ASYMMETRIC TIE-POINTS AND ALMOST CLOPEN SUBSETS OF $\mathbb{N}^*$

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Abstract. A tie-point of a compact space is analogous to a cut-point: the complement of the point falls apart into two relatively clopen non-compact subsets. Set-theoretically, a tie-point of $\mathbb{N}^*$ is an ultrafilter whose dual maximal ideal can be generated by the union of two non-principal mod finite orthogonal ideals. We review some of the many consistency results that have depended on the construction of tie-points of $\mathbb{N}^*$. One especially important application, due to Velickovic, was to the existence of non-trivial involutions on $\mathbb{N}^*$. A tie-point of $\mathbb{N}^*$ has been called symmetric if it is the unique fixed point of an involution. We define the notion of an almost clopen set to be the closure of one of the proper relatively clopen subsets of the complement of a tie-point. We explore asymmetries of almost clopen subsets of $\mathbb{N}^*$ in the sense of how may an almost clopen set differ from its natural complementary almost clopen set.

1. Introduction

In this introductory section we review some background to motivate our interest in further study of tie-points and almost clopen sets. The Stone-Čech compactification of the integers $\mathbb{N}$, is denoted as $\beta\mathbb{N}$ and, as a set, is equal to $\mathbb{N}$ together with all the free ultrafilters on $\mathbb{N}$. The remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ can be topologized as a subspace of the Stone space of the power set of $\mathbb{N}$ as a Boolean algebra and, in particular, for a subset a of $\mathbb{N}$, the set a^* of all free ultrafilters with a as an element, is a basic clopen subset of $\mathbb{N}^*$. Set-theoretically it is sometimes more convenient to work with the set of ordinals ω in place of the natural numbers $\mathbb{N}$, and the definitions of ω^* and a^* for $a \subset \omega$ are analogous.

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A point x of a space X is a butterfly point (or b-point [23]) if there are sets $D, E \subset X \setminus \{x\}$ such that $\{x\} = \overline{D} \cap \overline{E}$. In [5], the authors introduced the tie-point terminology.

Definition 1.1. *A point x is a tie-point of a space X if there are closed sets A, B of X such that $X = A \cup B$, $\{x\} = A \cap B$ and x is a limit point of each of A and B. We picture (and denote) this as $X = A \underset{x}{\bowtie} B$ where A, B are the closed sets which have a unique common accumulation point x and say that x is a tie-point as witnessed by A, B.*

In this note the focus is on the local properties of x with respect to each of the closed sets A and B such that $A \underset{x}{\bowtie} B$ in the case when A, B witness that x is a tie-point. For this reason we introduce the notion of an almost clopen subset of $\mathbb{N}^*$.

Definition 1.2. *A set $A \subset \mathbb{N}^*$ is almost clopen if A is the closure of an open subset of $\mathbb{N}^*$ and has a unique boundary point, which we denote x_A.*

Proposition 1.3. *If A is an almost clopen subset of $\mathbb{N}^*$, then $B = \{x_A\} \cup (\mathbb{N}^* \setminus A)$ is almost clopen and $x_B = x_A$. In addition x_A is a tie-point as witnessed by A, B.*

Definition 1.4. [5] *A tie-point x is a symmetric tie-point of $\mathbb{N}^*$ if there is a pair A, B witnessing that x is a tie-point and if there is a homeomorphism $h : A \to B$ satisfying that $h(x) = x$.*

If A is almost clopen, then we refer to $B = \{x_A\} \cup (\mathbb{N}^* \setminus A)$ as the almost clopen complement of A. A more set-theoretically inclined reader would surely prefer a staightforward translation of almost clopen to properties of ideals of subsets of $\mathbb{N}$ and the usual mod finite ordering $\subset^*$.

Definition 1.5. *If A is any subset of $\mathbb{N}^*$, then $\mathcal{I}_A$ is defined as the set $\{a \subset \mathbb{N} : a^* \subset A\}$.*

For any family $\mathcal{A}$ of subsets of $\mathbb{N}$ (or ω), we define $\mathcal{A}^\perp$ to be the orthogonal ideal $\{b \subset \mathbb{N} : (\forall a \in \mathcal{A})\ b \cap a =^* \emptyset\}$. Let us note that if $\mathcal{I}$ is an ideal that has no $\subset^*$-maximal element, then the ideal generated by $\mathcal{I} \cup \mathcal{I}^\perp$ is a proper ideal.

Lemma 1.6. *If A is an almost clopen subset of $\mathbb{N}^*$ with almost clopen complement B, then $\mathcal{I}_A \cap \mathcal{I}_B$ is the Frechet ideal $[\mathbb{N}]^{<\aleph_0}$, $\mathcal{I}_B = \mathcal{I}_A^\perp$, and x_A is the unique ultrafilter that is disjoint from $\mathcal{I}_A \cup \mathcal{I}_B$.*

Almost clopen sets (and tie-points) first arose implicitly in the work of Fine and Gillman [11] in the investigation of extending continuous functions on dense subsets of $\mathbb{N}^*$. A subset Y of a space X is C^*-embedded if every bounded continuous real-valued function on Y can be continuously extended to all of X. The character of a point $x \in \mathbb{N}^*$ is the minimal cardinality of a filter base for x as an ultrafilter on $\mathbb{N}$.

Proposition 1.7. *([11]) If x is a tie-point of $\mathbb{N}^*$, then $\mathbb{N}^* \setminus \{x\}$ is not C^*-embedded in $\mathbb{N}^*$. Every point of character $\aleph_1$ is a tie-point of $\mathbb{N}^*$.*

It was shown [4] to be consistent with ZFC that $\mathbb{N}^* \setminus \{x\}$ is C^*-embedded for all $x \in \mathbb{N}^*$. It was also shown by Baumgartner [1] that their result holds in models of the Proper Forcing Axiom (PFA).

Proposition 1.8. *([1, 4]) The proper forcing axiom implies $\mathbb{N}^* \setminus \{x\}$ is C^*-embedded in $\mathbb{N}^*$ for all $x \in \mathbb{N}^*$*

Corollary 1.9. *PFA implies that there are no almost clopen sets and no tie-points in $\mathbb{N}^*$.*

Almost clopen sets arise in the study of minimal extensions of Boolean algebras ([16]) and in the application of this method of construction for building a variety of counterexamples (e.g. [7, 13, 17, 22]). The next application of almost clopen subsets of $\mathbb{N}^*$ were to the study of non-trivial automorphisms of $\mathcal{P}(\mathbb{N})/fin$, or non-trivial autohomeomorphisms of $\mathbb{N}^*$. Katětov [15] proved that the set of fixed points of an autohomeomorphism of $\beta\mathbb{N}$ will be a clopen set. It is immediate from Fine and Gillman's work in [11] that every P-point of character of $\aleph_1$ is a fixed point of a non-trivial autohomeomorphism of $\mathbb{N}^*$.

Definition 1.10. *A point x of $\mathbb{N}^*$ is a P-point if the ultrafilter x is countably complete mod finite. For a cardinal κ, an ultrafilter x on $\mathbb{N}$ is a simple P_κ-point if x has a base well-ordered by mod finite inclusion of order type κ.*

Proposition 1.11. [11] *If A is an almost clopen subset of $\mathbb{N}^*$ and x_A is a simple $P_{\aleph_1}$-point of $\mathbb{N}^*$, then*

(1) A is homeomorphic to $\mathbb{N}^*$,
(2) x_A is a symmetric tie-point,
(3) there is an autohomeomorphism f on N^* such that $\{x\}$ is the only fixed point of f.

As we have seen above, PFA implies that there are no almost clopen subsets of $\mathbb{N}^*$, and of course, PFA also implies that all autohomeomorphisms of $\mathbb{N}^*$ are trivial [24]. However Velickovic utilized the simple P-point trick (motivating our definition of symmetric tie-point) in order to prove that this is not a consequence of Martin's Axiom (MA).

Proposition 1.12. [27] *It is consistent with MA and $\mathfrak{c} = \aleph_2$ that there is an almost clopen set A of $\mathbb{N}^*$ such that x_A is a simple $P_{\aleph_2}$-point and,*

(1) x_A is a symmetric tie-point,
(2) there is an autohomeomorphism f on N^* such that $\{x\}$ is the only fixed point of f.

Velickovic's result and approach was further generalized in [25, 26]. It is very interesting to know if an almost clopen subset of $\mathbb{N}^*$ is itself homeomorphic to $\mathbb{N}^*$ ([9, 14]). This question also arose in the authors' work on two-to-one images of $\mathbb{N}^*$ [6]. Velickovic's method was slightly modified in [6] to produce a complementary pair of almost clopen sets so that neither is homeomorphic to $\mathbb{N}^*$, but it is not known if there is a symmetric tie-point $A \underset{x}{\bowtie} B$ where A is not a copy of $\mathbb{N}^*$.

Our final mention of recent interest in almost clopen subsets of $\mathbb{N}^*$ is in connection to the question [8, 19] of whether the Banach space ℓ_∞/c_0 is necessarily primary. It was noted by Koszmider [20, p577] that a special almost clopen subset of $\mathbb{N}^*$ could possibly resolve the problem. For a compact space K, we let $C(K)$ denote the Banach space of continuous real-valued functions on K with the supremum norm. It is well-known that $C(\mathbb{N}^*)$ is isomorphic (as a Banach space) to ℓ_∞/c_0. Naturally if a space A is homeomorphic to $\mathbb{N}^*$, then $C(A)$ is isomorphic to $C(\mathbb{N}^*)$.

Proposition 1.13. [20, p577] *Suppose that A is an almost clopen subset of $\mathbb{N}^*$ and that B is its almost clopen complement. If $C(\mathbb{N}^*)$ is not homeomorphic to either of $C(A)$ or $C(B)$, then ℓ_∞/c_0 is not primary.*

2. Asymmetric tie-points

In many of the applications mentioned in the introductory section, the tie-points utilized were symmetric tie-points. In other applications, for example the primariness of ℓ_∞/c_0, it may be useful to find examples where the witnessing sets A, B for a tie-point are quite different. There are any number of local topological properties that x_A may enjoy as a point in A that it may not share as a point in B. We make the following definition in connection with simple P_κ-points.

Definition 2.1. *Let κ be a regular cardinal. An almost clopen set A is simple of type κ if $\mathcal{I}_A$ has a $\subset^*$-cofinal $\subset^*$-increasing chain $\{a_\alpha : \alpha \in \kappa\}$ of type κ.*

If $\{a_\alpha : \alpha \in \kappa\}$ is strictly $\subset^*$-increasing and $\subset^*$-cofinal in $\mathcal{I}_A$ for an almost clopen set A, then the family $\{a_{\alpha+1} \setminus a_\alpha : \alpha \in \kappa\}$ can not be *reaped*. A family $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$ is reaped by a set $c \subset \mathbb{N}$ if $|a \setminus c| = |a \cap c|$ for all $a \in \mathcal{A}$. The reaping number $\mathfrak{r}$ is the minimum cardinal of a family that can not be reaped [12]. For any infinite set $a \subset \mathbb{N}$, let $\mathrm{next}(a, \cdot)$ be the function in $\mathbb{N}^{\mathbb{N}}$ defined by $\mathrm{next}(a, k) = \min(a \setminus \{1, \ldots, k\})$. As usual, for $f, g \in \mathbb{N}^{\mathbb{N}}$, we say that $f <^* g$ if $\{k : g(k) \leq f(k)\}$ is finite.

Proposition 2.2. [12] *If $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$ and if there is some $g \in \mathbb{N}^{\mathbb{N}}$ such that $\mathrm{next}(a, \cdot) <^* g$ for all $a \in \mathcal{A}$, then $\mathcal{A}$ can be reaped. In particular, $\mathfrak{b} \leq \mathfrak{r}$.*

Again, if $\{a_\alpha : \alpha \in \kappa\}$ is strictly $\subset^*$-increasing and $\subset^*$-cofinal in $\mathcal{I}_A$ for an almost clopen set A, then the family $\{a_{\alpha+1} \setminus a_\alpha : \alpha \in \kappa\}$ is an example of a *converging* family of infinite sets.

Definition 2.3. *Let $\mathcal{A}$ be a family of infinite subsets of $\mathbb{N}$. We say that $\mathcal{A}$ converges if there is an ultrafilter x on $\mathbb{N}$ such that for each $U \in x$, the set $\{a \in \mathcal{A} : a \setminus U \neq^* \emptyset\}$ has cardinality less than that of $\mathcal{A}$.*

We say that $\mathcal{A}$ is hereditarily unreapable if each reapable subfamily of $\mathcal{A}$ has cardinality less than that of $\mathcal{A}$.

An ultrafilter x of $\mathbb{N}^*$ is said to be an almost P_κ-point if each set of fewer than κ many members of x has a pseudointersection (an infinite set mod finite contained in each of them). Certainly a converging family is hereditarily unreapable and converges to a point that is an almost P_κ-point where κ is the cardinality of the family. Clearly the cardinality of any hereditarily unreapable family will have cofinality less than the splitting number $\mathfrak{s}$. First we recall that a family $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$ is a *splitting* family if for all infinite $b \subset \mathbb{N}$, there is an $a \in \mathcal{A}$ such that $|b \cap a| = |b \setminus a|$. We say that b is split by a. The splitting number, $\mathfrak{s}$, is the least cardinality of a splitting family and $\mathfrak{s} \leq \mathfrak{d}$ ([12]). Therefore if, for example, $\mathfrak{s} = \aleph_1$ and $\mathfrak{r} = \mathfrak{c} = \aleph_2$, there will be no hereditarily unreapable family. If $\mathfrak{s} = \mathfrak{c}$, then there is a hereditarily unreapable family of cardinality $\mathfrak{s}$. In the Mathias model, of $\mathfrak{s} = \mathfrak{c} = \mathfrak{b} = \aleph_2$, there is no converging unreapable family because there is no almost $P_{\aleph_2}$-point. In the Goldstern-Shelah model [12] of $\mathfrak{r} = \mathfrak{s} = \aleph_1 < \mathfrak{u}$, there is (easily checked) no converging family of cardinality $\mathfrak{r}$. It might be interesting to determine if there is a hereditarily unreapable family in this model because that would imply there was a stronger preservation result for the posets used.

If there is a simple almost clopen set of type κ, are there restrictions on the behavior of its almost clopen complement and can there be simple almost clopen sets of different types (including the complement)? These are the types of questions that stimulated this study. The most compelling of these has been answered.

Theorem 2.4. *If A is a simple almost clopen set of type κ and if the complementary almost clopen set B is simple, then it also has type κ.*

Similarly, there is a restriction on what the type of a simple almost clopen set can be that is shared by simple P_κ-points (as shown by Nyikos (unpublished) see [2]).

Theorem 2.5. *If A is a simple almost clopen set of type κ, then κ is one of* $\{\mathfrak{b}, \mathfrak{d}\}$.

Now that we understand the limits on the behavior of a complementary pair of simple almost clopen sets, we look to the properties of the complement B when it is not assumed to be simple. The topological properties of character and tightness of x_B in B are natural cardinal invariants to examine. These correspond to natural properties of $\mathcal{I}_B$ as well. An indexed subset $\{y_\beta : \beta < \lambda\}$ of a space X is said to be a free sequence if the closure of each initial segment is disjoint from the closure of its complementary final segment. A λ-sequence $\{y_\beta : \beta < \lambda\}$ is converging if there is a point y such that every neighborhood of y contains a final segment of $\{y_\beta : \beta < \lambda\}$. A subset D of $\mathbb{N}^*$ is said to be *strongly* discrete [10, 21] if there is a family of pairwise disjoint clopen subsets of $\mathbb{N}^*$ each containing a single point of D.

Theorem 2.6. *If $\kappa < \lambda$ are uncountable regular cardinals with $\mathfrak{c} \leq \lambda$, then there is a ccc forcing extension in which there is a simple almost clopen set A of type κ such that the almost clopen complement B contains a free λ-sequence $\{y_\beta : \beta < \lambda\}$ that converges to x_A.*

We prove these theorems in the next section. We finish this section by formulating some open problems about almost clopen sets and possible asymmetries.

Question 2.1. Can there exist simple almost clopen sets of different types?

Question 2.2. If there is a simple almost clopen set of type κ is there a point of $\mathbb{N}^*$ of character κ? Is there a simple P_κ-point?

The next question is simply a special case of the previous.

Question 2.3. Is a simple almost clopen set of type $\aleph_1$ necessarily homeomorphic to $\mathbb{N}^*$?

Question 2.4. If A is a simple almost clopen set of type κ, is there a simple almost clopen set B' contained in the almost clopen complement B of A such that $x_A \in B'$? Is there a family of κ-many members of $\mathcal{I}_B$ that converges to x_A?

Let us note that Theorem 3.6 is pertinent to this question.

3. Proofs

Our analysis of simple almost clopen sets depends on the connection between the type of the clopen set and the ultrafilter ordering of functions from $\mathbb{N}$ to $\mathbb{N}$. For an ultrafilter x on $\mathbb{N}$ the ordering $<_x$ is defined on $\mathbb{N}^{\mathbb{N}}$ by the condition that $f <_x g$ if $\{n \in \mathbb{N} : f(n) < g(n)\} \in x$. Since x is an ultrafilter, a set $F \subset \mathbb{N}^{\mathbb{N}}$ is cofinal in $(\mathbb{N}^{\mathbb{N}}, <_x)$ if it is not bounded. Of course a subset of $\mathbb{N}^{\mathbb{N}}$ that is unbounded with respect to the $<_x$-ordering is also unbounded with respect to the mod finite ordering $<^*$.

Fix a $<^*$-unbounded family $\{f_\xi : \xi < \mathfrak{b}\} \subset \mathbb{N}^{\mathbb{N}}$ such that each f_ξ is strictly increasing and such that $f_\eta <^* f_\xi$ for all $\eta < \xi < \mathfrak{b}$. The following well-known fact will be useful.

Proposition 3.1. *For each infinite $b \subset \mathbb{N}$ and each unbounded $\Gamma \subset \mathfrak{b}$, the family $\{f_\xi \restriction b : \xi \in \Gamma\}$ is $<^*$-unbounded in $\mathbb{N}^b$.*

Proof. For each $\eta < \mathfrak{b}$, there is a $\xi \in \Gamma \setminus \eta$ such that $f_\eta <^* f_\xi$, hence $\{f_\xi : \xi \in \Gamma\}$ is $<^*$-unbounded. If $g \in \mathbb{N}^b$, then $g \circ \mathrm{next}(b, \cdot) \in \mathbb{N}^{\mathbb{N}}$. So there is a $\xi \in \Gamma$ such that $f_\xi \not<^* g \circ \mathrm{next}(b, \cdot)$. Since f_ξ is strictly increasing, $f_\xi \restriction b \not<^* g$. $\square$

Lemma 3.2. *If a family $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$ converges to an ultrafilter x and if $\{f_\xi : \xi \in \mathfrak{b}\}$ is bounded mod $<_x$, then $\mathcal{A}$ has cardinality $\mathfrak{b}$.*

Proof. Choose $g \in \mathbb{N}^{\mathbb{N}}$ so that $f_\xi <_x g$ for all $\xi < \mathfrak{b}$. Since $\mathcal{A}$ can not be reaped, Proposition 2.2 implies that $\mathfrak{b} \leq |\mathcal{A}|$. For each ξ, let $U_\xi = \{n \in \mathbb{N} : f_\xi(n) < g(n)\} \in x$. If $\mathfrak{b} < |\mathcal{A}|$, then there is a $b \in \mathcal{A}$ such that $b \subset^* U_\xi$ for all $\xi < \mathfrak{b}$ (i.e. x is an almost $P_{\mathfrak{b}^+}$-point). However we would then have that $f_\xi \restriction b <^* g \restriction b$ for all $\xi < \mathfrak{b}$, and by Proposition 3.1, there is no such set b. This completes the proof. $\square$

Lemma 3.3. *If a family $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$ converges to an ultrafilter x and if $\{f_\xi : \xi \in \mathfrak{b}\}$ is unbounded mod $<_x$, then if $\mathcal{A}$ has regular cardinality, that cardinal is equal to $\mathfrak{d}$.*

Proof. Since we are assuming that $\{f_\xi : \xi \in \mathfrak{b}\}$ is $<_x$-unbounded, it is actually $<_x$-cofinal. We check that the family $\{f_\xi \circ \mathrm{next}(a, \cdot) : \xi < \mathfrak{b}, a \in \mathcal{A}\}$ is a $<^*$-dominating family. Take any strictly increasing $g \in \mathbb{N}^{\mathbb{N}}$ and choose $\xi < \mathfrak{b}$ such that $U = \{n : g(n) < f_\xi(n)\} \in x$. Since $\mathcal{A}$ converges to x, there is an $a \in \mathcal{A}$ such that $a \subset^* U$. Since g is strictly increasing, it is clear that $g < f_\xi \circ \mathrm{next}(U, \cdot) <^* f_\xi \circ \mathrm{next}(a, \cdot)$. Again, since $\mathcal{A}$ can not be reaped, we have $\mathfrak{b} \leq |\mathcal{A}|$ and this implies that $\mathfrak{d} \leq |\mathcal{A}|$. Assume that $\{g_\beta : \beta < \mathfrak{d}\} \subset \mathbb{N}^{\mathbb{N}}$ is a $<^*$-dominating family. For each $a \in \mathcal{A}$, there is a $\beta_a < \mathfrak{d}$ such that $\mathrm{next}(a, \cdot) <^* g_{\beta_a}$.

Now since $\mathcal{A}$ is hereditarily unreapable, Proposition 2.2 implies that if $\mathcal{A}$ has regular cardinality, the mapping $a \mapsto \beta_a$ is $<|\mathcal{A}|$-to-1. This implies that $|\mathcal{A}| \leq \mathfrak{d}$. □

Corollary 3.4. *Suppose that A is a simple almost clopen subset of $\mathbb{N}^*$ of type κ. If $\{f_\xi : \xi < \mathfrak{b}\}$ is $<_{x_A}$-bounded, then $\kappa = \mathfrak{b}$; otherwise $\kappa = \mathfrak{d}$.*

Proof. Let $\{a_\alpha : \alpha \in \kappa\}$ be the family contained in $\mathcal{I}_A$ witnessing that A has type κ. Set $\mathcal{A}$ equal to the family $\{a_{\alpha+1} \setminus a_\alpha : \alpha \in \kappa\}$ which converges to x_A. If $\{f_\xi : \xi < \mathfrak{b}\}$ is $<_{x_A}$-bounded, then by Lemma 3.2, $\kappa = \mathfrak{b}$. Otherwise, since κ is a regular cardinal, we have by Lemma 3.3, $\kappa = \mathfrak{d}$. □

Proof of Theorem 2.4. Assume that A and its complementary almost clopen set B are both simple and let $x = x_A$. If $\{f_\xi : \xi < \mathfrak{b}\}$ is $<_x$-bounded then, by Corollary 3.4 they both have type $\mathfrak{b}$; otherwise they both have type $\mathfrak{d}$. □

Proof of Theorem 2.5. Immediate from Corollary 3.4. □

We can improve Theorem 2.4.

Proposition 3.5. *There is no almost $P_{\mathfrak{s}^+}$-point in $\mathbb{N}^*$.*

Proof. Let $\mathcal{A}$ be a splitting family of cardinality $\mathfrak{s}$. We may assume that $\mathcal{A}$ is closed under complements. Let x be any point of $\mathbb{N}^*$. It is easily seen that any pseudointersection of $x \cap \mathcal{A}$ is not split by any member of $\mathcal{A}$. Since $\mathcal{A}$ is splitting, $x \cap \mathcal{A}$ has no pseudointersection, and so x is not an almost $P_{\mathfrak{s}^+}$-point. □

Now we improve Theorem 2.4.

Theorem 3.6. *If A is a simple almost clopen set of type κ then x_A is not an almost P_{κ^+}-point.*

Proof. We first note that by Proposition 3.5 we must have that $\kappa < \mathfrak{d}$. Therefore, by Lemma 3.3, $\{f_\xi : \xi < \mathfrak{b}\}$ is $<_{x_A}$-bounded. Choose any $g \in \mathbb{N}^\mathbb{N}$ so that $f_\xi <_{x_A} g$ for all $\xi < \mathfrak{b}$. For each ξ, let $U_\xi = \{n \in \mathbb{N} : f_\xi(n) < g(n)\}$. By Proposition 3.1, we have that the collection $\{U_\xi : \xi < \mathfrak{b}\} \subset x$ has no pseudointersection. By Theorem 2.4, $\mathfrak{b} \leq \kappa$ and this proves the theorem. □

Now we prove Theorem 2.6. We first prove the easier special case when $\kappa = \aleph_1$. An α-length finite support iteration sequence of posets, denoted $(\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle)$, will mean that $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ is an increasing chain of posets, $\dot{Q}_\beta$ is a $\mathbb{P}_\beta$-name of a poset for each $\beta < \alpha$, and members p of $\mathbb{P}_\alpha$ will be functions with domain a finite subset, $\mathrm{supp}(p)$, of α satisfying that $p\restriction\beta \in \mathbb{P}_\beta$ forces that $p(\beta) \in \dot{Q}_\beta$ for $\beta \in \mathrm{supp}(p)$. As usual, $p_2 < p_1$ providing $p_2\restriction\beta \Vdash_{\mathbb{P}_\beta}$ "$p_2(\beta) < p_1(\beta)$" for all $\beta \in \mathrm{supp}(p_1)$. Since $\mathbb{P}_0$ is the trivial poset, we will allow ourselves to simply specify a poset Q_0 in such an iteration sequence rather than the $\mathbb{P}_0$-name of that poset.

Definition 3.7. *Let $\mathcal{A} = \{a_\beta : \beta < \alpha\}$ be a $\subset^*$-increasing chain of subsets of ω, and let $\mathcal{I}$ be an ideal contained in $\mathcal{A}^\perp$. We define the poset $Q = Q(\mathcal{A}; \mathcal{I})$ where $q \in Q$ if $q = (F_q, \sigma_q, b_q)$ where*

(1) $F_q \in [\omega]^{<\aleph_0}$,
(2) $b_q \in \mathcal{I}$ is disjoint from b_q,
(3) $\sigma_q : H_q \to \omega$ and $H_q \in [\alpha]^{<\aleph_0}$,
(4) for each $\beta \in H_q$, $a_\beta \setminus \sigma_q(\beta)$ is disjoint from b_q.

For $r, q \in Q$ we define $r < q$ providing $F_r \supset F_q$, $\sigma_r \supset \sigma_q$, and $b_r \supset b_q$.

Lemma 3.8. *If $\mathcal{A} = \{a_\beta : \beta < \alpha\}$ is a $\subset^*$-increasing chain of subsets of ω and if $\mathcal{I}$ is an ideal contained in $\mathcal{A}^\perp$, then $Q(\mathcal{A}; \mathcal{I})$ is ccc whenever $\mathrm{cf}(\alpha)$ is not equal to ω_1. In addition, $Q(\mathcal{A}; [\omega]^{<\omega})$ is ccc for any infinite $\subset^*$-increasing chain $\mathcal{A}$.*

Proof. The proofs are standard. The first statement is basically the same as Theorem 4.2 of [1]. For the last, $Q(\mathcal{A}; [\omega]^{<\omega})$ is ccc since conditions $p, q \in Q(\mathcal{A}; [\omega]^{<\omega})$ are compatible so long as $F_p = F_q$, $b_p = b_q$, and $\sigma_p \cup \sigma_q$ is a function. □

Definition 3.9. *If Q is $Q(\mathcal{A}; \mathcal{I})$ for some $\subset^*$-increasing chain of subsets of ω, and $\mathcal{I} \subset \mathcal{A}^\perp$ is an ideal, then the Q-generic set $\dot{a}_Q$ is defined as the natural name $\{(\check{F}_q, q) : q \in Q\}$, i.e. for each Q-generic filter G, $\mathrm{val}_G(\dot{a}_Q)$ is equal to the union of the family $\{F_q : q \in G\}$.*

Lemma 3.10. *If λ is a regular cardinal with $\mathfrak{c} \le \lambda$, then there is a ccc forcing extension in which there is a simple almost clopen set A of type ω_1 such that there is a strongly discrete free λ-sequence converging to x_A.*

Proof. There are ccc posets of cardinality λ that add a strictly $\subset^*$-increasing sequence $\{b_\zeta : \zeta < \lambda\}$ of infinite subset of ω (e.g. [18, II Ex. 22]). Alternatively, by Definition 3.7 and Lemma 3.8, we could let Q_0 be a λ-length finite support sequence of posets of the form $Q(\{b_\beta : \beta < \zeta\}; [\omega]^{<\omega})$ and recursively let b_ζ be the resulting $\dot{a}_Q$ as in Definition 3.9.

For convenience we now work in such a ccc forcing extension and we construct a finite support ccc iteration sequence of cardinality λ and length ω_1 that will add a strictly $\subset^*$-increasing sequence $\{a_\alpha : \alpha \in \omega_1\}$ of infinite subsets of ω so that the closure, A, of $\bigcup\{a_\alpha^* : \alpha \in \omega_1\}$ is almost clopen. Suppose that we do this in such a way that $\{b_\zeta : \zeta < \lambda\}$ is contained in $\{a_\alpha : \alpha \in \omega_1\}^\perp$ and, for all $U \in x_A$, and all $\zeta < \lambda$, there is an $\eta < \lambda$ such that $U \cap (b_\eta \setminus b_\zeta)$ is infinite. We check that there is then a strongly discrete free λ-sequence converging to x_A. Let $\{U_\zeta : \zeta < \lambda\}$ enumerate the members of x_A. There is a cub $C \subset \lambda$ satisfying that for each $\delta \in C$, the family $\{U_\xi : \xi < \delta\}$ is closed under finite intersections. Recursively define a strictly increasing function g from C into λ satisfying that $U_\zeta \cap (b_{g(\delta)} \setminus b_\delta)$ is infinite for all $\zeta < \delta \in C$. Now, for each $\delta \in C$, let x_δ be an ultrafilter extending the family $\{U_\zeta \cap (b_{g(\delta)} \setminus b_\delta) : \zeta < \delta\}$. Pass to a cub subset $C_1 \subset C$ satisfying that $g(\eta) < \delta$ for all $\delta \in C$ and $\eta \in \delta \cap C$. It follows immediately that $\{x_\delta : \delta \in C_1\}$ is strongly discrete and free. Similarly, the sequence converges to x_A since $U_\zeta \in x_\delta$ for all $\zeta < \delta \in C_1$.

Now we construct the iteration sequence to define the $\subset^*$-increasing chain $\{a_\alpha : \alpha \in \omega_1\}$ that will be cofinal in $\mathcal{I}_A$. We will use iterands of the form $\dot{Q}_\alpha = Q(\{\dot{a}_\beta : \beta < \alpha\}; \dot{\mathcal{I}}_\alpha)$ for $0 < \alpha < \omega_1$, and will recursively let $\dot{a}_\alpha$ be the standard $\mathbb{P}_{\alpha+1}$-name for $a_{\dot{Q}_\alpha}$ (as in Definition 3.9). Clearly the only choices we have for the construction are the definition of a_0 and, by recursion, the definition of $\dot{\mathcal{I}}_\alpha$. We will recursively ensure that $\Vdash_{\mathbb{P}_\alpha}$ "$\{\dot{a}_\beta : \beta < \alpha\} \subset \{b_\zeta : \zeta < \lambda\}^\perp$" simply by ensuring that $\Vdash_{\mathbb{P}_\alpha}$ "$\{b_\zeta : \zeta < \lambda\} \subset \dot{\mathcal{I}}_\alpha$".

To start the process, we let $\mathcal{I}_1$ be any maximal ideal extending the ideal generated by $\{b_\zeta : \zeta < \lambda\} \cup \{b_\zeta : \zeta < \lambda\}^\perp$. Very likely $\{b_\zeta : \zeta < \lambda\}^\perp$ is simply $[\omega]^{<\omega}$, so we let a_0 be exceptional and equal the emptyset. We now have our definition (working in the extension by Q_0) of $Q_1 = Q(\{a_0\}; \mathcal{I}_1)$ and the generic set $\dot{a}_1 = \dot{a}_{Q_1}$ is forced to be almost disjoint from every member of $\mathcal{I}_1$ (it is a pseudointersection of the ultrafilter dual to $\mathcal{I}_1$). Now assume that $\alpha < \omega_1$ and that we have defined $\dot{\mathcal{I}}_\beta$ for all $\beta < \alpha$. We recursively also ensure that, for $\beta < \gamma < \alpha$, the $\mathbb{P}_\beta$-name $\dot{\mathcal{I}}_{\beta+1}$ is a subset of the $\mathbb{P}_\gamma$-name $\dot{\mathcal{I}}_\gamma$, and that $\mathbb{P}_\gamma$ forces $\dot{\mathcal{I}}_\gamma$ is contained in $\{\dot{a}_\beta : \beta < \gamma\}^\perp$. For the definition of $\dot{\mathcal{I}}_\alpha$ we break into three cases. If α is a limit ordinal, then we define $\dot{\mathcal{I}}_\alpha$ to be the $\mathbb{P}_\alpha$-name of the ideal $\{\dot{a}_\beta : \beta < \alpha\}^\perp$. By induction, we have, for $\gamma < \alpha$, that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathcal{I}}_{\gamma+1} \subset \{\dot{a}_\beta : \beta < \alpha\}^\perp = \dot{\mathcal{I}}_\alpha$", as required. In the case that $\alpha = \beta + 1$ for a successor β, we note that $\mathbb{P}_{\beta+1}$ forces that (by the genericity of $\dot{a}_\beta$) the family $\{\dot{a}_\beta\} \cup \dot{\mathcal{I}}_\beta$ generates a proper ideal $\dot{\mathcal{J}}_\alpha$. In the case that β is a limit and $\alpha = \beta + 1$, we note that $\mathbb{P}_{\beta+1}$ forces that the family $\{\dot{a}_\beta\} \cup \bigcup\{\dot{\mathcal{I}}_{\gamma+1} : \gamma < \beta\}$ also generates a proper ideal $\dot{\mathcal{J}}_\alpha$. Then, in either case where $\alpha = \beta + 1$, we let $\dot{\mathcal{J}}'_\alpha$ be the $\mathbb{P}_\alpha$-name of any maximal ideal that contains $\dot{\mathcal{J}}_\alpha \cup (\dot{\mathcal{J}}_\alpha)^\perp$. The definition of $\dot{\mathcal{I}}_\alpha$ is then the $\mathbb{P}_\alpha$-name of $\dot{\mathcal{J}}'_\alpha \cap \{\dot{a}_\beta\}^\perp$. For convenience, let $\dot{y}_{\beta+1}$ denote the $\mathbb{P}_{\beta+1}$-name of this ultrafilter, and let us notice that $\{\omega \setminus (\dot{a}_\beta \cup b) : b \in \dot{\mathcal{I}}_{\beta+1}\}$ is forced to be a base for $\dot{y}_{\beta+1}$. The set $\dot{a}_{\beta+1} \setminus \dot{a}_\beta$ will be a pseudointersection of $\dot{y}_{\beta+1}$.

This completes the definition of the poset $\mathbb{P}_{\omega_1}$. Now we establish some properties. Let $\dot{A}$ denote $\mathbb{P}_{\omega_1}$-name of the closure in ω^* of the open set $\bigcup\{\dot{a}^*_\alpha : \alpha \in \omega_1\}$.

Claim 1. For each $\beta < \alpha < \omega_1$, $\mathbb{P}_{\alpha+1}$ forces that $\dot{a}_\alpha \setminus \dot{a}_\beta$ is a pseudointersection of the filter $\dot{y}_{\beta+1}$.

Proof of Claim: We proceed by induction on $\alpha \geq \beta+1$. For $\alpha = \beta+1$, $\dot{a}_\alpha$ is almost disjoint from each member of $\dot{\mathcal{I}}_\alpha$, and so $\dot{a}_\alpha \setminus \dot{a}_\beta$ is almost disjoint from every member of $\dot{\mathcal{J}}'_\alpha$. Thus $\dot{a}_\alpha \setminus \dot{a}_\beta$ is forced to be mod finite contained in every member of the dual filter, namely $\dot{y}_{\beta+1}$. Similarly, for $\alpha > \beta + 1$, $\dot{a}_\alpha$ is forced to be almost disjoint from each member of $\dot{\mathcal{I}}_\alpha$. This means that $\dot{a}_\alpha$ is almost disjoint from each member of $\dot{\mathcal{I}}_{\beta+1}$, and so $\dot{a}_\alpha \setminus \dot{a}_{\beta+1}$ is also almost disjoint from every member of $\dot{\mathcal{J}}'_{\beta+1}$. □

Claim 2. The family $\{\dot{y}_{\beta+1} : \beta < \omega_1\}$ is a family of $\mathbb{P}_{\omega_1}$-names and the union is forced to generate an ultrafilter $\dot{x}_{\dot{A}}$ that is indeed the unique boundary point of $\dot{A}$.

Proof of Claim: Since $\dot{\mathcal{I}}_{\beta+1}$ is contained in $\dot{\mathcal{I}}_{\alpha+1}$, and $\mathbb{P}_{\omega_1}$ forces that $\dot{a}_\beta \subset^* \dot{a}_\gamma$, we have that $\mathbb{P}_{\omega_1}$ forces that $\bigcup\{\dot{y}_{\beta+1} : \beta < \omega_1\}$ is a filter. Furthermore, since $\mathbb{P}_{\omega_1}$ is ccc, every $\mathbb{P}_{\omega_1}$-name of a subset of ω is equal to a $\mathbb{P}_\beta$-name for some $\beta < \omega_1$. The fact that, for each $\beta < \omega_1$, $\mathbb{P}_{\beta+1}$ forces that $\dot{y}_{\beta+1}$ is an ultrafilter implies that $\mathbb{P}_{\omega_1}$ forces that $\dot{x}_{\dot{A}}$ is an ultrafilter. Finally, it follows immediately from the previous claim that $\dot{x}_{\dot{A}}$ is the unique boundary point of $\dot{A}$. □

Claim 3. For each $0 < \alpha < \omega_1$, $\mathbb{P}_{\alpha+1}$ forces that $\{b_\zeta : \zeta < \lambda\}$ is $\subset^*$-unbounded in $\dot{\mathcal{I}}_{\alpha+1}$.

Proof of Claim: We prove this by induction on α. We know that $\mathbb{P}_{\alpha+1}$ forces that $\dot{a}_\alpha$ is almost disjoint from every member of $\{b_\zeta : \zeta < \lambda\}$. Therefore, if $\mathbb{P}_{\beta+1}$ forces that $\{b_\zeta : \zeta < \lambda\}$ is $\subset^*$-unbounded in $\dot{\mathcal{I}}_{\beta+1}$ for each $\beta < \alpha$, it follows that $\{b_\zeta : \zeta < \lambda\}$ is $\subset^*$-unbounded in what we called $\dot{\mathcal{J}}_{\alpha+1}$ above. In addition, we have that $\dot{\mathcal{J}}^\perp_{\alpha+1} \subset \{b_\zeta : \zeta < \lambda\}^\perp$, so we have that $\mathbb{P}_{\alpha+1}$ forces that $\{b_\zeta : \zeta < \lambda\}$ is $\subset^*$-unbounded in $\dot{\mathcal{I}}_{\alpha+1}$. □

To finish the proof of the Lemma, we have to verify that $\mathbb{P}_{\omega_1}$ forces that for all $U \in \dot{x}_{\dot{A}}$, and all $\zeta < \lambda$, there is an $\eta < \lambda$ such that

$U \cap (b_\eta \setminus b_\zeta)$ is infinite. Let $\zeta < \lambda$ be given and suppose that $\dot{U}$ is a $\mathbb{P}_\alpha$-name of a member of $\dot{x}_{\dot{A}}$ for some $\alpha < \omega_1$. Of course this means that $\dot{U}$ is a member of $\dot{y}_{\alpha+1}$. Now consider the $\mathbb{P}_{\alpha+1}$-name, $\dot{b}$, of the set $b_\zeta \cup \Big(\omega \setminus (\dot{U} \cup \dot{a}_{\alpha+1})\Big)$. Evidently, $\dot{b}$ is forced to be disjoint from $\dot{a}_{\alpha+1}$ and also is forced to not be in $\dot{y}_{\alpha+2}$. It follows that $\dot{b}$ is an element of $\dot{\mathcal{I}}_{\alpha+2}$. By Claim 3, there is an η and a condition $p \in \mathbb{P}_{\alpha+2}$ such that p forces that $b_\eta \setminus \dot{b}$ is infinite. Since $b_\eta \setminus \dot{b}$ is mod finite equal to $(b_\eta \setminus b_\zeta) \cap \dot{U}$, this completes the proof. $\square$

To prove Theorem 2.6, we want to continue the recursive construction of Lemma 3.10 to a κ-length iteration of posets of the same form, namely $Q(\{\dot{a}_\beta : \beta < \alpha\}; \dot{\mathcal{I}}_\alpha)$. It turns out that with the exact construction of Lemma 3.10, $\mathbb{P}_{\omega_1}$ forces that $Q(\{\dot{a}_\beta : \beta < \omega_1\}; \{\dot{a}_\beta : \beta < \omega_1\}^\perp)$ is not ccc. For limit ordinals α of uncountable cofinality, it is likely that we have to use $\{\dot{a}_\beta : \beta < \alpha\}^\perp$ as our choice for $\dot{\mathcal{I}}_\alpha$. However, we do have more flexibility at limits of countable cofinality and this is critical for extending the construction to any length κ.

Definition 3.11. *We say that $\mathcal{A}$ is a pre-ccc sequence if*

(1) *$\mathcal{A} = \{a_\beta : \beta < \alpha\}$ for some increasing $\subset^*$-chain of subsets of ω with $\mathrm{cf}(\alpha) = \omega_1$,*
(2) *for each increasing sequence $\{\gamma_\xi : \xi \in \omega_1\}$ cofinal in α, and each sequence $\{b_\xi : \xi \in \omega_1\} \subset \mathcal{A}^\perp$, such that $a_{\gamma_\xi} \cap b_\xi = \emptyset$ for all ξ, there are $\xi < \eta$ such that $a_{\gamma_\xi} \subset a_{\gamma_\eta}$ and $b_\xi \cap a_{\gamma_\eta}$ is empty.*

Lemma 3.12. *If $\mathcal{A}$ is a pre-ccc sequence, then $Q(\mathcal{A}; \mathcal{I})$ is ccc for any ideal $\mathcal{I} \subset \mathcal{A}^\perp$.*

Proof. Let $\mathcal{A} = \{a_\beta : \beta < \alpha\}$. Let $\{q_\xi : \xi \in \omega_1\} \subset Q = Q(\mathcal{A}; \mathcal{I})$. By passing to a subcollection we can suppose there is a single $F \in [\omega]^{<\aleph_0}$ such that $F_{q_\xi} = F$ for all ξ. For each ξ, let $b_\xi = b_{q_\xi}$, $\sigma_\xi = \sigma_{q_\xi}$, and $H_\xi = \mathrm{dom}(\sigma_\xi)$. By the standard Δ-system lemma argument, we may assume that $\sigma_\xi \cup \sigma_\eta$ is a function for all $\xi, \eta \in \omega_1$.

For each ξ, let γ_ξ be the maximum element of H_ξ. By a trivial density argument we can assume that $\{\gamma_\xi : \xi \in \omega_1\}$ is a strictly increasing sequence that is cofinal in α.

Next, we choose an integer $\bar{m}$ sufficiently large so that there is again an countable $I \subset \omega_1$ and a subset $\bar{b}$ of $\bar{m}$ such that, for all $\xi \in I$ and all $\beta \in H_\xi$

(1) $\sigma_\xi(\beta) < \bar{m}$,
(2) $a_\beta \setminus \bar{m} \subset a_{\gamma_\xi}$,
(3) $b_\xi \cap \bar{m} = \bar{b}$,

Now we apply the pre-ccc property for the family $\{\gamma_\xi : \xi \in I\}$ and the sequence $\{b_\xi \setminus \bar{m} : \xi \in I\}$. Thus, we may choose $\xi < \eta$ from I so that $a_{\gamma_\xi} \subset a_{\gamma_\eta}$ and $b_\xi \setminus \bar{m}$ is disjoint from a_{γ_η}. We claim that $r = (F, \sigma_\xi \cup \sigma_\eta, b_\xi \cup b_\eta)$ is in Q and is an extension of each of q_ξ and q_η. It suffices to prove that for $\beta \in H_\xi$, $a_\beta \setminus \sigma_\xi(\beta)$ is disjoint from b_η, and similarly, that $a_\beta \setminus \sigma_\eta(\beta)$ is disjoint from b_ξ for $\beta \in H_\eta$. Since $b_\xi \cap \bar{m} = b_\eta \cap \bar{m} = \bar{b}$, it suffices to consider $a_\beta \setminus \bar{m}$ in each case. For $\beta \in H_\xi$, we have $a_\beta \setminus \bar{m} \subset a_{\gamma_\xi} \subset a_{\gamma_\eta}$, and a_{γ_η} is disjoint from $b_\eta \setminus \bar{m}$. For $\beta \in H_\eta$, we have $a_\beta \setminus \bar{m} \subset a_{\gamma_\eta}$ and a_{γ_η} is disjoint from $b_\xi \setminus \bar{m}$. □

Definition 3.13. *$\mathfrak{A}$ is the class of triples $(\mathfrak{P}, \mathcal{A}, \mathfrak{I})$ such that, there is an ordinal α, and the following holds for each $\beta < \alpha$:*

1. *$\mathfrak{P} = (\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle)$ is a finite support iteration sequence of ccc posets,*
2. *$\mathcal{A}$ is an α-sequence $\{\dot{a}_\beta : \beta < \alpha\}$, and $\Vdash_{\mathbb{P}_{\beta+1}}$ "$\dot{a}_\beta \subset \omega$",*
3. *$\mathfrak{I}$ is an α-sequence $\{\dot{\mathcal{I}}_\beta : \beta < \alpha\}$,*
4. *$\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathcal{I}}_\beta \subset \{\dot{a}_\xi : \xi < \beta\}^\perp$ is an ideal"*
5. *for $\beta < \gamma < \alpha$, $\Vdash_{\mathbb{P}_{\gamma+1}}$ "$\dot{a}_\beta \subset^* \dot{a}_\gamma$" and $\Vdash_{\mathbb{P}_\gamma}$ "$\dot{\mathcal{I}}_\beta \subset \dot{\mathcal{I}}_\gamma$"*
6. *$\Vdash_{\mathbb{P}_\beta}$ "$\dot{Q}_\beta = Q(\{\dot{a}_\xi : \xi < \beta\}; \dot{\mathcal{I}}_\beta)$",*
7. *$\dot{a}_\beta$ is the $\mathbb{P}_{\beta+1}$-name for $\dot{a}_{Q(\{\dot{a}_\xi:\xi<\beta\};\dot{\mathcal{I}}_\beta)}$*
8. *if $\mathrm{cf}(\beta) = \omega$, then there is a sequence $\{\dot{\mathcal{I}}_{\beta,\xi} : \xi < \beta\}$ such that $\dot{\mathcal{I}}_{\beta,\xi}$ is a $\mathbb{P}_\xi$-name and $\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathcal{I}}_\beta = \bigcup\{\dot{\mathcal{I}}_{\beta,\xi} : \xi < \beta\}$".*

Lemma 3.14. *If α is an ordinal with cofinality ω_1, and if $(\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle), \{\dot{a}_\beta : \beta < \alpha\}, \{\dot{\mathcal{I}}_\beta : \beta < \alpha\})$ is in $\mathfrak{A}$ then $\Vdash_{\mathbb{P}_\alpha}$ "$\{\dot{a}_\beta : \beta < \alpha\}$ is a pre-ccc sequence".*

Proof. Let $\dot{\mathcal{A}}$ denote the $\mathbb{P}_\alpha$-name of the sequence $\{\dot{a}_\beta : \beta < \alpha\}$. Let $\{\dot{\gamma}_\xi : \xi \in \omega_1\}$ and $\{\dot{b}_\xi : \xi \in \omega_1\}$ be sequences of $\mathbb{P}_\alpha$-names such that there is some $p_0 \in \mathbb{P}_\alpha$ forcing that, for each $\xi < \omega_1$, $\dot{\gamma}_\xi \in \alpha$, $\dot{b}_\xi \in \dot{\mathcal{A}}^\perp$ and $\dot{b}_\xi \cap \dot{a}_{\gamma_\xi}$ is empty. Suppose also that p_0 forces that $\{\dot{\gamma}_\xi : \xi \in \omega_1\}$ is strictly increasing and cofinal in α. We may assume that p_0 decides the value, γ_0, of $\dot{\gamma}_0$. For each $\xi < \omega_1$, choose any $p_\xi < p_0$ that decides a value, γ_ξ, of $\dot{\gamma}_\xi$ and that $\dot{b}_\xi$ is a $\mathbb{P}_\beta$-name for some $\beta \in \mathrm{supp}(p_\xi)$.

Let g be a continuous strictly increasing function from ω_1 into α with cofinal range. Since $\mathbb{P}_\alpha$ is ccc we have, for each $\delta \in \omega_1$, the set $\{\xi : \gamma_\xi < g(\delta)\}$ is countable. Therefore there is a cub $C \subset \omega_1$ such that $g(\delta) \leq \gamma_\delta$ for all $\delta \in C$. We may also arrange that, for each $\delta \in C$ and $\xi < \delta$, $\mathrm{supp}(p_\xi) \subset g(\delta)$.

For each $\delta \in C$, we may extend p_δ so as to ensure that each of $g(\delta)$ and γ_δ are in $\mathrm{supp}(p_\delta)$, and such that there is a $\beta \in \mathrm{supp}(p_\xi)$ such that $\dot{b}_\xi$ is a $\mathbb{P}_\beta$-name. We also extend each p_δ so that we can arrange a list of special properties (referred to as “determined” in many similar constructions). Specifically, for each $\beta \in \mathrm{supp}(p_\delta)$,

(1) there are $F^\delta_\beta \in [\omega]^{<\aleph_0}$, $H^\delta_\beta \in [\beta]^{<\aleph_0}$, $\sigma^\delta_\beta : H^\delta_\beta \to \omega$, and a $\mathbb{P}_\beta$-name $\dot b^\delta_\beta$ such that $p_\delta\restriction\beta \Vdash_{\mathbb{P}_\beta}$ “$p_\delta(\beta) = (F^\delta_\beta, \sigma^\delta_\beta, \dot b^\delta_\beta)$”,
(2) if $g(\delta) < \gamma_\delta$, then $g(\delta) \in H^\delta_{\gamma_\delta}$,
(3) $H^\delta_\beta \subset \mathrm{supp}(p_\delta)$,
(4) if β is a limit with countable cofinality, then there is a $\mu^\delta_\beta < \beta$ such that $\dot b^\delta_\beta$ is a $\mathbb{P}_{\mu^\delta_\beta}$-name and $\mathrm{supp}(p_\delta) \cap \beta \subset \mu^\delta_\beta$,
(5) if $\iota < \beta$ is in $\mathrm{supp}(p_\delta)$, then $H^\delta_\iota \supset H^\delta_\beta \cap \iota$,

By the pressing down lemma, there is a stationary set $S \subset C$ and a $\mu < \alpha$ such that $\mu^\delta_{g(\delta)} < \mu$ for all $\delta \in S$. Now let $G_{\mu+1}$ be $\mathbb{P}_{\mu+1}$-generic filter satisfying that, in the extension, there is a stationary set $S_1 \subset S$ so that $p_\xi\restriction\mu \in G_{\mu+1}$ for all $\xi \in S_1$. We may also arrange that the values of the pair $\{F^\xi_{g(\xi)}, F^\xi_{\gamma_\xi}\}$ is the same for all $\xi \in S_1$. For all $\beta \in \mu+1$, we let a_β denote the valuation of $\dot a_\beta$ by $G_{\mu+1}$. By further shrinking S_1 we may suppose there is an $\bar m \in \omega$ and a $\bar b \subset \bar m$, satisfying that, for all $\delta \in S_1$,

(1) for all $\beta \in \mathrm{supp}(p_\delta)$ $F^\delta_\beta \subset \bar m$, and, for all $\iota \in H^\delta_\beta$, $\sigma^\delta_\beta(\iota) < \bar m$,
(2) for all $\beta \in \mathrm{supp}(p_\delta) \cap \mu$, $a_\iota \setminus a_\mu \subset \bar m$
(3) $\bar b = \bar m \cap b^\delta_{g(\delta)}$, where $b^\delta_{g(\delta)}$ is the valuation of $\dot b^\delta_{g(\delta)}$ by $G_{\mu+1}$,
(4) $b^\delta_{g(\delta)} \cap a_\mu \subset \bar m$.

Fix any $\xi < \eta$ from S_1. Define q_ξ so that $\mathrm{supp}(q_\xi) = \mathrm{supp}(p_\xi) \setminus \mu+1$, and, for $\beta \in \mathrm{supp}(q_\xi)$,

$$q_\xi(\beta) = \begin{cases} (F^\xi_{g(\xi)}, \sigma^\xi_{g(\xi)} \cup \{(\mu, \bar m)\}, \dot b^\xi_{g(\xi)} \cup \dot b^\eta_{g(\eta)}) & \text{if } \beta = g(\xi) \\ (F^\xi_\beta, \sigma^\xi_\beta, \dot b^\xi_\beta \cup b^\eta_{g(\eta)} \setminus \bar m) & \text{if } g(\xi) < \beta \ . \end{cases}$$

We prove by induction on $\beta \in \mathrm{supp}(q_\xi)$, that there is a condition $r^\xi_\beta \in G_{\mu+1}$ such that $r^\xi_\beta \cup (q_\xi\restriction(\beta+1)) \le p_\xi\restriction(\beta+1)$. Evidently, for the case $\beta = g(\xi)$, $F^\xi_{g(\xi)}$ and $\bar m \cap a_\iota$ are disjoint from $\bar b$ and so there is some condition in $G_{\mu+1}$ that forces that, they are disjoint from $\dot b^\eta_{g(\eta)}$. Similarly, for $\iota \in H^\xi_{g(\xi)}$, $a_\iota \setminus \bar m \subset a_\mu$, and since $a_\mu \setminus \bar m \cap (b^\xi_{g(\xi)} \cup b^\eta_{g(\eta)})$ is empty, there is a condition r in $G_{\mu+1}$ that forces that $q_\xi(g(\xi)) \in \dot Q_{g(\xi)}$ and that $q_\xi(g(\xi)) < p_\xi(g(\xi))$. In addition, $r \cup q_\xi\restriction(g(\xi)+1)$ forces that $\dot a_{g(\xi)}$ is disjoint from $b^\eta_{g(\eta)} \setminus \bar m$. Now, suppose that $g(\xi) < \beta \in \mathrm{supp}(p_\xi)$, and that $r \cup q\restriction\beta$ is a condition in $\mathbb{P}_\beta$ that is below $p_\xi\restriction\beta$. We recall that $H^\xi_\beta \subset \mathrm{supp}(p_\xi)$, and so it follows that $r \cup q_\xi\restriction\beta$ forces that $\dot a_\iota$ is disjoint from $b^\eta_{g(\eta)} \setminus \bar{m}$ for all $\iota \in H^\xi_\beta$. This is the only thing that needs verifying when checking that $r \cup q_\xi \restriction (\beta+1) < p_\xi \restriction (\beta+1)$.

Now that we have that $r \cup q_\xi$ forces that $\dot{a}_{\gamma_\xi}$ is disjoint from $b^\eta_{g(\eta)} \setminus \bar{m}$, we can add $\{(\gamma_\xi, \bar{m})\}$ to $\sigma^\eta_{g(\eta)}$ and still have a condition. Similarly, for all $\iota \in \mathrm{supp}(p_\eta) \cap \mu$, $a_\iota \setminus \bar{m}$ is contained in a_μ, and $r \cup q_\xi$ forces that $a_\mu \setminus \bar{m}$, being a subset of $\dot{a}_{\gamma_\xi}$, is disjoint from $\dot{b}_\xi$. This implies that $r \cup q_\xi$ forces that $(F^\eta_{g(\eta)}, \sigma^\eta_{g(\eta)} \cup \{(\gamma_\xi, \bar{m})\}, b^\eta_{g(\eta)} \cup (\dot{b}_\xi \setminus \bar{m}))$ is a condition in $\dot{Q}_{g(\eta)}$ and is less than $p_\eta(g(\eta))$. Now we define a condition q_η so that $\mathrm{supp}(q_\eta) = \mathrm{supp}(p_\eta) \setminus \mu + 1$, and, for $\beta \in \mathrm{supp}(q_\eta)$,

$$q_\eta(\beta) = \begin{cases} (F^\eta_{g(\eta)}, \sigma^\eta_{g(\eta)} \cup \{(\gamma_\xi, \bar{m})\}, \dot{b}^\eta_{g(\eta)} \cup \dot{b}_{\gamma_\xi}) & \text{if } \beta = g(\xi) \\ (F^\eta_\beta, \sigma^\eta_\beta, \dot{b}^\eta_\beta \cup (\dot{b}_{\gamma_\xi} \setminus \bar{m})) & \text{if } g(\eta) < \beta\ . \end{cases}$$

It again follows, by induction on $\beta \in \mathrm{supp}(q_\eta)$, that $r \cup q_\xi$ forces that $r \cup q_\xi \cup (q_\eta \restriction (\beta+1))$ is a condition in $\mathbb{P}_{\beta+1}$ and is below $p_\eta \restriction (\beta+1)$. Finally, we observe that $r \cup q_\xi \cup q_\eta$ forces that $\dot{a}_{\gamma_\xi} \subset \dot{a}_{\gamma_\eta}$ because it forces that $\dot{a}_{\gamma_\xi} \cap \bar{m} = \dot{a}_{\gamma_\eta} \cap \bar{m}$ and that $\dot{a}_{\gamma_\xi} \setminus \bar{m} \subset \dot{a}_{g(\eta)} \setminus \bar{m} \subset \dot{a}_{\gamma_\eta}$. Similarly $r \cup q_\xi \cup q_\eta$ forces that $\dot{a}_{\gamma_\eta}$ is disjoint from $\dot{b}_\xi$ because $\dot{b}_\xi \cap \bar{m} = \bar{b}$ and $\dot{a}_{\gamma_\eta}$ is disjoint from $\dot{b}_\xi \setminus \bar{m}$. $\square$

Corollary 3.15. *If* $(\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle), \{\dot{a}_\beta : \beta < \alpha\}, \{\dot{\mathcal{I}}_\beta : \beta < \alpha\}$ *is in* $\mathfrak{A}$, *then* $\Vdash_{\mathbb{P}_\alpha}$ *"*$Q(\{\dot{a}_\beta : \beta < \alpha\}; \dot{\mathcal{I}})$ *is ccc" for each* $\mathbb{P}_\alpha$*-name* $\dot{\mathcal{I}}$ *satisfying that* $\Vdash_{\mathbb{P}_\alpha}$ *"*$\dot{\mathcal{I}} \subset \{\dot{a}_\beta : \beta < \alpha\}^\perp$ *is an ideal".*

Proof of Theorem 2.6. We simply adapt the proof of Lemma 3.10 so as to ensure that, for each $\alpha < \kappa$,

$$(\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle), \{\dot{a}_\beta : \beta < \alpha\}, \{\dot{\mathcal{I}}_\beta : \beta < \alpha\}) \text{ is in } \mathfrak{A}\ .$$

The only change to the proof is that when $\beta < \kappa$ is a limit with countable cofinality, the definition of $\dot{\mathcal{I}}_\beta$ is equal to the union of the sequence $\{\dot{\mathcal{I}}_{\xi+1} : \xi < \beta\}$. With this change, we recursively have ensured that our iterations remain in $\mathfrak{A}$, and by Corollary 3.15, our iteration is ccc. All the details showing that the closure, $\dot{A}$, of the union of the chain $\{\dot{a}^*_\alpha : \alpha < \kappa\}$ is forced to be almost clopen go through as in Lemma 3.10. Similarly, it follows by recursion that the initial family $\{b_\zeta : \zeta < \lambda\}$ is a $\subset^*$-unbounded subfamily of the ideal $\{\dot{a}_\alpha : \alpha < \kappa\}^\perp$. $\square$

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