# AN EFIMOV SPACE FROM MARTIN'S AXIOM

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Abstract. We show from a weak set-theoretic hypothesis that there is an Efimov space.

## 1. Introduction

A topological space is an *Efimov space* if it is infinite, compact and contains no infinite converging sequence and no homeomorphic copy of $\beta\omega$. It is a long-standing problem in set-theoretic topology to determine if there is an Efimov space. A Boolean algebra will be called Efimov if its Stone space of ultrafilters is an Efimov space. A compact space $X$ contains a homeomorphic copy of $\beta\omega$ if and only if it has a closed subspace which maps onto the product space $2^{\mathfrak{c}}$. For the Stone space of a Boolean algebra $B$ this is equivalent to containing an independent family of cardinality $\mathfrak{c}$.

Efimov spaces are known to exist in various models of set-theory, for example, the continuum hypothesis implies they exist (see [3, 4]). It is not known if they exist in ZFC, and, as was asked in [4], it was not even known if their existence was consistent with Martin's Axiom and the failure of CH. We prove that the hypothesis $\mathfrak{b} = \mathfrak{c}$ is sufficient to imply their existence. Of course $\mathfrak{b}$ is the minimum cardinal such that there is a mod finite unbounded family of functions from $\omega^\omega$ of that cardinality.

We construct a very special kind of Efimov Boolean algebra, one that is minimally generated (see [6]). The paper by Koszmider [7, 2.7] introduces a special kind of minimally generated Boolean algebra that are called T-algebras. We introduce an alternative description as a game strategy through a device that we find useful that we call coherently minimally generated Boolean algebras. These give rise to superatomic Boolean algebras, and we begin by recalling several basic notions. Complete coverage of superatomic Boolean algebras can be found in [6]. Minimal Boolean algebras are introduced in [5] and the genesis of the notion of coherently minimally generated Boolean algebras (and T-algebras) is from [5, 7]. We limit our discussion to some basics chosen to make the paper more self-contained.

Let $B$ be a Boolean algebra. The Stone space, $S(B)$, is the space consisting of all ultrafilters on $B$ with the topology generated by the clopen sets $\{b^* : b \in B\}$ where $b^* = \{u \in S(B) : b \in u\}$ for $b \in B$. If $A$ is a subalgebra of $B$, then $S(B)$

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canonically maps onto $S(A)$ by sending $u \in S(B)$ to $u \cap A \in S(A)$. If $X$ is a non-empty subspace of $S(B)$, then there is a natural homomorphism $\varphi_X$ on $B$ induced by $X$ where, for $a, b \in B$, $\varphi_X(a) = \varphi_X(b)$ if and only if $a^* \cap X = b^* \cap X$.

**Definition 1.1.** *A Boolean algebra is superatomic if every homomorphic image is atomic.*

**Proposition 1.2.** *If a Boolean algebra $B$ is superatomic then every subspace of $S(B)$ has a dense set of isolated points.*

*Proof.* Assume that $B$ is superatomic and that $X$ is a subspace of $S(B)$. To show that $X$ has a dense set of isolated points, we choose any $b \in B$ such that $b^* \cap X$ is non-empty and show that it contains an isolated point of $X$. Note that $\varphi_X(b) \neq 0$. Choose $a \in B$ so that $\varphi_X(a) \leq \varphi_X(b)$ is an atom in the homomorphic image of $B$. Since $\varphi_X(a)$ is an atom it is immediate that $a^* \cap X$ is a single point of $X$. □

## 2. Coherently minimally generated Boolean algebras

$B$ is a minimal extension of $A$ if $A$ is a proper subalgebra of $B$ and has no proper strictly larger extension. If $B$ is a minimal extension of $A$, then for all $b \in B \setminus A$ and all $a \in A$, exactly one of $b \cap a$ and $b \setminus a$ is a member of $A$. To see this, suppose that $b \in B \setminus A$ and $a \in A$. If $b \setminus a \notin A$, then $B$ is equal to $\langle A, b \setminus a\rangle$. Thus, there is a $c \in A$ such that $b = (b \setminus a) \cup (c \cap a)$. This of course implies that $b \cap a = c \cap a \in A$. It follows then that there is a unique ultrafilter $u_A$ on $A$ which does not generate an ultrafilter on $B$, and that $a \in u_A$ if and only if for each $b \in B \setminus A$, $b \setminus a \in A$. The definition of minimal extension does not preclude the case when $B = \langle A, b\rangle$ and $b$ is an atom of $B$ (hence $b \cap a = 0$ for all $a \in A \setminus u_A$), but our constructions will avoid this case.

A minimally generated Boolean algebra will mean that we have an ordinal $\lambda$ and a Boolean algebra $B_\lambda$ with a sequence of generators $\{a_\alpha : \alpha \in \lambda\}$ such that for each $\alpha < \lambda$, $B_{\alpha+1} = \langle B_\alpha, a_\alpha\rangle$ is a minimal extension of $B_\alpha$, where $B_\alpha$ is the Boolean algebra generated by $\{a_\beta : \beta < \alpha\}$. The notation $u_\alpha$ will denote the filter on $B_\alpha$ witnessing that $B_{\alpha+1}$ is the minimal extension of $B_\alpha$ by $a_\alpha$. This can all be defined abstractly, but in this paper we will assume that we are simply dealing with subalgebras of $\mathcal{P}(\omega)$. We will henceforth assume without mention that for each $n \in \omega$, $a_n = \omega \setminus n$ and, therefore that $B_\omega$ is the Boolean subalgebra of finite and cofinite sets and $u_\omega$ is the cofinite filter.

We will say that a minimally generated Boolean algebra is *coherently* minimally generated by the sequence $\langle a_\alpha : \alpha \in \lambda\rangle$ if for all $\alpha$, $u_\alpha$ is the filter in $B_\alpha$ generated by the set $\{a_\beta : \beta < \alpha\}$. This is equivalent to the condition of having $u_\beta = u_\alpha \cap B_\beta$ for all $\beta < \alpha < \lambda$. In the case of a coherently minimally generated $B_\lambda$, it makes sense to talk about the ultrafilter $u_\lambda$ on $B_\lambda$.

**Definition 2.1.** *If $B_\lambda$ is coherently minimally generated by the sequence $\{a_\alpha : \alpha \in \lambda\}$, then, for $\beta < \alpha$, let $x_\beta$ denote the filter on $B_\alpha$ generated by $u_\beta \cup \{\omega \setminus a_\beta\}$.*

**Proposition 2.2.** *If $B_\lambda$ is coherently minimally generated by the sequence $\{a_\alpha : \alpha \in \lambda\}$, then for each $\beta < \alpha < \lambda$, $a_\alpha \setminus a_\beta$ is a member of the algebra $B_\beta$, and $x_\beta$ is an ultrafilter on $B_\lambda$.*

*Proof.* Fix $\beta < \lambda$ and prove by induction on $\alpha$ that $a_\alpha \setminus a_\beta$ is a member of $B_\beta$. Fix any $\alpha < \lambda$ and assume that $a_\gamma \setminus a_\beta \in B_\beta$ for all $\gamma < \alpha$. It follows that for each

$b \in B_\alpha$, $b \setminus a_\beta$ is a member of $B_\alpha$. Since $B_{\alpha+1}$ is minimal over $B_\alpha$ and $u_\alpha$ is the ultrafilter witnessing this, since $a_\beta \in u_\alpha$, we have that $b = a_\alpha \setminus a_\beta$ is a member of $B_\alpha$, and so $b \setminus a_\beta = b$ is also a member of $B_\beta$.

The fact that $x_\beta$ is an ultrafilter on $B_\lambda$ now follows directly. The fact that $B_{\beta+1}$ is minimal over $B_\beta$ ensures that $x_\beta$ is an ultrafilter on $B_{\beta+1}$. For each $\beta < \alpha$, $a_\alpha \setminus a_\beta = a_\alpha \cap (\omega \setminus a_\beta)$ is a member of $B_{\beta+1}$, and so, either contains, or is disjoint from, a member of $x_\beta$. □

**Proposition 2.3.** [5] *Every coherently minimally generated Boolean algebra is superatomic.*

*Proof.* Let $B$ be coherently minimally generated by the sequence $\{a_\alpha : \alpha < \lambda\}$. Suppose the $\varphi$ is a homomorphism from the Boolean algebra $B$ onto $A$. Choose $\alpha$ minimal so that $\varphi(a_\alpha)$ is not equal to 1. Then we check that $\varphi((\omega \setminus a_\alpha))$ is an atom. By the minimality of $\alpha$, we have that if $b \in B_\alpha \setminus u_\alpha$, then $\varphi(b) = 0$. Since $x_\alpha$ generates an ultrafilter on $B$, we then have that $\varphi(b \setminus a_\alpha) > 0$ if and only if $b \in x_\alpha$. It then also follows immediately that $\varphi(b \setminus a_\alpha) = \varphi(\omega \setminus a_\alpha)$ for all $b \in x_\alpha$. This shows that $\varphi(\omega \setminus a_\alpha)$ is an atom of $A$. □

It is also proven in [5] that every superatomic Boolean algebra can be coherently minimally generated but we shall not need that result.

**Definition 2.4.** *Let $B$ be a Boolean algebra and let $u$ be an ultrafilter on $B$. A sequence $\{b_n : n \in \omega\} \subset B$ is said to converge to $u$ if for all $a \in u$, the set $\{n : b_n < a\}$ is cofinite. Similarly, a sequence $\{x_n : n \in \omega\}$ of ultrafilters of $B$ is said to converge to $u$ if for all $a \in u$, the set $\{n : a \in x_n\}$ is cofinite.*

A topological space is *sequentially compact* if every infinite subset contains a converging sequence.

**Proposition 2.5.** *If $B$ is a coherently minimally generated, then $S(B)$ is sequentially compact.*

*Proof.* Let $\{\xi_n : n \in \omega\}$ be an infinite subset of $\lambda$, hence $\{x_{\xi_n} : n \in \omega\}$ is an infinite set of ultrafilters on $B_\lambda$. We must show there is an infinite $I \subset \omega$ and an ultrafilter $u$ on $B_\lambda$ such that $\{x_{\xi_n} : n \in I\}$ converges to $u$. We may assume that $\{x_{\xi_n} : n \in \omega\}$ does not converge to $u_\lambda$. Choose $\alpha < \lambda$ minimal such that $I = \{n \in \omega : a_\alpha \in x_{\xi_n}\}$ is infinite. Clearly, $\{n \in I : (\omega \setminus a_\alpha) \in x_{\xi_n}\}$ is a cofinite subset of $I$. Since $\alpha$ was minimal, we then have that for all $a \in x_\alpha$, $\{n \in I : a \in x_{\xi_n}\}$ is a cofinite subset of $I$. □

**Proposition 2.6.** *If $B_\lambda$ is a coherently minimally generated Boolean algebra, then $S(B_\lambda) \setminus \{u_\lambda\}$ is sequentially compact if and only if there is no infinite sequence from $S(B_\lambda)$ which converges to $u_\lambda$.*

## 3. The Scarborough-Stone game

We define a game, which we will call the Scarborough-Stone game, which calls for the construction of coherently minimally generated Boolean algebras. The main object to construct is a sequence $\langle a_\alpha : \alpha < \lambda \rangle$ ($\lambda \leq \mathfrak{c}$) of subsets of $\omega$. The sequence will be required to be the generators of a coherently minimally generated Boolean algebra. We reiterate our convention that $a_n = \omega \setminus n$ for $n \in \omega$. The *game* aspect will be that Player I will propose a set $b_\alpha$ and (for $\alpha \geq \omega$) Player II will choose to let $a_\alpha$ to be one of $\{b_\alpha, \omega \setminus b_\alpha\}$. Player I will win at any stage $\lambda$ if $S(B_\lambda) \setminus \{u_\lambda\}$ is sequentially compact. At stage $\alpha$, if Player I has not already won, then Player II will choose a sequence converging to $u_\alpha$, and Player II will win at stage $\alpha+1$ if that sequence still converges to $u_{\alpha+1}$. For simpler notation, Player II can simply choose any set $S \in [\alpha]^\omega$ such that $\{x_\xi : \xi \in S\}$ converges to $u_\alpha$.

Our interest is not really to explore this game, but rather to show that if $\mathfrak{b} = \mathfrak{c}$ then Player I has a winning strategy $\sigma$ which will win at some stage $\lambda \leq \mathfrak{c}$. It is not known if, in ZFC, Player I has a winning strategy for this game.

The reason we call this the Scarborough-Stone game is that if Player I has a winning strategy, then the Scarborough-Stone problem has an negative answer. The Scarborough-Stone question asks if the product of sequentially compact spaces is necessarily countably compact. It is known that the answer is negative if, for each free ultrafilter $u$ on $\omega$, there is a sequentially compact space, $X_u$, containing $\omega$ such that for each $x \in X_u$, $x$ has a neighborhood $W_x$ satisfying that $W_x \cap \omega \notin u$. If Player II always chooses $a_\alpha$ so that $a_\alpha \notin u$, then $X_u = S(B_\lambda) \setminus \{u_\lambda\}$ is such a space (when Player I has won at stage $\lambda$).

Let $\mathcal{S} = \{S_\xi : \xi \in \mathfrak{c}\}$ be a fixed enumeration of $[\mathfrak{c}]^\omega$, in effect a well-ordering of $[\mathfrak{c}]^\omega$. This sequence can be used to ensure that if the play of the game lasts until $\lambda = \mathfrak{c}$, then Player I has ensured that $S(B_\mathfrak{c}) \setminus \{u_\mathfrak{c}\}$ is sequentially compact. We will simplify the game (the $\mathcal{S}$-game) and assume that the sequence played by Player II at stage $\alpha$ will always be $S_{\xi_\alpha}$ where $\xi_\alpha$ is the minimal $\xi$ such that $S_\xi \subset \alpha$ and $\{x_\zeta : \zeta \in S_\xi\}$ converges to $u_\alpha$. Thus we can regard the elements of the tree $\mathcal{T} = ([\omega]^\omega)^{\leq \mathfrak{c}}$, (ordered by extension) as partial plays of the game. For each $t \in \mathcal{T}$, let $o(t)$ denote the domain of $t$ (which is an ordinal less than or equal to $\mathfrak{c}$). Also let $B_t$ denote the Boolean algebra generated by the elements of $t$ together with the cofinite subsets of $\omega$, and let $u_t$ denote the (possibly degenerate) filter extending the cofinite filter generated in $B_t$ by the elements of $t$. For each $\beta < o(t)$, let $x^t_\beta$ denote the (possibly degenerate) filter generated by $u_{t\restriction\beta} \cup \{\omega \setminus t(\beta)\}$.

**Definition 3.1.** *A strategy $\sigma$, for Player I, is a function from $\mathcal{T}$ into 2-element partitions of $\omega$. The subtree $T^\sigma \subset \mathcal{T}$ would be, recursively, the collection of $t = \langle a_\beta : \beta < \alpha\rangle \in T^\sigma$ which satisfy that, for all $n \in \omega$, $a_n = \omega \setminus n$, and for all $\omega \leq \gamma < \alpha$, $\sigma(t \restriction \gamma) = \{a_\gamma, \omega \setminus a_\gamma\}$ and $S(B_{t\restriction\gamma}) \setminus \{u_{t\restriction\gamma}\}$ is not sequentially compact. If $S(B_t) \setminus \{u_t\}$ is sequentially compact, then $t$ is a maximal node in $T^\sigma$.*

The following is a straightforward reformulation of what a winning strategy for Player I must satisfy if we are playing the $\mathcal{S}$-game. We omit the routine verification.

**Lemma 3.2.** *A strategy $\sigma$ for Player I is a winning $\mathcal{S}$-strategy if for each $t \in T^\sigma$ which is not maximal, each of the following holds for each $a \in \sigma(t)$:*

1. *$\langle B_t, a\rangle$ is a minimal extension of $B_t$,*
2. *$u_t \cup \{a\}$ generates a proper filter on $\omega$*
3. *the set of $\gamma \in S_{\xi_t}$, such that $a \in x^t_\gamma$ is infinite, where $\xi_t$ is minimal such that $\{x^t_\gamma : \gamma \in S_{\xi_t}\}$ converges to $u_t$.*

**Theorem 3.3.** *There is an Efimov space if Player I has a winning $\mathcal{S}$-strategy.*

*Proof.* Let $\sigma$ be a winning $\mathcal{S}$-strategy for Player I. Let $T^\sigma$ be defined as above, and let $B$ be the Boolean subalgebra of $\mathcal{P}(\omega)$ generated by the family $\{\sigma(t) : t \in T^\sigma$ is not maximal $\}$. We check that $S(B)$ is Efimov.

The main tool to prove this is a structure theorem on the ultrafilters developed by Koszmider. For each maximal $t \in T^\sigma$, the ultrafilter $u_t$ on the Boolean subalgebra $B_t$ generates an ultrafilter on $B$. To see this, assume that $s \in T^\sigma$ and that $\gamma < o(s)$. Thus, $a^s_\gamma$ can be regarded as an arbitrary member of the family that generates $B$. If $s \subset t$, then $a^s_\gamma \in B_t$ so there is nothing to check. Otherwise, choose $\beta$ minimal such that $s(\beta) \neq t(\beta)$. It follows that $a^s_\beta$ and $a^t_\beta$ are complementary sets. We want to show that $a^s_\gamma$ either contains, or is disjoint from, a member of $u_t$. If $\gamma < \beta$, then $a^s_\gamma \in u^t_\beta$, and so, is a member of $u_t$. If $\gamma \geq \beta$, then $a^s_\gamma \cap a^s_\beta$ is disjoint from $a^t_\beta$, which is a member of $u_t$. Thus we finish by proving that $a^s_\gamma \setminus a^s_\beta$ is either a member of $u_t$, or is disjoint from a member of $u_t$. But remember (Lemma 2.2) that $a^s_\gamma \setminus a^s_\beta$ is a member of $B_{s\restriction\beta}$, which is a subset of $B_t$.

Conversely, if $u$ is an ultrafilter on $S(B)$, we show there is a maximal $t \in T^\sigma$ such that $u$ is the ultrafilter generated by $u_t$. This is proven by a simple recursion by which $a^t_\gamma$ is the unique member of $\sigma(t \restriction \gamma)$ which is a member of $u$.

Another useful tool is to note that for each maximal element $t \in T^\sigma$, we have the canonical continuous map $\varphi_t$ from $S(B)$ onto $S(B_t)$. It is easily checked that this map is given by $\varphi(u_t) = u_t$ and, for $s \neq t$, $\varphi(u_s) = x^t_\gamma$ where $\gamma$ is minimal such that $s(\gamma) \neq t(\gamma)$.

Let $\{t_n : n \in \omega\}$ be a collection of pairwise distinct maximal members of $T^\sigma$, and let $U = \{u_{t_n} : n \in \omega\}$. Since $S(B)$ is compact, we may choose a maximal element $t \in T^\sigma$ so that $u_t$ is a limit of $U$, which means that, for all $\gamma < o(t)$, $\{n : a^t_\gamma \in u_{t_n}\}$ is infinite. We may assume $t \neq t_n$ for all $n$.

First we prove that $U$ is not a converging sequence in $S(B)$. Since $u_t$ is a limit of the sequence $\{u_{t_n} : n \in \omega\}$, it follows that it is a limit of the set $S = \{\varphi_t(u_{t_n}) : n \in \omega\} \subset S(B_t) \setminus \{u_t\}$. Since $S(B_t) \setminus \{u_t\}$ is sequentially compact, the set $S$ has a limit distinct from $u_t$. Therefore $\{u_{t_n} : n \in \omega\}$ has more than one limit point.

Now we prove that $u_t$ has a countable local $\pi$-base in the closure of $U$. A collection of non-empty open sets in a space $X$ is a local $\pi$-base at $x$ if every open set containing $x$ contains a set from the collection. Let $\{\gamma_k : k \in \omega\}$ be the set of ordinals $\gamma < o(t)$ satisfying that $x^t_\gamma$ is an isolated point of the subspace $\{\varphi_t(u_{t_n}) : n \in \omega\}$. By Lemma 1.2, $\{x^t_{\gamma_k} : k \in \omega\}$ is dense (and thus infinite) in $\varphi_t[U]$. For each $k$, $U_k = \overline{U} \cap \varphi_t^{-1}(\{x^t_{\gamma_k}\})$ is a relatively open subset of $\overline{U}$. Let $a \in u_t$ and choose $k$ so that $x^t_{\gamma_k} \in a^*$, i.e. $a \in x^t_{\gamma_k}$. Choose $b \in B_{t\restriction(\gamma_k+1)} \cap x^t_{\gamma_k}$ so that $b \subset a$. For $u \in U_k$, we have that $x^t_{\gamma_k} \cap B_{t\restriction(\gamma_k+1)}$ is contained in $u$, hence $a \supset b \in u$. We have shown that $a^*$ contains $U_k$, and thus, the family $\{U_k : k \in \omega\}$ is a local $\pi$-base at $u_t$ in $\overline{U}$.

Since $\beta\omega$ has separable subspaces in which no point has a countable local $\pi$-base, we have shown that $\beta\omega$ is not a subspace of $S(B)$. In fact, of course, we have shown that $S(B)$ does not map onto $2^{\omega_1}$ and so $B$ does not contain an uncountable independent family. □

Let us again remark that the algebra $B$ in the proof of Theorem 3.3 can be shown to be a T-algebra in the sense of Koszmider [7]. Koszmider showed that each T-algebra is minimally generated, and Koppelberg [5] established that no minimally generated algebra contains an uncountable independent family. However, it does seem easier to prove directly in Theorem 3.3 that the Stone space does not contain a copy of $\beta\omega$ rather than to review all the terminology of T-algebras. Nonetheless, we certainly acknowledge that our method was based from the outset on the fact that we were constructing a T-algebra.

## 4. Building a strategy from $\mathfrak{b} = \mathfrak{c}$

The construction of a strategy $\sigma$ will require that we maintain some auxillary objects and inductive assumptions. Specifically, we proceed by induction on $\alpha$ and define $\sigma$ for members of $t \in T^\sigma \cap [\mathfrak{c}]^{<\alpha}$. If $t \in [\mathfrak{c}]^{<\alpha} \setminus T^\sigma$, then $\sigma(t)$ can be set to be $\{\omega, \emptyset\}$. We suppress mention of the $t \in T$ and simply assume that we have followed the recursive definition of the strategy in defining $\{a_\beta : \beta < \alpha\}$. Recall that we have a fixed sequence $\mathcal{S} = \{S_\xi : \xi \in \mathfrak{c}\}$ which enumerates $[\mathfrak{c}]^\omega$. Player I's choice will always consist of a partition of the form $\{b_0^\alpha, \bigcup_{\ell>0} b_\ell^\alpha\}$. For $\ell > 0$, $b_\ell^\alpha$ will be a member of $B_\alpha \setminus u_\alpha$ and the sequence will converge to $u_\alpha$. So long as we have ensured that, in addition, $b_0^\alpha$ is not a member of $B_\alpha$, then items (1) and (2) of Lemma 3.2 will be fulfilled. It will be convenient to use the notation from Stone duality theory. For $b \in B_\alpha$, the standard notation $b^*$ denotes the clopen set of ultrafilters on $B_\alpha$ which have $b$ as a member. If $\beta \le \alpha$ and $b \in B_\beta \setminus u_\beta$, then we may interpret $b^*$ as the set of $\xi < \beta$ such that $b \in x_\xi$. It is important to observe that this is unambiguous, since if $\beta \le \xi$, then $u_\beta \subset x_\xi$ and so $b \notin x_\xi$ for all $b \in B_\beta \setminus u_\beta$. For each $\alpha$, let $X_\alpha$ denote the space we obtain on the set $\alpha$ by using the family $\{b^* : b \in B_\alpha \setminus u_\alpha\}$ as a base for the open sets.

**Definition 4.1.** *Let $\Gamma_\alpha$ be the set of $\xi \in \alpha$ that $x_\xi$ does not have a countable filter base. Set $\Lambda_\alpha$ to be the family of limit ordinals $\lambda$ of cofinality $\omega$ which are less than $\alpha + 1$ satisfying that for each $a \notin u_\alpha$, $\Gamma_\alpha \setminus a^*$ is cofinal in $\lambda$.*

We observe that for $\beta < \alpha$, $\Gamma_\beta = \Gamma_\alpha \cap \beta$ and $\Lambda_\beta \supset \Lambda_\alpha \cap \beta$. If $a_\alpha$ is set to be $b_0^\alpha$, then refer to this as that $\alpha$ is Case 0. Otherwise, $a_\alpha = \bigcup_{\ell>0} b_\ell^\alpha$, and we are in Case 1. In each case we are assuming that the condition (1) and (2) of Lemma 3.2 are fulfilled.

**Claim 1.** If $\alpha$ is Case 0, then $x_\alpha$ has a countable base given by $\{\bigcup_{\ell>n} b_\ell^\alpha : n \in \omega\}$.

**Claim 2.** If $\alpha$ is Case 1, in particular if $\alpha \in \Gamma_{\alpha+1}$, then $X_{\alpha+1}$ is $\sigma$-compact.

*Proof.* We have that $X_{\alpha+1}$ is covered by the collection $\{(b_\ell^\alpha)^* : \ell \in \omega\}$. □

Next we have the very useful observation.

**Lemma 4.2.** *For each $\gamma \in \Gamma_\alpha$, $\Lambda_\alpha \cap \gamma$ is countable.*

*Proof.* We prove this by induction on $\gamma \in \Gamma_\alpha$. So assume that $\Lambda_\alpha \cap \delta$ is countable for all $\delta \in \Gamma_\alpha \cap \gamma$ and that $\Lambda_\alpha \cap \gamma$ is uncountable. Let $\zeta \le \gamma$ be minimal such that $\Lambda_\alpha \cap \zeta$ is uncountable. Since $\Gamma_\alpha$ is cofinal in each $\lambda \in \Lambda_\alpha$, we also have that $\Gamma_\alpha \cap \zeta$ cofinal in $\zeta$. Let $\gamma'$ be the minimal element of $\Gamma_\alpha \setminus \zeta$, hence $\zeta \le \gamma' \le \gamma$. The key to the proof is that with this minimal $\gamma'$ being in $\Gamma_{\gamma'+1} \subset \Gamma_\alpha$ results in a drastic change in the membership of $\Lambda_{\gamma'} \supset \Lambda_\alpha \cap \zeta$ from that of $\Lambda_\delta$ for $\delta < \gamma'$ as we now discuss.

We claim that $b^* \cap \zeta$ is bounded in $\zeta$ for all $b \in B_{\gamma'} \setminus u_{\gamma'}$. This is immediate for all $b \in B_\zeta \setminus u_\zeta$ since $B_\zeta$ is coherently minimal. If $\zeta < \gamma'$, then we prove this for $b \in B_\beta \setminus u_\beta$ by induction on $\beta \in [\zeta, \gamma']$. For limit $\beta$ there is nothing to check, so assume it holds for some $\beta < \gamma'$ and we check $\beta + 1$. By our assumptions on the game strategy, we will have that $(b_\ell^\beta)^* \cap \zeta$ is bounded in $\zeta$ for each $\ell > 0$. It should be clear that if $\beta \notin \Gamma_\alpha$, then it must be the case that $a_\beta = b_0^\beta$, and that $a_\beta^*$ contains a final segment of $\zeta$. For each $b \in B_{\beta+1} \setminus u_{\beta+1}$ we will have that $b \cap a_\beta$ will be a member of $B_\beta \setminus u_\beta$ and so the induction hypothesis still holds.

Now since we have that $\gamma' \in \Gamma_\alpha$, this time it follows that $a_{\gamma'}$ is chosen to be $\bigcup_{\ell>0} b_\ell^{\gamma'}$. In addition, we have that $a = (\omega \setminus a_{\gamma'})$ is not in $u_{\gamma'+1}$ (note also that $a \notin u_\alpha$), and $a^*$ contains a final segment of $\zeta$. Fix any $\delta < \zeta$ so that $\zeta \setminus a^*$ is contained in $\delta$. Of course this implies that $(\Gamma_\alpha \setminus a^*) \cap \zeta$, and thus $\Lambda_\alpha \cap \zeta$, is also contained in $\delta + 1$. This contradicts the original assumption that $\Lambda_\alpha \cap \zeta$ is cofinal in $\zeta$. □

Now we define the extra induction hypotheses we impose when defining our sequence. For each $\lambda$ such that $\lambda \in \Lambda_\lambda$, we select a countable family, $\mathcal{Z}_\lambda$, of pairwise disjoint compact $G_\delta$'s of $X_\lambda$; and we will impose inductive assumptions on this family that must be maintained. Of course, in the case that $S(B_\alpha) \setminus \{u_\alpha\}$ is sequentially compact, there is no need to proceed.

**Definition 4.3.** *$b \in B_\alpha \setminus u_\alpha$ is $\mathcal{Z}_\lambda$-saturated (for $\lambda \leq \alpha$) if $\mathcal{Z}_\lambda$ refines the cover $\{b^*, X_\alpha \setminus b^*\}$. A $\mathcal{Z}_\lambda$-saturated open set from $B_\alpha$ will always mean a set of the form $b^*$ for some $b \in B_\alpha \setminus u_\alpha$ which is $\mathcal{Z}_\lambda$-saturated.*

**Definition 4.4.** *Inductive Hypotheses (IH$_\alpha$) on the family of $\{\mathcal{Z}_\lambda : \lambda < \alpha\}$ at stage $\alpha$ for all $\beta < \alpha$ and $\lambda < \alpha$:*

1. *$\mathcal{Z}_\lambda$ is a countable family of pairwise disjoint compact $G_\delta$'s of $X_\lambda$ whose union covers $\Gamma_\lambda$,*
2. *$\lambda \in \Lambda_\lambda$ implies each $Z \in \mathcal{Z}_\lambda$ has a base of $\mathcal{Z}_\lambda$-saturated open subsets from $B_\lambda$,*
3. *for each $\ell > 0$, $b_\ell^\beta$ must be chosen to be $\mathcal{Z}_\lambda$-saturated if $\lambda \in \Lambda_\beta$,*
4. *if $\lambda \in \Lambda_\alpha$, and $\beta \notin \bigcup \mathcal{Z}_\lambda$, then $x_\beta$ has a base of $\mathcal{Z}_\lambda$-saturated elements from $B_{\beta+1}$,*
5. *if $\lambda' < \lambda$ are both in $\Lambda_\lambda$ then $\mathcal{Z}_{\lambda'} \subset \mathcal{Z}_\lambda$, and for $Z \in \mathcal{Z}_\lambda \setminus \mathcal{Z}_{\lambda'}$, $Z \cap \lambda'$ is empty.*
6. *if $\Lambda_\alpha$ is uncountable, then there is $\bar{\lambda}_\beta \in \Lambda_\alpha$ such that $(b_\ell^\beta)^* \cap (\bigcup \Lambda_\alpha) \subset \bar{\lambda}_\beta$.*

Now we consider "stage $\alpha$". We first define $\mathcal{Z}_\alpha$ and then give the construction of the sequence $\{b_\ell^\alpha : \ell \in \omega\}$ along with the selection of $\bar{\lambda}_\alpha$. We then have to check that regardless of the choice of $a_\alpha$ from $\{b_0^\alpha, \bigcup\{b_\ell^\alpha : \ell > 0\}\}$ that IH$_{\alpha+1}$ and the conditions in Lemma 3.2 will hold. If $\alpha \notin \Lambda_\alpha$, for example if $\alpha$ is not a limit with cofinality $\omega$, then $\mathcal{Z}_\alpha$ is empty. If $\alpha$ has countable cofinality and $\alpha \in \Lambda_\alpha$, then we have two cases when making the choice of $\mathcal{Z}_\alpha$. Let $\mu_\alpha$ be the supremum of $\Lambda_\alpha \setminus \{\alpha\}$.

**Lemma 4.5.** *If $\Lambda_\alpha$ is countable and cofinal in $\alpha$, then we set $\mathcal{Z}_\alpha$ to be $\bigcup\{\mathcal{Z}_{\lambda'} : \lambda' \in \Lambda_\alpha\}$, and the hypotheses of Definition 4.4 will hold.*

*Proof.* It bears checking that the family is pairwise disjoint. Suppose $\lambda_1 < \lambda_2$ are each in $\Lambda_\alpha$ and $Z_1 \in \mathcal{Z}_{\lambda_1}$ and $Z_2 \in \mathcal{Z}_{\lambda_2} \setminus \{Z_1\}$. By definition, $\lambda_1 \in \Lambda_\alpha$, which as mentioned above, means that $\lambda_1 \in \Lambda_{\lambda_2}$. Therefore, by induction hypothesis (5), $\mathcal{Z}_{\lambda_1} \subset \mathcal{Z}_{\lambda_2}$ and we have by (1) that $Z_1 \cap Z_2$ is empty. Properties (1)–(5) are now immediate by the inductive assumptions. Of course (6) is vacuous. □

Now for the construction of the family $\mathcal{Z}_\alpha$ in the case that $\alpha \in \Lambda_\alpha$ and $\mu_\alpha < \alpha$. Recall that $X_{\gamma+1}$ is $\sigma$-compact and open in $X_\alpha$ for each $\gamma \in \Gamma_\alpha$. In fact, as stated in Claim 2 above, $\{(b_\ell^\gamma)^* : \ell \in \omega\}$ is a cover of $X_{\gamma+1}$ by compact open sets.

Furthermore, if $\gamma > \mu_\alpha$, this collection is $\mathcal{Z}_{\mu_\alpha}$-saturated. Furthermore, since $\Gamma_\alpha$ has a countable subset cofinal in $\mu_\alpha$, $X_{\mu_\alpha}$ is also $\sigma$-compact and open in $X_\alpha$.

Let $\{\gamma_n : n \in \omega\}$ be any cofinal subsequence of $\Gamma_\alpha \setminus \mu_\alpha$. Now we have that $\{(b_\ell^{\gamma_n})^* : n < \omega, \ell < \omega\}$ is a cover of $X_\alpha \setminus X_{\mu_\alpha}$ by compact $\mathcal{Z}_{\mu_\alpha}$-saturated open sets. Fix an enumeration, $\{c_m : m \in \omega\}$, of this collection. Set $d_0 = c_0$ and, for each $m > 0$, let $d_m = c_m \setminus (c_0 \cup \cdots \cup c_{m-1})$. Finally, for each $m$, let $Z_m = d_m \setminus X_{\mu_\alpha}$ – a $G_\delta$. Evidently each $Z_m$ has a countable base of $\mathcal{Z}_{\mu_\alpha}$-saturated open sets.

Therefore $\{Z_m : m \in \omega\}$ is a pairwise disjoint sequence of compact $G_\delta$ subsets of $X_\alpha$. In addition $Z_n \cap X_{\mu_\alpha}$ is empty for all $n$. Therefore, $\mathcal{Z}_\alpha = \mathcal{Z}_{\mu_\alpha} \cup \{Z_m : m \in \omega\}$ is a sequence that fulfills item (1) of $\mathrm{IH}_{\alpha+1}$ for $\lambda = \alpha$. Items (2) and (5) of $\mathrm{IH}_{\alpha+1}$ are also immediate. Similarly there are no new instances of items (3) or (6) to check. We must verify that (4) of $\mathrm{IH}_{\alpha+1}$ is satisfied for $\lambda = \alpha$. Since $\mu_\alpha \in \Lambda_\alpha$, conditions (2) and (4) applied to $\lambda = \mu_\alpha$ ensures that $X_{\mu_\alpha}$ can be covered by $\mathcal{Z}_{\mu_\alpha}$-saturated sets. Since $X_\alpha \setminus X_{\mu_\alpha}$ is contained in $\bigcup_m Z_m$, we have that item (4) holds.

This completes the construction of $\mathcal{Z}_\alpha$. The next step is to examine the choice of $\{b_\ell^\alpha : \ell \in \omega\}$. Recall that $S(B_\alpha) \setminus \{u_\alpha\}$ is not sequentially compact, and we are handed the sequence $\{\xi_n^\alpha : n \in \omega\} \subset \alpha$ from the listing $\mathcal{S}$ which (equivalent to converging to $u_\alpha$) has no limit points in $X_\alpha$. If $\Lambda_\alpha$ is uncountable, then, by Lemma 4.2, it has order type $\omega_1$ and $\mu_\alpha$ is the supremum. If $\Lambda_\alpha$ is countable, we let $\bar\lambda_\alpha$ denote its maximum element. If $\Lambda_\alpha$ is uncountable we will define some $\bar\lambda_\alpha \in \Lambda_\alpha$ to fulfill the condition (6) of $\mathrm{IH}_{\alpha+1}$.

For each $n$, if there is any $\lambda_n \in \Lambda_\alpha$ such that $x_{\xi_n^\alpha} \in Z_n$ for some $Z_n \in Z_{\lambda_n}$, then, by Induction Hypothesis (5) and (1), there is a unique such $Z_n$. If there are infinitely many $n$ such that such a $Z_n$ and $\lambda_n$ exists, then we may pass to a subsequence and assume $Z_n$ exists for each $n$. Furthermore, since each $Z_n$ is compact and the sequence $\{\xi_n^\alpha : n \in \omega\}$ has no limit in $X_\alpha$, we may assume that the $Z_n$'s are pairwise distinct. If $\Lambda_\alpha$ is countable then we have, by induction hypothesis (5), that $\{Z_n : n \in \omega\} \subset \mathcal{Z}_{\bar\lambda_\alpha}$. Otherwise, again by passing to a subsequence, if we let $\bar\lambda_\alpha \in \Lambda_\alpha$ be minimal such that $\lambda_n \leq \bar\lambda_\alpha$ for infinitely many $n$, then we may assume that $\{Z_n : n \in \omega\} \subset \mathcal{Z}_{\bar\lambda_\alpha}$. In either case, by inductive hypothesis (2), we have that for each $n$, $Z_n$ has a base of $\mathcal{Z}_{\bar\lambda_\alpha}$-saturated sets from $B_{\bar\lambda_\alpha}$.

Next we assume, by passing to a subsequence, that no $\xi_n^\alpha$ is a member of any element of $\bigcup\{\mathcal{Z}_\lambda : \lambda \in \Lambda_\alpha\}$. We can adopt a uniform notation by setting $Z_n = \{\xi_n^\alpha\}$. If there are infinitely many $n$ for which $\xi_n^\alpha < \lambda_n$ for some $\lambda_n \in \Lambda_\alpha$, then we may suppose there is a value $\bar\lambda_\alpha \in \Lambda_\alpha$ such that $\xi_n^\alpha < \bar\lambda_\alpha$ for all $n$. Finally, we are faced with the case that $\Lambda_\alpha$ is uncountable and $\mu_\alpha < \xi_n^\alpha$ for all $n$. We then apply induction hypothesis (6) in turn for $\beta = \xi_n^\alpha$ for each $n$, and are able to choose $\bar\lambda_\alpha \in \Lambda_\alpha$ such that for all $\ell > 0$ and all $\beta \in \{\xi_n^\alpha : n \in \omega\}$, $(b_\ell^\beta)^* \cap \mu_\alpha \subset \bar\lambda_\alpha$.

In each of the above cases, we have selected a value $\bar\lambda_\alpha \in \Lambda_\alpha$ satisfying that each $Z_n$ has a descending base, generated by $\{c_{n,k}^\alpha : k \in \omega\} \subset B_\alpha \setminus u_\alpha$, of $\mathcal{Z}_{\bar\lambda_\alpha}$-saturated open sets from $B_\alpha$, each satisfying that $(c_{n,k}^\alpha)^* \cap \lambda$ is contained in $\bar\lambda_\alpha$ for all $\lambda \in \Lambda_\alpha$. Since the original sequence $\{\xi_n^\alpha : n \in \omega\}$ has no limit points, we may assume that, for each $n$, $(c_{n,0}^\alpha)^* \cap \{\xi_m^\alpha : m \in \omega\} = \{\xi_n^\alpha\}$. Furthermore, since these bases are all $\mathcal{Z}_{\bar\lambda_\alpha}$-saturated, it follows that $(c_{n,0}^\alpha)^* \cap Z_m = \emptyset$ for all $m \neq n$. Therefore, we can also assume that $\{c_{n,0}^\alpha : n \in \omega\}$ is a pairwise disjoint family. For each $\gamma < \alpha$, there is a $b_\gamma \in x_\gamma \cap B_\alpha \setminus u_\alpha$ and a function $f_\gamma \in \omega^\omega$ such that $b_\gamma \cap c_{n,f_\gamma(n)}^\alpha = \emptyset$ for all but at most one $n$. Using that $\alpha < \mathfrak{b}$, we may choose $f \in \omega^\omega$ so that $f_\gamma$ is mod finite below $f$ for all $\gamma \in \alpha$.

Finally we define our sequence $\{b^\alpha_\ell : \ell > 0\}$ by setting $b^\alpha_\ell = c^\alpha_{2\ell, f(2\ell)}$; and we let $b^\alpha_0$ be the complement of $\bigcup\{b^\alpha_\ell : \ell > 0\}$. Naturally for each $n \in \omega$ (after all the reductions), $\bigcup\{(b^\alpha_\ell)^* : \ell > 0\} \cap \{\xi^\alpha_{2n}, \xi^\alpha_{2n+1}\} = \{\xi^\alpha_{2n}\}$. This ensures that item (3) of Lemma 3.2 will be fulfilled in this strategy. We check that $\{b^\alpha_\ell : \ell > 0\}$ converges to $u_\alpha$. Let $a \in u_\alpha$ and consider the compact set $K = (\omega \setminus a)^* \subset X_\alpha$. For each $\gamma \in K$, since $f_\gamma <^* f$, we have chosen the element $b_\gamma \in x_\gamma \cap B_\alpha \setminus u_\alpha$ such that $b_\gamma \cap b^\alpha_\ell = \emptyset$ for all but finitely many $\ell$. Since $K$ is compact, there is a finite cover of $K$ by elements of $\{(b_\gamma)^* : \gamma \in K\}$. Thus there is an $n$ such that $b^\alpha_\ell \subset a$ for all $\ell > n$.

By the construction, each member of the sequence $\{b^\alpha_\ell : 0 < \ell \in \omega\}$ is $\mathcal{Z}_{\bar\lambda_\alpha}$-saturated. In case that $\Lambda_\alpha$ is countable, it is immediate that item (3) of the Induction Hypotheses $\mathrm{IH}_{\alpha+1}$ is satisfied. If $\bar\lambda_\alpha < \mu_\alpha$, then item (5) of the Induction Hypotheses combined with the fact that $(b^\alpha_\ell)^* \cap \mu_\alpha \subset \bar\lambda_\alpha$ ensures that item (3) holds in this case as well. Of course the choice of $\bar\lambda_\alpha$ also ensures that item (6) of the Induction Hypotheses also holds for $\beta = \alpha$.

The only thing remaining is to prove that induction hypothesis (4) will hold for $\mathrm{IH}_{\alpha+1}$. Also this only needs to be checked for $\beta = \alpha$.

**Lemma 4.6.** *If $a_\alpha$ is chosen to be $b^\alpha_0$, then the induction hypotheses (4) of $IH_{\alpha+1}$ holds.*

*Proof.* In case 0, the filter base for $x_\alpha$ is $\{\bigcup_{\ell>n} b^\alpha_\ell : n \in \omega\}$ and $\Lambda_{\alpha+1} = \Lambda_\alpha$. This collection is $\mathcal{Z}_{\bar\lambda_\alpha}$-saturated. Furthermore, for $\lambda \in \Lambda_\alpha$ and $\ell > 0$, $(b^\alpha_\ell)^* \cap \lambda \subset \bar\lambda_\alpha$, hence, by inductive hypothesis (3), this collection is $\mathcal{Z}_\lambda$-saturated for all $\lambda \in \Lambda_\alpha$. □

**Lemma 4.7.** *If $a_\alpha$ is chosen to be $\bigcup\{b^\alpha_\ell : \ell > 0\}$, then the induction hypotheses (4) of $IH_{\alpha+1}$ holds.*

*Proof.* Let $\mathfrak{B}_\alpha$ denote the set of elements of $B_\alpha \setminus u_\alpha$ which are $\mathcal{Z}_{\bar\lambda_\alpha}$-saturated. The filter $x_\alpha$ has $b^\alpha_0$ as an element and $(b^\alpha_0)^*$ is a compact open subset of $X_{\alpha+1}$. By construction $b^\alpha_0$ is $\mathcal{Z}_\lambda$-saturated for all $\lambda \in \Lambda_\alpha$. By induction hypotheses (2) and (4), $X_{\alpha+1} \setminus \{x_\alpha\}$ can be expressed as a union of $\mathcal{Z}_{\bar\lambda_\alpha}$-saturated sets. Thus the family $\{(b^\alpha_0 \setminus b) : b \in \mathfrak{B}_\alpha\}$ is a base for $x_\alpha$ which is $\mathcal{Z}_{\bar\lambda_\alpha}$-saturated. We must show that it is $\mathcal{Z}_\lambda$-saturated for all $\lambda \in \Lambda_{\alpha+1}$. If $\Lambda_\alpha$ is countable, then $\bar\lambda_\alpha$ is the maximum element of $\Lambda_\alpha$, and we are done by induction hypothesis (5). Otherwise, in this so-called Case 1, we have a potentially big change from $\Lambda_\alpha$ to $\Lambda_{\alpha+1}$. By construction, the interval $(\bar\lambda_\alpha, \mu_\alpha)$ is contained in $(b^\alpha_0)^*$. In particular, $(\Gamma_{\alpha+1} \setminus (b^\alpha_0)^*) \setminus \bar\lambda_\alpha$ is finite. This implies that $\Lambda_{\alpha+1}$ is contained in $\Lambda_\alpha \cap \bar\lambda_\alpha{+}1$; which completes the verification of induction hypothesis (4). □

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