EFIMOV'S PROBLEM AND BOOLEAN ALGEBRAS

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Abstract. We continue the study, started by P. Koszmider, of a class of Boolean algebras, the so called $\mathcal{T}$-*algebras*. We prove the following.

(1) All superatomic Boolean algebras belong to this class.
(2) This class is contained properly in Koppelberg's class of minimally generated Boolean algebras.
(3) The existence of an Efimov $\mathcal{T}$-algebra (i.e., a $\mathcal{T}$-algebra whose Stone space is infinite and contains no converging sequence and no copy of $\beta\omega$) implies a negative answer to Scarborough-Stone's problem.
(4) There is an Efimov $\mathcal{T}$-algebra of countable tightness in the generic extension obtained by a finite support iteration of length ω_2 of Hechler's poset over a model of CH.

1. Introduction

This paper focuses on the study of $\mathcal{T}$-algebras (a class of Boolean algebras which was introduced by P. Koszmider in [22]) and specially on their conection with Efimov's problem: is there an infinite compact Hausdorff space which contains no infinite converging sequence and no copy of $\beta\omega$, the Stone-Čech compactification of the integers? Such a space will be called an *Efimov space*. An Efimov space is clearly not sequential and so one which has countable tightness is an example of a *Moore-Mrówka space*, i.e., a countably tight compact space which fails to be sequential.

$\mathcal{T}$-algebras were introduced and developed in [22] as a special method of building minimally generated Boolean algebras [21] with a generating family indexed by a tree. Koszmider notes that $\mathcal{T}$-algebras have the very special feature of presenting a natural correspondence between the maximal branches of the underlying tree and the ultrafilters of the Boolean algebra generated. Minimally generated Boolean algebras have their origins in Fedorchuk's method of resolutions [12] and have been utilized many times to solve fundamental problems exploring the connections between countable tightness, hereditary density, the character, and the abundance or absence of converging sequences in compact spaces (see [13, 5, 24, 18] for excellent examples).

All superatomic Boolean algebras are minimal ([21]) and are also $\mathcal{T}$-algebras (see Proposition 4.1). However we produce the first example of a minimal Boolean algebra which is not a $\mathcal{T}$ -algebra in Theorem 4.2. Minimally generated Boolean algebras have a special connection to Efimov's problem since Koppelberg ([21])

2010 *Mathematics Subject Classification.* Primary: 06E15; Secondary: 54A35, 03E35.

Key words and phrases. Efimov's problem, minimally generated Boolean Algebras, finite support iteration, Hechler's poset.

The research of the first author was supported by NSF Grant No. NSF-DMS-0901168.

The second author acknowledges financial support by Programa de Becas Posdoctorales en la UNAM 2012-2013.

showed that their Stone spaces can not contain $\beta\omega$. In fact, the algebra has no uncountable independent family, which is to say the free algebra on ω_1 generators cannot be embedded as a subalgebra of a minimally generated Boolean algebra. We establish another omitting subalgebra result by showing that the Stone space of an Efimov $\mathcal{T}$-algebra will not map onto the product space $(\omega_1 + 1) \times (\omega + 1)$ (see Theorem 5.4). This is intended as an illustration of an obstruction in possible inductive constructions of Efimov $\mathcal{T}$-algebras.

Let us recall that the cardinal $\mathfrak{s}$ is the least cardinality of a splitting family of subsets of ω and that $\mathfrak{d}$ (resp., $\mathfrak{b}$) is the least cardinality of a dominating (resp., unbounded) family of functions from ω to ω (see [10]).

One of the original motivations for the introduction of $\mathcal{T}$-algebras is that Fedorchuk's classical example of a compact S-space from $\diamondsuit$ (see [13]) is a $\mathcal{T}$-algebra. This space is also an Efimov space of countable tightness. It was, along with Ostaszewski's space, the earliest example of the consistency of the existence of a Moore-Mrówka space. The main result of the paper (in section 6) is to establish that in the standard Hechler model for $\mathfrak{b} = \mathfrak{c} = \omega_2$, which is obtained by a finite support iteration of a ccc poset adding a dominating real, there is an Efimov $\mathcal{T}$-algebra. There are two additional special properties of this example. In the first place, its Stone space has countable tightness, and so is also a Moore-Mrówka space. The proper forcing axiom, PFA, implies that $\mathfrak{b} = \mathfrak{c} = \omega_2$ and that Moore-Mrówka spaces do not exist (see [2]). While it is not known if there is a ZFC example of an Efimov space, it is evident that PFA implies they do not have countable tightness.

Our space is the first example of an Efimov $\mathcal{T}$-algebra in a model of $\mathfrak{b} > \omega_1$. It is constructed in the same manner as the example in [22, Theorem 4.7] which is done in the usual Cohen model. The second special property of our example is that the height of the tree is only ω_1 (as it is in the Cohen model).

The character of an ultrafilter on a $\mathcal{T}$-algebra is bounded by the cardinality of the associated branch (Corollary 3.7). In a topological space, any point with character less than $\mathfrak{p}$ which is a limit point of a countable set will have sequences converging to it (see *Proof that* $\mathfrak{p} \leq \mathfrak{p}_\chi$ on page 130 of [10]). Therefore in models of Martin's Axiom, for example, the minimum height of an Efimov $\mathcal{T}$-algebra will be $\mathfrak{c}$. In fact, if we denote by $\mathfrak{h}$ the distributivity degree of the Boolean algebra $\mathcal{P}(\omega)/fin$ (see [1]), it can be shown (but this will be material for a subsequent paper) that the minimum height of an Efimov $\mathcal{T}$-algebra is at least $\mathfrak{h}$ and so can be greater than $\mathfrak{p}$. Analysis of the minimum height led to the paper [9] where it is shown that the assumption $\mathfrak{b} = \mathfrak{c}$ implies the existence of an Efimov $\mathcal{T}$-algebra (of height $\mathfrak{c}$ and uncountable tightness). We expect to explore restrictions on the possible heights of Efimov $\mathcal{T}$-algebras in a subsequent paper, for example, we have established that in some models of $\mathfrak{h} = \mathfrak{b} < \mathfrak{s}$, such as [3], there is no Efimov $\mathcal{T}$-algebra of height $\mathfrak{h}$.

In Section 4 we show that the existence of an Efimov $\mathcal{T}$-algebra implies a negative answer to the celebrated Scarborough-Stone problem. The Scarborough-Stone problem asks if the product of sequentially compact spaces is necessarily countably compact. It is interesting that this problem, like the Efimov space problem, is also open in ZFC. The Scarborough-Stone problem has been resolved under $\mathfrak{b} = \mathfrak{c}$ by van Douwen [10] and in special models of $\mathfrak{b} < \mathfrak{d}$ by Nyikos and Vaughan [25]. Efimov spaces are known to exist in a variety of models. The hypothesis $\mathfrak{b} = \mathfrak{c}$ mentioned above and the hypotheses $\mathrm{cf}([\mathfrak{s}]^\omega) = \mathfrak{s}$ plus $2^\mathfrak{s} < 2^\mathfrak{c}$ (see [14, 15, 8, 6]) may cover all known cases. However, the existence of Efimov $\mathcal{T}$-algebras is not known in all such models. An interesting plausible conjecture, which would be new for both problems, is that $\mathfrak{d} = \omega_1$ (even for specific models) may imply that there is an Efimov $\mathcal{T}$-algebra of height ω_1.

The structure of this article is as follows, section 3 presents material on $\mathcal{T}$-algebras that will be used in later sections. Section 4 is dedicated to three examples. The fifth section deals with two consequences of the existence of an Efimov $\mathcal{T}$-algebra (i.e., a $\mathcal{T}$-algebra whose Stone space is an Efimov space): (1) Scarborough-Stone's problem would have a negative answer and (2) the Stone space of such an algebra does not map continuously onto the product $(\omega_1 + 1) \times (\omega + 1)$. Finally, section 6 establishes that it is consistent with $\neg\mathsf{CH}$ that Efimov $\mathcal{T}$-algebras (of countable tightness) do exist.

2. Notation and Definitions

For a function f and a set S, the symbol $f \restriction S$ will denote the restriction of f to S.

If t is a function whose domain, $\mathrm{dom}(t)$, is an ordinal, then we define, for any i, $t^\frown i := t \cup \{(\mathrm{dom}(t), i)\}$. Equivalently, if α is the domain of t, then $t^\frown i$ is the only function satisfying $\mathrm{dom}(t^\frown i) = \alpha + 1$, $(t^\frown i) \restriction \alpha = t$ and $(t^\frown i)(\alpha) = i$.

Given an ordinal ε, we will denote by $2^{<\varepsilon}$ the collection of all functions whose domain is an ordinal $< \varepsilon$ and whose image is a subset of $\{0, 1\}$. For all $s, t \in 2^{<\varepsilon}$ we define $s \leq t$ iff t extends s, i.e., $s \subseteq t$. This relation turns out to be a tree ordering (as defined in [23, Section III.5]) for $2^{<\varepsilon}$. Note that any subset of $2^{<\varepsilon}$ with the tree ordering described above is itself a tree. A similar discussion applies to $2^{\leq\varepsilon}$.

Definition 2.1. If $t \in 2^{<\varepsilon}$ and $\mathrm{dom}(t) = \alpha + 1$, we let $t^* := (t \restriction \alpha)^\frown(1 - t(\alpha))$.

Note that if t and α are as in the previous definition, then $t \restriction \alpha = t^* \restriction \alpha$, but $t(\alpha) \neq t^*(\alpha)$.

A *branch* in a tree T is a maximal chain in T. The height of a node $t \in T$ will be represented by $\mathrm{ht}(t, T)$ or by $\mathrm{ht}(t)$ when the tree is clear from the context.

To simplify notation, all Boolean algebras discussed in this paper are assumed to be subalgebras of the power set of some set Z. For similar reasons, $-a$ will denote the complement of $a \subseteq Z$ with respect to Z.

We follow closely the notation and terminology used in [20], except for some minor details. For example, we use $\mathrm{St}(A)$ to denote the Stone space of a Boolean algebra A. Also, the collection of all clopen subsets of a topological space X will be denoted by $CO(X)$.

A *Boolean space* is a compact T_2 zero-dimensional topological space; equivalently, any space homeomorphic to the Stone space of a Boolean algebra.

All topological notions should be understood as in [11].

3. $\mathcal{T}$-Algebras

Let A be a Boolean subalgebra of B. We say that B is *minimal over* A (in symbols, $A \leq_m B$) if no proper subalgebra of B contains A as a proper subset. This notion was introduced in [21] by S. Koppelberg, where she also proves that $A \leq_m B$ iff (i) B is the Boolean algebra generated by $A \cup \{x\}$, for some $x \in B$, and (ii) at most one ultrafilter of A is contained in two different ultrafilters of B (see [21, Lemma 1.2]). When $x \notin A$, there is exactly one ultrafilter u with this characteristic and, in this case, we will say that *x is minimal for* (A, u).

A Boolean algebra A is *minimally generated* if there exist an ordinal ε and a family $\{A_\alpha : \alpha < \varepsilon\}$ of subalgebras of A such that (1) A_0 is the two-element algebra; (2) if $\alpha + 1 < \varepsilon$, then $A_\alpha \leq_m A_{\alpha+1}$; (3) if $\alpha < \varepsilon$ is a limit ordinal, then $A_\alpha = \bigcup_{\xi<\alpha} A_\xi$; and (4) $\bigcup_{\alpha<\varepsilon} A_\alpha = A$. Informally speaking, a Boolean algebra is minimally generated if one can construct it by small, indivisible steps.

In order to establish a topological translation of minimality some concepts are needed. First, given two Boolean spaces, X and Y, we will say that Y is a *simple extension* of X if there is a continuous map $f : X \to Y$ in such a way that, for some point $y \in Y$, the fiber $f^{-1}[y]$ contains exactly two points while all sets of the form $f^{-1}[x]$, $x \in Y \setminus \{y\}$, are singletons. Then $A \leq_m B$ iff $\mathrm{St}(B)$ is a simple extension of $\mathrm{St}(A)$ and, moreover, X is a simple extension of Y iff $CO(Y)$ is minimal over a subalgebra which is isomorphic to $CO(X)$.

Now the second concept: a *simplistic system* is an inverse system, $\langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \varepsilon\rangle$, of Boolean spaces where (1) X_0 is a singleton, (2) $X_{\alpha+1}$ is a simple extension of X_α, whenever $\alpha + 1 < \varepsilon$, and (3) if $\gamma < \varepsilon$ is a limit ordinal, then X_γ is homeomorphic to the inverse limit of $\langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \gamma\rangle$. A topological space will be called *simplistic* if it is homeomorphic to the inverse limit of a simplistic system. Straightforward arguments show that X is simplistic iff $CO(X)$ is minimally generated and vice versa, a Boolean algebra A is minimally generated iff $\mathrm{St}(A)$ is simplistic.

It is proved in [21, Example 2.4]) and [21, Corollary 1.7], respectively, that (1) the free Boolean algebra on ω_1 generators is not minimally generated and (2) every subalgebra of a minimally generated Boolean algebra is itself minimally generated. Hence no simplistic space maps onto the topological product 2^{ω_1} and so we obtain:

Remark 3.1. No simplistic space contains a copy of $\beta\omega$.

The following notions appeared first in [22] (recall Definition 2.1).

Definition 3.2. Let ε be an ordinal. A set $T \subseteq 2^{<\varepsilon}$ will be called an *acceptable tree* if the following holds

(1) The domain of each member of T is a succesor ordinal.
(2) For all $t \in 2^{<\varepsilon}$, $t \in T$ iff $t^* \in T$.

Definition 3.3. Let T be an acceptable tree and let A be a Boolean algebra. A is a *T-algebra* if

(1) There is a function $a : T \to A$ whose range, $\{a(t) : t \in T\}$, generates A (it will be a common practice to write a_t instead of $a(t)$).
(2) For each $t \in T$, a_t is minimal for (A_t, u_t), where A_t denotes the Boolean algebra generated by $\{a_s : s < t\}$ in A and u_t is the filter generated by $\{a_s : s < t\}$ in A_t.
(3) For any $t \in T$ we have $a(t^*) = -a_t$. Equivalently, $s^\frown 0 \in T$ implies $a_{s^\frown 0} = -a_{s^\frown 1}$.

Naturally, a collection $\{a_t : t \in T\}$ as the one described in the definition *witnesses that A is a T-algebra*.

There are two comments we need to make. First, we are commiting an abuse of notation: $\{a_s : s < t\}$ really means $\{a_s : s \in T \wedge s < t\}$. Secondly, it is implicit in condition (2) that $\{a_s : s < t\}$ has the finite intersection property and that u_t turns out to be an ultrafilter in A_t because it contains all its generators.

To simplify things we will adopt the following convention: the phrase "B is a $\mathcal{T}$-algebra" means that there is an acceptable tree T such that B is a T-algebra.

When we refer to the height of a $\mathcal{T}$-algebra A, we mean the minimum height of a tree T witnessing that A is a T-algebra.

Any $\mathcal{T}$-algebra is minimally generated as proved in [22, Fact 2.10 on p. 3081].

Lemma 3.4. *Let T be an acceptable tree and let $\{a_t : t \in T\}$ be a set of generators for the Boolean algebra A. Then condition* (2) *in Definition 3.3 holds iff for each $t \in T$ we have that*

($2'$) *$\{a_s : s < t\}$ has the finite intersection property and*

($2''$) *$a_t - a_s \in A_t$ for all $s < t$.*

Proof. Assume that condition (2) holds. Only ($2''$) needs an argument, so let us start by fixing $s < t$. Given that a_t is minimal for (A_t, u_t), we apply [22, Proposition 2.2] to obtain $u_t = \{x \in A_t : x \cap a_t \notin A_t\}$. Since u_t is an ultrafilter in A_t and $a_s \in u_t$, we get $-a_s \notin u_t$, i.e., $(-a_s) \cap a_t \in A_t$.

Suppose now that ($2'$) and ($2''$) hold and let A_t and u_t be as defined in condition (2). Then u_t is an ultrafilter in A_t. The set $I := \{x \in A_t : x \cap a_t \in A_t\}$ is, according to [21, Lemma 1.1], an ideal in A_t. Since the dual filter of I is $F := \{x \in A_t : (-x) \cap a_t \in A_t\}$ and $u_t \subseteq F$ (as a consequence of ($2''$)), we have that $u_t = F$. In particular, F is an ultrafilter and therefore, for each $x \in A_t$, $x \in F$ iff $-x \notin F$, i.e., $x \in u_t$ iff $x \cap a_t \notin A_t$. So we invoke [22, Proposition 2.2] to conclude that a_t is minimal for (A_t, u_t). □

The following was proved in [22, Lemma 2.8].

Proposition 3.5. *Let A be a T-algebra as witnessed by $\{a_t : t \in T\}$. Then*

(1) *For each $u \in \mathrm{St}(A)$ there is a branch $b \subseteq T$ such that u is the ultrafilter generated by $\{a_t : t \in b\}$ in A.*

(2) *If b is a branch in T, the set $\{a_t : t \in b\}$ generates an ultrafilter in A.*

To prove the first part of this proposition one constructs, by transfinite induction, a branch as follows: we start by selecting the only node on level 0 of T, let us say, t_0 which satisfies $a(t_0) \in u$ and at stage α we select, if possible, an upper bound for $\{t_\xi : \xi < \alpha\}$ on level α, let us say, t_α in such a way that $a(t_\alpha) \in u$. This argument and Stone's representation theorem ([20, Theorem 7.8]) prove the following:

Remark 3.6. Let X be a Boolean space whose clopen algebra is a T-algebra as witnessed by $\{a_t : t \in T\}$ and let b be a branch in T. If $t \in T$ satisfies

$$\bigcap\{a_s : s \in b\} \cap \bigcap\{a_s : s \leq t\} \neq \emptyset,$$

then $\{s \in T : s \leq t\} \subseteq b$.

Our next result will be used several times.

Corollary 3.7. *Let X be a Boolean space for which $CO(X)$ is a T-algebra as witnessed by $\{a_t : t \in T\}$. Then:*

(1) *For each $x \in X$ there is a branch $b \subseteq T$ such that the following three equivalent conditions hold.*

 (a) *The ultrafilter $\{c \in CO(X) : x \in c\}$ is generated by $\{a_t : t \in b\}$.*

 (b) *$\{\bigcap\{a_t : t \in F\} : F \in [b]^{<\omega} \setminus \{\emptyset\}\}$ is a local base for X at x.*

 (c) *$\bigcap\{a_t : t \in b\} = \{x\}$.*

(2) *If b is a branch in T, $\bigcap\{a_t : t \in b\}$ is a singleton.*

Proof. Since X is a Boolean space, the map $f : X \to \mathrm{St}(CO(X))$ given by

$$f(z) := \{c \in CO(X) : z \in c\}$$

is a homeomorphism.

By Proposition 3.5, T has a branch b such that $f(x)$ is the ultrafilter generated by $\{a_t : t \in b\}$. Thus we get condition 1-(c) because $\bigcap f(x) = \{x\}$.

To prove that all conditions listed in (1) are equivalent notice that (b) is a consequence of (a) because X is zero-dimensional and that (b) implies (c) because X is Hausdorff. To finish the argument let us assume (c) and let $x \in c \in CO(X)$. Then $\bigcap\{a_t : t \in b\} \subseteq c$ and since X is compact, there is a finite set $F \subseteq b$ so that $\{a_t : t \in F\} \subseteq c$. Therefore we obtain (a).

To prove (2), observe that if we let u be the ultrafilter generated by $\{a_t : t \in b\}$, then there is $z \in X$ with $f(z) = u$ and so z is the only element of $\bigcap\{a_t : t \in b\}$. □

Lemma 3.8. *Let X be a Boolean space for which $CO(X)$ is a T-algebra as witnessed by $\{a_t : t \in T\}$.*

(1) *If $s, t \in T$ are comparable and $W := \bigcap\{a_r : r \leq s^*\}$, then either a_t and W are disjoint or $W \subseteq a_t$.*
(2) *Assume that $\langle t_n : n \in \omega\rangle$ is an increasing sequence in T and that for each $n \in \omega$ we have $\{x_n, y_n\} \subseteq \bigcap\{a_r : r \leq t_n^*\}$. Then*

$$\overline{\{x_k : k \in \omega\}} \cap \overline{\{y_k : k \in \omega\}} \neq \emptyset.$$

Proof. To prove (1) let us assume that $W \cap a_t \neq \emptyset$. Fix a branch b in T satisfying $s, t \in b$. Note that $T' := b \cup \{r^* : r \in b\}$ is an acceptable tree and B, the Boolean algebra generated by $\{a_r : r \in b\}$ in $CO(X)$, is a T'-algebra. Moreover, $d := \{r \in T : r \leq s^*\}$ is a branch in T' and therefore $\{a_r : r \in d\}$ generates an ultrafilter in B (Proposition 3.5) which will be denoted by u.

Our assumption implies that $\{a_t\} \cup \{a_r : r \in d\}$ has the finite intersection property, so $a_t \in u$ and, in particular, there is a finite set $F \subseteq d$ such that

$$W \subseteq \bigcap\{a_r : r \in F\} \subseteq a_t.$$

Now let us prove the second part of our Lemma. Set $S := \{x_k : k \in \omega\}$ and $a_n := \bigcap\{a_s : s \leq t_n\}$ for each integer n. Note that our hypotheses imply that if $m \in \omega$, then $S \setminus \{x_k : k \leq m\} \subseteq a_m$ and therefore $\{a_n \cap \overline{S} : n \in \omega\}$ is a decreasing sequence of nonempty closed subsets of X. Let $z \in \overline{S} \cap \bigcap\{a_n : n < \omega\}$ and let b be a branch in T such that $\bigcap\{a_s : s \in b\} = \{z\}$ (Proposition 3.5).

We will show that $z \in \overline{\{y_n : n \in \omega\}}$ with the aid of Proposition 3.7, so let F be a finite subset of b. Then there is an integer m such that $x_m \in \bigcap\{a_s : s \in F\}$ and, on the other hand, an straightforward application of Remark 3.6 produces $t_m \in b$. Hence $\bigcap\{a_s : s \leq t_m^*\} \subseteq a_t$, for all $t \in F$ (part (1) of the Lemma). Which gives $y_m \in \bigcap\{a_s : s \in F\}$. □

4. Examples

It is proved in [21, Example 2.3] that all superatomic Boolean algebras are minimally generated. Our next proposition strengthens this result by showing that all superatomic Boolean algebras are, in fact, $\mathcal{T}$-algebras (see [20, Remark 17.2]).

Recall that a topological space is *scattered* if every subspace of it has an isolated point.

Proposition 4.1. *If X is a compact Hausdorff scattered space, then there is an acceptable tree T for which $CO(X)$ is a T-algebra.*

Proof. For each ordinal α let X_α be the set of isolated points of $X \setminus \bigcup\{X_\xi : \xi < \alpha\}$. Since X is compact scattered, there exists δ in such a way that $X = \bigcup\{X_\alpha : \alpha \leq \delta\}$ and $X_\delta \neq \emptyset$.

Fix a well-ordering $\prec$ on X for which $\alpha < \beta \leq \delta$, $x \in X_\alpha$, and $y \in X_\beta$ imply $x \prec y$. Since X is compact, X_δ is finite so let z be the $\prec$-last element of X_δ (note that z is actually the $\prec$-maximum of X). Denote by ε the order type of $(X \setminus \{z\}, \prec)$ and let $h : \varepsilon \to X \setminus \{z\}$ be an order isomorphism.

Given $\alpha \leq \delta$ and $x \in X_\alpha \setminus \{z\}$ let us fix W_x, a clopen subset of X, such that $W_x \setminus \bigcup\{X_\xi : \xi < \alpha\} = \{x\}$. Observe that $W_z := X \setminus \bigcup\{W_x : x \in X_\delta \setminus \{z\}\}$ is clopen in X and $W_z \cap X_\delta = \{z\}$.

Let $f : \varepsilon \to 2$ be the constant zero function, i.e., $f(\alpha) = 0$ for all $\alpha < \varepsilon$. Then $T := \{(f \restriction \alpha)^\frown i : \alpha < \varepsilon \text{ and } i < 2\}$ is an acceptable tree and $T \subseteq 2^{<\varepsilon}$. For each $\alpha < \varepsilon$ define

$$a((f \restriction \alpha)^\frown 0) := X \setminus W_{h(\alpha)} \text{ and } a((f \restriction \alpha)^\frown 1) := W_{h(\alpha)}.$$

Note that if $\alpha < \varepsilon$, then $\{W_{h(\alpha)} \setminus \bigcup\{W_{h(\xi)} : \xi \in H\} : H \in [\alpha]^{<\omega}\}$ is a local base for X at $h(\alpha)$; therefore $CO(X)$ is generated by $\{a_t : t \in T\}$.

Let $\alpha < \varepsilon$ be arbitrary and set $t := (f \restriction \alpha)^\frown 0$. To finish the proof we will show that conditions $(2')$ and $(2'')$ in Lemma 3.4 hold for t. This will suffice because a simple modification of our argument proves that the same is true for t^*.

Notice that if $s \in T$ satisfies $s < t$, then $s = (f \restriction \beta)^\frown 0$ for some $\beta < \alpha$. Hence $h(\alpha) \in a_s$ and therefore $\{a_r : r < t\}$ has the finite intersection property. On the other hand, $a_t - a_s$ is compact open and $a_t - a_s = W_{h(\beta)} \setminus W_{h(\alpha)}$ so there are finite sets $F \subseteq \beta + 1$ and $H_\xi \subseteq \xi$, for each $\xi \in F$, satisfying $a_t - a_s = \bigcup\{W_{h(\xi)} \setminus \bigcup\{W_{h(\eta)} : \eta \in H_\xi\} : \xi \in F\}$; thus $a_t - a_s \in A_t$. □

Theorem 4.2. *There is a minimally generated Boolean algebra which is not a $\mathcal{T}$-algebra.*

Proof. Our strategy is to construct a simplistic system so that the clopen algebra of its limit is as required in the statement of the theorem.

Enumerate all rational numbers in the Cantor set, $2^\omega \cap \mathbb{Q} = \{q_n : n \in \omega\}$, and define, by induction, a sequence $\langle Z_m, g_m : m \in \omega\rangle$ of topological spaces and mappings so that

(1) $Z_0 := 2^\omega$,
(2) $Z_{m+1} = Z_m \oplus \{(q_0, m)\}$, and
(3) $g_m : Z_{m+1} \to Z_m$ is continuous, $g_m \restriction Z_m$ is the identity map, and $g_m(q_0, m) = q_0$.

In other words, Z_{m+1} is obtained from Z_m by splitting q_0 into two points and making one of them isolated.

Notice that the sequence we just defined is an inverse system based on simple extensions. Let Y_1 be its limit and let $h_0 : Y_1 \to 2^\omega$ be the corresponding projection map (i.e., $h_0(x)$ is the 0th coordinate of x).

Y_1 is homeomorphic to the subspace $(2^\omega \times \{\omega\}) \cup (\{q_0\} \times \omega)$ of the topological product $2^\omega \times (\omega + 1)$, i.e., Y_1 results of adding a converging sequence with limit q_0 to the space $Y_0 := 2^\omega$.

The process described in the second and third paragraph of this argument applied to Y_1 and q_1 (instead of 2^ω and q_0) produces Y_2 and, in general, we obtain an inverse system $\langle Y_m, h_m : m \in \omega\rangle$, where each Y_m is homeomorphic to the subspace $(2^\omega \times \{\omega\}) \cup (\{q_i : i \le m\} \times \omega) \subseteq 2^\omega \times (\omega+1)$ and h_m collapses the new converging sequence to a point: $h_m(q_m, i) = (q_m, \omega)$ for all $i < \omega$. Let X_0 be the limit of this inverse system.

X_0 is homeomorphic to the space obtained by endowing the set $2^\omega \cup ((\mathbb{Q} \cap 2^\omega) \times \omega)$ with the following topology: each (q, m) is isolated and a local base for $r \in 2^\omega$ is given by all sets of the form

$$W \cup ((W \cap \mathbb{Q}) \times \omega) \setminus F),$$

where W is an arbitrary clopen subset of 2^ω which contains r and F is a finite set (moreover, when $r \notin \mathbb{Q}$ one can take $F = \emptyset$). For this reason we will assume, for the rest of the proof, that X_0 is actually the space we just described.

Let $\{(r_\alpha, m_\alpha) : \alpha < \mathfrak{c}\}$ be an enumeration of all pairs (r, m) so that

(a) $r : 2^{<\omega} \to \mathbb{Q} \cap 2^\omega$ and $m : 2^{<\omega} \to \omega$.
(b) For all $g \in 2^\omega$ the sequence $\langle r(g \restriction n) : n \in \omega\rangle$ converges.
(c) If $f, g \in 2^\omega$ satisfy $f \ne g$ then

$$\lim_{n\to\infty} r(f \restriction n) \ne \lim_{n\to\infty} r(g \restriction n).$$

For each $\alpha < \mathfrak{c}$ we will obtain, by transfinite induction, a function $g_\alpha \in 2^\omega$ so that

$$(*) \qquad x_\alpha := \lim_{n\to\infty} r_\alpha(g_\alpha \restriction n) \notin \{x_\xi : \xi < \alpha\} \cup \mathbb{Q}$$

and a topology τ_α for $X_\alpha := X_0 \cup \{(x_\xi, 0) : \xi < \alpha\}$ together with mappings $f_{\beta\alpha}$, $\beta < \alpha$, in such a way that $S := \langle X_\beta, f_{\beta\gamma} : \beta < \gamma < \mathfrak{c}\rangle$ ends up being a continuous inverse system which satisfies

(1α) If $\beta < \alpha$, then $f_{\beta\alpha} \restriction X_\beta$ is the identity map and $f_{\beta\alpha}(x_\xi, 0) = x_\xi$ whenever $\beta \le \xi < \alpha$.
(2α) The sequence $e_\alpha := \{(r_\alpha(g_\alpha \restriction n), m_\alpha(n)) : n \in \omega\}$ converges to x_α in τ_α.
(3α) $\tau_\alpha \cup \{e_\alpha \cup \{(x_\alpha, 0)\}, X_\alpha \setminus e_\alpha\}$ is a subbase for $\tau_{\alpha+1}$.

Observe that, according to this prescription, the inverse system is based on simple extensions. More precisely, at stage $\alpha + 1$ the point x_α is doubled and e_α becomes a converging sequence to the 'twin' of x_α, namely, $(x_\alpha, 0)$.

We only have to explain how to get $\tau_{\alpha+1}$ from τ_α. Condition (c) above implies that $|\{x_\xi : \xi < \alpha\}| < \mathfrak{c} = \left|\left\{\lim_{n\to\infty} r_\alpha(g \restriction n) : g \in 2^\omega\right\}\right|$ and therefore we can find $g_\alpha \in 2^\omega$ satisfying (*). As one can verify, a local base at any given point $z \in X_\alpha$ in τ_α is given by all sets of the form

$$(W \setminus \bigcup\{e_\xi : \xi \in F\}) \cup ((W \cap \{x_\xi : \xi \in \alpha \setminus F\}) \times \{0\}),$$

where W is a clopen set in X_0 containing z and F is an arbitrary finite subset of α. Therefore, our choice of x_α is in complete agreement with (2α). This completes the induction.

Let X be the limit of S. To prove that X is simplistic we only need to concatenate all inverse systems involved in the construction of this space. Hence $A := CO(X)$ is minimally generated.

X will be identified with $X_0 \cup \{(x_\alpha, 0) : \alpha < \mathfrak{c}\}$ in such a way that X_0 is a subspace of X and if $\alpha < \mathfrak{c}$, then $\{\{(x_\alpha, 0)\} \cup (e_\alpha \setminus F) : F \in [e_\alpha]^{<\omega}\}$ is a local base of clopen sets at $(x_\alpha, 0)$.

Seeking a contradiction let us assume that $\{a_t : t \in T\}$ witnesses that A is a T-algebra for some acceptable tree T. For each $t \in 2^{<\omega} \setminus \{\emptyset\}$ we will inductively define $f(t) \in T$, $q(t) \in \mathbb{Q}$, $\ell(t) \in \omega$, and $W(t) \in CO(2^\omega)$ in such a way that the following is true for all t and all $i < 2$.

(1t) f is increasing: $f(s) < f(t)$ whenever $s < t$.
(2t) $f(t^*) = f(t)^*$.
(3t) $W(t)$ has diameter $< 1/2^{|t|}$.
(4t) $q(t) \in W(t) \subseteq a(f(t^*))$.
(5t) If $s \in T$ and $s < f(t^\frown i)$, then $W(t^\frown i) \subseteq W(t) \subseteq a_s$.
(5t) $a(f(t^\frown i)) \cap W(t) \neq \emptyset$.
(6t) $(q(t), \ell(t)) \in a(f(t^*))$.

For the base of the induction: let $u \in T$ be so that $a(u) \cap 2^\omega \neq \emptyset \neq 2^\omega \setminus a(u)$ but $2^\omega \subseteq a_s$ for all $s < u$. Define $f(\emptyset^\frown 0) = u$ and $f(\emptyset^\frown 1) = u^*$. Given $t \in \{\emptyset^\frown 0, \emptyset^\frown 1\}$, let $q(t) \in a(f(t^*)) \cap \mathbb{Q}$ and $\ell(t) \in \omega$ be such that $(q(t), \ell(t)) \in a(f(t^*))$. Finally, let $W(t)$ be a clopen subset of the Cantor set whose diameter is less than $1/2$ and such that $q(t) \in W(t) \subseteq a(f(t^*))$.

Assume that for some $n \in \omega$ and for all $t \in 2^{\leq n}$ we have defined $f(t)$, $q(t)$, $\ell(t)$, and $W(t)$ as required. Fix $t \in 2^n$ and let $u_0 \in T$ be so that $f(t) < u_0$, $W(t) \cap a(u_0) \neq \emptyset \neq W(t) \setminus a(u_0)$, and $W(t) \subseteq a_s$ for all $s < u_0$. Set $f(t^\frown 0) = u_0$ and $f(t^\frown 1) = u_0^*$. As before, for each $i < 2$ we can find $q(t^\frown i)$, $\ell(t^\frown i)$, and $W(t^\frown i)$ satisfying all the requirements and this completes the induction.

Consider the functions $q : 2^{<\omega} \to 2^\omega \cap \mathbb{Q}$ and $\ell : 2^{<\omega} \to \omega$ given by $t \mapsto q(t)$ and $t \mapsto \ell(t)$, respectively. Properties (1t), (2t), and (4t) imply that, for some $\alpha < \mathfrak{c}$, we obtain $(q, \ell) = (r_\alpha, m_\alpha)$. Let $t_n := g_\alpha \restriction n$ for all $n \in \omega$. Thus $H := \{q(t_n) : n \in \omega\}$ and e_α have disjoint closures in X. Also note that conditions (4t) and (6t) give

$$\{q(t_n), (q(t_n), \ell(t_n))\} \subseteq \bigcap\{a_s : s < f(t_n)\} - a(f(t_n))$$

and since $\{f(t_n) : n \in \omega\}$ is an increasing sequence in T, Lemma 3.8 guarantees that $\overline{H} \cap \overline{e_\alpha} \neq \emptyset$. A contradiction. □

Recall that Alexandroff's double arrow space is the subspace $[0,1] \times \{0,1\}$ of the square $[0,1] \times [0,1]$ endowed with the topology given by the lexicographic order (alternatively, split each point x of $[0,1]$ into two points, x^+ and x^-, and define a total order by declaring $x^- < x^+$ and using the induced order of $[0,1]$ otherwise). By identifying 2^ω with the Cantor Middle Third Set, we can consider $2^\omega \times \{0,1\}$ as a subspace of Alexandroff's double arrow; this space will be called *Alexandroff's double arrow on* 2^ω.

In [19, Example 1], Koppelberg proves that the topological product of 2^ω with Alexandroff's double arrow on 2^ω is not simplistic (she actually uses Alexandroff's double arrow, but the same argument works). Since both spaces are simplistic (see [21, Example 2.1]), this shows that the class of simplistic spaces is not closed under products. Equivalently, the class of minimally generated Boolean algebras is not closed under free products.

Proposition 4.3. *There are two acceptable trees, T and T', together with two Boolean algebras, B and B', such that B is a T-algebra, B' is a T'-algebra, and the free product $B \oplus B'$ fails to be a $\mathcal{T}$-algebra.*

Proof. Let $B' := CO(2^\omega)$ and $T' := 2^{<\omega} \setminus \{\emptyset\}$. For each $t \in T'$ define

$$a(t^\frown 0) := \{f \in 2^\omega : t \subseteq f\} \text{ and } a(t^\frown 1) := 2^\omega \setminus a(t^\frown 0).$$

According to [22, Example 2.9], $\{a_t : t \in T'\}$ witnesses that B' is a T'-algebra.

Claim. If X denotes Alexandroff's double arrow on 2^ω and $T := T' \cup 2^{\omega+1}$, then $CO(X)$ is a T-algebra.

In order to prove the Claim let us define, for each $t \in T'$, $c_t := a_t \times \{0,1\}$. Thus B' is isomorphic to the Boolean algebra generated by $\{c_t : t \in T'\}$ in $\mathcal{P}(2^\omega \times \{0,1\})$.

Observe that if one identifies 2^ω with Cantor's Middle Third Set in the canonical way, then each $x \in 2^\omega$ represents a real number in $[0,1]$ so one can consider the closed interval $[0,x]$ and, moreover, the intersection $2^\omega \cap [0,x]$. Keeping this in mind, define

$$c(x^\frown 0) := ((2^\omega \cap [0,x]) \times \{0\}) \cup ((2^\omega \cap [0,x)) \times \{1\})$$

and $c(x^\frown 1) := (2^\omega \times \{0,1\}) \setminus c(x^\frown 0)$.

It is straightforward to verify the following.

(1) $\{c_t : t \in T\}$ generates $CO(X)$.
(2) For each $x \in 2^\omega$: if $S := \{c(x \restriction n) : 0 < n < \omega\}$, then $c(x^\frown 0)$ is minimal for (B_x, u_x), where B_x is the Boolean algebra generated by S and u_x is the filter generated by S in B_x.

Therefore $CO(X)$ is a T-algebra and the Claim is proved.

As we mentioned in the paragraph preceding Proposition 4.3, $B \oplus B'$ is not minimally generated. On the other hand, [22, Fact 2.10 on p. 3081] guarantees that if S is an acceptable tree, then all S-algebras are minimally generated and so the proof of the proposition is complete. $\square$

5. Efimov $\mathcal{T}$-Algebras

Let T be an acceptable tree. A T-algebra whose Stone space is an Efimov space will be called an *Efimov T-algebra*.

The following is a consequence of Stone's representation theorem.

Remark 5.1. The existence of an Efimov T-algebra is equivalent to the existence of a zero-dimensional Efimov space X for which $CO(X)$ is a T-algebra.

As we did before, the phrase "B is an Efimov $\mathcal{T}$-algebra" means that B is an Efimov T-algebra for some acceptable tree T.

Note that, according to Remark 3.1, a $\mathcal{T}$-algebra is Efimov iff its Stone space contains no copy of $\omega + 1$.

One of the long-standing problems in Set-Theoretic Topology (it was posed by C.T. Scarborough and A.H. Stone in 1966) is Scarborough-Stone's question: *Must every product of sequentially compact spaces be countably compact?* As one may expect, the literature related to this question is vast so we refer the reader interested in the topic to [27].

The first part of this section will be dedicated to this problem.

First of all we establish some definitions. As usual, ω^* will denote the collection of all nonprincipal ultrafilters in ω.

Now assume that $s : \omega \to X$ is a sequence in a topological space X and $r \in \omega^*$. A point $x \in X$ is an *r-limit of s* if for each neighborhood U of x we obtain that $\{n \in \omega : s(n) \in U\} \in r$. X will be called *r-compact* if every sequence in X has an r-limit. Notice that $\{\overline{\{s(n) : n \in a\}} : a \in r\}$ has the finite intersection property and therefore when X is compact, all sequences in X have an r-limit. Moreover, if X is T_2, this r-limit is unique.

A straightforward argument shows that r-limits are preserved by continuous functions, i.e., that if $f : X \to Y$ is continuous and s is a sequence in X which has x as an r-limit, then $f(x)$ is an r-limit of the sequence $f \circ s$.

Theorem 5.2. *The existence of an Efimov $\mathcal{T}$-algebra implies a negative answer to Scarborough-Stone's question.*

Proof. We only need to exhibit a family $\{X_r : r \in \omega^*\}$ of sequentially compact spaces such that each X_r is not r-compact because, according to [27, Lemma 2.1], the topological product of such a family is not countably compact.

Let X be a zero-dimensional Efimov space such that, for some acceptable tree T, $CO(X)$ is a T-algebra as witnessed by $\{a_t : t \in T\}$ (Remark 5.1). X possesses an accumulation point p so we can apply Corollary 3.7 to obtain a branch $b \subseteq T$ satisfying $\bigcap\{a_t : t \in b\} = \{p\}$. Notice that if b were finite, p would be an isolated point so b is infinite. Using this fact let us fix an increasing sequence of nodes $\{t_n : n < \omega\} \subseteq b$.

For each integer n there is a branch $b_n \subseteq T$ satisfying $\{t_k : k < n\} \cup \{t_n^*\} \subseteq b_n$. According to Corollary 3.7-(2), $\bigcap\{a_s : s \in b_n\}$ contains a single point we will call w_n.

Set $W := \{w_k : k \in \omega\}$ and note that the equality $W \cap \bigcap\{a_s : s < t_n\} - a(t_n) = \{w_n\}$ holds for all $n < \omega$. In particular, W is infinite discrete.

For the rest of the proof we will fix $r \in \omega^*$.

Let w_r be the unique r-limit of the sequence $\langle w_n : n < \omega\rangle$ in X and let b_r be a branch in T which satisfies $\bigcap\{a_t : t \in b_r\} = \{w_r\}$ (we are using Corollary 3.7). Denote by B_r the Boolean algebra generated by $\{a_t : t \in b_r\}$ in $CO(X)$.

The map $f : X \to \mathrm{St}(B_r)$ given by $f(x) := \{c \in B_r : x \in c\}$ is onto and continuous. Note that if $x \in X$ satisfies $f(x) = f(w_r)$, then $\{a_t : t \in b_r\} \subseteq f(x)$ and therefore $x \in \bigcap\{a_t : t \in b_r\}$, which gives $x = w_r$. On the other hand, the fact that W is infinite discrete, implies that $w_r \notin W$. These two remarks show that the sequence $\langle f(w_n) : n < \omega\rangle$ has no r-limit in the subspace $X_r := \mathrm{St}(B_r) \setminus \{f(w_r)\}$.

It remains to show that X_r is sequentially compact. According to [26, Theorem 5.7] we only need to prove that X_r is scattered and countably compact (notice that X_r is T_3).

Let $\{x_n : n \in \omega\}$ be an infinite subset of X_r. For each $n \in \omega$ there is $y_n \in X$ so that $f(y_n) = x_n$. Since X is Efimov, $\{y_n : n \in \omega\}$ possesses more than one accumulation point. In particular, $\{y_n : n \in \omega\}$ accumulates to some $y \in X \setminus \{w_r\}$ and thus $f(y)$ is an accumulation point of $\{x_n : n \in \omega\}$ in X_r. Hence X_r is countably compact. It is worth mentioning that this is the only part of the proof where being Efimov is used.

One can prove that $T' := b_r \cup \{t^* : t \in b_r\}$ is an acceptable tree and that B_r is a T'-algebra as witnessed by $\{a_t : t \in T'\}$. Notice that if b' is a branch in T', then $b' = b_r$ or $b' = \{s \in T' : s < t\} \cup \{t^*\}$ for some $t \in b_r$. Therefore (Proposition 3.5), for each $y \in X_r$ there exists $t_y \in b_r$ so that y is the ultrafilter generated by $\{a_s : s < t_y\} \cup \{-a(t_y)\}$. We are ready to show that X_r is scattered: let E

be a nonempty subset of X_r; since b_r is well-ordered, there exists $z \in E$ so that $t_z = \min\{t_y : y \in E\}$. By definition, $U := \{u \in \mathrm{St}(B_r) : -a(t_z) \in u\}$ is a clopen subset of $\mathrm{St}(B_r)$ and our choice of t_z guarantees that $U \cap E = \{z\}$ so E has an isolated point. $\square$

Since $CO(2^{\omega_1})$ is not minimally generated, it cannot be embedded as a subalgebra of any minimally generated Boolean algebra. Our next theorem shows that no Efimov $\mathcal{T}$-algebra contains a copy of the clopen algebra of $(\omega_1 + 1) \times (\omega + 1)$, even though this Boolean algebra is a $\mathcal{T}$-algebra (Proposition 4.1).

Recall that a continuous mapping between topological spaces is called *perfect* if it is closed and all its fibers are compact. Also, a continuous mapping f from X onto Y is called *irreducible* if no proper closed subset of X is mapped by f onto Y.

Remark 5.3. Assume that $f : X \to Y$ is a continuous closed map.

(1) If f is irreducible and H is a regular closed subset of X, then $f[H]$ is a regular closed subset of Y.
(2) If f is perfect, $S \subseteq Y$, and $p \in \mathrm{cl}_Y S$, then $f^{-1}[p] \cap \mathrm{cl}_X f^{-1}[S] \neq \emptyset$.

To prove (1): observe that if U is an open subset of X, then $F := X \setminus (U \cap f^{-1}[\mathrm{int} f[X \setminus U]])$ is closed and $f[F] = Y$, which implies that $f[U] \cap \mathrm{int} f[X \setminus U] = \emptyset$. Therefore
$$Y \setminus f[X \setminus U] \subseteq f[U] \subseteq \overline{Y \setminus f[X \setminus U]}.$$
Hence, since f is a closed mapping, $f[\overline{U}]$ is a regular closed subset of Y.

The proof of (2) can be done by contradiction and it is a routine argument so we omit it.

Theorem 5.4. *If X is the Stone space of an Efimov T-algebra, for some acceptable tree T, then X does not map continuously onto $Y := (\omega_1 + 1) \times (\omega + 1)$.*

Proof. Stone's representation theorem guarantees that $CO(X)$ is a T-algebra so let $\{a_t : t \in T\}$ be a witness to this fact.

Seeking a contradiction, assume that $f : X \to Y$ is continuous and onto.

Let K be a closed subset of X such that $f[K] = Y$ and $f \restriction K$ is irreducible ([11, Exercise 3.1.C-(a)]). Since $f \restriction K$ is a perfect mapping, we apply Remark 5.3 to obtain a point $q \in K \cap \overline{f^{-1}[\{\omega_1\} \times \omega]}$ such that $f(q) = (\omega_1, \omega)$. Notice that our choice of q guarantees that if U is a neighborhood of q in X, then the set $\{n < \omega : (\omega_1, n) \in f[U \cap K]\}$ is infinite. Fix a branch $b \subseteq T$ for which $\bigcap\{a_s : s \in b\} = \{q\}$.

We claim that there are two sequences, $\{n_k : k < \omega\} \subseteq \omega$ and $\{t_{n_k} : k < \omega\} \subseteq b$, such that if $k < \omega$, then $t_{n_{k+1}}$ is the least node in b (recall that b is a well-ordered subset of T) for which there is an integer $n_{k+1} > n_k$ satisfying
$$(\omega_1, n_{k+1}) \in f\left[a(t^*_{n_{k+1}}) \cap \bigcap\{a(t_{n_i}) : i \leq k\} \cap K\right]$$
and n_{k+1} is the smallest integer having these properties.

To prove the claim we will use induction. For each $x \in K \cap f^{-1}[\{\omega_1\} \times \omega]$ let b_x be a branch in T with $\bigcap\{a_s : s \in b_x\} = \{x\}$. Clearly $b \neq b_x$ so, for some $s_x \in b$, we obtain $s^*_x \in b_x$. Let $z \in K \cap f^{-1}[\{\omega_1\} \times \omega]$ be such that s_z is the first element of $\{s_x : x \in K \cap f^{-1}[\{\omega_1\} \times \omega]\}$. Thus $(\omega_1, m) \in f[a(s^*_z) \cap K]$, for some integer m, so we let n_0 be the least integer satisfying this property and $t_{n_0} := s_z$ (notice that this choice of t_{n_0} works because, according to Corollary 3.7-(1), each $t \in b$ is of the form s_x for some $x \in X \setminus \{q\}$).

Now assume that $\{t_{n_i} : i \le k\}$ and $\{n_i : i \le k\}$ have been defined. The fact that $U := \bigcap\{a(t_{n_i}) : i \le k\}$ is an open set in X which contains q implies, as we noticed before, that there are infinitely many integers ℓ such that $(\omega_1, \ell) \in f[U \cap K]$. An immediate consequence of this observation is that the set

$$(\star) \qquad \{s_x : x \in U \cap K \cap f^{-1}[\{\omega_1\} \times (\omega \setminus (n_k + 1))]\}$$

is non-empty so there exist $w \in U \cap K$ and $\ell > n_k$ such that s_w is the first element of $(\star)$ and $f(w) = (\omega_1, \ell)$. Finally, let n_{k+1} be the least integer such that $n_{k+1} > n_k$ and $(\omega_1, n_{k+1}) \in f[a(s_w^*) \cap U \cap K]$ and define $t_{n_{k+1}} := s_w$.

We are going to establish some notation which will be used throughout the rest of the proof: denote by I the set $\{n_k : k < \omega\}$ and for each $n \in I$ define

(i) $c_n := a(t_n^*) \cap \bigcap\{a(t_k) : k \in I \cap n\}$ (of course, $c_{n_0} := a(t_{n_0}^*)$),
(ii) $W := \mathrm{int}_Y \left(\bigcup_{k \in I} f[c_k \cap K] \cap (\omega_1 \times \{k\})\right)$,
(iii) $V_n := (f \restriction K)^{-1}[W \cap (\omega_1 \times \{n\})] = K \cap f^{-1}[W \cap (\omega_1 \times \{n\})]$, and
(iv) $K_0 := \mathrm{cl}_K \left(\bigcup_{k \in I} V_k\right)$.

Also, for each $t \in T$ set $\Delta(t) := \bigcap\{a_s : s \le t\}$.

Let us show that $\langle t_n : n \in I\rangle$ is an increasing sequence. Assume that $m, n \in I$ are such that $n < m$. Then $c_m \subseteq a(t_m^*) \cap \bigcap\{a(t_k) : k \in I \cap n\}$ and therefore

$$(\omega_1, m) \in f[c_m \cap K] \subseteq f\left[a(t_m^*) \cap \bigcap\{a(t_k) : k \in I \cap n\} \cap K\right],$$

which, together with t_n's minimality, gives $t_n \le t_m$. On the other hand, if $t_n = t_m$, then $c_m \subseteq a(t_m^*) \cap a(t_n) = \emptyset$, a contradiction to $(\omega_1, m) \in f[c_m \cap K]$; so we obtain $t_n < t_m$. Observe that an immediate consequence of this result is that $\Delta(t_k^*) \subseteq c_k$ for all $k \in I$.

We claim that $\overline{V_n} \cap \Delta(t_n^*) \cap f^{-1}[(\omega_1, n)] \neq \emptyset$ for all $n \in I$. It suffices to show that $\{\overline{V_n} \cap a_s \cap f^{-1}[(\omega_1, n)] : s \le t_n^*\}$ has the finite intersection property, so let F be a finite nonempty subset of $\{s \in T : s \le t_n^*\}$ and define $d := K \cap c_n \cap \bigcap\{a_s : s \in F\}$. Since $g := f \restriction K$ is irreducible and d is a clopen subset of K, $g[d]$ is a regular closed subset of Y which contains (ω_1, n) (because $K \cap \Delta(t_n^*) \subseteq d$). Set $U := \mathrm{int}(g[d] \cap (\omega_1 \times \{n\}))$. Then $U \subseteq W \cap (\omega_1 \times \{n\})$ and $(\omega_1, n) \in \overline{U}$; an application of Remark 5.3 gives

$$\emptyset \neq g^{-1}[(\omega_1, n)] \cap \mathrm{cl}_K g^{-1}[U] \subseteq f^{-1}[(\omega_1, n)] \cap \overline{V_n}.$$

On the other hand, $g^{-1}[U] \subseteq g^{-1}[\mathrm{int}\ g[d]] \subseteq d \subseteq \bigcap\{a_s : s \in F\}$ because $H := K \setminus (g^{-1}[\mathrm{int}\ g[d]] \setminus d)$ is a closed subset of K such that $g[H] = Y$. Thus we obtain

$$f^{-1}[(\omega_1, n)] \cap \overline{V_n} \cap \bigcap\{a_s : s \in F\} \neq \emptyset,$$

as we wanted.

For each $n \in I$, let us fix a point $x_n \in \overline{V_n} \cap \Delta(t_n^*) \subseteq K_0 \cap \Delta(t_n^*)$ satisfying $f(x_n) = (\omega_1, n)$. Since $\langle t_k : k \in I\rangle$ is increasing, we get

Claim 1. If $n \in I$, then $\{x_k : k \in I \setminus (n+1)\} \subseteq \bigcap\{a_s : s < t_n\}$.

Now we will construct two functions, $e : 2^{<\omega} \to T$ and $H : 2^{<\omega} \to [I]^\omega$, so that the following conditions hold for each $t \in 2^{<\omega}$.

(1) For all $n \in I$, $t_n < e(\emptyset)$.
(2) If $s < e(\emptyset)$, then $\{n \in I : x_n \in a(s^*)\}$ is finite.
(3) $H(\emptyset) := \{n \in I : x_n \in a(e(\emptyset))\}$.
(4) $I \setminus H(\emptyset) = \{n \in I : x_n \in a(e(\emptyset)^*)\}$ is infinite.
(5) $e(r) < e(t)$ whenever $r < t$.

(6) $e(t^*) = e(t)^*$.
(7) $H(t^\frown i) = \{n \in H(t) : x_n \in a(e(t^\frown i))\}$ for each $i < 2$.
(8) If $i < 2$ and $e(t) < s < e(t^\frown i)$, then $\{n \in H(t) : x_n \in a(s^*)\}$ is finite.

Before we embark on the construction let us prove that if $\{t_n : n \in I\}$ is cofinal with b, then $\langle x_n : n \in I\rangle$ converges to q. We will use Corollary 3.7 to prove it: let F be a finite nonempty subset of b. There exists $n \in I$ satisfying $t_n > \max F$ and hence (Claim 1) $x_k \in \bigcap\{a_s : s \in F\}$, for all $k > n$.

Since X is an Efimov space, $\{t_n : n \in I\}$ has an upper bound in b and therefore it has one with minimum height, let us say $t_\omega \in b$.

Given $s \in T$ and $E \in [I]^\omega$, we will say that s *splits* E if $\{n \in E : x_n \in a_r\}$ is infinite for all $r \in \{s, s^*\}$.

e and H will be built recursively. If t_ω does not split I, let $r_0 \in \{t_\omega, t_\omega^*\}$ be such that $\{n \in I : x_n \in a(r_0^*)\}$ is finite. In general, assume that for some ordinal γ we have $\{r_\xi : \xi < \gamma\} \subseteq T$ so that the following holds for all $\xi < \gamma$.

(a) $r_\eta < r_\xi$ whenever $\eta < \xi$.
(b) $\mathrm{ht}(r_\xi) = \mathrm{ht}(t_\omega) + \xi$.
(c) $\{n \in I : x_n \in a(r_\xi^*)\}$ is finite.

If $\{r_\xi : \xi < \gamma\}$ has an upper bound $u \in T$ such that $\mathrm{ht}(u) = \mathrm{ht}(t_\omega) + \gamma$ and u does not split I, then we define $r_\gamma \in \{u, u^*\}$ as the only one for which $\{n \in I : x_n \in a(r_\gamma^*)\}$ is finite.

Clearly, the process described above has to stop. Let δ be such that $\{r_\xi : \xi < \delta\}$ cannot be extended. Seeking a contradiction let us assume that the process stopped because $\{r_\xi : \xi < \delta\}$ does not have an upper bound on level $\mathrm{ht}(t_\omega) + \delta$. Set $b' := \{s \in T : \exists \xi < \delta (s \le r_\xi)\}$. Condition (a) guarantees that b' is a chain in T. Moreover, if $b' \cup \{s\}$ is a chain for some $s \in T \setminus b'$, then it must be the case that $r_\xi < s$, for all $\xi < \delta$. Therefore $\mathrm{ht}(s) \ge \mathrm{ht}(t_\omega) + \delta$, but this gives the existence of an upper bound for b' on level $\mathrm{ht}(t_\omega) + \delta$. In other words, b' is a branch in T. We will prove that $\langle x_n : n \in I\rangle$ converges to the only element of $\bigcap\{a_s : s \in b'\}$ by showing that

$$\{n \in I : x_n \in a(s^*)\} \text{ is finite} \tag{†}$$

for all $s \in b'$ (Corollary 3.7). If $s < t_\omega$, then $s < t_m$ for some $m \in I$ and therefore $\{n \in I : x_n \in a(s^*)\} \subseteq m + 1$ (Claim 1). When $t_\omega \le s$, we have that $s = r_\xi$ for some $\xi < \delta$ and thus condition (c) implies (†).

Hence $\{r_\xi : \xi < \delta\}$ has an upper bound $t \in T$ such that $\mathrm{ht}(t) = \mathrm{ht}(t_\omega) + \delta$. Since the process did stop, t splits I so we define $e(\emptyset) := t$ and $H(\emptyset)$ as in condition (3).

Now assume that for some $t \in 2^{<\omega}$ we have constructed $\{e(r) : r \le t\}$ and $\{H(r) : r \le t\}$. Let us start by proving the following.

Claim 2. If $s \le e(t)$, then

$$\{n \in H(t) : x_n \in a(s^*)\} \text{ is finite.} \tag{††}$$

When $s < e(\emptyset)$, condition (2) guarantees that (††) holds. If $e(\emptyset) < s$ and $s \notin \{e(r) : r \le t\}$, we invoke condition (8) to obtain (††). So the case $s = e(r)$, for some $r \le t$, is the only one which needs an argument: an straightforward consequence of condition (7) is that $H(t) \subseteq H(r)$, which together with conditions (3) and (6) implies that $H(r)$ and the set given in (††) are disjoint. Therefore this set is empty.

In view of Claim 2, we proceed as we did in the construction of $e(\emptyset)$, replacing I with $H(t)$, to obtain a node $v \in T$ such that the following holds: $e(t) < v$, v splits $H(t)$, and condition (8) holds when one sets $e(t^\frown 0) := v$ and $e(t^\frown 1) := v^*$. To complete the recursion we only need to define $H(t^\frown i)$, $i < 2$, as prescribed in (7).

To simplify notation we will use e_t and H_t to denote $e(t)$ and $H(t)$, respectively.

For each function $r \in 2^\omega$ define $\overline{r} := \{s \in T : \exists n < \omega (s \leq e_{r\restriction n})\}$ and $[r] := \bigcap\{a_s : s \in \overline{r}\}$. Equivalently (see condition (5)), $[r] = \bigcap_{n<\omega} \Delta(e_{r\restriction n})$.

We claim that $\{a_s \cap \mathrm{cl}_K(\{x_n : n \in I\}) : s \in \overline{r}\}$ has the finite intersection property for each $r \in 2^\omega$. If F is a finite nonempty subset of $\overline{r}$, there is an integer m such that $e_{r\restriction m} > \max F$ and therefore $\{x_n : n \in H_{r\restriction m}\} \setminus a_s$ is finite for all $s \in F$ (Claim 2). Hence $\overline{\{x_n : n \in H_{r\restriction m}\}} \cap \bigcap\{a_s : s \in F\} \neq \emptyset$. An immediate consequence of this result is

Claim 3. For all $r \in 2^\omega$: $\overline{\{x_n : n \in I\}} \cap [r] \neq \emptyset$.

We will prove by contradiction that the set

$$\{r \in 2^\omega : \exists \alpha < \omega_1 (K_0 \cap [r] \cap f^{-1}[(\alpha+1) \times (\omega+1)] \neq \emptyset)\}$$

is finite. So assume that it is infinite. In this case, the set contains an infinite sequence S which converges to some $\rho \in 2^\omega$ (we are using the product topology here). For each $r \in S$ fix a point $y_r \in [r] \cap K_0$ and an ordinal $\alpha_r < \omega_1$ such that $f(y_r) \in (\alpha_r + 1) \times (\omega + 1)$. Since S is countable, there exists $\alpha < \omega_1$ for which $\{f(y_r) : r \in S\} \subseteq (\alpha+1) \times (\omega+1)$. In particular, $\{x_n : n \in I\}$ and $\{y_r : r \in S\}$ have disjoint closures in X. On the other hand, the fact that S is infinite and converges to ρ implies the existence of two sequences, $\{r_k : k \in \omega\} \subseteq S$ and $\{n_k : k \in \omega\} \subseteq \omega$, such that $m_k \neq m_\ell$ whenever $k \neq \ell$ and $(\rho \restriction (m_k+1))^* = r_k \restriction (m_k+1)$ for all $k \in \omega$ (i.e., m_k is the first integer where $r_k(m_k) \neq \rho(m_k)$). For each $k \in \omega$ define $s_k := e(r_k \restriction (m_k+1))$ and fix a point $z_k \in \overline{\{x_n : n \in I\}} \cap [r_k]$ (Claim 3). Note that condition (6) above gives $s_k^* = e(r_k \restriction (m_k+1))$ and therefore

$$\{z_k, y_{r_k}\} \subseteq [r_k] \subseteq \Delta(e(r_k \restriction (m_k+1))) = \Delta(s_k^*)$$

so we apply Lemma 3.8-(2) to obtain

$$\emptyset \neq \overline{\{y_{r_k} : k \in \omega\}} \cap \overline{\{z_k : k \in \omega\}} \subseteq \overline{\{y_r : r \in S\}} \cap \overline{\{x_n : n \in I\}},$$

a contradiction.

For the rest of the proof let us fix $g \in 2^\omega$ such that

$$\bigcap_{n<\omega} f^{-1}[(\alpha+1) \times (\omega+1)] \cap \Delta(e_{g\restriction n}) \cap K_0 = \emptyset,$$

for all $\alpha < \omega_1$. Condition (5) gives $\Delta(e_{g\restriction(n+1)}) \subseteq \Delta(e_{g\restriction n})$ and therefore, using K_0's compactness, we obtain, for each $\alpha < \omega_1$, an integer m_α such that

$$f^{-1}[(\alpha+1) \times (\omega+1)] \cap \Delta(e_{g\restriction m_\alpha}) \cap K_0 = \emptyset.$$

Let $n \in \omega$ be so that $\{\xi < \omega_1 : m_\xi = n\}$ is uncountable and set $r := g \restriction n$. Observe that $f[K_0 \cap \Delta(e_r)] \subseteq \{\omega_1\} \times (\omega+1)$.

Claim 4. There is a countable ordinal γ such that for any integer ℓ, for any increasing sequence $\{s_k : k \leq \ell\} \subseteq 2^{<\omega}$, and for each $m \in H(s_\ell)$ the set

$$f[V_m \cap \bigcap\{a(e_{s_k}) : k \leq \ell\}] \cap ((\gamma+1) \times \{m\})$$

is infinite.

Let $\{s_k : k \leq \ell\}$ be as in the Claim. We will show that if $d := \bigcap\{a(e_{s_k}) : k \leq \ell\}$ and $m \in H(s_\ell)$, then $f[V_m \cap d] \cap (\omega_1 \times \{m\})$ is uncountable. This suffices because there are only countably many sequences as the one described in Claim 4.

For each $k \leq \ell$ condition (7) gives $H(s_\ell) \subseteq H(s_k)$ and therefore $x_m \in \overline{a(e_{s_k})}$. Hence we get $x_m \in d \cap \overline{V_m} \cap K$. Since d is a clopen subset of X, $d \cap \overline{V_m} = \overline{d \cap V_m}$ and so $x_m \in \mathrm{cl}_K(d \cap V_m)$. Then (ω_1, m) belongs to the closure of $\mathrm{int} f[K \cap V_m \cap d]$ (Remark 5.3 applied to $f \upharpoonright K$) and, in particular, $f[V_m \cap d] \cap (\omega_1 \times \{m\})$ is uncountable as we needed.

Define $K_1 := \overline{\bigcup\{V_m : m \in H_r\}} \cap f^{-1}[(\gamma+1) \times (\omega+1)]$ and observe that if $x \in K_1 \cap \Delta(e_r)$, then our choice for n gives $f(x) \in \{\omega_1\} \times (\omega+1)$ (because $K_1 \subseteq K_0$); a contradiction to $x \in K_1$.

For each $x \in K_1$ let t_x be the least element of $\{s \in T : s \leq e_r \wedge x \in a(s^*)\}$ (the argument given above proves that this sest is nonempty) and for each $C \subseteq K_1$ define $C^\sharp := \{t_x : x \in C\}$.

We affirm that if C is a closed nonempty subset of K_1, then $C^\sharp$ has a maximum element. Indeed, by definition, $\{a(t_x^*) : x \in C\}$ covers C so there exists a finite nonempty set $F \subseteq C^\sharp$ satisfying $C \subseteq \bigcup\{a(t_x^*) : x \in F\}$. Let $z \in F$ be so that $t_z = \max F$. Note that if $x \in C$, then $x \in a(t_y^*)$ for some $y \in F$ and therefore, given t_x's minimality, we obtain $t_x \leq t_y \leq t_z$.

Let $y_0 \in K_1$ be such that $t_{y_0} = \max K_1^\sharp$. Fix s_0, an immediate successor of r (i.e., $s_0 = r^\frown i$ for some $i < 2$), such that $y_0 \notin a(e_{s_0})$. Now, if $K_1 \cap a(e_{s_0})$ is non-empty, let y_1 be an element of this set satisfying $t_{y_1} = \max(K_1 \cap a(e_{s_0}))^\sharp$ and let s_1 be an immediate successor of s_0 so that $y_1 \notin a(e_{s_1})$. And so forth: y_2 will be an element of $K_1 \cap a(e_{s_0}) \cap a(e_{s_1})$ (assuming this set is not empty) such that t_{y_2} is the maximum of $(K_1 \cap a(e_{s_0}) \cap a(e_{s_1}))^\sharp$. Given that $t_{y_0} > t_{y_1} > t_{y_2} > \dots$, there must be an integer ℓ for which $K_1 \cap \bigcap\{a(e_{s_k}) : k \leq \ell\} = \emptyset$, giving us the contradiction to Claim 4 that finishes the proof. □

6. Consistency Results

In this section: for unexplained notation, definitions and results on Forcing cf. [23, Chapters IV and V]; also, *space* will mean *Hausdorff space*.

Let us start by recalling that

Definition 6.1. *Hechler forcing* is the set $\omega^{<\omega} \times \omega^\omega$ ordered by $(s,f) \leq (t,g)$ iff

(1) $t \subseteq s$,
(2) $g(n) \leq f(n)$ for all $n \in \omega$, and
(3) $g(i) < s(i)$ whenever $i \in \mathrm{dom}\, s \setminus \mathrm{dom}\, t$.

This notion of forcing was introduced in [16] and, as one easily verifies, it is ccc so, in particular, it preserves ω_1. Moreover, it adds a *dominating real*: if G is a generic filter, then $g := \bigcup\{s : \exists f[(s,f) \in G]\}$ is a member of $\omega^\omega \cap V[G]$ satisfying $f \leq^* g$, for all $f \in \omega^\omega \cap V$. For this reason, Hechler's poset is also called *the dominating forcing* or *forcing adding a dominating real*.

The main result of this section is the following.

Theorem 6.2. *There is an Efimov $\mathcal{T}$-algebra of countable tightness in the generic extension yield by the finite support iteration of length ω_2 of Hechler forcing over a model of* CH.

Let us note that in the model described above, $\mathfrak{h}$, $\mathfrak{s}$, and $\mathfrak{t}$ are all equal to ω_1 and $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \omega_2$ (see [4]).

A standard feature of such iterated forcing constructions is the need for preservation results. In our case, we need to insure that further forcing will not introduce undesired converging sequences. For this purpose we introduce the following new notion.

Definition 6.3. Let X be a topological space. We say that X has the *stationary set property* (*X has the SSP*, for short) if it possesses a cover of compact open subsets, $\{c_\alpha : \alpha < \omega_1\}$, so that

1. each c_α is countable,
2. for any stationary set $S \subseteq \omega_1$, $X \setminus \bigcup\{c_\xi : \xi \in S\}$ is a compact subspace of X.

Lemma 6.4. *Any space having the SSP is countably compact.*

Proof. Assume that X has the SSP and let Y be an infinite countable subset of X. If $Y \cap c_\beta$ were infinite for some $\beta < \omega_1$, Y would have an accumulation point in X. So let us assume that $Y \cap c_\alpha$ is finite for each $\alpha < \omega_1$. Since Y is countable, there is a finite set $F \subseteq Y$ for which the set $S := \{\alpha < \omega_1 : Y \cap c_\alpha = F\}$ is stationary. Thus $Y \setminus F$ is an infinite subset of the compact subspace $X \setminus \bigcup_{\alpha \in S} c_\alpha$ and therefore Y has an accumulation point in X. $\square$

Definition 6.5. Let $\mathbb{P}$ be a notion of forcing which preserves ω_1. We say that $\mathbb{P}$ *preserves the SSP* if whenever a family $\{c_\alpha : \alpha < \omega_1\}$ witnesses the SSP for a topological space X in the ground model, the same family witnesses the SSP for X after forcing with $\mathbb{P}$.

The following result seems to be a well-known theorem of K. Kunen, but since we could not find a reference for it, we are including a proof here.

Lemma 6.6. *Let $\mathbb{P}$ be a notion of forcing. If X is a compact scattered topological space in the ground model, then $\mathbb{P} \Vdash$ "X is compact."*

Proof. Working in V, the ground model, set $\mathcal{B} := CO(X)$. Denote by X_α the αth scattered level, i.e., X_α is the set of isolated points of $X \setminus \bigcup_{\beta<\alpha} X_\beta$. Also, for each $x \in X_\alpha$, let us fix $W_x \in \mathcal{B}$ satisfying $W_x \setminus \bigcup_{\beta<\alpha} X_\alpha = \{x\}$ and let us denote by δ the only ordinal for which $X = \bigcup_{\xi\leq\delta} X_\xi$ and $X_\delta \neq \emptyset$.

Observe that X is covered by a finite subset of $\{W_x : x \in X\}$ and therefore we only need to show that each W_x is compact in the generic extension yield by $\mathbb{P}$. We will do this by transfinite induction. More accurately, given G, a $\mathbb{P}$-generic filter over V, we claim that for each $\alpha \leq \delta$: if $x \in X_\alpha$, then W_x is compact in $V[G]$.

When $\alpha = 0$, each W_x is a singleton so let us assume that for some $0 < \alpha \leq \delta$, W_x is compact in $V[G]$ whenever $\beta < \alpha$ and $x \in X_\beta$. Let $z \in X_\alpha$ be arbitrary and let $\mathcal{U} \subseteq \mathcal{B}$ be a cover for W_x in $V[G]$ (recall that $\mathcal{B}$ is a base for X in $V[G]$). Thus there exists $U \in \mathcal{U}$ with $z \in U$ and so $W_z \setminus U$ is a compact subset of $\bigcup_{\beta<\alpha} X_\beta$ in V. Hence there is a finite set $F \subseteq \bigcup_{\beta<\alpha} X_\beta$ for which $K := \bigcup_{x\in F} W_x \supseteq W_z \setminus U$ and therefore, our inductive hypothesis implies that $W_z \setminus U$ is compact in $V[G]$. To finish our argument, note that if $\mathcal{U}_0 \in [\mathcal{U}]^{<\omega}$ covers $W_z \setminus U$, then $\mathcal{U}_0 \cup \{U\}$ covers W_z $\square$

As a consequence, if $\{c_\alpha : \alpha < \omega_1\}$ is as in Definition 6.3 and S is a stationary subset of ω_1, then $X \setminus \bigcup\{c_\alpha : \alpha \in S\}$ is contained in $\bigcup\{c_\alpha : \alpha \in F\}$, for some finite set $F \subseteq \omega_1$. Since each c_α is compact scattered, we get the following.

Remark 6.7. If $\{c_\alpha : \alpha < \omega_1\}$ witnesses that X has the SSP and S is a stationary subset of ω_1, then the subspace $X \setminus \bigcup\{c_\alpha : \alpha \in S\}$ is compact scattered in any generic extension.

Lemma 6.8. *Hechler forcing preserves the SSP.*

Proof. Assume that X is a locally countable locally compact topological space in the ground model which has the SSP as witnessed by $\{c_\alpha : \alpha < \omega_1\}$.

For each set $A \subseteq \omega_1$, define $A^\sharp := \bigcup\{c_\alpha : \alpha \in A\}$.

Let $\mathbb{P}$ be Hechler's poset and let $\dot{S}$ be a $\mathbb{P}$-name for a stationary subset of ω_1. Set $E := \{\alpha < \omega_1 : \exists p \in \mathbb{P}\ (p \Vdash \check{\alpha} \in \dot{S})\}$ and for each $\alpha \in E$ fix a condition (s_α, f_α) which forces $\check{\alpha} \in \dot{S}$. Since $\mathbb{P}$ is ccc, E is stationary ([17, Lemma 22.5]) and so there is $s \in \omega^{<\omega}$ for which $S_0 := \{\alpha \in E : s_\alpha = s\}$ is stationary.

Our assumptions on X imply that this space is zero-dimensional and locally compact so there exists K, a compact clopen subset of X, satisfying $X \setminus (S_0)^\sharp \subseteq K$.

For each $t \in \omega^{<\omega}$ let $S_t := \{\alpha \in S_0 : \forall i \in \operatorname{dom} t\ (f_\alpha(i) \leq t(i))\}$.

Claim. If $t \in \omega^{<\omega}$ satisfies $X \setminus K \subseteq (S_t)^\sharp$, then $X \setminus K \subseteq (S_{t^\frown m})^\sharp$ for some integer m.

In order to prove the Claim let us set $U_k := \bigcup\{c_\alpha : \alpha \in S_t \wedge f_\alpha(|t|) < k\}$, for each $k \in \omega$. Then $\{U_k : k \in \omega\}$ is an increasing sequence of open sets in X which covers $X \setminus K$. According to Lemma 6.8, there exists $m \in \omega$ such that $X \setminus K \subseteq U_m$ and therefore $t^\frown m$ is as required.

Since $S_\emptyset = S_0$, we use the Claim to inductively construct a function $h : \omega \to \omega$ so that $X \setminus K \subseteq (S_{h\restriction n})^\sharp$ for all $n \in \omega$.

We will prove that $(s, h) \Vdash X \setminus K \subseteq \bigcup\{c_\alpha : \alpha \in \dot{S}\}$ by showing that for each $y \in X \setminus K$ the set $D_y := \{p \in \mathbb{P} : \exists \alpha < \omega_1 (y \in c_\alpha \wedge p \Vdash \check{\alpha} \in \dot{S})\}$ is dense below (s, h). Let $y \in X \setminus K$ and $(t, g) \leq (s, h)$ be arbitrary. Our choice for h guarantees that $y \in c_\alpha$, for some $\alpha \in S_{h\restriction|t|}$. Thus $p := (t, f_\alpha + g)$ satisfies $p \in D_y$ (because $p \leq (s, f_\alpha)$) and $p \leq (t, g)$.

The previous paragraph shows that $X \setminus \bigcup\{c_\alpha : \alpha \in \dot{S}\}$ is forced by (s, h) to be contained in K and since K is compact in the generic extension (Lemma 6.6) this finishes the proof. □

Notice that if $\mathbb{P}$ is a notion of forcing which preserves the SSP and $\mathbb{Q}$ completely embedds into $\mathbb{P}$, then $\mathbb{Q}$ also preserves the SSP. In particular, since $\omega^{<\omega}$ is completely embedded into Hechler's poset, we obtain that the notion of forcing which adjoins one Cohen real also preserves the SSP.

Lemma 6.9. *Let $\mathbb{P}$ be a ccc poset and let E be a stationary subset of ω_1. If $p \in \mathbb{P}$ and $\{p_\alpha : \alpha \in E\} \subseteq \mathbb{P}$ satisfy $p_\xi \leq p$, for all $\xi \in E$, then there is $p' \leq p$ such that $p' \Vdash \{\alpha \in E : p_\alpha \in \dot{G}\}$ is stationary.*

Proof. Seeking a contradiction, let us assume that there is no such condition p'. Define $D \subseteq \mathbb{P}$ by $q \in D$ iff there is a club $C_q \subseteq \omega_1$ such that $q \Vdash p_\alpha \notin \dot{G}$, whenever $\alpha \in C_q$. Our assumption and the fact that $\mathbb{P}$ is ccc, imply that D is dense below p (see [17, Lemma 22.5]).

Let A be a maximal antichain in D. Then $C := \bigcap\{C_a : a \in A\}$ is a club and so there is $\beta \in C \cap E$. Our choice for A guarantees the existence of $q \in A$ and $r \in \mathbb{P}$

satisfying $r \le q$ and $r \le p_\beta$. Clearly, $r \Vdash p_\beta \in \dot{G}$. On the other hand, $\beta \in C \subseteq C_q$ and $r \le q$ imply $r \Vdash p_\beta \notin \dot{G}$, which is the contradiction we were looking for. $\square$

For our next two results we assume the reader is familiar with elementary submodels (cf. [23, Section III.8]).

Lemma 6.10. *Let E be a stationary subset of ω_1. If $\{F_\alpha : \alpha \in E\}$ is a family of finite subsets of ω_1, there exist a stationary set $E' \subseteq E$ and $\mu < \omega_1$ for which $\{F_\alpha \setminus \mu : \alpha \in E'\}$ is pairwise disjoint.*

Proof. We divide the proof into two cases. First, if $E_0 := \{\alpha \in E : F_\alpha \cap \alpha \neq \emptyset\}$ is stationary, then the map $f : E_0 \to \omega_1$ given by $f(\alpha) := \max(F_\alpha \cap \alpha)$ is regressive and so there is $\beta < \omega_1$ for which $E' := \{\alpha \in E_0 : f(\alpha) = \beta\}$ is stationary. To finish this case, set $\mu := \beta + 1$.

Now assume that $E_1 := \{\alpha \in E : F_\alpha \cap \alpha = \emptyset\}$ is stationary. Let θ be a cardinal such that H_θ has a continuous $\in$-chain, $\langle M_\alpha : \alpha < \omega_1\rangle$, of countable elementary submodels with $\{F_\alpha : \alpha \in E_1\} \in M_0$. Define $C := \{\alpha < \omega_1 : M_\alpha \cap \omega_1 = \alpha\}$. Then C is a club and so $E' := C \cap E_1$ is stationary. Moreover, if $\alpha, \beta \in E_0$ satisfy $\alpha < \beta$, then $F_\alpha \in M_\beta$ and $F_\beta \cap M_\beta = F_\beta \cap \beta = \emptyset$; thus $F_\alpha \cap F_\beta = \emptyset$.

Finally, observe that the equality $E = E_0 \cup E_1$ implies that E_i is stationary for some $i < 2$ and therefore the argument given in the two previous paragraphs proves our lemma. $\square$

Theorem 6.11. *Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \varepsilon\rangle$ be a finite support iteration of ccc posets and let $\mathbb{P}$ be its limit. If, for each $\alpha < \varepsilon$, $P_\alpha \Vdash$ "$\dot{Q}_\alpha$ preserves the SSP", then $\mathbb{P}$ preserves the SSP.*

Proof. The result will be proved by induction on ε. Let us fix, in V, a space X which has the SSP as witnessed by $\{c_\alpha : \alpha < \omega_1\}$ and let $\dot{S}$ be a $\mathbb{P}$-name which is forced by $p \in \mathbb{P}$ to be a stationary subset of ω_1.

Define $S_0 := \{\alpha < \omega_1 : \exists q \le p\ (q \Vdash \check{\alpha} \in \dot{S})\}$ and for each $\alpha \in S_0$ fix a condition $p_\alpha \le p$ in such a way that $p_\alpha \Vdash \check{\alpha} \in \dot{S}$. Notice that $p \Vdash \dot{S} \subseteq \check{S}_0$ and therefore S_0 is stationary (ω_1 is not collapsed because $\mathbb{P}$ is ccc). The proof will be divided into three cases.

Case 1. ε has countable cofinality.

Since p and each condition p_α ($\alpha \in S_0$) have finite support, there is $\mu < \omega_1$ for which $p \in P_\mu$ and $S_1 := \{\alpha \in S_0 : p_\alpha \in P_\mu\}$ is stationary. Lemma 6.9 provides us with a P_μ-name, $\dot{E}$, and a condition $p' \in P_\mu$ such that $p' \le p$ and $p' \Vdash$ "$\dot{E} = \{\alpha \in S_1 : p_\alpha \in \dot{G}_\mu\}$ is stationary," where $\dot{G}_\mu$ is a name for the P_μ-generic filter. Observe that p' forces in $\mathbb{P}$ that $\dot{E} \subseteq \dot{S}$.

Our inductive hypothesis guarantees that P_μ preserves the SSP so there is $q \in P_\mu$ such that $q \le p'$ and $q \Vdash$ "$X \setminus \bigcup\{c_\alpha : \alpha \in \dot{E}\}$ is compact." Thus (see Remark 6.7) $X \setminus \bigcup\{c_\alpha : \alpha \in \dot{S}\}$ is forced by q to be compact.

Case 2. $\mathrm{cf}(\varepsilon) = \omega_1$.

For each condition $r \in \mathbb{P}$, $s(r)$ will denote its support. Apply Lemma 6.10 to $\{s(p_\alpha) : \alpha \in S_0\}$ to obtain a stationary set $S_0' \subseteq S_0$ and an ordinal $\mu < \varepsilon$ in such a way that $\{s(p_\alpha) \setminus \mu : \mu \in S_0'\}$ is pairwise disjoint. Then (see Lemma 6.9) there exist $p' \in P_\mu$ and $\dot{S}_1$, a P_μ-name, in such a way that $p' \Vdash$ "$\dot{S}_1 = \{\alpha \in S_0' : p_\alpha \restriction \mu \in \dot{G}_\mu\}$ is stationary."

Fix a P_μ-name, $\dot{B}$, satisfying $p' \Vdash \dot{B} = \{x \in X : |\{\xi \in \dot{S}_1 : x \in c_\xi\}| < \omega\}$ and define $\overline{p} := p' \cup p \upharpoonright (\varepsilon \setminus \mu) \in \mathbb{P}$. We will show that

$$(\star) \qquad \overline{p} \Vdash X \setminus \dot{B} \subseteq \bigcup\{c_\alpha : \alpha \in \dot{S}\}.$$

Let $q \le \overline{p}$ be an arbitrary condition in $\mathbb{P}$ and let G_μ be a P_μ-generic filter over V with $q \upharpoonright \mu \in G_\mu$. Denote by S_1 and B the valuations of $\dot{S}_1$ and $\dot{B}$ with respect to G_μ, respectively, and let $x \in X \setminus B$ be arbitrary. Working in $V[G_\mu]$, since $s(q)$ is finite and S_1 is stationary, there is $\alpha \in S_1$ so that $x \in c_\alpha$ and $s(q)$ is disjoint from $s(p_\alpha) \setminus \mu$. Hence there exists $r \in G_\mu$ satisfying $r \le p_\alpha \upharpoonright \mu$ and $r \le q \upharpoonright \mu$. Therefore $\overline{r} := r \cup p_\alpha \upharpoonright (\varepsilon \setminus \mu)$ is a condition in $\mathbb{P}$ which extends q and p_α; in particular, $\overline{r} \Vdash \check{\alpha} \in \dot{S}$.

Fix G, a $\mathbb{P}$-generic filter over V, with $\overline{p} \in G$ and define $G_\mu := \{q \upharpoonright \mu : q \in G\}$ to obtain a P_μ-generic filter over V.

The discussion in this paragraph takes place in $V[G_\mu]$. Let θ be a cardinal for which H_θ has a continuous $\in$-chain, $\langle M_\alpha : \alpha < \omega_1\rangle$, of countable elementary submodels satisfying $\{c_\alpha : \alpha < \omega_1\}, S_1 \in M_0$. Then $S_2 := \{\alpha \in S_1 : M_\alpha \cap \omega_1 = \alpha\}$ is stationary.

Now we will show that $\{B \cap c_\alpha : \alpha \in S_2\}$ is pairwise disjoint. Given $\alpha \in S_2$ and $x \in B \cap c_\alpha$, set $\beta := \min\{\xi \in S_2 : x \in c_\xi\}$. Assume, seeking a contradiction, that $\beta < \alpha$. Then $\beta \in M_\alpha$ and therefore $c_\beta \in M_\alpha$; since c_β is countable, $x \in M_\alpha$. The fact $x \in B$ implies that $\{\xi \in S_1 : x \in c_\xi\} \subseteq M_\alpha$ and, in particular, $x \notin c_\alpha$. This contradiction proves that $\beta = \alpha$ and hence the collection is indeed pairwise disjoint.

Our inductive hypothesis applied to μ guarantees that $X \setminus \bigcup\{c_\alpha : \alpha \in S_2\}$ is compact and so there is $F \in [\omega_1]^{<\omega}$ in such a way that the set $U := \bigcup\{c_\alpha : \alpha \in F\}$ satisfies $X = U \cup \bigcup\{c_\alpha : \alpha \in S_2\}$. Then $\{B \cap c_\alpha \setminus U : \alpha \in S_2\}$ is a discrete family in X. According to Lemma 6.8, there is a finite set $F_0 \subseteq S_2$ such that $\bigcup\{B \cap c_\alpha : \alpha \in S_2 \setminus F_0\} \subseteq U$. Thus B is a subset of the compact scattered subspace $U \cup \bigcup\{c_\alpha : \alpha \in F_0\}$ and so we get that $X \setminus \bigcup\{c_\alpha : \alpha \in \dot{S}_G\}$ is compact (see the paragraph preceeding Remark 6.7 and ($\star$)).

Case 3. $\mathrm{cf}(\varepsilon) > \omega_1$.

Here we can assume, without loss of generality, that $\dot{S}$ is a nice name for a subset of $\check{\omega}_1$ (see [23, Definition IV.3.8]) and therefore, for some $\mu < \varepsilon$, $\dot{S}$ is a P_μ-name and $p \in P_\mu$. Hence the inductive hypothesis applied to μ takes care of the conclusion. □

We will need two lemmas concerning countable tightness.

Lemma 6.12. *If A is a T-algebra, then* $\mathrm{St}(A)$ *has countable tightness provided* $\mathrm{St}(A_b)$ *has countable tightness for each maximal branch $b \subset T$.*

Proof. Let $u \in \mathrm{St}(B)$ be any point and $Y \subset \mathrm{St}(B)$ be any set such that u is a limit point of Y. According to Corollary 3.5, there is a unique maximal branch b such that the ultrafilter u is generated $\{a_t : t \in b\}$. Assuming that $\mathrm{St}(A_b)$ has countable tightness, we may choose a countable family $\{y_n : n \in \omega\} \subset Y$, so that the ultrafilter $u \cap A_b \in \mathrm{St}(A_b)$ is a limit of the family $\{y_n \cap A_b : n \in \omega\}$. We finish by showing that u is a limit of $\{y_n : n \in \omega\}$. Choose any $a \in u$, we must show there is an n such that $a \in y_n$. Since $u \cap A_b$ is a base for $u \in A$ we may assume that $a \in u \cap A_b$. Therefore there is an $n \in \omega$ such that $a \in y_n \cap A_b$. Of course this means that $a \in y_n$. □

Lemma 6.13. *If B is a Boolean algebra whose Stone space has countable tightness, then any finite support iteration of σ-centered posets will preserve that* $\mathrm{St}(B)$ *has countable tightness.*

Proof. Assume that the Stone space of B has countable tightness. Recall that $\{b_\xi : \xi \in \omega_1\} \subset B$ is an algebraic free sequence if for each $\alpha \in \omega_1$, the family $\{b_\xi : \alpha \leq \xi\} \cup \{1 - b_\xi : \xi < \alpha\}$ generates a filter. It suffices to show B contains no algebraic free sequences in the forcing extension ([20]). Suppose that Q is a finite support iteration of σ-centered posets. Let us note that each uncountable subset of Q will have an uncountable centered subset. We show that no uncountable free sequence is added. For each $\alpha \in \omega_1$ suppose that $\dot{b}_\alpha$ is a Q-name of a member of B and that some $q \in Q$ forces that $\{\dot{b}_\alpha : \alpha \in \omega_1\}$ is a free sequence. For each α, choose $q_\alpha < q$ so that there is a $b_\alpha \in B$ such that $q_\alpha \Vdash \dot{b}_\alpha = \check{b}_\alpha$. Let $I \subset \omega_1$ be an uncountable set such that $\{q_\alpha : \alpha \in I\}$ is a centered subset of Q. It follows easily that $\{b_\alpha : \alpha \in I\}$ must be an uncountable free sequence – which is a contradiction. □

Using a similar proof, the following result is well-known.

Proposition 6.14. *If Q is a finite support iteration of σ-centered posets, then forcing with Q adds no new cofinal branches to the tree $2^{<\omega_1}$.*

The proof of Theorem 6.2 will be split into a series of lemmas so, for simplicity, we will establish notation that will be followed for the rest of the section: $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\rangle$ denotes a finite support iteration of Hechler forcing whose limit is $\mathbb{P}$ and V is a model of CH. Also, let us define $E := \{0\} \cup \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$.

Given two ordinals, α and β, their product will be denoted by $\alpha \cdot \beta$. In particular, if γ is an ordinal, $\gamma^2 = \gamma \cdot \gamma$.

Lemma 6.15. *There are $\{\dot{t}_\alpha : \alpha \in E\}$ and $\delta : E \to \omega_1$ such that the following conditions hold, for each $\alpha \in E$,*

1. $\{\dot{t}_\xi : \xi \in E \cap \gamma^2\} = \{\check{s} : s \in 2^{<\gamma} \cap V\}$*, whenever* $\gamma \in \{\omega, \omega_1\}$*,*
2. $\dot{t}_\alpha$ *is a* P_α*-name,*
3. $P_\alpha \Vdash \dot{t}_\alpha \in 2^{\delta(\alpha)} \setminus \{\dot{t}_\xi : \xi \in E \cap \alpha\}$*,*
4. $P_\alpha \Vdash \{\dot{t}_\alpha \restriction \beta : \beta < \delta(\alpha)\} \subseteq \{\dot{t}_\xi : \xi \in E \cap \alpha\}$*, and*
5. $\mathbb{P} \Vdash 2^{<\omega_1} = \{\dot{t}_\xi : \xi \in E\}$.

Proof. Let E_0 be the set of all limit ordinals in E; in other words, $\alpha \in E_0$ iff $\alpha \in E$ and $\beta + \omega < \alpha$ for all $\beta \in E \cap \alpha$.

Since CH holds in the ground model, a lexicographical ordering of the limit levels of the tree $2^{<\omega_1} \cap V$ provides us with an enumeration $\{s_\alpha : \alpha < \omega_1^2\}$ of $2^{<\omega_1} \cap V$ in such a way that for all $\alpha < \omega_1^2$: $\{s_\alpha \restriction \beta : \beta < \mathrm{dom}\, s_\alpha\} \subseteq \{s_\xi : \xi < \alpha\}$ and if α is a limit ordinal, then $\{s_\alpha {}^\frown r : r \in 2^{<\omega}\} = \{s_{\alpha+n} : n \in \omega\}$. Note that these conditions imply $\{s_n : n \in \omega\} = 2^{<\omega}$ and $s_0 = \emptyset$.

Let us denote by S the set consisting of all triples $(p, \dot{r}, \beta)$, where $p \in \mathbb{P}$, $\beta < \omega_1$, and $\dot{r}$ is a nice $\mathbb{P}$-name for a subset of $\beta \times 2$ with $p \Vdash \dot{r} \in 2^\beta$.

Given that $\mathbb{P}$ is ccc, we have that for each $(p, \dot{r}, \beta) \in S$ there is $\gamma < \omega_2$ satisfying

$$(\star) \qquad \dot{r} \text{ is a } P_\gamma\text{-name and } p \in P_\gamma.$$

On the other hand, the fact that V models CH implies that for any fixed $\gamma < \omega_2$ there are at most ω_1 triples $(p, \dot{r}, \beta) \in S$ for which $(\star)$ holds. As a consequence of these remarks, we get that there is an enumeration $\{(p_\alpha, \dot{r}_\alpha, \beta_\alpha) : \alpha < \omega_2\}$ of

S in such a way that, for each $\alpha < \omega_2$, $p_\alpha \in P_\alpha$ and $\dot{r}_\alpha$ is a P_α-name. Moreover, $\dot{r}_\alpha = \check{s}_\alpha$, whenever $\alpha < \omega_1^2$.

We will obtain $\{\dot{t}_\alpha : \alpha \in E\}$ and δ by transfinite induction on E. Let us start by setting, for each $\alpha < \omega_1^2$, $\dot{t}_{\omega\cdot\alpha} := \check{s}_\alpha$ and $\delta(\omega\cdot\alpha) := \operatorname{dom} s_\alpha$. This produces $\{\dot{t}_\xi : \xi \in E \cap \omega_1^2\}$ and $\delta \restriction (E \cap \omega_1^2)$ as required in conditions (1)–(4).

Now assume that for some $\alpha \in E \setminus \omega_1^2$ we have defined $\{\dot{t}_\xi : \xi \in E \cap \alpha\}$ and $\delta \restriction (E \cap \alpha)$ satisfying conditions (1)–(4) and in such a way that for any $\xi \in E_0 \cap \alpha$ and $n < \omega$, we obtain the following: $\xi + \omega \cdot n < \alpha$; $\delta(\xi)$ is a limit ordinal; $\delta(\xi + \omega \cdot n) = \delta(\xi) + |s_n|$; and $P_{\xi+\omega\cdot n} \Vdash \dot{t}_{\xi+\omega\cdot n} = \dot{t}_\xi {}^\frown \check{s}_n$. Notice that a consequence of these hypotheses is $\alpha \in E_0$.

Let $\{\gamma_n : n \in \omega\}$ be an strictly increasing sequence whose supremum is α and let G_α be a P_α-generic filter over V. For each integer n, set $G_{\gamma_n} := G_\alpha \cap P_{\gamma_n}$ and note that the quotient forcing $P_{\gamma_{n+1}}/G_{\gamma_n}$ adds a new real to $V[G_{\gamma_n}]$ because each $\dot{Q}_{\gamma_n}$ is forced by 1 to be Hechler's poset. This proves the following claim.

Claim 1. $2^\omega \cap V[G_\alpha] \setminus \bigcup_n V[G_{\gamma_n}] \neq \emptyset$.

Claim 2. There is $\eta < \omega_2$ with $p_\eta \in P_\alpha$ and

$$p_\eta \Vdash \dot{r}_\eta \notin \{\dot{t}_\xi : \xi \in E \cap \alpha\} \wedge (\{\dot{r}_\eta \restriction \beta : \beta < \beta_\eta\} \subseteq \{\dot{t}_\xi : \xi \in E \cap \alpha\}).$$

Let us start the proof by noticing that Claim 1 implies the existence of $\dot{r}$, a nice P_α-name for a subset of $\omega \times 2$, with $P_\alpha \Vdash \dot{r} \in 2^\omega \setminus \{\dot{t}_\xi : \xi \in E \cap \alpha\}$. Hence we obtain $(1, \dot{r}, \omega) = (p_\eta, \dot{r}_\eta, \beta_\eta)$, for some $\eta < \omega_2$. Finally, note that the equality $\{s_n : n < \omega\} = 2^{<\omega}$ and our definition of $\dot{t}_{\omega\cdot n}$, $n \in \omega$, guarantee that η is as needed.

Let η be the least ordinal satisfying all conditions in Claim 2.

Let us show that β_η is a limit ordinal. Indeed, given $\gamma < \beta_\eta$, our choice for η gives $q \le p_\eta$ and $\xi \in E \cap \alpha$ such that $q \Vdash \dot{r}_\eta \restriction \gamma = \dot{t}_\xi$. Thus, according to the inductive hypothesis,

$$q \Vdash \dot{r}_\eta \restriction (\gamma + 1) = (\dot{r}_\eta \restriction \gamma)^\frown \dot{r}_\eta(\gamma) \in \{\dot{t}_\xi : \xi \in E \cap \alpha\},$$

which implies $\gamma + 1 \neq \beta_\eta$.

Define $\delta(\alpha) := \beta_\eta$ and fix W, a maximal antichain in P_α with $p_\eta \in W$. We will obtain, for each $q \in W$, a P_α-name $\dot{t}_q$ such that

$$q \Vdash (\dot{t}_q \in 2^{\delta(\alpha)} \setminus \{\dot{t}_\xi : \xi \in E \cap \alpha\}) \wedge (\{\dot{t}_q \restriction \beta : \beta < \delta(\alpha)\} \subseteq \{\dot{t}_\xi : \xi \in E \cap \alpha\}).$$

Notice that once this is done, we would be able to use [23, Lemma IV.7.2] to get a P_α-name which is forced by each $q \in W$ to be equal to its corresponding $\dot{t}_q$. By letting $\dot{t}_\alpha$ be this name, the induction will be complete except for the verification of condition (5).

Let $q \in W$ be arbitrary. When $q = p_\eta$, it suffices to set $\dot{t}_q := \dot{t}_\eta$, so assume $q \neq p_\eta$ and fix an strictly increasing sequence $\langle \alpha_n : n < \omega \rangle$ whose supremum is $\delta(\alpha)$. Working in V, let $\{e_r : r \in 2^{<\omega}\} \subseteq 2^{\delta(\alpha)}$ be such that for all $r, s \in 2^{<\omega}$: $r < s$ implies $e_r < e_s$ and $\operatorname{dom}(e_r) = \alpha_{|r|}$, i.e., a cofinal copy of the Cantor tree.

The discussion in this paragraph takes place in $V[G_\alpha]$. As usual, we drop the dots to indicate the valuation of the corresponding name with respect to G_α. According to Claim 1, there is $f \in 2^\omega \setminus \bigcup_n V[G_{\gamma_n}]$ and hence $t_q := \bigcup_n e_{f\restriction n}$ is an element of $2^{\delta(\alpha)} \setminus \{t_\xi : \xi \in E \cap \alpha\}$. On the other hand, if $\beta < \delta(\alpha)$, then $\beta < \alpha_n$, for some $n \in \omega$, and therefore $t_q \restriction \beta = e_{f\restriction n} \restriction \beta \in \{t_\xi : \xi \in E \cap \alpha\}$. This proves the existence of $\dot{t}_q$ as discussed above.

Before we embark on the proof of (5), notice that at stage $\alpha \in E \setminus \omega_1^2$ of the induction we selected an ordinal $\eta = \eta(\alpha)$ as the least one satisfying the conditions of Claim 2. This gives a map $\alpha \mapsto \eta(\alpha)$ which is strictly increasing: $\alpha < \beta$ implies $\eta(\alpha) < \eta(\beta)$. Also, our definition of $\dot{t}_\alpha$ was done in such a way that $p_{\eta(\alpha)} \Vdash \dot{t}_\alpha = \dot{r}_{\eta(\alpha)}$.

To verify (5) let us suppose, seeking a contradiction, that there is a $\mathbb{P}$-name $\dot{s}$ and a condition $p \in \mathbb{P}$ such that $p \Vdash \dot{s} \in 2^{<\omega} \setminus \{\dot{t}_\xi : \xi \in E\}$. There is no loss of generality in assuming that, for some $\gamma < \omega_1$, we get

$$p \Vdash \dot{s} \in 2^\gamma \wedge (\{\dot{s} \restriction \beta : \beta < \gamma\} \subseteq \{\dot{t}_\xi : \xi \in E\}).$$

Hence $(p, \dot{s}, \gamma) = (p_\nu, \dot{r}_\nu, \beta_\nu)$, for some $\nu < \omega_2$. Since β_ν is countable and P_ν is ccc, there exists $\alpha \in E$ for which $p_\nu \in P_\alpha$ and $p_\nu \Vdash \{\dot{r}_\nu \restriction \beta : \beta < \beta_\nu\} \subseteq \{\dot{t}_\xi : \xi \in E \cap \alpha\}$. Note that condition (1) gives $\alpha \geq \omega_1^2$.

From the two previous paragraphs we obtain $\eta(\alpha) \leq \nu$; moreover, the equality $\eta(\alpha) = \nu$ would imply $p_\nu \Vdash \dot{t}_\alpha = \dot{r}_\nu$, i.e., $p \Vdash \dot{t}_\alpha = \dot{s}$. Hence $\eta(\alpha) < \nu$ and therefore, by letting $\overline{\alpha} \in E \setminus \omega_1^2$ be so that $\nu < \eta(\overline{\alpha})$, we get $\alpha < \overline{\alpha}$. This, in turn, implies that if one replaces η and α with ν and $\overline{\alpha}$, respectively, in Claim 2, then all conditions are fulfilled and so $\eta(\overline{\alpha}) \leq \nu$, contradicting our choice for $\overline{\alpha}$. □

To simplify notation we will set $\delta_\alpha := \delta(\alpha)$, for each $\alpha \in E$.

The argument given in [22, Example 2.9] shows that there is a collection $Y = \{a(s^\frown k) : s \in 2^{<\omega} \wedge k < 2\} \subseteq \mathcal{P}(\omega) \cap V$ in such a way that the Boolean algebra generated by Y in $\mathcal{P}(\omega)$ is (a) isomorphic to $CO(2^\omega)$ and (b) a $(2^{<\omega} \setminus \{\emptyset\})$-algebra as witnessed by Y.

On the other hand, condition (1) in our previous lemma gives $\{\check{s} : s \in 2^{<\omega}\} = \{\dot{t}_\xi : \xi \in E \cap \omega^2\}$; therefore, by letting $\dot{a}(\check{s}^\frown k)$ be the canonical name for $a(s^\frown k)$ we get a family $\{\dot{a}(\dot{t}_\xi{}^\frown k) : \xi \in E \cap \omega^2 \wedge k < 2\}$ in such a way that for each $\alpha \in E \cap \omega^2$:

(1$\boldsymbol{\alpha}$) $\dot{a}(\dot{t}_\alpha{}^\frown k)$, $k < 2$, is a $P_{\alpha+\omega}$-name for a subset of ω and

$$P_{\alpha+\omega} \Vdash \dot{a}(\dot{t}_\alpha{}^\frown 1) = \omega \setminus \dot{a}(\dot{t}_\alpha{}^\frown 0).$$

We plan to obtain, by transfinite induction on E, a collection of names $\{\dot{a}(\dot{t}_\alpha) : \alpha \in E \wedge k < 2\}$ satisfying (1α) together with conditions (2α)–$(4\alpha')$ below. In order to understand the meaning of these last four conditions, some remarks and definitions are needed.

First, observe that at stage $\alpha \in E$ of our induction we will be assuming that $\{\dot{a}(\dot{t}_\xi) : \xi \in E \cap \alpha \wedge k < 2\}$ is given. By noticing that $\xi + \omega \leq \alpha$, for all $\xi \in E \cap \alpha$, we obtain $\{a(t_\xi) : \xi \in E \cap \alpha \wedge k < 2\} \subseteq V[G_\alpha]$, whenever G_α is a P_α-generic filter over V. Hence $Y_\alpha := \{a(t_\alpha \restriction (\beta + 1)) : \beta + 1 < \delta_\alpha\}$ turns out to be an element of $V[G_\alpha]$ and so are A_α, the Boolean algebra generated by Y_α in $\mathcal{P}(\omega)$, and u_α, the ultrafilter generated by Y_α in A_α.

Working in $V[G_\alpha]$, define R_α as follows, $p \in R_\alpha$ iff there exist $p_0, p_1 \in R_\alpha \setminus u_\alpha$ with $p = (p_0, p_1)$ and $p_0 \cap p_1 = \emptyset$. We order R_α by $p \leq q$ iff $q_0 \subseteq p_0$ and $q_1 \subseteq p_1$. This poset was introduced by Koszmider in [22] to force a minimal element, i.e., whenever H_α is an R_α-generic filter over $V[G_\alpha]$, $\bigcup\{p_0 : p \in H_\alpha\}$ is minimal for (A_α, u_α). The following relations between R_α and A_α will be of use.

Lemma 6.16. *When A_α is an atomless Boolean algebra, the following holds.*

1. *R_α is an atomless poset.*
2. *If H_α is R_α-generic over $V[G_\alpha]$ and $x := \bigcup\{p_0 : p \in H_\alpha\}$, then the Boolean algebra generated by $A_\alpha \cup \{x\}$ is atomless.*

Proof. Let $p \in R_\alpha$ be arbitrary, Then $-(p_0 \cup p_1) \in u_\alpha$ and thus $-(p_0 \cup p_1) \neq \emptyset$; so there are $q_0, q_1 \in A_\alpha \setminus \{\emptyset\}$ such that $q_0 \cap q_1 = \emptyset$ and $q_0 \cup q_1 = -(p_0 \cup p_1) \in u_\alpha$. Hence we can assume, without loss of generality, that $q_0 \in u_\alpha$ and $q_1 \notin u_\alpha$. Once again, the fact that q_1 is not an atom implies the existence of $r_0, r_1 \in A_\alpha \setminus \{\emptyset\}$ with $q_1 = r_0 \cup r_1$ and $r_0 \cap r_1 = \emptyset$. In particular, $r_0, r_1 \notin u_\alpha$ and therefore $(p_0 \cup r_0, p_1 \cup r_1)$ and $(p_0 \cup r_1, p_1 \cup r_0)$ are two incompatible extensions of p in R_α. This proves (1).

For (2), let $b \neq \emptyset$ be an element of $A_\alpha(x)$, the Boolean algebra generated by $A_\alpha \cup \{x\}$. Then there are $b_0, b_2 \in A_\alpha$ with $b = (b_0 \cap x) \cup (b_1 - x)$. Thus $b_0 \cap x \neq \emptyset$ or $b_1 - x \neq \emptyset$. In the first case, $p_0 \cap b_0 \neq \emptyset$, for some $p_0 \in R_\alpha$, and so $p_0 \cap b_0$ is not an atom in A_α, which implies that b is not an atom in $A_\alpha(x)$. A similar argument can be used when $b_1 - x \neq \emptyset$. □

Observe that A_{ω^2} is atomless because 2^ω has no isolated points. We want to keep this property in our induction:

(2$\boldsymbol{\alpha}$) $P_\alpha \Vdash$ "The Boolean algebra generated by $\{\dot{a}(\dot{t}_\alpha \restriction (\beta+1)) : \beta + 1 < \delta_\alpha\}$, is atomless."

Note that, as a consequence of (2α), R_α is a countable atomless poset and so (see [23, Exercise III.3.70]) the poset $(\omega^{<\omega}, \supseteq)$ embedds densely into R_α. On the other hand, the quotient forcing $P_{\alpha+\omega}/G_\alpha$ adjoins a Cohen real to $V[G_\alpha]$, i.e., $V[G_{\alpha+\omega}]$ always contains an R_α-generic filter over $V[G_\alpha]$. Therefore, by letting $\dot{a}(\dot{t}_\alpha{}^\frown 0)$ be a $P_{\alpha+\omega}$-name for the generic object added by R_α to $V[G_\alpha]$, conditions (1α) and ($2\alpha + \omega$) hold. Moreover, if $\dot{A}_\alpha$ and $\dot{u}_\alpha$ are P_α-names for the corresponding objects discussed above, then

(3$\boldsymbol{\alpha}$) For each $k < 2$, $P_{\alpha+\omega} \Vdash$ "$\dot{a}(\dot{t}_\alpha{}^\frown k)$ is minimal for $(\dot{A}_\alpha, \dot{u}_\alpha)$."

Up to this point our induction is guaranteed to produce a $\mathcal{T}$-algebra, but since we are interested in getting an Efimov $\mathcal{T}$-algebra, our next result is needed.

Lemma 6.17. *Assume that, in $V[G_\alpha]$, u_α is an accumulation point of a countable set $S \subseteq \mathrm{St}(A_\alpha)$. If $k < 2$, then $x \cup \{a(t_\alpha{}^\frown k)\}$ has the finite intersection property for infinitely many $x \in S$.*

Proof. We will discuss only the case $k = 0$ because the same argument, mutatis mutandis, works for $k = 1$. Start by enumerating $S \setminus \{u_\alpha\}$ as $\{x_n : n \in \omega\}$ in such a way that $x_m \neq x_n$, whenever $m \neq n$.

Now, for each integer m, define $D_m := \{p \in R_\alpha : \exists n \geq m \ (p_0 \in x_n)\}$. We claim that D_m is dense in R_α: given $p \in R_\alpha$, we get $p_0 \cup p_1 \notin u_\alpha$ and therefore, our assumption on u_α implies that $p_0 \cup p_1 \notin x_n$, for some $n \geq m$. Since $x_n \neq u_\alpha$, there is $q \in x_n \setminus u_\alpha$ with $q \cap (p_0 \cup p_1) = \emptyset$. Finally, $(p_0 \cup q, p_1)$ is an element of D_m which extends p.

To conclude the proof: if $m < \omega$, there is $p \in H_\alpha \cap D_m$ and so $p_0 \in x_n$ and $p_0 \subseteq a(t_\alpha{}^\frown 0)$, for some $n \geq m$. Hence $x_n \cup \{a(t_\alpha{}^\frown 0)\}$ has the finite intersection property. □

A set S satisfying the hypothesis of the Lemma will be called *unbounded in* $V[G_\alpha]$.

As discussed at the beginning of Section 3, u_α is the only ultrafilter in A_α that can be extended to more than one ultrafilter in the Boolean algebra generated by $A_\alpha \cup \{a(t_\alpha{}^\frown 0)\}$. Thus, for any $x \in \mathrm{St}(A_\alpha) \setminus \{u_\alpha\}$, $x \cup \{a(t_\alpha{}^\frown k)\}$ has the finite intersection property for exactly one $k \in \{0, 1\}$. Keeping this in mind, one may paraphrase the lemma as $a(t_\alpha{}^\frown 0)$ *splits* any unbounded subset of $\mathrm{St}(A_\alpha)$ from

$V[G_\alpha]$, i.e., $a(t_\alpha{}^\frown 0)$ "grabs" infinitely many elements of S and "leaves" infinitely many. This will be our official statement for the last condition in our induction:

(4$\boldsymbol{\alpha}$) $P_{\alpha+\omega} \Vdash$ "$\dot{a}(\dot{t}_\alpha{}^\frown 0)$ splits all unbounded subsets of $\mathrm{St}(\dot{A}_\alpha)$ in $V[\dot{G}_\alpha]$."

(4$\boldsymbol{\alpha'}$) $P_{\alpha+\omega} \Vdash$ "each unbounded subset of $\mathrm{St}(\dot{A}_\alpha)$ from $V[\dot{G}_\alpha]$ remains unbounded."

Thus the induction can be carried through, giving us the following (recall Lemma 6.15):

Lemma 6.18. *Let G be a $\mathbb{P}$-generic filter over V and, in $V[G]$, let B be the Boolean algebra generated by $\{a(t_\alpha{}^\frown k) : \alpha \in E \wedge k < 2\}$. Then B is a T-algebra, where*

$$T := \{s^\frown k : s \in 2^{<\omega_1} \wedge k < 2\}.$$

The only thing left now is to verify that B is, indeed, Efimov and has countable tightness. For Efimov, by Remark 3.1 it suffices to show that $\mathrm{St}(B)$ has no converging sequences. To do this we will prove that no point in $\mathrm{St}(B)$ is the limit of an infinite sequence in the Stone space of B. So let $z \in \mathrm{St}(B)$ be arbitrary.

According to Proposition 3.5, there is $f \in 2^{\omega_1} \cap V[G]$ in such a way that z is the ultrafilter generated by $\{a(f \restriction (\alpha+1)) : \alpha < \omega_1\}$ in B. Define

$$T_f := \{(f \restriction \alpha)^\frown k : \alpha < \omega_1 \wedge k < 2\},$$

denote by B_f the Boolean algebra generated by $\{a(s) : s \in T_f\}$, and let u_f be the ultrafilter generated by $\{a(s) : s \in T_f\}$ in B_f. Also, set $X_f := \mathrm{St}(B_f) \setminus \{u_f\}$.

For each $\alpha < \omega_2$, define $G_\alpha := G \cap P_\alpha$. Since $\mathbb{P}$ is ccc and the iteration has length ω_2, $f \in V[G_\lambda]$, for some $\lambda < \omega_2$.

Condition (5) in Lemma 6.15 implies that for each $\beta < \omega_1$ there is $\alpha \in E$ with $t_\alpha = f \restriction \beta$. Moreover, if $\overline{\alpha} \in E \setminus (\alpha+1)$ satisfies $t_{\overline{\alpha}} = f \restriction \beta$, then $t_{\overline{\alpha}} = t_\alpha \in \{t_\xi : \xi \in E \cap \overline{\alpha}\}$, contradicting Lemma 6.15-(3). This remark proves that $\mu := \sup\{\alpha \in E : \exists \beta < \omega_1 (t_\alpha = f \restriction \beta)\} < \omega_2$. It follows from Lemma 6.14, that $\lambda \le \mu$. In summary, we have that $f \in V[G_\mu]$ and for each $\beta < \omega_1$, there is a $\gamma < \mu$ such that $\{a(f \restriction (\alpha+1)) : \alpha < \beta\} \in V[G_\gamma]$.

From the last paragraph we deduce that T_f, B_f, u_f, and the topological space X_f are all elements of $V[G_\mu]$. More can be proved:

Lemma 6.19. *X_f has countable tightness.*

Proof. Let $E_f = \{\alpha \in E : t_\alpha \subset f\}$. Choose any $\alpha \in E_f$ and any unbounded $S \subset \mathrm{St}(A_\alpha)$ in $V[G_\alpha]$ as described above. We show that it follows from induction condition $(4\alpha')$ and induction on $\beta \in E \setminus \alpha$, we have that if $t_\alpha \subset t_\beta \subset f$, then S can be regarded as a subset of $\mathrm{St}(A_\beta)$, and that S remains unbounded in $\mathrm{St}(A_\beta)$. If $\beta \in E \setminus E_f$, then the introduction of $a(t_\beta)$ has no effect on the unboundedness of S, while for $\beta \in E_f \setminus \alpha$, the fact that S remains unbounded is assumption $(4\alpha')$. Therefore, for each $\alpha \in E_f$ and $S \in V[G_\alpha]$ which is unbounded in $\mathrm{St}(A_\alpha)$, we have that z is a limit point of S.

By Lemma 6.13, it suffices to show that for each uncountable $S \subset \mathrm{St}(A_f)$ which is a member of $V[G_\mu]$, there is an $\alpha \in E_f$ such that $S \cap \mathrm{St}(A_\alpha)$ is unbounded in $\mathrm{St}(A_\alpha)$ and is a member of $V[G_\alpha]$. For each $s \in S$, there is a minimal $\xi_s \in \mu$ such that $s \in \mathrm{St}(A_{\xi_s+1})$. The special nature of T-algebras also ensures that for each $\xi < \omega_1$, there is a unique $s \in \mathrm{St}(A_f)$ such that $\xi_s = \xi$. By the definition of μ, the set $\{\xi_s : s \in S\}$ is cofinal in μ. Fix a P_μ-name $\dot{Y}$ for the set $\{\xi_s : s \in S\}$. It will be most convenient to use a countable elementary submodel argument. Fix any countable elementary submodel $M \prec H(\aleph_3)$ such that $\dot{Y}, P_\mu, \{\dot{t}_\beta : \beta \in \mu\}$ are all in M. Let $\gamma = \sup(M \cap \mu)$ and let $\dot{S}_M$ be the name $\dot{S} \cap M$. It follows easily that $\dot{S}_M$ is a P_γ-name, and that $\Vdash_{P_\mu}$ "$\dot{S} \supset \dot{S}_M$". An easy elementary argument shows that $\mathrm{val}_{G_\gamma}(\dot{S}_M) = S_M$ is cofinal in γ. In addition, if $\alpha = \min(E_f \setminus \gamma)$, then $\{s : \xi_s \in S_M\}$ is unbounded in $\mathrm{St}(A_\alpha)$. □

By an argument very much the same as in Lemma 6.19, one can show the following.

Lemma 6.20. *In $V[G_\mu]$, X_f is sequentially compact.*

Unfortunately we need to prove something stronger because we need this to hold in $V[G]$ and it is not true that Hechler forcing preserves the sequential compactness of general spaces. This is the reason we needed to introduce the SSP property and to finish the paper by proving the following.

Lemma 6.21. *In $V[G_\mu]$, X_f has the SSP property.*

Proof. For each $\xi \in \omega_1$, let x_ξ be the point in X_f with filter base equal to $\{a(t) : t < f \restriction \xi\} \cup \{\omega \setminus a(f \restriction \xi + 1)\}$. For each $\xi \in \omega_1$, let $\alpha_\xi \in \mu$ be the unique value so that $f \restriction \xi = t_{\alpha_\xi}$. Also, let $c_\xi = \omega \setminus a(f \restriction \xi + 1)$. To show that X_f has the SSP property, we must show that for each stationary set $S \subset \omega_1$, there is a $b \in B_f \setminus u_f$ such that $X_f \setminus \bigcup\{c_\xi : \xi \in S\}$ is contained in the compact open set corresponding to b. We may assume that we have a P_μ-name $\dot{f}$ for f.

To do so, let $\dot{S}$ be a P_μ-name and assume there is a $p \in P_\mu$ forcing that S is stationary. Let S_1 denote the set of $\xi \in \omega_1$ such that there is some $p_\xi < p$ in P_μ with $p_\xi \Vdash \xi \in \dot{S}$. It follows immediately that S_1 is stationary. Naturally, we select such a $p_\xi < p$ for each $\xi \in S_1$. We may suppose additionally, that p_ξ forces a value on α_ξ so that $p_\xi \Vdash \dot{t}_{\alpha_\xi} = \dot{f} \restriction \xi + 1$. For each $\xi \in S_1$, $\dot{c}_\xi$ will denote the canonical $P_{\alpha_\xi+\omega}$-name for c_ξ, which is simply the complement of $\dot{a}(\dot{t}_{\alpha_\xi})$. By passing to a stationary subset of S_1, and by symmetry, we may suppose that $p_\xi \Vdash \dot{f}(\xi) = 1$ for each $\xi \in S_1$. There is no loss of generality to assume that $\alpha_\xi \in \mathrm{dom}(p_\xi)$ for all $\xi \in S_1$.

For each $\xi \in S_1$, we have that $p_\xi \restriction \alpha_\xi$ forces that $p_\xi \restriction [\alpha_\xi, \alpha_\xi + \omega)$ corresponds to a certain pair (b_0^ξ, b_1^ξ) in the poset R_{α_ξ}. Let us understand this better. In the extension $V[G_{\alpha_\xi}]$ (with $p \restriction \alpha_\xi \in G_{\alpha_\xi}$), we have that (b_0^ξ, b_1^ξ) is a disjoint pair in $A_{\alpha_\xi} \setminus u_{\alpha_\xi}$ and that, for any extension $(c_0^\xi, c_1^\xi) < (b_0^\xi, b_1^\xi)$ in R_{α_ξ}, we can find an extension of $p_\xi \restriction [\alpha_\xi, \alpha_\xi + \omega)$ which forces that $a(t_{\alpha_\xi+1})$ contains c_0^ξ and is disjoint from c_1^ξ. In particular, if there is some $\zeta < \xi$ such that $b_0^\xi \notin x_\zeta$, then we can extend $p_\xi \restriction [\alpha_\xi, \alpha_\xi + \omega)$ to force that $x_\zeta \in c_\xi$ (where c_ξ is defined as above).

For each $\xi \in S_1$, choose a finite $L_\xi \subset \xi$ such that $p_\xi \restriction \alpha_\xi$ forces that b_0^ξ and b_1^ξ are in the Boolean algebra generated by $\{a(f \restriction \gamma + 1) : \gamma \in L_\xi\}$. By the pressing down lemma, we may choose a stationary set $S_2 \subset S_1$ such that for $\xi, \xi' \in S_2$, $L_\xi = L_{\xi'}$ and $(b_0^\xi, b_1^\xi) = (b_0^{\xi'}, b_1^{\xi'})$. Let (b_0, b_1) denote this common pair.

For each $\xi \in S_2 \setminus \min(S_2)$, let $g(\xi) \in S_2 \cap \xi$ be chosen so that $\mathrm{dom}(p_\xi) \cap \max(\mathrm{dom}(p_\beta)) \subset \mathrm{dom}(p_\xi) \cap \max(\mathrm{dom}(p_{g(\xi)}))$ for all $\beta \in S_2 \cap \xi$. In words, just pick a witness to the largest intersection of $\max(\mathrm{dom}(p_\beta))$ with $\mathrm{dom}(p_\xi)$. By the pressing down lemma, there is a $\delta \in \omega_1$ and a stationary $S_3 \subset S_2$ such that $g(\xi) < \delta$ for all $\xi \in S_3$. We may also assume that $L_\xi \subset \delta$ for all $\xi \in S_3$. Let $\mu_\delta < \mu$ be such that $\mathrm{dom}(p_\beta) \subset \mu_\delta$ for all $\beta < \delta$. Using that P_{μ_δ} is ccc, there is a generic filter

G_{μ_δ} for P_{μ_δ} such that $S_4 = \{\xi \in S_3 : p_\xi \restriction \mu_\delta \in G_{\mu_\delta}\}$ is stationary (in the model $V[G_{\mu_\delta}]$).

Now we prove that if G_μ is P_μ-generic, and $G_\mu \cap P_{\mu_\delta} = G_{\mu_\delta}$, then $X_f \setminus \bigcup\{c_\xi : \xi \in S\}$ is contained in the clopen set given by b_1. Choose any condition $q \in P_\mu$ such that $q \restriction \mu_\delta \in G_{\mu_\delta}$. Assume that $q \Vdash b \notin x_\zeta$. Choose any $\beta \in S_3$ so that $\mathrm{dom}(q) \subset \alpha_\beta$. Then choose any $\xi \in S_4 \setminus \beta$. Therefore we have that $\mathrm{dom}(p_\xi) \cap \max(\mathrm{dom}(p_\beta)) \subset \mu_\delta$, which implies that q and $p_\xi \restriction \alpha_\xi$ are compatible conditions in P_{α_ξ}. By the discussion above, there is a common extension of q and p_ξ which forces that $c_\xi \in x_\zeta$. $\square$

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