) \cap \lambda$), that (wlog) $\rho(\delta) = 0$, and that there is an integer m such that $p \Vdash \dot{Y} \cap \dot{a}_{\rho\restriction\delta+1} \subset m$. By a finite descending induction, we can arrange that p also satisfies that for each $\alpha \in \mathrm{dom}(p)$, either $\delta \in H^\alpha_p$ or $p \Vdash \delta \notin \mathrm{dom}(\dot{t}_\alpha)$.

Of course $\gamma_\delta \notin M$, but for each $\xi \in H^\delta_p$, there is a $\beta \in \mathrm{dom}(p) \cap M \cap \gamma_\delta$ such that p forces that $\dot{t}_\beta$ is equal to $\dot{\rho} \restriction \xi$. Similarly, for each limit $\alpha \in \mathrm{dom}(p) \cap M$ and each $\xi \in h^\alpha_p \cup H^\alpha_p$ there is a $\beta \in \mathrm{dom}(p) \cap M$ such that p forces that $\dot{t}_\beta = \dot{\rho}_\alpha \restriction \xi$ and p decides the value of $\dot{\rho}_\alpha(\xi)$.

Since p forces $\dot{Y}$ is almost disjoint from $\dot{a}_t$ for all $t \in \{\dot{t}_{\gamma_\delta} \restriction \xi + 1 : \xi \in H^\delta_p\}$ and $H^\delta_p \subset M$, we can choose, in M, a determined condition p_1 so that p_1 is compatible with p and there is an integer $k > \max(m, m^\delta_p)$ so that $p_1 \Vdash k \in \dot{Y}$ and $p_1 \Vdash k \notin \dot{a}_t$ for all $t \in \{\dot{t}_{\gamma_\delta} \restriction \xi + 1 : \xi \in H^\delta_p\}$. We may assume that $\mathrm{dom}(p) \cap M \subset \mathrm{dom}(p_1)$ and that, for each $\alpha \in \mathrm{dom}(p) \cap M$, $H^\alpha_p \cap M \subset H^\alpha_{p_1}$ and $h^\alpha_p \cap M \subset h^\alpha_{p_1}$.

Let q be any common extension of p and p_1. We construct another condition $\bar{q}$ below each of p_1 and p so that $\bar{q} \Vdash k \in \dot{a}_{\dot{\rho}\restriction\delta+1}$. The main difficulty in ensuring that $\bar{q}$ is below each of p_1 and p is that for each limit $\alpha \in \mathrm{dom}(p) \cap M$, $\bar{q} \restriction \alpha$ must force that $h^\alpha_{p_1}$ is disjoint from $a[\rho_\alpha, \zeta]$ for each $\zeta \in H^\alpha_p \setminus M$. We can use q to help with this. We may view this as a demand on the construction of $\dot{a}_{t_\beta 0} = a[\rho_\alpha, \zeta]$ or $\dot{a}_{t_\beta 1} = a[\rho_\alpha, \zeta]$, where p forces that $\dot{t}_\beta = \dot{\rho}_\alpha \restriction \zeta$. Let us again note that this β is in $\mathrm{dom}(p)$. In particular, by Lemma 3.18, for each such ζ with $p \Vdash \dot{\rho}_\alpha(\zeta) = 1$ we will have that $\zeta \cap h^\alpha_{p_1} \setminus h^\alpha_p \subset h^\beta_q$. Similarly, for each $\zeta \in H^\alpha_p$ such that $p \Vdash \dot{\rho}_\alpha(\zeta) = 0$ we will have that $\zeta \cap h^\alpha_{p_1} \setminus h^\alpha_p \subset H^\beta_q \setminus h^\beta_q$.

For each successor $\beta \in \mathrm{dom}(p)$ and $p \Vdash \dot{t}_\beta \in 2^{\zeta_\beta}$, define H^β to be the union of H^β_p together with all $\zeta_\beta \cap h^\alpha_{p_1} \setminus h^\alpha_p$ (α a limit) such that $\zeta_\beta \in H^\alpha_p$. Also let h^β be the union of h^β_p plus the union of those $\zeta_\beta \cap h^\alpha_{p_1} \setminus h^\alpha_p$ for which $\zeta_\beta \in H^\alpha_p$ and $p \Vdash \rho_\alpha(\zeta_\beta) = 1$.

Claim 1. If $\xi \in h^\beta$ and $\xi \in h^\alpha_{p_1} \setminus h^\alpha_p$ for some limit α and $\zeta \in H^\alpha_p$ with $p \Vdash \dot{t}_\beta = \dot{\rho}_\alpha \restriction \zeta$, then $p \Vdash \dot{\rho}_\alpha(\zeta) = 1$.

This holds by Lemma 3.18 because q forces that $\xi \notin a[\rho_\alpha, \zeta]$.

Claim 2. For each successor $\beta \in \mathrm{dom}(p_1)$, $H^\beta \subset H^\beta_{p_1}$ and $h^\beta \subset h^\beta_{p_1}$.

This follows because $p_1 \not\perp p$ and so if there is an $\alpha \in \mathrm{dom}(p_1)$ and a $\zeta \in H^\alpha_p$ with $p \Vdash \dot{\rho}_\alpha \restriction \zeta = \dot{t}_\beta$, then $\zeta \in H^\alpha_p \cap M \subset H^\alpha_{p_1}$. Since $p_1 \Vdash h^\alpha_{p_1} \setminus h^\alpha_p$ is disjoint from $a[\dot{\rho}_\alpha, \zeta]$ we have that $h^\alpha_{p_1} \setminus h^\alpha_p \subset H^\beta_{p_1}$ and $h^\alpha_{p_1} \setminus h^\alpha_p$ intersected with $h^\beta_{p_1}$ is the appropriate value.

Now we define $\bar{q}$ with $\mathrm{dom}(\bar{q}) = \mathrm{dom}(p_1) \cup \mathrm{dom}(p)$ by induction on $\alpha \in \mathrm{dom}(\bar{q})$. To start, $\bar{q}(0) = p_1(0)$. For α a successor in $\mathrm{dom}(p_1)$ we set $\bar{q}(\alpha) = p_1(\alpha)$. For α a limit in $\mathrm{dom}(p_1)$, we set $\bar{q}(\alpha)$ equal to $(h^\alpha_{p_1}, H^\alpha_{p_1} \cup H^\alpha_p)$.

We make note of a trivial claim:

Claim 3. For $\alpha \in \mathrm{dom}(p_1)$, $p_1 \Vdash \bar{q}(\alpha) \leq p_1(\alpha)$.

For α a limit in $\mathrm{dom}(p) \setminus M$, $\bar{q}(\alpha) = p(\alpha)$. Let L be the set of successors α in $\mathrm{dom}(p) \setminus M$ such that $p \Vdash \dot{t}_{\gamma_\delta} \subset \dot{t}_\alpha$. Note that that for each $\alpha \in \mathrm{dom}(p)$, p has decided if $\dot{t}_{\gamma_\delta} \subset \dot{t}_\alpha$ since p is determined and $\delta \in H^\alpha_p$ if $p \Vdash \delta \in \mathrm{dom}(\dot{t}_\alpha)$. Notice that γ_δ is the minimum element of L.

For α a successor in $\mathrm{dom}(p) \setminus M$ that is not in L, then

$$\bar{q}(\alpha) \quad \text{is set equal to} \quad q(\alpha, H^\alpha \setminus H^\alpha_p, h^\alpha \setminus H^\alpha_p, \dot{b}^\alpha_p, \dot{c}^\alpha_p)\ .$$

and by Lemma 3.16

Claim 4. For each successor $\alpha \in \mathrm{dom}(p) \setminus (M \cup L)$, $p \restriction \alpha$ forces $\bar{q}(\alpha) \in \dot{Q}_\alpha$ and $\bar{q}(\alpha) \leq p(\alpha)$.

Now we consider $\alpha \in L$.

If $\alpha = \gamma_\delta$, then $\bar{q}(\alpha) = q(\alpha, H^\alpha \setminus H^\alpha_p, h^\alpha \setminus H^\alpha_p, \dot{b}^\alpha_p, \dot{c}^\alpha_p, k)$. It is still clear that $p \restriction \gamma_\delta$ forces that $\bar{q}(\gamma_\delta) < p(\gamma_\delta)$, but also since p_1 forces that $k \notin c^\alpha_p$, we have

Claim 5. For any condition $r \in \mathbb{P}_\alpha$ such that $r \leq p_1 \restriction \alpha$ and $r \leq p \restriction \alpha$, we have that r forces that $k \in \dot{a}_{\rho\restriction\delta+1}$.

If $\gamma_\delta < \alpha \in L$ then we again define $\bar{q}(\alpha)$ to simply be $q(\dot{t}_\alpha, H^\alpha \setminus H^\alpha_p, h^\alpha \setminus H^\alpha_p, \dot{b}^\alpha_p, \dot{c}^\alpha_p)$; but we note in passing that $p \restriction \alpha + 1$ does not determine if $k \in \dot{a}_{\dot{t}_\alpha 0}$.

Similar to Claim 4

Claim 6. For each $\alpha \in L$, p forces $\bar{q}(\alpha) \leq p(\alpha)$.

It remains to prove (by induction)

Claim 7. For each limit $\alpha \in \mathrm{dom}(p) \setminus M$, $\bar{q} \restriction \alpha \Vdash \bar{q}(\alpha) < p(\alpha)$.

Proof of Claim 7: By induction we have that $\bar{q} \restriction \alpha$ is below $p_1 \restriction \alpha$ and $p \restriction \alpha$. To prove that $\bar{q}(\alpha) = (h^\alpha_{p_1}, H^\alpha_p \cup H^\alpha_{p_1})$ is below $p(\alpha)$ it suffices to prove that $\bar{q} \restriction \alpha$ forces that $\xi \notin a[\rho_\alpha, \zeta]$ for each $\xi \in h^\alpha_{p_1} \setminus h^\alpha_p$ and $\zeta \in H^\alpha_p$. Choose $\beta \in \mathrm{dom}(p)$ so that $p \restriction \alpha \Vdash \dot{t}_\beta = \rho_\alpha \restriction \zeta$. For $\zeta \in H^\alpha_p \cap M$, $\xi \notin a[\rho_\alpha, \zeta]$ follows from the fact that $p_1 \not\perp p$. For $\zeta < \xi$, this is vacuous in the sense that $a[\rho_\alpha, \zeta] \subset \zeta$ but also since we then have that $\zeta \leq \xi \in M$. Therefore we may assume that $\xi < \zeta$, and so we have that $\xi \in H^\beta$. Again, since p is determined, p forces a value on $\dot{\rho}_\alpha(\zeta)$. By symmetry, assume that $p \Vdash \dot{\rho}_\alpha(\zeta) = 1$. Assume first that $\xi \in H^\beta_p$. It then follows from Lemma 3.18 that p decides the statement $\xi \in a[\dot{\rho}_\alpha, \zeta]$. Since p_1 is compatible with p, we have that $\xi \notin a[\dot{\rho}_\alpha, \zeta]$ as required. Now suppose that $\xi \in H^\beta \setminus H^\beta_p$. Since $p \Vdash \dot{\rho}_\alpha(\xi) = 1$, we have that $\xi \in h^\beta \setminus H^\beta_p$. By the definition of $\bar{q}(\beta)$, it is apparent that $\bar{q} \restriction \beta + 1$ forces that $\xi \notin a[\dot{t}_\beta 1, \zeta]$ and so $\bar{q} \restriction \alpha$ forces that $\xi \notin a[\dot{\rho}_\alpha, \zeta]$. □

This completes the proof of Lemma 3.19 □

Question 1. Does $\mathfrak{p} > \omega_1$ imply that compact C-closed spaces are sequential?

Question 2. Is there a weak $\diamondsuit$-principle from [6] that implies that C-closed compact spaces are sequential?

Question 3. If $K \subset [0,1]^{\omega_1}$ has countable tightness, is $\overline{K}$ sequentially compact in the forcing extension by adding uncountably many Cohen reals?

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