S-SPACES AND LARGE CONTINUUM

ALAN DOW AND SAHARON SHELAH

Abstract. We prove that it is consistent with large values of the continuum that there are no S-spaces. We also show that we can also have that compact separable spaces of countable tightness have cardinality at most the continuum.

1. Introduction

An S-space is a regular hereditarily separable space that is not Lindelöf. If an S-space exists it can be assumed to be a topology on ω_1 in which initial segments are open [12]. The continuum hypothesis implies that S-spaces exist [9] and the existence of a Souslin tree implies that S-spaces exist [15]. Therefore it is consistent with any value of $\mathfrak{c}$ that S-spaces exist. Todorcevic [17] proved the major result that it is consistent with $\mathfrak{c} = \aleph_2$ that there are no S-spaces. He also remarks that this follows from PFA. We prove that it is consistent with arbitrary large values of $\mathfrak{c}$ that there are no S-spaces. Our method adapts the approach used in [17] and incorporates ideas, such as *the Cohen real trick* in Lemma 2.15, first introduced in [1, 2].

The outline of the proof (of Theorem 4.3) is that we choose a regular cardinal κ in a model of GCH. We construct a preparatory mixed support iteration sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa,\ \beta < \kappa\rangle$ consisting of iterands that are Cohen posets and cardinal preserving subposets of Jensen's poset for adding a generic cub. Following methods first introduced in [13], but more closely those of [17], the poset P_κ is shown to be cardinal preserving. We then extend the iteration sequence to one of length $\kappa+\kappa$ with iterands that are ccc posets of cardinality less than κ. These iterands are the same as those used in [17]. For cofinally many $\beta < \kappa$, $\dot{Q}_{\kappa+\beta}$ is constructed so as to add an uncountable discrete subset to a P_β-name of an S-space. The bookkeeping is routine to ensure that

Date: June 27, 2022.

2020 *Mathematics Subject Classification.* 54A35, 03E35 .

Key words and phrases. S-spaces, forcing .

The research of the second author was supported by the United States-Israel Binational Science Foundation (BSF Grant no. 1838/19), and by the NSF grant No. NSF-DMS 1833363. Paper 1228 on Shelah's list.

$P_{\kappa+\kappa}$ forces there are no S-spaces. The challenging part of the proof is to prove that these $\dot{Q}_\beta$ ($\kappa \le \beta < \kappa+\kappa$) are ccc in this new setting. In the final section, we use similar techniques to produce a model in which compact separable spaces of countable tightness have cardinality at most $\mathfrak{c}$.

2. Constructing P_κ

Throughout the paper we assume that GCH holds and that $\kappa > \aleph_2$ is a regular uncountable cardinal.

Definition 2.1. *The Jensen poset $\mathfrak{J}$ is the set of pairs (a, A) where a is a countable closed subset of ω_1 and $A \supset a$ is an uncountable closed subset of ω_1. The condition (a, A) is an extension of $(b, B) \in \mathfrak{J}$ providing a is an end-extension of b and $A \subset B$.*

We use $\mathbf{E}$ to denote the set $\{\lambda + 2k : \lambda < \kappa \text{ a limit}, k \in \omega\}$. We also choose a family $\mathfrak{I} = \{I_\gamma : \gamma \in \mathbf{E}\}$ of subsets of κ such that, for each $\mu < \gamma \in \mathbf{E}$

1. $\gamma \in I_\gamma \subset \gamma + 1$ and $|I_\gamma| \le \aleph_1$,
2. if $\gamma < \omega_2$, then $I_\gamma = \gamma + 1$,
3. if $\mu \in I_\gamma \cap \mathbf{E}$, then $I_\mu \subset I_\gamma$
4. for all $I \in [\kappa]^{\aleph_1}$, the set $\{\gamma : I \subset I_\gamma\}$ is unbounded in κ.

Say that a set $I \subset \kappa$ is $\mathfrak{I}$-saturated if it satisfies that $I_\mu \subset I$ for all $\mu \in I \cap \mathbf{E}$. Of course, each $I_\gamma \in \mathfrak{I}$ is $\mathfrak{I}$-saturated.

Definition 2.2. ***A.*** *We define a mixed support iteration sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \le \kappa,\ \ \beta < \kappa\rangle$:*

(1) $P_0 = \emptyset$,

(2) $p \in P_\alpha$ *is a function with* $\mathrm{dom}(p)$, *a countable subset of* α, *such that* $\mathrm{dom}(p) \cap \mathbf{E}$ *is finite,*

(3) for all $p \in P_\alpha$ *and* $\beta \in \mathrm{dom}(p)$, $p(\beta)$ *is a* P_β*-name forced by* 1_{P_β} *to be an element of* $\dot{Q}_\beta$,

(4) the support of a P_α*-name* τ, $\mathrm{supp}(\tau)$, *is defined, by recursion on* α *to be the union of the set* $\{\mathrm{supp}(\sigma) \cup \mathrm{dom}(q) : (\sigma, q) \in \tau\}$,

(5) for $\alpha \in \mathbf{E}$, $\dot{Q}_\alpha$ *is the trivial* P_α*-name for* $\mathcal{C}_{\omega_1} = \mathrm{Fn}(\omega_1, 2)$ *(i.e. each element of* $\dot{Q}_\alpha$ *has empty support),*

(6) for $\alpha \in \mathbf{E}$, $\dot{Q}_{\alpha+1}$ *is the subposet of the standard* $P_{\alpha+1}$*-name for* $\mathfrak{J}$ *consisting of the* $P_{\alpha+1}$*-names that are forced to have the form* $(\dot{a}, \dot{A})$ *where* $\mathrm{supp}(\dot{a}) \subset \mathbf{E} \cap I_\alpha$, $\mathrm{supp}(\dot{A}) \subset \alpha$, *and* $1_{P_{\alpha+1}}$ *forces that* $(\dot{a}, \dot{A}) \in \mathcal{J}$. $\dot{Q}_{\alpha+1}$ *is chosen so as to be sufficiently rich in names in the sense that if* $p \in P_{\alpha+1}$ *and* $\dot{q}$ *is a* $P_{\alpha+1}$*-name*

such that $p \Vdash_{P_\alpha} \dot{q} \in \dot{Q}_{\alpha+1}$*, then there is a* $\dot{q}_1 \in \dot{Q}_{\alpha+1}$ *such that* $p \Vdash \dot{q} = \dot{q}_1$*.*

B. *For each* $\alpha \in \mathbf{E}$*, we let* $\dot{C}_\alpha$ *denote the* $P_{\alpha+2}$*-name of the generic subset of* ω_1 *added by* $\dot{Q}_{\alpha+1}$*.*

Remark 1. Since we defined the family $\mathcal{I}$ to have the property that $I_\gamma = \gamma + 1$ for all $\gamma \in \omega_2 \cap \mathbf{E}$, it follows that P_{ω_2} is isomorphic to that used in [17]. It also follows that for all $\beta \in \omega_2 \cap \mathbf{E}$, $P_{\beta+1} \Vdash \dot{Q}_{\beta+1}$ is countably closed. We necessarily lose this property for $\omega_2 \le \beta$ for any family $\mathcal{I}$ satisfying our properties (1)-(4). Nevertheless, our development of the properties of P_κ will closely follow that of [17].

Remark 2. We prove in Lemma 2.13 that, for each $\alpha \in \mathbf{E}$, $\dot{C}_\alpha$ is forced, as hoped, to be a cub. However, even though, for $\beta \ge \omega_2$, $P_{\beta+1}$ does not force that $\dot{Q}_{\beta+1}$ is countably closed, we make note of subsets of the iteration sequence that have special properties, such as in Lemma 2.9.

For any ordered pair (a, b), let $\pi_0((a, b)) = a$ and $\pi_1((a, b)) = b$. For convenience, for an element v of V and any $\alpha < \kappa$, we identify the usual trivial P_α-name for v with v itself. In particular, if $s \in \mathcal{C}_{\omega_1}$ and $\alpha \in \mathbf{E}$, then $s \in \dot{Q}_\alpha$. Similarly, if $(\dot{a}, \dot{A})$ is a pair of the form specified in Definition 2.2(6), then again $(\dot{a}, \dot{A})$ can be regarded as an element of $\dot{Q}_{\alpha+1}$. We will say that a P-name τ for a subset of an ordinal λ and poset P is canonical if it is a subset of $\lambda \times P$ (and optionally, if $\{p : (\alpha, p) \in \tau\}$ is an antichain for all $\alpha \in \lambda$). Let $\mathcal{D}_\beta$ denote the set of canonical P_β-names of closed and unbounded subsets of ω_1.

Definition 2.3. *For each* $\alpha < \kappa$*, let* P'_α *denote the subset of* P_α*, where* $p \in P'_\alpha$ *providing for all* $\beta \in \mathrm{dom}(p) \cap \mathbf{E}$*,* $p(\beta)$ *is, literally, an element of* $\mathcal{C}_{\omega_1}$*.*

Lemma 2.4. *For all* $\alpha \le \kappa$*,* P'_α *is a dense subset of* P_α*.*

Proof. Assume $\alpha \le \kappa$ and that, by induction, P'_β is a dense subset of P_β for all $\beta < \alpha$. Consider any $p \in P_\alpha$. If α is a limit, choose any $\beta < \alpha$ such that $\mathrm{dom}(p) \cap \mathbf{E} \subset \beta$. Choose any $p' \in P'_\beta$ so that $p' < p \restriction \beta$. We then have that $p' \cup p \restriction (\alpha \setminus \beta)$ is a condition in P_α that is below p.

Now let $\alpha = \beta + 1$. If $\beta \in \mathbf{E}$, then choose $p' \in P'_\beta$ so that there is an $s \in \mathcal{C}_{\omega_1}$ such that $p' \Vdash_{P_\beta} p(\beta) = s$. Then the desired extension of p in P'_α is $p' \cup \langle \beta, s \rangle$. Similarly, if $\beta \notin \mathbf{E}$ and $p' \in P'_\beta$ with $p' < p \restriction \beta$, then $p' \cup \langle \beta, p(\beta) \rangle \in P'_\alpha$. □

Proposition 2.5. *If* $p \in P_\kappa$ *then for every* $I \subset \kappa$*,* $p \restriction I \in P_\kappa$ *and* $p \le p \restriction I$*.*

Definition 2.6. *For a subset $I \subset \kappa$ and $\alpha \leq \kappa$, let $P_\alpha(I)$ denote the subset $\{p \in P'_\alpha : \mathrm{dom}(p) \subset I\}$.*

Recall that for posets $(P, <_P)$ and $(R, <_R)$, P is a complete subposet of R, i.e. $P \subset_c R$, providing

1. $P \subset R$, $<_P = <_R \cap (P \times P)$,
2. $\perp_P = \perp_R \cap (P \times P)$, where $\perp$ is the incompatibility relation,
3. for each $r \in R$, the set of projections, $\mathrm{proj}_P(r)$, is not empty, where $\mathrm{proj}_P(r) = \{p \in P : (\forall q \in P)(q <_P p \Rightarrow q \not\perp_R r)\}$.

If $P \subset_c R$, then R/P is often used to denote the P-name of the poset satisfying that $R \simeq P \star R/P$. In fact, R/P can be defined so that simply if $G \subset P$ is a generic filter, then $\mathrm{val}_G(R/P) = \{r \in R : \mathrm{proj}_P(r) \cap G \neq \emptyset\}$ with the ordering inherited from $<_R$. With this view, $\mathrm{val}_G(R/P) = G^+$ where, as is standard, $G^+ = \{r \in R : (\forall p \in G) r \not\perp p\}$. Of course it follows that for $\beta < \alpha \leq \kappa$, $P_\beta \subset_c P_\alpha$.

It is clear that $P_\alpha(\mathbf{E})$ is isomorphic to (the usual dense subset of) a finite support iteration of the Cohen poset $\mathcal{C}_{\omega_1}$.

Proposition 2.7. *For each $\alpha \leq \kappa$, the set $P_\alpha(\mathbf{E}) \subset_c P_\alpha$ and is ccc.*

Definition 2.8. *For each $\alpha \in \mathbf{E}$, let $Q'_{\alpha+1}$ be the subset of $\dot{Q}_{\alpha+1}$ consisting of those pairs $(\dot{a}, \dot{A})$ as in Definition 2.2(6).*

We may note that, for each $(\dot{a}, \dot{A}) \in Q'_{\alpha+1}$, $\dot{a}$ is a $P_{\alpha+1}(I_\alpha \cap \mathbf{E})$-name and $\dot{A}$ is a P_α-name that is forced by 1_{P_α} to be a cub subset of ω_1. Also, for every $p \in P_{\alpha+1}$, $p \restriction \alpha \Vdash p(\alpha+1) \in Q'_{\alpha+1}$.

Lemma 2.9. *If $\alpha \in \mathbf{E}$ and, and $\{(\dot{a}_n, \dot{A}_n) : n \in \omega\} \subset Q'_{\alpha+1}$ is a sequence that satisfies, for each $n \in \omega$, $1 \Vdash_{P_{\alpha+1}} (\dot{a}_{n+1}, \dot{A}_{n+1}) \leq (\dot{a}_n, \dot{A}_n)$, then there is a condition $(\dot{a}, \dot{A}) \in Q'_{\alpha+1}$ such that*

(1) $1 \Vdash_{P_{\alpha+1}}$ forces that $\dot{a}$ is the closure of $\bigcup\{\dot{a}_n : n \in \omega\}$,
(2) 1_{P_α} forces that $\dot{A}$ equals $\bigcap\{\dot{A}_n : n \in \omega\}$,
(3) $1 \Vdash_{P_{\alpha+1}}$ forces that $(\dot{a}, \dot{A}) = \bigwedge\{(\dot{a}_n, \dot{A}_n) : n \in \omega\}$.

Proof. In the forcing extension by a $P_{\alpha+1}$-generic filter G, it is clear that $(\mathrm{cl}\,(\bigcup\{\mathrm{val}_G(\dot{a}_n)\}), \bigcap\{\mathrm{val}_G(\dot{A}_n) : n \in \omega\})$ is the meet in $\mathcal{J}$ of the sequence $\{(\mathrm{val}_G(\dot{a}_n), \mathrm{val}_G(\dot{A}_n)) : n \in \omega\}$. We just have to be careful about the supports of the names for these objects. Each $\dot{a}_n$ is a $P_{\alpha+1}(I_\alpha)$-name and so it is clear that there is a $P_{\alpha+1}(I_\alpha \cap \mathbf{E})$-name, $\dot{a}$, such that $1 \Vdash_{P_{\alpha+1}} \dot{a} = \mathrm{cl}\,(\bigcup\{\dot{a}_n : n \in \omega\})$. This is the only subtle point. Any P_α-name, $\dot{A}$, for $\bigcap\{\dot{A}_n : n \in \omega\}$ is adequate (although we are using that each $\dot{A}_n$ is a P_α-name forced by 1 to be a cub). $\square$

When we have a sequence $\{(\dot{a}_n, \dot{A}_n) : n \in \omega\} \subset \dot{Q}'_{\alpha+1}$ as in the hypothesis of Lemma 2.9, we will use $\bigwedge\{(\dot{a}_n, \dot{A}_n) : n \in \omega\}$ to denote the element $(\dot{a}, \dot{A})$ in the conclusion of the Lemma.

Let $<_E$ denote the relation on P_κ defined by $p_1 <_E p_0$ providing

(1) $p_1 \leq p_0$,
(2) $p_1 \restriction \mathbf{E} = p_0 \restriction \mathbf{E}$,
(3) for $\beta \in \mathrm{dom}(p_0)$, $\mathbf{1}_{P_\beta} \Vdash p_1(\beta) < p_0(\beta)$.

For $r \in P_\kappa(\mathbf{E})$ and compatible $p \in P_\kappa$, let $p \wedge r$ denote the condition with domain $\mathrm{dom}(p) \cup \mathrm{dom}(r)$ satisfying $(p \wedge r)(\beta) = p(\beta) \cup r(\beta)$ for $\beta \in \mathrm{dom}(r)$ and $(p \wedge r)(\beta) = p(\beta)$ for $\beta \in \mathrm{dom}(p) \setminus \mathrm{dom}(r)$. For convenience, let $p \wedge r$ equal p if $r \in P_\kappa$ is not compatible with p.

Lemma 2.10. *Assume that $\{p_n : n \in \omega\} \subset P'_\kappa$ is a $<_E$-descending sequence. Then there is a $p_\omega \in P'_\kappa$ such that $\mathrm{dom}(p) = \bigcup_n \mathrm{dom}(p_n)$ and $p_\omega <_E p_n$ for all $n \in \omega$.*

Proof. We let $J = \bigcup\{\mathrm{dom}(p_n) : n \in \omega\}$. We define $p_\omega \restriction \beta$ by induction on $\beta \in \mathbf{E}$ so that $\mathrm{dom}(p_\omega \restriction \beta) = J \cap \beta$. For limit α, simply $p_\omega \restriction \alpha = \bigcup_{\beta<\alpha} p_\omega \restriction \beta$. If $p_\omega \restriction \beta <_E p_n \restriction \beta$ for all $n \in \omega$ and $\beta < \alpha$, then we have $p_\omega \restriction \alpha <_E p_n \restriction \alpha$ for all $n \in \omega$. Now let $\alpha = \beta + 2$ with $\beta \in \mathbf{E}$ and assume that we have defined $p_\omega \restriction \beta$ as above. If $\beta \in J$, then let $p_\omega(\beta) = p_0(\beta)$. If $\beta + 1 \in J$, then $\mathbf{1}_{P_{\beta+1}}$ forces that $\{p_n(\beta + 1) : n \in \omega\}$ is a descending sequence in $\dot{Q}_{\beta+1}$. We define $p_\omega(\beta + 1)$ to equal $\bigwedge\{p_n(\beta + 1) : n \in \omega\}$. It follows by the definition of $\bigwedge\{p_n(\beta + 1) : n \in \omega\}$, that $\mathbf{1}_{P_{\beta+1}} \Vdash p_\omega(\beta + 1) < p_n(\beta + 1)$ for all $n \in \omega$. ☐

Lemma 2.11. *For every $p_0 \in P'_\kappa$ and dense subset D of P_κ, there is a $p <_E p_0$ satisfying that the set $D \cap \{p \wedge r : r \in P_\kappa(\mathbf{E})\}$ is predense below p. Moreover, there is a countable subset of $D \cap \{p \wedge r : r \in P_\kappa(\mathbf{E})\}$ that is predense below p.*

Proof. Let $r_0 = p_0 \restriction \mathbf{E}$. There is nothing to prove if $p_0 \in D$ so assume that it is not. By induction on $0 < \eta < \omega_1$, we choose, if possible, conditions p_η, r_η such that, for all $\zeta < \eta$:

(1) $p_\zeta <_E p_\eta$ and $r_\zeta < r_0$,
(2) $p_\zeta \wedge r_\zeta \in D$,
(3) $(p_\eta \wedge r_\eta) \perp (p_\zeta \wedge r_\zeta)$.

Suppose that we have so chosen $\{p_\zeta, r_\zeta : \zeta < \eta\}$. Let $L_\eta = \bigcup\{\mathrm{dom}(p_\zeta) : \zeta < \eta\}$. If $\eta = \beta+1$, let $\bar{p}_\eta = p_\beta$. If η is a limit, then let $\bar{p}_\eta$ be a condition as in Lemma 2.10 for some cofinal sequence in η. If $\{p_\zeta \wedge r_\zeta : \zeta < \eta\}$ is predense below $\bar{p}_\eta$, we halt the induction and set $p = \bar{p}_\eta$. Otherwise we choose any $p_\eta <_E \bar{p}_\eta$ and an $r_\eta \supset r_0$ so that $p_\eta \wedge r_\eta$ in D. The induction will halt for some $\eta < \omega_1$ since the family $\{r_\zeta : \zeta < \eta\}$ is evidently an antichain in $P_\kappa(\mathbf{E})$. □

Corollary 2.12. *For each $\beta \in \mathbf{E}$, P_β is proper and $P_\beta/P_\beta(\mathbf{E}\cap\beta)$ does not add any reals.*

Proof. Let $P_\beta \in M$ where M is a countable elementary submodel of $H(\kappa^+)$. Let $\{D_n : n \in \omega\}$ be an enumeration of the dense open subsets of P_β that are members of M. By Lemma 2.11, we have that for each $q \in P_\beta \cap M$ and $n \in \omega$, there is a $\bar{q} <_E q$ also in $P_\beta \cap M$ so that $D_n \cap \{\bar{q} \wedge r : r \in P_\beta(\mathbf{E}) \cap M\}$ is predense below $\bar{q}$. Let $M \cap \omega_1 = \delta$. Fix any $p_0 \in P_\beta \cap M$. By a simple recursion, we may construct a $<_E$-descending sequence $\{p_n : n \in \omega\} \subset M$ so that, for each n, $D_n \cap \{p_{n+1} \wedge r : r \in P_\beta(\mathbf{E}) \cap M\}$ is predense below p_{n+1}. By Lemma 2.10, we have the (P_β, M)-generic condition p_ω. It is clear that for each P_β-name $\tau \in M$ for a subset of ω, p_ω forces that τ is equal to a $P_\beta(\mathbf{E})$-name. This implies that $P_\beta/P_\beta(\mathbf{E}\cap\beta)$ does not add reals. □

We can now prove that $P_{\beta+2}$ does indeed force that $\dot{C}_\beta$ is a cub.

Lemma 2.13. *For each $\beta \in \mathbf{E}$, $P_{\beta+2}$ forces that $\dot{C}_\beta$ is unbounded in ω_1.*

Proof. Let $p \in P_{\beta+2}$ be any condition and let $\gamma \in \omega_1$. By possibly strengthening p we can assume that $p(\beta+1) \in Q'_{\beta+1}$. We find $q < p$ so that $q \Vdash \dot{C}_\beta \setminus \gamma$ is not empty. Let $p, P_{\beta+2}$ be members of a countable elementary submodel $M \prec H(\kappa^+)$. Let $\bar{p} < p \restriction \beta$ be (P_β, M)-generic and let $\dot{D} = \pi_1(p(\beta+1)) \in \mathcal{D}_\beta$. Since $p, \dot{D}$ are members of M and p forces that $\dot{D}$ is a cub, it follows that $\bar{p} \Vdash \delta \in \dot{D}$. It also follows that $\bar{p} \Vdash \dot{a} \subset \dot{D} \cap \delta$. Let $\dot{a}_1$ be the $P_{\beta+1}$-name that has support equal to the support of the name $\dot{a}$ and satisfies that $\mathbf{1}_{P_{\beta+1}} \Vdash \dot{a}_1 = \dot{a} \cup \{\delta\}$. Let $\dot{E}$ be the P_β-name for $\dot{D} \cup \{\delta\}$ and notice that, given that $(\dot{a}, \dot{D}) \in Q'_{\beta+1}$, we have that $(\dot{a}_1, \dot{E})$ is also in $Q'_{\beta+1}$. Now let $q \in P_{\beta+2}$ be defined according to $q \restriction \beta = \bar{p}$, $q(\beta) = p(\beta)$, and $q(\beta+1) = (\dot{a}_1, \dot{E})$. It is immediate that $q \restriction \beta+1 < p \restriction \beta+1$. Also, $q \restriction \beta+1$ forces that $\dot{a}$ is an initial segment of $\dot{a}_1$, that $\dot{a}_1 \subset \dot{D}$, and that $\dot{E} \subset \dot{D}$. Therefore, $q < p$ and $q \Vdash \delta \in \dot{C}_\beta$. □

Lemma 2.14. *For each $\beta \leq \kappa$, P_β satisfies the $\aleph_2$-cc.*

Proof. We prove the lemma by induction on β. If $\beta \in \mathbf{E}$ and P_β satisfies the $\aleph_2$-cc, then it is trival that $P_{\beta+1}$ does as well. Similarly $P_{\beta+2}$ satisfies the $\aleph_2$-cc since $P_{\beta+1} \star Q'_{\beta+1}$ clearly does, and this poset is dense in $P_{\beta+2}$. The argument for limit ordinals β with cofinality less than ω_2 is straightforward, so we assume that β is a limit with cofinality greater than ω_1. Let $\{p_\gamma : \gamma \in \omega_2\}$ be a subset of P'_β. Choose any elementary submodel M of $H(\kappa^+)$ such that $\{p_\gamma : \gamma \in \omega_2\} \in M$, $|M| = \aleph_1$, and $M^\omega \subset M$. Let $M \cap \omega_2 = \lambda$ and let $I = \mathrm{dom}(p_\lambda) \cap M$ and fix any $\mu \in M \cap \beta$ so that $I \subset \mu$. For each $\beta \in \mathbf{E}$ such that $\beta + 1 \in I$, let $\dot{a}_\beta \in M$ so that $\pi_0(p_\lambda(\beta+1)) = \dot{a}_\beta$. That is, $p_\lambda(\beta) = (\dot{a}_\beta, \dot{D}_\beta)$ for some $\dot{D}_\beta \in \mathfrak{D}_\beta$. Clearly the countable sequence $\{\dot{a}_\beta : \beta \in I \cap \mathbf{E}\}$ is an element of M. Therefore there is a $\gamma \in M$ so that $\mathrm{dom}(p_\gamma) \cap \mu = I$ and so that $\pi_0(p_\gamma(\beta+1)) = \dot{a}_\beta$ for all $\beta \in \mathbf{E}$ such that $\beta + 1 \in I$. It follows that $p_\gamma \not\perp p_\lambda$. □

Now we discuss the Cohen real trick, which, though simple and powerful, is burdened with cumbersome notation.

Lemma 2.15. *Let $\alpha \in \mathbf{E}$ and let $p_0 \in P_{\alpha+2} \in M$ be a countable elementary submodel of $H(\kappa^+)$ and let $\delta = M \cap \omega_1$. There is a $(P_{\alpha+2}, M)$-generic condition $p_1 < p_0$ satisfying that for all P_α-generic filters satisfying $p_1 \restriction \alpha \in G_0$ and $\dot{Q}_\alpha$-generic filters $p_1(\alpha) \in G_1$, the collection, in $V[G_0 \star G_1]$,*

$$p^{\uparrow}_{1\alpha} = \{p(\alpha+1) : p \in M \cap P_{\alpha+2},\ p \restriction (\alpha+1) \in G_0 \star G_1,\ p_1 < p\}$$

is $\mathrm{val}_{G_0 \star G_1}(\dot{Q}_{\alpha+1} \cap M)$-generic over $V[G_0 \star (G_1 \restriction \delta)]$.

Moreover, for any P_α-name $\dot{Q}$ of a ccc poset and $P_\alpha \star \dot{Q}$-generic filter $G_0 \star G_2$, $p^{\uparrow}_{1\alpha}$ is also generic over the model $V[G_0 \star G_2][G_1 \restriction \delta]$.

Proof. Let $\dot{Q}$ be any P_α-name of a ccc poset. Choose any $\bar{p}_1 < p_0 \restriction (\alpha+1)$ that is (M, P_α)-generic with $\bar{p}_1(\alpha) = p(\alpha)$. We will let $p_1 \restriction \alpha = \bar{p}_1 \restriction \alpha$ and then we simply have to choose a value for $p_1(\alpha+1)$. We may assume that $\bar{p}_1 \restriction \mathbf{E} = p_0 \restriction \mathbf{E}$. Let $\tilde{G}$ denote the filter $(G_0 \star G_1) \cap P_{\alpha+1}(I_\alpha \cap \mathbf{E})$ and let $R = (M \cap \dot{Q}_{\alpha+1})/\tilde{G}$. For $r \in R$ we may regard r in the extension $V[\tilde{G}]$ to have the form $(a_r, \dot{A}_r)$, with $a_r \subset \omega_1$, because, for each $(\dot{a}, \dot{A}) \in M \cap \dot{Q}_{\alpha+1}$, $\dot{a}$ has support contained in $P_{\alpha+1}(I_\alpha \cap \mathbf{E})$. We have no such reduction for $\dot{A}$. We adopt the subordering, $<_R$, on R where $(a, \dot{A}) <_R (b, \dot{B})$ in R will mean that $\mathbf{1}_{P_{\alpha+1}} \Vdash \dot{A} \subset \dot{B}$. The fact that $(a, \dot{A}) \in R$ already means that $\mathbf{1}_{P_{\alpha+1}} \Vdash a \subset \dot{A}$. If $p \in M \cap P_{\alpha+1}$ and $(a, \dot{A}_1) \in R$ is such that $p \Vdash (a, \dot{A}_1) < (b, \dot{B})$, then there is an $(a, \dot{A}) \in R$ such that $p \Vdash \dot{A} = \dot{A}_1$ and $(a, \dot{A}) <_R (b, \dot{B})$.

The quotient poset $(R/\tilde{G}, <_R)$ is isomorphic to $\mathcal{C}_\omega$. Let $\psi \in V[\tilde{G}]$ be an isomorphism from $\mathcal{C}_{(\delta,\delta+\omega)}$ to $(R/\tilde{G}, <_R)$. We regard $\mathcal{C}_{(\delta,\delta+\omega)}$ as the canonical subposet of $\dot{Q}_\alpha$ and let G^δ_α denote a generic filter for this subposet of $\dot{Q}_\alpha$. Now we have, in the extension $V[\tilde{G}][G^\delta_\alpha]$, a $<_R$-filter $R^\delta_\alpha \subset R$ given by $\{\psi(\sigma) : \sigma \in G^\delta_\alpha\}$. Let $a_\omega = \{\delta\} \cup \bigcup\{a_r : r \in R^\delta_\alpha\}$.

Note that $\bar{p}_1$ forces that $\delta \in \dot{C}$ for all $\dot{C} \in M \cap \mathcal{D}_\alpha$. By the construction, it follows that we may fix a $P_{\alpha+1}$-name, $\dot{a}_\omega$, for a_ω, that has support contained in $I_\alpha \cap \mathbf{E}$. Let $\dot{A}_\omega$ be the $P_{\alpha+1}$-name satisfying that $\bar{p}_1$ forces that $\dot{A}_\omega$ equals the intersection of all $\dot{C} \in \mathcal{D}_\alpha \cap M$ such that $\dot{a}_\omega \subset \dot{C}$. It follows that for $r \in R^\delta_\alpha$ and $\tilde{p} \restriction \alpha + 1 < \bar{p}_1$, $\tilde{p}(\alpha) \in G^\delta_\alpha$, and $\tilde{p}(\alpha + 1) = r$, we have that $\tilde{p} \wedge r \Vdash \dot{A}_\omega \subset \dot{A}_r$ (and this takes place in $V[\tilde{G}]$). We may choose $\dot{A}_\omega$ so that $\tilde{p} \Vdash \dot{A}_\omega = \omega_1$ for all $\tilde{p} \perp \bar{p}_1$ in $P_{\alpha+1}$. It then follows that $(\dot{a}_\omega, \dot{A}_\omega)$ is an element of $\dot{Q}_{\alpha+1}$. We now define p_1 so that $p_1 \restriction \alpha + 1 = \bar{p}_1$ and $p_1(\alpha + 1) = (\dot{a}_\omega, \dot{A}_\omega)$. The fact that p_1 is $(M, P_{\alpha+2})$-generic follows from the stronger claim below.

Claim 3. Let G_0 be a P_α-generic with $\bar{p}_1 \restriction \alpha \in G_0$ and let G_1 be a $\dot{Q}_\alpha$-generic filter with $\bar{p}_1(\alpha) \in M \cap G_1$. Also let $G_0 \star G_2$ be $P_\alpha \star \dot{Q}$-generic. Let $\sigma \in \mathcal{C}_{(\delta,\delta+\omega)}$ be arbitrary. Let $\dot{D}$ be a $P_{\alpha+1} \star \dot{Q}$-name of a dense subset of $\mathrm{val}_{G_0\star G_1}(\dot{Q}_{\alpha+1} \cap M)$. Then there is a $\tau \supset \sigma$ such that $\tau \Vdash p^{\uparrow}_{1\alpha} \cap \mathrm{val}_{G_0\star(G_1\times G_2)}(\dot{D}) \neq \emptyset$.

Proof of Claim. Fix the generic filter $\tilde{G} \subset G_0 \star G_1$ as used in the construction of $(\dot{a}_\omega, \dot{A}_\omega)$ and let $\psi : \mathcal{C}^\delta_\alpha \to (R/\tilde{G}, <_R)$ denote the above mentioned isomorphism. Let $(b, \dot{B}) = \psi(\sigma)$ and, using the density of $\mathrm{val}_{G_0\star(G_1\times G_2)}(\dot{D})$, choose $(a, A) < (b, \mathrm{val}_{G_0\star G_1}(\dot{B}))$, so that $(a, A) \in \mathrm{val}_{G_0\star(G_1\times G_2)}(\dot{D})$. By elementarity, choose $(\dot{a}, \dot{A}) \in M \cap \dot{Q}_{\alpha+1}$ such that $\mathrm{val}_{G_0\star G_1}((\dot{a}, \dot{A})) = (a, A)$. Again by elementarity and using that $\bar{p}_1$ is $(M, P_{\alpha+1})$-generic, there is a $p \in M \cap (G_0 \star G_1)$ such that $p \Vdash \dot{A} \subset \dot{B}$. Now choose $\tau \supset \sigma$ so that $\psi(\tau) = (a, \dot{A}_1)$ satisfies that $(a, \dot{A}_1) <_R (b, \dot{B})$ and $p \Vdash \dot{A}_1 = \dot{A}$. It follows that $\tau \Vdash (a, \dot{A}_1) \in \mathrm{val}_{G_0\star(G_1\times G_2)}(\dot{D})$. Since $p_1 \wedge \tau$ also forces that $p_1(\alpha + 1) < (a, \dot{A}_1)$ we have that $p_1 \wedge \tau \Vdash (a, \dot{A}_1) \in p^{\uparrow}_{1\alpha}$. □

This completes the proof of the Lemma. □

Lemma 2.16. *Let $\lambda < \kappa$ with $\lambda \in \mathbf{E}$ and let $\dot{Q}$ be a P_λ-name of a ccc poset. Then P_κ forces that $\dot{Q}$ is ccc.*

Proof. Let G be a P_λ-generic filter and let $Q = \mathrm{val}_G(\dot{Q})$. Since P_κ satisfies the $\aleph_2$-cc, we can assume that Q is of the form $(\omega_1, <_Q)$. We work in the extension $V[G]$ and we view, for each $\lambda < \alpha \leq \kappa$, $\bar{P}_\alpha = P_\alpha/G$ as a subset of P_α. We prove, by induction on $\lambda \leq \alpha \in \mathbf{E}$, that for any countable elementary submodel $\{Q, \lambda, \bar{P}_\alpha\} \in M$ and any $p \in \bar{P}_\alpha \cap M$, there is a $p_M <_E p$ such that $(1_Q, p_M)$ is $(M, Q \times \bar{P}_\alpha)$-generic. Note that this inductive hypothesis, i.e. the fact that it is $(1_Q, p_M)$ that is the generic condition rather than (q, p_M) for some other $q \in Q$, is equivalent to the statement that P_α preserves that Q is ccc.

The proof at limit steps follows the standard proof (as in [16]) that the countable support iteration of proper posets is proper. We feel that this can be skipped. So let $\alpha = \beta + 2$ for some $\beta \in \mathbf{E}$. Let M be a suitable countable elementary submodel and let $p \in P_\alpha \cap M$ (such that $p \restriction \lambda \in G$). Let $M \cap \omega_1 = \delta$. By the inductive hypothesis, we can assume that we have $\bar{p}_1 \in P_\beta$ so that, $\bar{p}_1 \restriction \lambda \in G$, $\bar{p}_1 <_E p \restriction \beta$ and so that $(1_Q, \bar{p}_1)$ is an $(M, Q \times P_\beta)$-generic condition. Of course it is also clear that $(1_Q, \bar{p}_1)$ is an $(M, Q \times P_{\beta+1})$-generic condition. Now let $p_1 \in P_{\beta+2}$ be chosen as in Lemma 2.15. That is, p_1 is chosen so that for any P_β-generic filter $G_\beta \supset G$ with $p_1 \restriction \beta \in G_\beta$, any $\mathcal{C}_{\omega_1}$-generic G_1 with $p_1(\beta) \in G_1$, and, since Q is ccc in $V[G_\beta]$, any Q-generic filter G_Q, we have that $p_{1\beta}^{\uparrow}$ is generic over $V[G_\beta \star (G_1 \times G_Q)]$. Let $G_{\beta+1} = G_\beta \star G_1$.

Let $D \in M$ be any dense open subset of $P_{\beta+2} \star Q$. Let R denote $\dot{Q}_{\beta+1}/(G_\beta \star G_1)$. It follows that $D/(G_\beta \star G_1)$ or

$$E = \{(r, q) : (\exists d \in D)\ (d{\restriction}\beta{+}1 \in G_\beta \star G_1 \ \&\ d = d{\restriction}\beta{+}1 \star (r, q))\}$$

is a dense open subset of $R \times Q$ and $E \in M[G_{\beta+1}]$. By standard product forcing theory, we have that for each $r \in R$, $E_r = \{q \in Q : (\exists s \in R)(s < r \ \&\ (s, q) \in E\})$ is a dense subset of Q. For each $r \in R \cap M[G_{\beta+1}]$, $E_r \in M[G_{\beta+1}]$ and so, $E_r \cap M[G_{\beta+1}]$ is a predense subset of Q. This implies that, for each $\bar{q} \in Q$, the set $E(\bar{q}) = \{s \in R \cap M[G_{\beta+1}] : (\exists (s, q) \in E \cap M[G_{\beta+1}])(\bar{q} \not\perp q)\}$ is a dense subset of $R \cap M[G_{\beta+1}]$. Although $E(\bar{q})$ need not be an element of $M[G_{\beta+1}]$, it is an element of $V[G_\beta \star (G_1 \restriction \delta)]$. Therefore, by Lemma 2.15, $E(\bar{q}) \cap p_{1\beta}^{\uparrow}$ is not empty for all $\bar{q} \in G_Q$. By elementarity, it then follows that p_1 is an $(M, P_{\beta+2} \star Q)$-generic condition. □

3. S-space tasks

Following [1] and [17] we define a poset of finite subsets of ω_1 separated by a cub.

Definition 3.1. *For a family $\mathcal{U} = \{U_\xi : \xi \in \omega_1\}$ and a cub $C \subset \omega_1$, define the poset $Q(\mathcal{U}, C) \subset [\omega_1]^{<\aleph_0}$, to be the set of finite sets $H \subset \omega_1$ such that for $\xi < \eta$ both in H*

(1) $\xi \notin U_\eta$ and $\eta \notin U_\xi$,
(2) there is a $\gamma \in C$ such that $\xi < \gamma \leq \eta$.

$Q(\mathcal{U}, C)$ is ordered by $\supset$.

Definition 3.2. *A family $\mathcal{U} = \{U_\xi : \xi < \omega_1\}$ is an S-space task if it satisfies:*

(1) $\xi \in U_\xi \in [\omega_1]^{<\aleph_1}$,
(2) *every uncountable* $A \subset \omega_1$ *has a countable subset that is not contained in any finite union from the family* $\mathcal{U}$.

Remark 4. If $\mathcal{T}$ is a regular locally countable topology on ω_1 that contains no uncountable free sequence (see Definition 5.1), then each neighborhood assignment $\{U_\xi : \xi \in \omega_1\}$ consisting of open sets with countable closures, is an S-space task. An uncountable $A \subset \omega_1$ failing property (2) would contain an uncountable free sequence. Suppose that there is a cub $C \subset \omega_1$ such that $Q(\mathcal{U}, C)$ is ccc. Then, as usual, there is a $q \in Q(\mathcal{U}, C)$ such that any generic filter including q is uncountable. If $G \subset Q(\mathcal{U}, C)$ is a filter (even pairwise compatible), then $\bigcup G$ is a discrete subspace of $(\omega_1, \mathcal{T})$. Of course this cub C can be assumed to satisfy that if $\xi < \eta$ are separated by C, then $\eta \notin U_\xi$. This means that requirement (1) in the definition of $Q(\mathcal{U}, C)$ can be weakened to only require that $\xi \notin U_\eta$.

The following result is a restatement of Lemma 1 from [17]. It also uses the Cohen real trick. We present a proof that is more adaptable to the modifications needed for the consistency with $\mathfrak{c} > \aleph_2$.

Proposition 3.3. *Let* R *be a ccc poset and let* $\mathcal{U} = \{\dot{U}_\xi : \xi \in \omega_1\}$ *be a sequence of* R*-names such that* $\mathcal{U}$ *is forced to be an S-space task. Then* $R \times P_2$ *forces that for every* $n \in \omega$*, every uncountable pairwise disjoint subfamily* $\mathcal{H}$ *of* $Q(\mathcal{U}, \dot{C}_1) \cap [\omega_1]^n$*, has a countable subset* $\mathcal{H}_0$ *satisfying that, for some* $\delta \in \omega_1$ *and all* $F \in [\omega_1 \setminus \delta]^n$*, there is an* $H \in \mathcal{H}_0$ *such that* $H \cap \bigcup\{U_\xi : \xi \in F\} = \emptyset$*. In particular,* $R \times P_2$ *forces that* $Q(\mathcal{U}, \dot{C}_1)$ *is ccc.*

Proof. Of course P_2 is isomorphic to $\mathcal{C}_{\omega_1} \star \dot{\mathfrak{J}}$. Fix any $n \in \omega$ and let $\{\dot{H}_\xi : \xi \in \omega_1\}$ be $R \times P_2$-names of pairwise disjoint elements of $[\omega_1]^n \cap Q(\mathcal{U}, \dot{C}_1)$. Since we can pass to an uncountable subcollection of $\{\dot{H}_\xi : \xi \in \omega_1\}$ we may assume that for all $\xi \in \omega_1$, it is forced that there is a $\delta \in \dot{C}_1$ such that $\xi < \delta \le \min(\dot{H}_\xi)$.

For each $(r, p) \in R \times P_2$ and $H \in [\omega_1]^n$, let $\Gamma_\xi(H, (r, p))$ be the set $\{s \in R : (\exists q \in P_2)((s, q) < (r, p)\ \&\ (s, q) \Vdash H = \dot{H}_\xi)\}$. In other words, $\Gamma_\xi(H, (r, p))$ is not empty if and only if $(r, p) \nVdash H \ne \dot{H}_\xi$. We say that $\Gamma_\xi(H, (r, p))$ is ω_1-full simply if it is not empty.

Now we define what it means for $\Gamma_\xi(H, (r, p))$ to be ω_1-full for $H \in [\omega_1]^{n-1}$. We require that there is a set $\{\dot{\eta}_\zeta : \zeta \in \omega_1\}$ of canonical R-names such that $r \Vdash \dot{\eta}_\zeta \in \omega_1 \setminus \zeta$ and for $(\eta, s) \in \dot{\eta}_\zeta$, $s \le r$ and satisfies that $\Gamma_\xi(H \cup \{\eta\}, (s, p))$ is ω_1-full. It is worth noting that (r, p) has been changed to (s, p) rather than to some (s, q) with $q < p$. This definition generalizes to $H \in [\omega_1]^i$. We say that $\Gamma_\xi(H,(r,p))$ is ω_1-full if there is a set of canonical R-names $\{\dot\eta_\zeta : \zeta \in \omega_1\}$ such that, for each $\zeta \in \omega_1$, $r \Vdash \dot\eta_\zeta \in (\omega_1 \setminus \zeta)$, and for $(\eta, s) \in \dot\eta_\zeta$, $s \leq r$ and $\Gamma_\xi(H \cup \{\eta\},(s,p))$ is ω_1-full.

Claim 5. Suppose that $\Gamma_\xi(\emptyset,(r,p))$ is ω_1-full and that $M \prec H(\kappa^+)$ is countable and $\{\xi, \mathcal{U}, R, (r,p)\} \in M$. Then for any $\bar r < r \in R$ and finite $F \subset \omega_1 \setminus M$, there are $(s,q), H \in M$ such that

1. $(s,q) < (r,p) \in R \times P_2$,
2. $H \cap \bigcup\{\dot U_\zeta : \zeta \in F\}$ is empty,
3. $(s,q) \Vdash \dot H_\xi = H$,
4. $s \not\leq \bar r$.

Proof of Claim: Let $\dot W_F = \bigcup\{\dot U_\zeta : \zeta \in F\}$. Since $R \in M \prec H(\kappa^+)$ is ccc and forces that $\mathcal{U}$ is an S-space task, it follows that for each R-name $\dot A \in M$ for an uncountable subset of ω_1, the set $\dot A \cap M$ is forced to not be contained in $\dot W_F$. By induction on $1 \leq i \leq n$, we choose $(\eta_i, s_i) \in (\omega_1 \times R) \cap M$ and $\bar r_i < s_i$ so that $\bar r_i \Vdash \eta_i \notin \dot W_F$, $s_i \leq s_j \leq r$ and $\bar r_i \leq \bar r_j$ for $j < i$, and $\Gamma_\xi(\{\eta_j : 1 \leq j < i\},(s_i,p))$ is ω_1-full.

Let $\bar r_0 = \bar r$, $(s_0, q_0) = (r,p)$, $\emptyset = \{\eta_j : 1 \leq j < 1\}$ and we assume by induction that, at stage i, $\Gamma(\{\eta_j : 1 \leq j < i\},(s_i,p))$ is ω_1-full. Fix any sequence $\{\dot\eta_\zeta : \omega \leq \zeta \in \omega_1\} \in M$ witnessing that $\Gamma_\xi(\{\eta_j : j < i\},(s_i,p))$ is ω_1-full. We have that $\{\dot\eta_\zeta : \omega \leq \zeta \in \omega_1\} \in M$ is an R-name for an uncountable subset of ω_1. It follows that $\bar r_{i-1}$ forces that there is a $\zeta \in M$ such that $\dot\eta_\zeta \notin \dot W_F$. We find an extension $\bar r_{i+1}$ of $\bar r_i$ so that we may choose $\zeta \in M$ and $(\eta, s) \in \dot\eta_\zeta$ such that $\eta \notin \dot W_F$, $\bar r_{i+1} < s \leq s_i$. Therefore we set $(\xi_i, s_{i+1}, q_{i+1}) = (\eta, s, q)$ and this completes the construction.

Setting $H = \{\xi_i : 1 \leq i \leq n\}$ and $(s,q) = (s_n, q_n)$ completes the proof of the Claim. □

Claim 6. If $\Gamma_\xi(H,(r,p))$ is not ω_1-full, there is an $s < r$ in R and a $\zeta < \omega_1$ such that $\Gamma_\xi(H \cup \{\eta\},(s,p))$ is not ω_1-full for all $\zeta < \eta \in \omega_1$.

Proof of Claim: Since $\Gamma_\xi(H,(r,p))$ is not ω_1-full, there is some $\zeta \in \omega_1$ so that the suitable nice name $\dot\eta_\zeta$ does not exist. It follows immediately that $\dot\eta_\gamma$ does not exist for all $\zeta < \gamma \in \omega_1$. In addition, since $\dot\eta_\zeta$ fails to exist, it is because $\Gamma_\xi(H \cup \{\eta\},(s',r))$ is not ω_1-full for all $s' \not\leq s$. □

Claim 7. For every $(r,p) \in R \times P_2$, there is a δ so that $\Gamma_\delta(\emptyset,(r,p))$ is ω_1-full.

Proof of Claim: Let M_0 be a countable elementary submodel of $H(\kappa^+)$ so that $\{\mathcal{U},(r,p),R\} \in M_0$. Choose any $p_1 <_E p$ (i.e. $p_1(0) = p(0)$ and

$p_1(0) \Vdash p_1(1) < p(1))$ that is (M_0, P_2)-generic. Notice that (r, p_1) is therefore $(M, R \times P_2)$-generic since R is ccc. Let $\delta_0 = M_0 \cap \omega_1$. Choose any continuous $\in$-chain $\{M_\alpha : 0 < \alpha < \omega_1\}$ of countable elementary submodels of $H(\kappa^+)$ such that $p_1 \in M_1$. For each $\alpha \in \omega_1$, let $\delta_\alpha = M_\alpha \cap \omega_1$. We did not actually have to choose p_1 before choosing M_1 of course. Let C be the cub $\{\delta_\alpha : \alpha \in \omega_1\}$ and let $p_2 \in P_2$ be a common extension of p_1 and $(\emptyset, (\emptyset, \delta_0 \cup (C \setminus \delta_0)))$ (or equivalently $p_2(0) \leq p_1(0)$ and $p_2(0) \Vdash p_2(1) \leq (\pi_0(p_1(1)), \pi_1(p_1(1)) \cap C))$. It follows that $p_2 \Vdash \dot{C}_1 \setminus \delta_0 \subset C$.

Assume $\Gamma_{\delta_0}(\emptyset, (r, p))$ is not ω_1-full. Choose $s_0 < r$ and $\zeta_0 \in \omega_1$ as in Claim 5. By elementarity we may assume that s_0, ζ_0 are in M_1.

Now choose any $\bar{s}_0 < s_0$ so that there is a $q_0 < p_1$ and an $H \in [\omega_1 \setminus \delta_0]^n$ such that $(\bar{s}_0, q_0) \Vdash \dot{H}_{\delta_0} = H$. Of course this implies that $\Gamma_{\delta_0}(H, (r, p))$ is not empty and therefore, it is ω_1-full. Let H be enumerated in increasing order $\{\eta_i : 1 \leq i \leq n\}$.

Since $(\bar{s}_0, q) \Vdash \dot{H}_{\delta_0} \in Q(\mathcal{U}, \dot{C}_1)$, we can assume that q has already determined the members of $\dot{C}_1$ that separate the elements of $\{\delta_0\} \cup H$. In other words, there is a set $\{\alpha_i : 1 \leq i \leq n\} \subset \omega_1$ so that $\{\delta_{\alpha_i} : 1 \leq i \leq n\} \subset \pi_0(q(1)) \subset C$ such that, for each $1 \leq i < n$, $\delta_0 \leq \delta_{\alpha_{i-1}} \leq \eta_i$. Therefore, $\{\eta_j : 1 \leq j < i\} \in M_{\alpha_i}$ for all $i < n$ and $\Gamma_{\delta_0}(\{\eta_j : 1 \leq j \leq n\}, (r, p))$ is ω_1-full. Clearly, for all $s' < \bar{s}_0$, $\Gamma_{\delta_0}(\{\eta_j : 1 \leq j \leq n\}, (s', p))$ is also ω_1-full.

By the choice of s_0 and ζ_0, we have that $\Gamma_{\delta_0}(\{\eta_1\}, (s_0, p)) \in M_{\alpha_2}$ is not ω_1-full. We note that $\bar{s}_0$ is (M_{α_2}, R)-generic condition. There is therefore, by Claim 5, a $\zeta_1 \in M_{\alpha_2}$ and a pair $\bar{s}_1 < s_1$ so that $s_1 \in M_{\alpha_2}$, $\bar{s}_1 < \bar{s}_0$ and $\Gamma_{\delta_0}(\{\eta_1, \eta\}, (s_1, p))$ is not ω_1-full for all $\eta > \zeta_1$. Following this procedure we can recursively choose a pair of descending sequences $\{s_i : 1 \leq i \leq n\} \subset R$ and $\{\bar{s}_i : 1 \leq i \leq n\} \subset R$ so that

(1) $s_{i-1} \in M_{\alpha_i}$ and $\bar{s}_i < s_i$,
(2) $\Gamma_{\delta_0}(\{\eta_1, \ldots, \eta_i\}, (s_i, p))$ is not ω_1-full.

We now have a contradiction that completes the proof. We noted above that since $\bar{s}_n < \bar{s}_0$, $\Gamma_{\delta_0}(\{\eta_1, \ldots, \eta_i\}, (\bar{s}_n, p))$ is ω_1-full. However since $\bar{s}_n < s_n$, this contradicts that $\Gamma_{\delta_0}(\{\eta_1, \ldots, \eta_n\}, (s_n, p))$ is not ω_1-full. □

Now we complete the proof of the Proposition. Consider any countable elementary submodel M as in Claim 5 and let $\delta = M \cap \omega_1$. Let p_1 be a condition as in Lemma 2.15 applied to the case $\alpha = 0$. Let G_R be any R-generic filter and let $G_1 \subset \mathcal{C}_{\omega_1}$ be any generic filter, which is generic over the model $V[G_R]$. Pass to the extension $V[G_R]$.

Fix any $F \in [\omega_1 \setminus \delta]^n$. It follows from Claim 5 and Claim 6, that the set $\mathcal{W}_F$ of those $(t, (\dot{b}, \dot{B})) \in M \cap (\mathcal{C}_{\omega_1} \star \dot{\mathfrak{J}})$ for which

$$(\exists \xi \in \delta)(\exists s \in G_R) \quad (s \Vdash H \cap \dot{W}_F = \emptyset \ \& \ (s, (t, (\dot{b}, \dot{B}))) \Vdash H = \dot{H}_\xi)$$

is a dense subset of $M \cap (\mathcal{C}_{\omega_1} \star \dot{\mathfrak{J}})$. The proof is that Claim 6 provides a potential $\xi \in M$ to strive for, and Claim 5 provides an (s, q) to yield an element of $\mathcal{W}_F$.

It then follows easily that, in the extension $V[G_R \times G_1]$, the set

$$\mathrm{val}_{G_1 \restriction \delta}(\mathcal{W}_F) = \{\mathrm{val}_{G_1}(\,(\dot{b}, \dot{B})\,) : (\exists t \in G_1)\ ((t, (\dot{b}, \dot{B}))) \in \mathcal{W}_F\}$$

is a dense subset of $\mathrm{val}_{G_1}(M \cap \dot{\mathfrak{J}})$ which is an element of $V[G_R \times (G_1 \restriction \delta)]$. Since p_1 forces that the generic filter meets $\mathrm{val}_{G_1 \restriction \delta}(\mathcal{W}_F)$, this completes the proof. □

For any $\alpha \leq \kappa$ and subset $I \subset \alpha$, we will say that a P_α-name $\dot{E}$ is a $P_\alpha(I)$-name if it is a $P_\alpha(I)$-name in the usual recursive sense. This definition makes technical sense even if $P_\alpha(I)$ is not a complete subposet of P_α.

Corollary 3.4. *Let $\lambda \in \mathbf{E}$ and let $\dot{R}_0$ be a $P_\lambda(I_\lambda)$-name that is forced by P_λ to be ccc poset. Let $\dot{R}$ be a P_λ-name of a ccc poset such $\mathbf{1}_{P_\lambda}$ forces that $\dot{R}_0 \subset_c \dot{R}$. Assume that $\mathcal{U} = \{\dot{U}_\xi : \xi \in \omega_1\}$ is a sequence of $P_\lambda(I_\lambda) \star \dot{R}_0$-names of subsets of ω_1 such that $P_\lambda \star \dot{R}$ forces that $\mathcal{U}$ is an S-space task. Then the $P_{\lambda+2}$-name $Q(\mathcal{U}, \dot{C}_\lambda)$ satisfies that $P_{\lambda+2}$ forces that $\dot{R} \times Q(\mathcal{U}, \dot{C}_\lambda)$ is ccc.*

Proof. Let G_λ be a P_λ-generic filter and pass to the extension $V[G_\lambda]$. Let $R = \mathrm{val}_{G_\lambda}(\dot{R})$ and observe that we may now regard $\mathcal{U}$ as a family of R-names of subsets of ω_1 that is forced to be an S-space task. We would like to simply apply Lemma 3.3 but unfortunately, $P_{\lambda+2}$ is not isomorphic to $P_\lambda \star P_2$. Naturally the difference is that $\dot{Q}_{\lambda+1}$ is a proper subset of $\dot{\mathfrak{J}}$. It will suffice to identify the three key places in the proof of Lemma 3.3 that depended on consequences of the properties of $\mathfrak{J}$ and to verify that the consequences also hold for $\dot{Q}_{\lambda+1}$. The first was in the proof of Claim 7 where we selected a condition $p_2(1) \in \mathfrak{J}$ that satisfied that $\pi_1(p_2(1))$ was forced to be a subset of $C \cup \delta_0$ for the cub C. Since, in this proof, C will be an cub set in the model $V[G_\lambda]$, it follows from condition (6) of Definition 2.2, this can be done. The next property of P_2 that we used was that Lemma 2.15 holds, but of course this also holds for $P_{\lambda+2}$. The third is in the proof and statement of Claim 5. When choosing the pair (s, q) in $R \times P_2$ we require that it satisfies condition (2) in Claim 5. In the current situation, each $\dot{U}_\zeta$ is not simply an R-name but rather it is a $P_\lambda(I_\lambda) \star \dot{R}_0$-name. Therefore, there is a $P_\lambda(I_\lambda)$-name for a suitable q so that $(s, q) \Vdash H \cap \bigcup\{\dot{U}_\zeta : \zeta \in F\}$ is empty. This causes no difficulty since $P_\lambda(I_\lambda)$-names for elements of $\dot{Q}_{\lambda+1}$ are, in fact, elements of $\dot{Q}_{\lambda+1}$. That is, a choice for (s, q) in $R \times (\dot{Q}_\lambda \star \dot{Q}_{\lambda+1})$ can be made in $V[G_\lambda]$ as required in Claim 5. □

4. Building the final model

In this section we present the construction of the iteration sequence of length $\kappa + \kappa$ extending that of Definition 2.2 that will be used to prove the main theorem.

We introduce more terminology.

Definition 4.1. *Fix any $\mu \leq \lambda \leq \kappa$ and define $\mathcal{Q}(\lambda, \mu)$ to be the set of all iterations $\mathbf{q}$ of the form $\langle P^{\mathbf{q}}_\alpha, \dot{Q}^{\mathbf{q}}_\beta : \alpha \leq \lambda + \mu, \beta < \lambda + \mu\rangle \in H(\kappa^+)$ satisfying that*

1. *$\langle P^{\mathbf{q}}_\alpha, \dot{Q}^{\mathbf{q}}_\beta : \alpha \leq \lambda, \beta < \lambda\rangle$ is our sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \lambda, \beta < \lambda\rangle$ from Section 2,*
2. *for all $\lambda \leq \beta < \lambda + \mu$, $\dot{Q}^{\mathbf{q}}_\beta \in H(\kappa)$ is a $P^{\mathbf{q}}_\beta$-name of a ccc poset,*
3. *for all $\alpha \leq \mu$ and $p \in P^{\mathbf{q}}_\alpha$, $p \restriction \lambda \in P^{\mathbf{q}}_\lambda$ and $\mathrm{dom}(p) \setminus \lambda$ is finite,*
4. *if $\lambda < \kappa$, then $\mathbf{q} \in H(\kappa)$.*

For $\mathbf{q} \in \mathcal{Q}(\lambda, \mu)$, let $\mathbf{q}(\kappa)$ denote the element of $\mathcal{Q}(\kappa, \mu)$ where $\dot{Q}^{\mathbf{q}(\kappa)}_{\kappa+\beta} = \dot{Q}^{\mathbf{q}}_{\lambda+\beta}$ for all $\beta < \mu$.

Lemma 4.2. *Let $\mu < \kappa$ and let $\mathbf{q} \in \mathcal{Q}(\kappa, \mu)$ and let $\mathcal{U} = \{\dot{U}_\xi : \xi \in \omega_1\}$ be a sequence of $P^{\mathbf{q}}_{\kappa+\mu}$-names. Assume that $P^{\mathbf{q}}_{\kappa+\mu}$ forces that $\mathcal{U}$ is an S-space task. Let $\bar{M}$ be an elementary submodel of $H(\kappa^+)$ of cardinality $\aleph_1$ that is closed under ω-sequences and contains $\{\mathcal{U}, \mathbf{q}\}$. Choose any $\lambda \in \mathbf{E} \cap \kappa$ so that $\bar{M} \cap \kappa \subset I_\lambda$. Then $P^{\mathbf{q}}_{\kappa+\mu}$ forces that $Q(\mathcal{U}, \dot{C}_\lambda)$ is ccc.*

Proof. Since $\mu \in \bar{M}$, it follows that $\mu \leq \lambda$. Furthermore, by the assumptions on $\mathbf{q} \in \mathcal{Q}$ and $\mathbf{q} \in \bar{M}$, it follows that there is a $\gamma \in \bar{M} \cap \kappa$ such that $\dot{Q}_\beta$ is a P_γ-name for all $\kappa \leq \beta < \kappa + \mu$. In addition, for each $\beta \in \bar{M} \cap \mu$, $\dot{Q}_\beta$ is a $P_\gamma(\bar{M} \cap \gamma)$-name. Since $\gamma < \lambda$, there is a P_λ-name, $\dot{R}$, of a finite support iteration of length μ such that $P_\kappa \star \dot{R}$ is isomorphic to $P^{\mathbf{q}}_{\kappa+\mu}$. More precisely, the β-th iterand for $\dot{R}$ is the name $\dot{Q}_{\kappa+\beta}$. Similarly, let $\dot{R}_0$ be the set of conditions in $\dot{R}$ with support contained in $\bar{M} \cap \mu$ and values taken in $\bar{M} \cap \dot{Q}_{\kappa+\beta}$ for each β in the support. Then we have that $\mathbf{1}_{P_\lambda} \Vdash \dot{R}_0 \subset_c \dot{R}$. By minor re-naming, we may treat $\mathcal{U}$ as a sequence of $P_\lambda(I_\lambda) \star \dot{R}_0$-names. Since $P^{\mathbf{q}}_{\kappa+\mu}$ forces that $\mathcal{U}$ is an S-space task, it follows that $P_\lambda \star \dot{R}$ also forces that $\mathcal{U}$ is an S-space task. By Corollary 3.4, $P_{\lambda+2}$ forces that $\dot R \times Q(\mathcal{U}, \dot C_\lambda)$ is ccc. By Lemma 2.16, P_κ forces that $\dot R \times Q(\mathcal{U}, \dot C_\lambda)$ is ccc. Since $P^{\mathbf{q}}_{\kappa+\mu}$ is isomorphic to $P_\kappa \star \dot R$, this completes the proof. □

Theorem 4.3. *Let $\kappa > \aleph_2$ be a regular cardinal in a model of GCH. There is an iteration sequence $\langle P_\alpha, \dot Q_\beta : \alpha \le \kappa + \kappa,\ \beta < \kappa + \kappa\rangle$ such that $P_{\kappa+\kappa}$ forces that there are no S-spaces and, for all $\mu < \kappa$, $\langle P_\alpha, \dot Q_\beta : \alpha \le \kappa + \mu,\ \beta < \kappa + \mu\rangle$ is in $\mathcal{Q}(\kappa, \mu)$. It therefore follows that $P_{\kappa+\kappa}$ is cardinal preserving and forces that $\kappa^{<\kappa} = \kappa = \mathfrak{c}$.*

The iteration can be chosen so that, in addition, Martin's Axiom holds in the extension.

Proof. Fix a sequence $\mathfrak{I} = \{I_\gamma : \gamma \in \kappa\}$ as described in the construction of the sequence $\langle P_\alpha, \dot Q_\beta : \alpha \le \kappa,\ \beta < \kappa\rangle$. Also let $\mathcal{Q}(\lambda, \mu)$ for $\mu \le \lambda < \kappa$ be defined as in Definition 4.1.

We introduce still more notation. For all $\alpha \le \lambda < \kappa$, let P^λ_α simply denote P_α and $\dot Q^\lambda_\alpha = \dot Q_\alpha$. Also for any $\mu \le \lambda < \kappa$ and sequence $\mathbf{q}' = \langle \dot Q'_\beta : \beta < \mu\rangle \in H(\kappa)$, let $\dot Q^\lambda_{\lambda+\beta}(\mathbf{q}')$ denote $\dot Q'_\beta$. By recursion on $\alpha < \mu$, let $P^\lambda_{\lambda+\alpha}(\mathbf{q}')$ denote the limit of the iteration sequence $\langle P^\lambda_\zeta(\mathbf{q}'), \dot Q^\lambda_\beta(\mathbf{q}') : \zeta < \alpha,\ \beta < \alpha\rangle$ so long as this sequence (and its limit) is in $\mathcal{Q}(\lambda, \alpha)$. Say that a sequence $\mathbf{q}' = \langle \dot Q'_\beta : \beta < \lambda\rangle \in H(\kappa)$ is suitable if for all $\alpha \in \mathbf{E} \cap \lambda{+}1$, $\langle P^\lambda_\zeta(\mathbf{q}'), \dot Q^\lambda_\beta(\mathbf{q}') : \zeta \le \alpha,\ \beta < \alpha\rangle$ is in $\mathcal{Q}(\lambda, \alpha)$. We state for reference two properties of suitable sequences.

Fact 1. If λ is a limit ordinal, then $\langle \dot Q'_\beta : \beta \in \lambda\rangle \in H(\kappa)$ is suitable so long as $\langle \dot Q'_\beta : \beta < \mu\rangle$ is suitable for all $\mu < \lambda$.

Fact 2. If $\mathbf{q}' = \langle \dot Q'_\beta : \beta \in \lambda\rangle \in H(\kappa)$ is suitable, then $\langle \dot Q'_\beta : \beta \in \lambda + 1\rangle$ is suitable for any $P^\lambda_{\lambda+\lambda}(\mathbf{q}')$-name $\dot Q'_\lambda$ of a ccc poset of cardinality at most $\aleph_1$.

Now that we have this cumbersome, but necessary, notation out of the way, the proof of the theorem is a routine consequence of the prior results. Let $\sqsubset$ be a well ordering of $H(\kappa)$ in type κ. We recursively define a sequence $\langle \dot Q'_\beta : \beta < \kappa\rangle$ and a 1-to-1 sequence $\langle \mathcal{U}_\beta : \beta < \kappa\rangle$. One inductive assumption is that every initial segment of $\langle \dot Q'_\beta : \beta < \kappa\rangle$ is a suitable sequence. The list $\{\mathcal{U}_\beta : \beta < \kappa\}$ will contain the list the potential S-space tasks as we deal with them.

Let $\lambda < \kappa$ and assume that $\langle \dot Q'_\beta, \mathcal{U}_\beta : \beta < \lambda\rangle \in H(\kappa)$ has been chosen. If $\lambda \notin \mathbf{E}$, then $\dot Q'_\lambda$ is the trivial poset and $\mathcal{U}_\lambda = \lambda$. Now let $\lambda \in \mathbf{E}$ and let $\mathbf{q}' = \langle \dot Q'_\beta : \beta < \lambda\rangle$. Consider the set of all $P^\lambda_{\lambda+\lambda}(\mathbf{q}')$-names $\mathcal{U} = \{\dot U_\xi : \xi \in \omega_1\}$ that are forced to be S-space tasks. Consider only those $\mathcal{U}$ for which there is an elementary submodel $\bar{M}$ of $H(\kappa^+)$ as in Lemma 4.2. More specifically, such that $\bar{M}\cap\lambda\subset I_\lambda$, $\{\mathcal{U}, P^{\lambda}_{\lambda+\lambda}(\mathbf{q}')\}\in\bar{M}$, $|\bar{M}|=\aleph_1$, and $\bar{M}^\omega\subset\bar{M}$. The final requirement of such $\mathcal{U}$ is that they are not in the set $\langle\mathcal{U}_\beta:\beta<\lambda\rangle$. If any such $\mathcal{U}$ exist, then let $\mathcal{U}_\lambda$ be the $\sqsubset$-minimal one. Loosely, $\mathcal{U}_\lambda$ is the $\sqsubset$-minimal S-space task that has not yet been handled and can be handled at this stage. Otherwise, let $\mathcal{U}_\lambda=\lambda$ (so as to preserve the 1-to-1 property). Now we choose $\dot{Q}'_\lambda$. If $\mathcal{U}_\lambda=\lambda$, then $\dot{Q}_\lambda$ is the trivial poset. Otherwise, of course, $\dot{Q}_\lambda$ is the $P^{\lambda+2}_{\lambda+\lambda}(\mathbf{q}')$-name for $Q(\mathcal{U}_\lambda,\dot{C}_\lambda)$. By Lemma 4.2 and Fact 2, $\langle\dot{Q}_\beta:\beta\leq\lambda\rangle$ is suitable.

This completes the recursive construction of the suitable sequence $\mathbf{q}'=\langle\dot{Q}'_\beta:\beta<\kappa\rangle$ and the listing $\langle\mathcal{U}_\beta:\beta<\kappa\rangle$. It remains only to prove that if $\mathcal{U}=\{\dot{U}_\xi:\xi\in\omega_1\}$ is a $P^{\kappa}_{\kappa+\kappa}(\mathbf{q}')$-name of an S-space task, then there is an $\alpha<\kappa$ such that $\mathcal{U}=\mathcal{U}_\alpha$. Fix any such $\mathcal{U}$ and elementary submodel $\bar{M}\prec H(\kappa^+)$ such that $\{\mathcal{U}, P^{\kappa}_{\kappa+\kappa}(\mathbf{q}')\}\in\bar{M}$, $|\bar{M}|=\aleph_1$, and $\bar{M}^\omega\subset\bar{M}$. Let Λ be the set of $\lambda\in\kappa$ such that $\bar{M}\cap\kappa\subset I_\lambda$. Let γ be the order type of the set of predecessors of $\mathcal{U}$ in the well ordering $\sqsubset$. Choose any $\lambda\in\Lambda$ such that the order type of $\Lambda\cap\lambda$ is greater than γ. Note that $\Lambda\subset\mathbf{E}$. For every $\mu\in\Lambda\cap\lambda$, $\mathcal{U}$ would have been an appropriate choice for $\mathcal{U}_\mu$ and if not chosen, then $\mu\neq\mathcal{U}_\mu\sqsubset\mathcal{U}$. Since the sequence is 1-to-1, there is therefore a $\mu\in\Lambda\cap\lambda$ such that $\mathcal{U}=\mathcal{U}_\mu$.

It should be clear that we can ensure that Martin's Axiom holds in the extension by making minor adjustments to the choice of $\dot{Q}'_\beta$ for $\beta\notin\mathbf{E}$ in the sequence $\langle\dot{Q}'_\beta:\beta<\kappa\rangle$ together with some additional bookkeeping, □

5. Moore-Mrowka tasks

The Moore-Mrowka problem asks if every compact space of countable tightness is sequential. A space has countable tightness if the closure of a set is equal to the union of the closures of all its countable subsets. A space is sequential providing that each subset is closed so long as it contains the limits of all its converging (countable) subsequences. To illustrate that a sequential space has countable tightness, note that a space has countable tightness if a set is closed so long as it contains the closures of all of its countable subsets. Say that a compact non-sequential space of countable tightness is a Moore-Mrowka space.

Results on the Moore-Mrowka problem have closely resembled those of the S-space problem. In particular, there are proofs that PFA implies there are no Moore-Mrowka spaces that have many similarities to the proof that PFA implies there are no S-spaces. While it is independent with CH as to whether Moore-Mrowka spaces exist [5], it is known that

$\diamondsuit$ implies there are (Cohen indestructible) Moore-Mrowka spaces of cardinality $\aleph_1$ [14]. In addition, $\diamondsuit$ implies there is a separable compact space of countable tightness with cardinality $2^{\aleph_1}$ (greater than $\mathfrak{c}$) [8]. It is also known that the addition of $\aleph_2$ Cohen reals over a model of $\diamondsuit + \aleph_2 < 2^{\aleph_1}$ results in a model in which there is a compact separable space of countable tightness that has cardinality greater than $\mathfrak{c}$ [6]. Of course these spaces are Moore-Mrowka spaces since every separable sequential space has cardinality at most $\mathfrak{c}$.

Here are two open problems and a third that we solve in the affirmative in this section.

Question 5.1. Is it consistent with $\mathfrak{c} > \aleph_2$ that every compact space of countable tightness is sequential?

Question 5.2. Is it consistent with $\mathfrak{p} > \aleph_2$ that there is a Moore-Mrowka space?

Question 5.3. Is it consistent with $\mathfrak{c} > \aleph_2$ that every separable Moore-Mrowka space has cardinality at most $\mathfrak{c}$?

A Moore-Mrowka task mentioned in the title of the section is similar to an S-space task. The difference will be that rather than using the poset $Q(\mathcal{U}, C)$ to force an uncountable discrete subset, we will hope to force an uncountable (algebraic) free sequence. We define these notions and indicate their relevance.

Definition 5.1. *A sequence $\{x_\alpha : \alpha \in \omega_1\}$ is a free sequence in a space X if, for every $\delta < \omega_1$, the initial segment $\{x_\alpha : \alpha \in \delta\}$ and the final segment $\{x_\beta : \beta \in \omega_1 \setminus \delta\}$ have disjoint closures.*

A sequence $\{x_\alpha, U_\alpha, W_\alpha : \alpha \in \omega_1\}$ is an algebraic free sequence in a space X providing

(1) $x_\alpha \in U_\alpha$ and W_α are open sets with $\overline{U_\alpha} \subset W_\alpha$,
(2) for every $\alpha < \delta \in \omega_1$, $x_\delta \notin W_\alpha$ and there is a finite $H \subset \delta+1$ such that $\{x_\eta : \eta \leq \delta\} \subset \bigcup\{U_\beta : \beta \in H\}$.

Free sequences were introduced by Arhangelskii. Algebraic free sequences were introduced by Todorcevic in a slightly different formulation. The advantage of an algebraic free sequence is that the only reference to the (second order) closure property is with the pairs U_α, W_α. If $\{x_\alpha, U_\alpha, W_\alpha : \alpha \in \omega_1\}$ is an algebraic free sequence, then the set $\{x_{\alpha+1} : \alpha < \omega_1\}$ is a free sequence. This follows from the fact that for all $\delta \in \omega_1$, there is a finite $H \subset \delta + 1$ satisfying that $\{x_\alpha : \alpha \leq \delta\} \subset U_H = \bigcup\{U_\alpha : \alpha \in H\}$ and $\{x_\beta : \delta < \beta \in \omega_1\}$ is disjoint from $W_H = \bigcup\{W_\alpha : \alpha \in H\}$. The free sequence property now follows from the fact that U_H and $X \setminus W_H$ have disjoint closures. This was crucial in Balogh's proof [4] that PFA implies there are no Moore-Mrowka spaces.

Proposition 5.2 ([3])**.** *A compact space has countable tightness if and only if it contains no uncountable free sequence.*

Definition 5.3. *A sequence $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ is a Moore-Mrowka task if, for all $\alpha \in \omega_1$, $\alpha \in A_\alpha \subset \alpha + 1$, and*

1. *for all $\beta < \alpha$ there is a γ such that $A_\gamma \cap \{\beta, \alpha\} = \{\alpha\}$, and*
2. *for all uncountable $A \subset \omega_1$, there is a $\delta \in \omega_1$ such that for all $\beta \in \omega_1 \setminus \delta$, $(A \cap \delta) \cap \bigcap_{\gamma \in H} A_\gamma$ is not empty for all finite $H \subset \{\gamma : \beta \in A_\gamma\}$.*

The idea behind a Moore-Mrowka task is that we identify ω_1 with a set of points in space X and so that there is a collection $\{U_\alpha, W_\alpha : \alpha \in \omega_1\}$ that is a neighborhood assignment for those points. For each α, $\overline{U_\alpha} \subset W_\alpha$ and $W_\alpha \cap \omega_1$ is also contained in $\alpha + 1$. Then we set $A_\alpha = U_\alpha \cap \omega_1$. Condition (1) is trivial to arrange but condition (2) is a $\diamondsuit$-like condition. A distinction with S-space task is that the non-existence of a Moore-Mrowka task extracted from a compact space of countable tightness does not imply that the space is sequential. The similarity with S-space task is that we will use a Moore-Mrowka task to generically introduce an algebraic free sequence.

Definition 5.4. *Let $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ be a Moore-Mrowka task and let $C \subset \omega_1$ be a cub. The poset $\mathcal{M}(\mathcal{A}, C)$ is the set of finite subsets of $\omega_1 \setminus \min(C)$ that are separated by C. For each $H \in \mathcal{M}(\mathcal{A}, C)$ and each $\beta \in H$, let $A(H, \beta)$ be the intersection of the family $\{A_\gamma : \gamma \in H,\ \beta \in A_\gamma\}$. We define $H < K$ from $\mathcal{M}(\mathcal{A}, C)$ providing $H \supset K$ and for each $\alpha \in H \cap \max(K)$, $\alpha \in A(K, \min(K \setminus \alpha))$.*

Lemma 5.5. *Let $\lambda \in \mathbf{E}$ and let $\dot{R}_0$ be a $P_\lambda(I_\lambda)$-name that is forced by P_λ to be ccc poset. Let $\dot{R}$ be a P_λ-name of a ccc poset such $\mathbf{1}_{P_\lambda}$ forces that $\dot{R}_0 \subset_c \dot{R}$. Assume that $\mathcal{A} = \{\dot{A}_\xi : \xi \in \omega_1\}$ is a sequence of $P_\lambda(I_\lambda) \star \dot{R}_0$-names of subsets of ω_1 such that $P_\lambda \star \dot{R}$ forces that $\mathcal{A}$ is a Moore-Mrowka task. Then the $P_{\lambda+2}$-name $\mathcal{M}(\mathcal{U}, \dot{C}_\lambda)$ satisfies that $P_{\lambda+2}$ forces that $\dot{R} \times \mathcal{M}(\mathcal{U}, \dot{C}_\lambda)$ is ccc.*

Proof. The proof proceeds much as it did in Lemma 3.3 and Corollary 3.4 for S-space tasks. To show that a poset of the form $\mathcal{M}(\mathcal{A}, C)$ is ccc, it again suffices to prove that, for each $n \in \omega$, there is no uncountable antichain consisting of pairwise disjoint sets of cardinality n. So we consider an arbitrary family of pairwise disjoint sets of cardinality n. Fix $P_{\lambda+2} \star \dot{R}$-names $\{\dot{H}_\xi : \xi \in \omega_1\}$ for a set of pairwise disjoint elements of $\mathcal{M}(\mathcal{A}, \dot{C}_\lambda) \cap [\omega_1]^n$. Following Lemma 3.3, we may assume that, for each $\xi \in \omega_1$, it is forced that $\xi < \min(\dot{H}_\xi)$ and that $\{\xi\} \cup \dot{H}_\xi$ is also separated by $\dot{C}_\lambda$. We prove that no condition forces this to be an antichain.

Let M be a countable elementary submodel containing all the above and let $p_1 \in P_{\lambda+2}$ be chosen as in Lemma 2.15 so that p_1 is $(M, P_{\lambda+2})$-generic and so that $p_1(\lambda) \in M$. Let $p_1 \restriction \lambda \in G_\lambda$ be a P_λ-generic filter and pass to the extension $V[G_\lambda]$. Let $R = \mathrm{val}_{G_\lambda}(\dot{R})$ and let $G_1 \subset \mathcal{C}_{\omega_1}$ so that $p_1 \restriction \lambda + 1 \in G_\lambda \star G_1$ is $P_{\lambda+1}$-generic. Let $\delta = M \cap \omega_1$. We will prove that p_1 forces that $\dot{H}_\delta$ is compatible with some element of $\{\dot{H}_\eta : \eta \in \delta\}$.

For each $\zeta \in \omega_1$, let, in $V[G_\lambda]$, $\dot{J}_\zeta$ denote the R-name for the set $\{\gamma : \zeta \in \dot{A}_\gamma\}$ and, for each finite $F \subset \omega_1$, also let $\dot{A}_F$ denote the R-name for $\bigcap_{\gamma \in F} \dot{A}_\gamma$. We leave the reader to check that it suffices to prove that p_1 forces that for each finite $F \subset \dot{J}_{\min(\dot{H}_\delta)}$, there is an $\eta < \delta$ such that $\dot{H}_\eta \subset \dot{A}_F$. For each $\zeta \in \omega_1$ and finite $F \subset \omega_1$, we will let J_ζ and A_F denote $\mathrm{val}_{G_R}(\dot{J}_\zeta)$ and $\mathrm{val}_{G_R}(\dot{A}_F)$ respectively. Also, for the remainder of the proof we will treat each $\dot{H}_\xi$ as the canonical $R \times (Q_\lambda \star \dot{Q}_{\lambda+1})$-name obtained from the evaluation of the original $P_{\lambda+2} \star \dot{R}$-name by G_λ. For each $\xi \in \omega_1$ and $H \in [\omega_1]^n$, let $\Gamma_\xi(H)$ be the (possibly empty) set of conditions in $R \times (Q_\lambda \star \dot{Q}_{\lambda+1})$ that force H to equal $\dot{H}_\xi$.

We need an updated version of ω_1-full. Say that a countable set B, in $V[G_\lambda][G_R]$, is $\mathcal{A}$-large if there is a $\gamma \in \omega_1$ such that $B \cap A_F \neq \emptyset$ for all $\beta \in \omega_1 \setminus \gamma$ and finite $F \in J_\beta$. We may interpret this as that $\overline{B}$ contains $\omega_1 \setminus \gamma$.

For $\xi \in \omega_1$ and $(r, p) \in R \times (Q_\lambda \star \dot{Q}_{\lambda+1})$, let $\Gamma_\xi(H, (r, p))$ be the set of conditions in $\Gamma_\xi(H)$ that are below (r, p). In other words, $\Gamma_\xi(H, (r, p))$ is not empty if and only if $(r, p) \not\Vdash H \neq \dot{H}_\xi$. Similarly, for each $0 < i < n$ and $H \in [\omega_1]^i$, let $\Gamma_\xi(H, (r, p)) = \bigcup \{\Gamma_\xi(H \cup \{\eta\}, (r, p)) : \eta \in \omega_1\}$. For $H \in [\omega_1]^n$, say that $\Gamma_\xi(H, (r, p))$ is full if $\Gamma_\xi(H, (\bar{r}, p))$ is not empty for all $\bar{r} \leq r$. For $0 < i < n$ and $H \in [\omega_1]^{n-i}$, say that $\Gamma_\xi(H, (r, p))$ is full if there is a R-name $\dot{B}$ that is forced to be an $\mathcal{A}$-large set of $\eta \in \omega_1$ and, for each η and $s \Vdash \eta \in \dot{B}$, $\Gamma_\xi(H \cup \{\eta\}, (s, p))$ is full.

Claim 8. Suppose that $\xi, r, p \in M[G_\lambda]$ and that $\Gamma_\xi(\emptyset, (r, p))$ is full. Suppose also that $\bar{r} \in R$ forces that F is a finite subset of $\dot{J}_\zeta$ for some $\delta \leq \zeta \in \omega_1$. Then there are $(s, q), H \in M[G_\lambda]$ and $\bar{s} < \bar{r}$ such that

(1) $(s, q) < (r, p)$ in $R \times (Q_\lambda \star \dot{Q}_{\lambda+1})$,
(2) $\bar{s} < s$,
(3) $\bar{s} \Vdash H \subset \dot{A}_F$,

(4) $(s,q) \Vdash \dot{H}_\xi = H$.

Proof of Claim: There is an $R\times Q_\lambda$-name $\dot{B}_0 \in M[G_\lambda]$ that is forced to be a $\mathcal{A}$-large subset of δ and witnesses that $\Gamma_\xi(\emptyset,(r,p))$ is full. Therefore there are $\eta < \delta$ and $r' < \bar{r}$ such that $\bar{r}_1 \Vdash \eta \in \dot{B}_0 \cap A_F$. There is no loss to assuming, by elementarity, that $\bar{r}_1$ extends some $r_1 \in M[G_\lambda]$ such that $r_1 \Vdash \eta \in \dot{B}_0$. Since $r_1 \Vdash \eta \in \dot{B}_0$, we have that $\Gamma_\xi(\{\eta\},(r_1,p))$ is full. Following a recursion of length n, there is an $\bar{r}_n < \bar{r}$ in R, an $H = \{\eta_1,\ldots,\eta_n\} \in M[G_\lambda]$, and an $\bar{r}_n < r_n \in M[G_\lambda]$ such that $\bar{r}_n \Vdash H \subset A_F$ and $\Gamma_\xi(H,(r_n,p))$ is full. Since $\bar{r}_n < r_n$, $\Gamma_\xi(H,(\bar{r}_n,p))$ is not empty. Therefore there is a pair $(\bar{s},\bar{q}) < (\bar{r},p)$ forcing that $H = \dot{H}_\xi$. By elementarity, since $\xi, H, p \in M[G_\lambda]$, the set of $\{s \in R\cap M : (\exists q)((s,q) < (r_n,p)\ \&(s,q) \Vdash H = \dot{H}_\xi)\}$ is predense below r_n. Therefore there is an $(s,q) < (r_n,p) \in M[G_\lambda]$ with $s \not\perp \bar{r}_n$ such that $(s,q) \Vdash H = \dot{H}_\xi$. Let $\bar{s}$ be any extension of $s, \bar{r}_n$. □

Claim 9. For every $(r,p) \in R\times(Q_\lambda\star\dot{Q}_{\lambda+1})$, there is a δ so and a $r_0 < r$ such that $\Gamma_\delta(\emptyset,(r_0,p))$ is full.

Proof of Claim: Let $(r,p) \in M_0$ be a countable elementary submodel of $H(\kappa^+)[G_\lambda]$ so that $\{\mathcal{A}, R, P_{\lambda+2}\} \in M_0$. Choose any $(\bar{r},\bar{p}) < (r,p)$ that is an $(M_0, R\times(Q_\lambda\star\dot{Q}_{\lambda+1}))$-generic condition. Let $\delta_0 = M_0\cap\omega_1$. Choose any continuous $\in$-chain $\{M_\alpha : 0 < \alpha < \omega_1\}$ of countable elementary submodels of $H(\kappa^+)[G_\lambda]$ such that $\{M_0,(\bar{r},\bar{p})\} \in M_1$.

For each $\alpha \in \omega_1$, let $\delta_\alpha = M_\alpha \cap \omega_1$. Let C^* be the cub $\{\delta_\alpha : \alpha \in \omega_1\}$. Choose any extension (r_n,p_n) of $(\bar{r},\bar{p})$ such that $\pi_1(p_2(\lambda+1)) \subset C^* \cup \delta_0$, and so that there is an $H = \{\xi_1,\ldots,\xi_n\} \in [\omega_1]^n$ with $(r_n,p_n) \Vdash H = \dot{H}_{\delta_0}$. Of course this implies that $\Gamma_{\delta_0}(H,(r_n,p)) \supset \Gamma_{\delta_0}(H,(r_n,p_n))$ is actually full. Okay, then $H_{n-1} = \{\xi_1,\ldots,\xi_{n-1}\}$ is in M_{α_n}. Let's take the R-name $\dot{E}_{n-1}$ to the set of $(\eta,\tilde{r})$ such that $\Gamma_{\delta_0}(\{\eta\}\cup H_{n-1},(\tilde{r},p))$ is full. The condition r_n forces that $\dot{E}_{n-1}$ is uncountable. Since $\mathcal{A}$ is a Moore-Mrowka task in $V[G_\lambda\star G_R]$, r_n forces that $\dot{E}_{n-1} \in M_{\alpha_n}$ contains an $\mathcal{A}$-large set. By elementarity and the fact that r_n is (M_{α_n},R)-generic, there is an r_{n-1} in M_{α_n} that forces $\dot{E}_{n-1}$ contains an $\mathcal{A}$-large set. Therefore, for such an $r_{n-1} \in M_{\alpha_n}$, we have that $\Gamma_{\delta_0}(H_{n-1},(r_{n-1},p))$ is full. This recursion continues as above and for each $i < n$, there is an $r_i \in M_{\alpha_i}$ such that $\Gamma_{\delta_0}(\{\xi_j : j < i\},(r_i,p))$ is full. Setting $\delta = \delta_0$, this completes the proof of the Claim. □

Following the proof of Corollary 3.4 we can complete the proof using that p_1 satisfied the conclusion of Lemma 2.15. Using Claim 9, it follows from Claim 8 that in $V[G_\lambda][G_R]$, for each $\delta \le \zeta \in \omega_1$ and finite $F \subset J_\zeta$, the set $\mathcal{W}_F$ consisting of those $p \in M[G_\lambda]\cap(Q_\lambda\star\dot{Q}_{\lambda+1})$ for which there is a $\bar{s} \in G_R$ and $\xi \in \delta$ such that $(\bar{s}, p) \Vdash \dot{H}_\xi \subset A_F$, is a dense subset of $M[G_\lambda] \cap (Q_\lambda \star \dot{Q}_{\lambda+1})$. By the genericity of $((G_1) \restriction \delta) \star (p_{1\lambda}^\uparrow)$ over the model $V[G_\lambda \star R]$ as in Lemma 2.15, it meets $\mathcal{W}_F$. It follows that p_1 forces that there is a $\xi \in \delta$ such that $\dot{H}_\xi \subset A_F$. Applying this fact to $\zeta = \min(H_\delta)$ completes the proof. □

Now we show that Moore-Mrowka tasks will arise that will allow us to prove there is a minor additional condition that we can place on the construction of $P_{\kappa+\kappa}$ (assuming an extra $\diamondsuit$-principle) that will force there are no separable Moore-Mrowka spaces of cardinality greater than $\mathfrak{c}$. Let S^κ_1 denote the set of $\lambda \in \kappa$ that have cofinality ω_1. We will assume there is a $\diamondsuit(S^\kappa_1)$-sequence.

We begin with this Lemma.

Lemma 5.6 ($\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$)**.** *Let X be a separable Moore-Mrowka space of cardinality greater than $\mathfrak{c}$. Let $X \in M$ be an elementary submodel of $H(\theta)$ for some sufficiently large θ such that $|M| = \mathfrak{c}$ and $M^\mu \subset M$ for all $\mu < \mathfrak{c}$. For any point $z \in X \setminus M$ there is a sequence $\{B_\eta : \eta < \mathfrak{c}\}$ of countable subsets of $M \cap X$ satisfying, for all $\eta < \zeta < \mathfrak{c}$,*

(1) $\overline{B_\eta}$ contains $B_\zeta \cup \{z\}$

(2) for all $A \subset M \cap X$ with $z \in \overline{A}$, there is an $\alpha < \mathfrak{c}$ such that $\overline{A}$ contains B_α.

Proof. Since X is separable, we can let $B_0 \in M$ be any countable dense subset. Fix an enumeration $\{S_\xi : \xi < \mathfrak{c}\}$ of all the countable subsets of $M \cap X$ that have z in their closure. Let $\mathcal{W} \in M$ be a base for the topology. Assume we have chosen $\{B_\xi : \xi < \eta\}$ for some $\eta < \mathfrak{c}$. Assume, by induction, that B_ξ is also a subset of $\overline{S_\xi}$. The set $\overline{S_\eta} \cup \{\overline{B_\xi} : \xi < \eta\}$ is an element of M and every member contains z. Let K_η denote the intersection of this family. Choose any neighborhood $U \in \mathcal{W}$ of z. Since $z \in W \cap K$, it follows from elementarity that $M \cap W \cap K_\eta$ is non-empty. Therefore, z is in the closure of some countable $B_\eta \subset M \cap K_\eta$. This completes the inductive construction of the family. We simply have to verify that property (2) holds. Let $z \in \overline{A}$ for some $A \subset M \cap X$. By countable tightness, there is an η such that $S_\eta \subset A$. Therefore $\overline{A} \supset B_\eta$. □

Remark 10. A compact separable space of cardinality at most $\mathfrak{c}$ will have a G_δ-dense set of points of character less than $\mathfrak{c}$. Therefore, in a model with $\mathfrak{p} = \mathfrak{c}$, any such space has the property that the sequential closure of any subset is countably compact. In particular, in such a model a Moore-Mrowka space necessarily has weight at least $\mathfrak{c}$ and will have a countably compact subset that is not closed. A space is said to be C-closed if it has no such subspace, see [7, 10].

Definition 5.7. *Say that a sequence $\langle y_\alpha, U_\alpha, W_\alpha : \alpha < \kappa\rangle$ is a κ-MM sequence of a space X if*

1. *U_α, W_α are open in X and $y_\alpha \in U_\alpha \subset \overline{U_\alpha} \subset W_\alpha$,*
2. *$y_\gamma \notin U_\alpha$ for all $\alpha < \gamma \in \kappa$,*
3. *for all $\beta < \alpha < \kappa$, $U_\gamma \cap \{y_\beta, y_\alpha\} = \{y_\alpha\}$ for some $\alpha \leq \gamma \in \kappa$,*
4. *for every $A \subset \kappa$, there is a countable $B \subset A$ and a $\gamma < \kappa$ such that the closure of $\{y_\alpha : \gamma < \alpha < \kappa\}$ is either contained in the closure of $\{y_\beta : \beta \in B\}$ or is disjoint from the closure of $\{y_\alpha : \alpha \in A\}$.*

Theorem 5.8. *Let $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa+\kappa,\ \beta < \kappa+\kappa\rangle$ be an iteration sequence in the sense of Theorem 4.3. In particular, assume that for all $\mu < \kappa$ there is a $\mathbf{q}_\mu \in \mathcal{Q}(\mu,\mu)$ satisfying that $P_{\kappa+\lambda}$ is equal to $P^{\mathbf{q}_\mu(\kappa)}_{\kappa+\mu}$.*

Let $\dot{X}$ be a $P_{\kappa+\kappa}$-name of a compact separable space of countable tightness. Assume also that $\langle \dot{y}_\alpha, \dot{U}_\alpha, \dot{W}_\alpha : \alpha < \kappa\rangle$ is forced to be a κ-MM sequence of $\dot{X}$. Then there is a cub $C_{\dot{X}} \subset \kappa$ such that for each $\lambda \in C_{\dot{X}} \cap S^\kappa_1$, there is an injection $f_\lambda : \omega_1 \to \lambda$ such that $\mathcal{A} = \langle \dot{A}_\eta : \eta < \omega_1\rangle$, where $\dot{A}_\eta = \{\xi : y_{f_\lambda(\xi)} \in \dot{U}_{f_\lambda(\eta)}\}$, is forced by $P^{\mathbf{q}_\lambda}_{\lambda+\lambda}$ to be a Moore-Mrowka task.

Proof. We may assume, since it is forced to be compact and separable, that $\dot{X}$ is a $P_{\kappa+\kappa}$-name of a closed subspace of $[0,1]^\kappa$. Let G be a $P_{\kappa+\kappa}$-generic filter so that we may make some observations about $\dot{X}$ and the κ-MM sequence $\langle y_\alpha, U_\alpha, W_\alpha : \alpha < \kappa\rangle$. There is a point $z \in \mathrm{val}_G(\dot{X})$ that is a κ-accumulation point of $\{y_\alpha : \alpha \in \kappa\}$. We check that z is the unique such point. If U, W are open neighborhoods of z with $\overline{U} \subset W$, then $A = \{\alpha \in \kappa : y_\alpha \in U\}$ is cofinal in κ. By condition (4) of the κ-MM property, there is a countable $B \subset A$ so that the closure of $\{y_\beta : \beta \in B\}$ contains $\{y_\alpha : \sup(B) < \alpha < \kappa\}$. It thus follows that that $\{y_\alpha : \sup(B) < \alpha < \kappa\}$ is contained in W and shows that $X \setminus W$ contains no κ-accumulation points of $\{y_\alpha : \alpha \in \kappa\}$. Now assume that z is in the closure of $\{y_\beta : \beta \in A\}$ for some $A \subset \kappa$. Since the second clause of condition (4) of the κ-MM property fails, it follows that there is a countable $B \subset A$ such that the closure of $\{y_\beta : \beta \in B\}$ contains a final segment of $\{y_\alpha : \alpha \in \kappa\}$. We will be interested in the subspace $X_\lambda = \{x \restriction \lambda : x \in X\}$ of $[0,1]^\lambda$. Since this space is a continuous image of X, it also has countable tightness. Let $\dot{z}$ be a canonical $P_{\kappa+\kappa}$-name for z.

Let $M \prec H(\kappa^+)$ so that $\sup(M \cap \kappa) = \lambda \in S^\kappa_1$ and $M^\omega \subset M$. We note that it follows from Corollary 2.14, and the fact that $P_{\kappa+\kappa}/P_\kappa$ is ccc, that every countable subset of $M \cap \kappa$ in $V[G]$ has a name in M. Assume also that $\dot{z}, \dot{X}, P_{\kappa+\kappa}$ and the κ-MM sequence are elements of

M. Choose any continuous $\in$-increasing sequence $\{M_\eta : \eta \in \omega_1\}$ of countable elementary submodels of M such that $Y_\lambda = \bigcup\{M_\eta \cap \lambda : \eta \in \omega_1\}$ is cofinal in λ. Define f_λ so that $f_\lambda(\eta) = \sup(M_\eta \cap \lambda)$. It should be clear that to show that $\mathcal{A}$, as in the statement of the Theorem, is forced by $P^{\mathbf{q}_\lambda}_{\lambda+\lambda}$ to be a Moore-Mrowka task it is sufficient to check that condition (2) of Definition 5.3 is forced to hold. Let $\dot{A}$ be any $P^{\mathbf{q}_\lambda}_{\lambda+\lambda}$-name of an uncountable subset of ω_1. We may regard $P^{\mathbf{q}_\lambda}_{\lambda+\lambda}$ as a complete subposet of $P_{\kappa+\kappa}$ and so consider $\mathrm{val}_G(\dot{A})$ in $V[G]$. In the space X_λ, it is clear that $z \restriction \lambda$ is in the closure of the set $\{y_{f_\lambda(\eta)} : \eta \in A\}$. Therefore, there is a countable $B \subset A$ such that $z \restriction \lambda$ is in the closure of the set $\vec{y}(f_\lambda(B)) = \{y_{f_\lambda(\eta)} : \eta \in B\}$. Now B is a countable subset of $M \cap \lambda$, and so there is a $P^{\mathbf{q}_\lambda}_{\lambda+\lambda}$-name $\dot{B}$ in M such that $\mathrm{val}_G(\dot{B})$ is B. Now we can apply elementarity (using that $f_\lambda \restriction B \in M$) and observe that $\dot{z}$ is forced to be in the closure of $\{\dot{y}_{f_\lambda(\beta)} : \beta \in \dot{B}\}$. Moreover, by elementarity and the κ-MM property, there is a $\gamma \in \kappa \cap M$ such that the closure of $\vec{y}(f_\lambda(\dot{B}))$ is forced to contain $\{\dot{y}_\alpha : \gamma < \alpha < \kappa\}$. For each $\gamma < \alpha < \kappa$, $\vec{y}(f_\lambda(\dot{B}))$ is forced to meet $\bigcap_{\zeta\in H} \dot{U}_\zeta$ for all finite $H \subset \{\zeta : \alpha \in \dot{U}_\zeta\}$. Of course there is an $\delta \in \omega_1$ such that $\gamma < f_\lambda(\delta)$. This completes the proof that, for all $\beta \in \omega_1 \setminus \delta$, $\dot{A} \cap \delta$ is forced to meet $\bigcap_{\zeta\in H} \dot{A}_\zeta$ for all finite $H \subset \{\zeta : \beta \in \dot{A}_\zeta\}$. □

Theorem 5.9. *It is consistent with Martin's Axiom and $\mathfrak{c} > \aleph_2$ that there are no S-spaces and that compact separable spaces of countable tightness have cardinality at most $\mathfrak{c}$.*

Proof. Let $\kappa > \aleph_2$ be a regular cardinal in a model of GCH. Using an iteration sequence as in Theorem 4.3, it follows from Theorem 5.8 and Lemma 5.6 that it suffices to ensure that for each $\dot{X}$ and κ-MM-sequence as in Theorem 5.8, there is a $\lambda \in C_{\dot{X}} \cap S^\kappa_1$ so that I_λ is chosen suitably and so that $\dot{Q}_{\kappa+\lambda}$ is chosen to be $\mathcal{M}(\mathcal{A}, \dot{C}_\lambda)$ for a sequence $\mathcal{A}$ as identified in Theorem 5.8. This is a somewhat routine application of $\diamondsuit(S^\kappa_1)$.

Since S^κ_1 is stationary, we may assume that $\diamondsuit(S^\kappa_1)$ holds in V. There are many equivalent formulations of $\diamondsuit(S^\kappa_1)$ and we choose this one: There is a sequence $\langle h_\alpha : \alpha \in S^\kappa_1\rangle$ satisfying

1. for each $\alpha \in S^\kappa_1$, $h_\alpha : \alpha \times \alpha \to \alpha$ is a function,
2. for all functions $h : \kappa \times \kappa \to \kappa$, the set $\{\alpha \in S^\kappa_1 : h_\alpha \subset h\}$ is stationary.

We will also have to recursively define our sequence $\mathcal{I} = \{I_\gamma : \gamma \in \mathbf{E}\}$ since special choices will have to be made for indices in S^κ_1 and which, due to conditions (3) and (4) impact all the subsequent choices. To assist with the condition (4) of the requirements on $\mathcal{I}$, we choose an enumeration $\{J_\xi : \xi \in \kappa\}$ of $[\kappa]^{\aleph_1}$ as follows. Let $D \subset \kappa$ be a cub consisting of λ such that $\mu + \mu^{\aleph_1} < \lambda$ for all $\mu < \lambda$. For each $\mu \in D$, the list $\{J_\xi : \mu \leq \xi < \mu + \mu^{\aleph_1}\}$ is an enumeration of $[\mu]^{\aleph_1}$.

Say that a sequence $\mathcal{I}_\lambda = \{I_\gamma : \gamma \in \mathbf{E} \cap \lambda\} \subset [\lambda]^{\leq \aleph_1}$ is an acceptable sequence if it satisfies the properties (1), (2), and (3) that we assume for the sequence $\mathcal{I}$ in section 2, and, it also satisfies that, for each $\xi < \mu \in \lambda$ such that $\mu + \mu^{\aleph_1} < \lambda$, there is a $\zeta \in \mathbf{E} \cap \mu + \mu^{\aleph_1}$ such that $J_\xi \subset I_\zeta$. If $\{\mathcal{I}_\lambda : \lambda \in D\}$ is an increasing sequence of acceptable sequences, then the union, $\mathcal{I}$, satisfies the requirements of section 2. Similarly, once we have chosen an acceptable sequence $\mathcal{I}_\lambda$, we will assume that the sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$ is defined as in Definition 2.2 using the sequence $\mathcal{I}_\lambda$.

In a similar fashion, we relativize the definition of $\mathcal{Q}(\lambda, \mu)$ from Definition 4.1. Given an acceptable sequence $\mathcal{I}_\lambda$, say that a sequence $\mathbf{q}' = \{\dot{Q}'_\beta : \beta < \lambda\} \in H(\kappa)$ is $\mathcal{I}_\lambda$-suitable providing (as in Theorem 4.3), by induction on $\beta < \lambda$, $\dot{Q}^\lambda_\beta(\mathbf{q}) = \dot{Q}'_\beta$ is a $P^\lambda_{\lambda+\beta}(\mathbf{q})$-name of a ccc poset, where $P^\lambda_\alpha(\mathbf{q}) = P_\alpha$ for $\alpha \leq \lambda$ and, for $\beta > 0$, $P^\lambda_{\lambda+\beta}(\mathbf{q})$ is the usual poset from the iteration sequence $\langle P^\lambda_\alpha(\mathbf{q}), \dot{Q}^\lambda_\zeta(\mathbf{q}) : \alpha \leq \beta, \zeta < \beta \rangle$.

Let f be any function from κ onto $H(\kappa)$. We recursively choose our sequences $\{\mathcal{I}_\lambda : \lambda \in D\}$ and $\{\dot{Q}'_\gamma : \gamma \in \kappa\}$. The critical inductive assumptions are, for $\lambda \in D$,

(1) $\mathcal{I}_\lambda$ extends $\mathcal{I}_\mu$ for all $\mu \in D \cap \lambda$,
(2) $\mathcal{I}_\lambda$ is acceptable,
(3) $\{\dot{Q}'_\gamma : \gamma < \lambda\}$ is $\mathcal{I}_\lambda$-suitable.

Now let $\lambda \in D$ and assume we have constructed, for each $\mu \in D \cap \lambda$, $\mathcal{I}_\mu$ and and $\{\dot{Q}'_\gamma : \gamma < \mu\}$. If $D \cap \lambda$ is cofinal in λ, then we simply let $\mathcal{I}_\lambda = \bigcup \{\mathcal{I}_\mu : \mu \in D \cap \lambda\}$ and there is nothing more to do. Otherwise, let μ be the maximum element of $D \cap \lambda$.

Case 1: $\mu \notin S^\kappa_1$. First choose any acceptable $\mathcal{I}_\lambda \supset \mathcal{I}_\mu$. Choose $\{\dot{Q}'_\beta : \mu \leq \beta < \lambda\}$ by induction as follows. For $\mu < \beta \notin \mathbf{E}$, let $\mathbf{q}$ denote $\{\dot{Q}'_\gamma : \gamma < \beta\}$. Let $\zeta < \kappa$ be minimal so that $\dot{Q}'_\beta = f(\zeta)$ is a $P^\mu_{\mu+\beta}(\mathbf{q})$-name of a ccc poset that is not in the list $\{\dot{Q}'_\gamma : \gamma < \beta\}$. For $\mu \leq \beta \in \mathbf{E}$, choose, if possible minimal $\zeta < \kappa$ so that $f(\zeta)$ is equal to $Q(\mathcal{U}, \dot{C}_\beta)$ for some S-space task that is not yet handled and let $\dot{Q}'_\beta = f(\zeta)$. Otherwise, let $\dot{Q}'_\beta = \mathcal{C}_\omega$.

The verification of the inductive hypotheses in Case 1 is routine. We also note that if the induction continues to κ, then $P^\kappa_{\kappa+\kappa}(\{\dot{Q}'_\beta : \beta < \kappa\})$ will force that there are no S-spaces and that Martin's Axiom holds.

Case 2: $\mu \in S^\kappa_1$. Let $\mathbf{q}$ denote $\{\dot{Q}'_\beta : \beta < \mu\}$. Now we decode the element h_μ from the $\diamondsuit$-sequence. If there is any $(\alpha, \xi) \in \mu \times \mu$ such that $f(h_\mu(\alpha, \xi))$ is not a $P^\mu_{\mu+\mu}(\mathbf{q})$-name, then proceed as in Case 1. For each $\alpha \in \mu$, if $f(h_\mu(\alpha, 0))$ is not a name of a finite subset of μ, then proceed as in Case 1, otherwise let $\dot{F}_\alpha = f(h_\mu(\alpha, 0))$. Similarly, if there is an $\alpha \in \mu$ such that $f(h_\mu(\alpha, 1))$ is not a name of a positive rational number, then proceed as in Case 1, otherwise let $\dot{\epsilon}_\alpha = f(h_\mu(\alpha, 1))$. If there is an $\alpha \in \mu$ and a $\xi > 1$ such that $f(h_\mu(\alpha, \xi))$ is not a name of a element of $[0, 1]$, then proceed as in Case 1, otherwise let

$$\text{for } (\alpha, \xi) \in \mu \times \mu \quad \dot{y}_\alpha(\xi) = \begin{cases} f(h_\mu(\alpha, \xi + 2)) & \text{if } \xi < \omega \\ f(h_\mu(\alpha, \xi)) & \text{if } \omega \leq \xi < \mu \end{cases}.$$

It now follows that $\dot{y}_\alpha$ is a name of an element of $[0, 1]^\mu$ and let the name $\{x \in [0, 1]^\mu : (\forall \beta \in \dot{F}_\alpha)|x(\beta) - \dot{y}_\alpha(\beta)| < \dot{\epsilon}_\alpha\}$ be denoted by $\dot{U}_\alpha$. Now we ask if there is a function $f_\mu : \omega_1 \to \mu$ as in Theorem 5.8. In particular, if there is an $I \in [\mu]^{\aleph_1}$ and such a function $f_\mu : \omega_1 \to \mu$ such that the sequence $\mathcal{A} = \{\dot{A}_\eta : \eta \in \omega_1\}$ as defined in the statement of Theorem 5.8 satisfies that $P^\mu_{\mu+\mu}(\mathbf{q})$ forces that $\mathcal{A}$ is a Moore-Mrowka task and each $\dot{A}_\alpha$ is a $P^\mu_{\mu+\mu}(\mathbf{q})(I) \star \dot{R}_0$-name in the sense of Lemma 5.5. If all this holds, then choose an appropriate I_μ so that $I \subset I_\mu$ and define $\dot{Q}'_\mu$ to be $\mathcal{M}(\mathcal{A}, \dot{C}_\mu)$. For the remaining choices proceed as in Case 1.

The construction of $P_{\kappa+\kappa} = P^\kappa_{\kappa+\kappa}(\mathbf{q})$ where $\mathbf{q} = \{\dot{Q}'_\beta : \beta < \kappa\}$ is complete. As explained at the beginning of the proof, it follows from Lemma 5.6 and Theorem 5.8, and that the fact that D is a cub, that separable Moore-Mrowka spaces in this model have cardinality at most $\mathfrak{c}$. □

References

[1] Uri Abraham, Matatyahu Rubin, and Saharon Shelah, *On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types*, Ann. Pure Appl. Logic **29** (1985), no. 2, 123–206, DOI 10.1016/0168-0072(84)90024-1. MR801036

[2] Uri Avraham and Saharon Shelah, *Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic*, Israel J. Math. **38** (1981), no. 1-2, 161–176, DOI 10.1007/BF02761858. MR599485

[3] A. V. Arhangel′skiĭ, *The structure and classification of topological spaces and cardinal invariants*, Uspekhi Mat. Nauk **33** (1978), no. 6(204), 29–84, 272 (Russian). MR526012

[4] Zoltán T. Balogh, *On compact Hausdorff spaces of countable tightness*, Proc. Amer. Math. Soc. **105** (1989), no. 3, 755–764, DOI 10.2307/2046929. MR930252

[5] Alan Dow and Todd Eisworth, *CH and the Moore-Mrowka problem*, Topology Appl. **195** (2015), 226–238, DOI 10.1016/j.topol.2015.09.025. MR3414886

[6] Alan Dow, *Compact spaces of countable tightness in the Cohen model*, Set theory and its applications (Toronto, ON, 1987), Lecture Notes in Math., vol. 1401, Springer, Berlin, 1989, pp. 55–67, DOI 10.1007/BFb0097331. MR1031765

[7] A. Dow, *Compact C-closed spaces need not be sequential*, Acta Math. Hungar. **153** (2017), no. 1, 1–15, DOI 10.1007/s10474-017-0739-x. MR3713559

[8] V. V. Fedorčuk, *Completely closed mappings, and the compatibility of certain general topology theorems with the axioms of set theory*, Mat. Sb. (N.S.) **99 (141)** (1976), no. 1, 3–33, 135 (Russian). MR0410631

[9] A. Hajnal and I. Juhász, *On hereditarily α-Lindelöf and α-separable spaces. II*, Fund. Math. **81** (1973/74), no. 2, 147–158, DOI 10.4064/fm-81-2-147-158. MR336705

[10] Mohammad Ismail and Peter Nyikos, *On spaces in which countably compact sets are closed, and hereditary properties*, Topology Appl. **11** (1980), no. 3, 281–292, DOI 10.1016/0166-8641(80)90027-9. MR585273

[11] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded. MR1940513 (2004g:03071)

[12] I. Juhász, *A survey of S- and L-spaces*, Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978), Colloq. Math. Soc. János Bolyai, vol. 23, North-Holland, Amsterdam-New York, 1980, pp. 675–688. MR588816

[13] William Mitchell, *Aronszajn trees and the independence of the transfer property*, Ann. Math. Logic **5** (1972/73), 21–46, DOI 10.1016/0003-4843(72)90017-4. MR313057

[14] A. J. Ostaszewski, *On countably compact, perfectly normal spaces*, J. London Math. Soc. (2) **14** (1976), no. 3, 505–516, DOI 10.1112/jlms/s2-14.3.505. MR438292

[15] Mary Ellen Rudin, *A normal hereditarily separable non-Lindelöf space*, Illinois J. Math. **16** (1972), 621–626. MR309062

[16] Saharon Shelah, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin-New York, 1982. MR675955

[17] Stevo Todorčević, *Forcing positive partition relations*, Trans. Amer. Math. Soc. **280** (1983), no. 2, 703–720, DOI 10.2307/1999642. MR716846

University of North Carolina at Charlotte, Charlotte, NC 28223
Email address: `adow@uncc.edu`

Department of Mathematics, Rutgers University, Hill Center, Piscataway, New Jersey, U.S.A. 08854-8019
Current address: Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem 91904, Israel
Email address: `shelah@math.rutgers.edu`