β**N**

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Mary Ellen Rudin is, of course, widely recognized as a major contributor to the knowledge of βN chiefly through her early involvment in the study of the partial orders of the types of ultrafilters. Her influential and well-known paper [28] gave wide publicity to the study of these orders and developed many of the basic properties (many of which had appeared in her earlier but poorly circulated [27]). The monograph [29] was the standard reference for set-theoretic topology for many years and, together with her Martin's Axiom survey article [30], helped break the floodgates of independence results about βN.

In this article we will first go back to the paper [28]; learn a few tricks about constructing filters and present the basic properties of the orders discussed, including the Rudin-Keisler and Rudin-Frolík orderings. Our discussion of the Rudin-Frolík ordering leads us to Kunen's construction of weak P-points and modifications of it by van Mill. We put a different perspective on the Rudin-Shelah construction of $2^{\mathfrak{c}}$ pairwise incomparable Rudin-Keisler ultrafilters and finish with a brief discussion of the combinatorial principle NCF which is related to Mary Ellen's work on composants of the Stone-Čech remainder of the real-line.

1 Partial Orderings on βN

βN is the space of ultrafilters on N, where the fixed ultrafilters are identified with the points in N, and the topology is the usual Stone space topology on the Boolean algebra $\mathcal{P}(N)$. That is, the closure of $A \subset N$, $\bar{A}$, consists of A together with the set of free ultrafilters including A, denoted A^*. Therefore, the topology can be characterized by the fact that a subset of N and its complement have complementary compact open closures.

Since βN is the Stone-Čech compactification of N, it follows that any continuous function f from N into a compact space X, extends uniquely to a continuous function βf from βN into X. Clearly $\beta f(p)$ will be a certain kind of limit point in X of the sequence $\{f(n) : n \in N\}$. We discuss this in more detail below by introducing Bernstein's concept of a p-limit [2].

Two ultrafilters are said to be of the same *type* (in βN) if there is a permutation f of N such that βf takes one to the other. Equivalently, p is of the same type as q if there is an autohomeomorphism of βN sending p to q. Since there are only $\mathfrak{c}$ such autohomeomorphisms of βN but $2^{\mathfrak{c}}$ many points in βN, βN is very much non-homogeneous in the sense that there are $2^{\mathfrak{c}}$ distinct types. Although there may be $2^{\mathfrak{c}}$ autohomeomorphisms of N^* (under CH for example), N^* was proven not to be homogeneous by studying partial orderings of the types of βN.

Definition 1.1 *$p \leq_{RK} q$ if there is a function $f: N \mapsto N$ such that $f^{-1}(I) \in q$ for each $I \in p$ – (equivalently, $\beta f(q) = p$).*

$p \leq_{RF} q$ if there is a function $f: N \mapsto \beta N$ such that $\beta f(p) = q$ and βf is an embedding.

To obtain a clearer picture let us recall some of the basic facts found in [28].

Proposition 1.2

1. *An injective function, $f: N \mapsto \beta N$, extends to an embedding precisely when its range is discrete.*
2. *If $p \leq_{RF} q$, then $p \leq_{RK} q$.*
3. *If $p < q$ and $q < r$, then $p < r$ (where $<$ is either $\leq_{RF}$ or $\leq_{RK}$).*
4. *If $f: N \mapsto N$ and $\beta f(p) = p$, then $\{n : f(n) = n\} \in p$.*
5. *If $p \leq_{RK} q$ and $q \leq_{RK} p$, then p and q have the same type.*

Proof. Let f be an injective function from N into βN with discrete range. Therefore, for each n, we may choose an element A_n of the ultrafilter $f(n)$ such that $\bar{A}_n \cap \{f(m) : m \in N - \{n\}\} = \emptyset$. Since $A_n - \bigcup_{m<n} A_m$ will still be a member of $f(n)$ we may assume that the A_n's are pairwise disjoint. Now if I and J are disjoint subsets of N, then $\{f(n) : n \in I\}$ and $\{f(n) : n \in J\}$ have disjoint closures in βN since they are contained in the closures of $\bigcup_{n\in I} A_n$ and $\bigcup_{n\in J} A_n$, respectively, and these sets, being disjoint subsets of N, have disjoint closures. It follows easily that βf is injective.

Now suppose that $\beta f(p) = q$ – i.e. $p \leq_{RF} q$. Define a function g from N to N as follows: $g(m) = n$ for all n and $m \in A_n$ and $g(m) = 0$ for all $m \in N - \bigcup_n A_n$. Now, since $\beta f(p) = q$, $q \in \overline{\{f(n) : n \in I\}} \subset \overline{g^{-1}(I)}$ for each $I \in p$. Hence it follows that $\beta g(q) = p$ and $p \leq_{RK} q$.

It is immediate that $\leq_{RF}$ and $\leq_{RK}$ are transitive. Suppose that $\beta f(p) = p$ for some $f: N \mapsto N$. Let I_0 be the set of fixed points of f. It is a classical result, that we can find a partition of N into I_1 , I_2 , I_3 such that $f[I_i - I_0] \cap I_i$ is empty for $i = 1, 2, 3$. Perhaps the simplest is to notice that one can apply Zorn's Lemma to the (non-empty) family of pairwise disjoint triples which have the above property and whose union is closed under f. A maximal triple is easily seen to be a partition. Now $I_i \in p$ for some i and, then so is $f(I_i) \cap I_i \subset I_0$.

The final item follows from the previous as follows. Fix functions f and g such that $\beta f(p) = q$ and $\beta g(q) = p$. Note that $\beta(g \circ f)(p) = p$. Therefore there is an $I \in p$ such that $f \restriction I$ is injective. We may assume that $N - (I \cup f[I])$ is infinite, hence $f \restriction I$ can be extended to a permutation whose extension takes p to q. $\square$

For some of us it is difficult to remember Definition 1.1 because the direction of the maps is reversed. There is a useful alternate view of these orders using the Bernstein notion of p-limit. Call a sequence of sets (or points) *discrete* if each member of the sequence has a neighbourhood whose closure is disjoint from all the other members of the sequence.

Definition 1.3 *Let $p \in N^*$ and let $\{A_n : n \in N\}$ be a collection of subsets of a space X. We call a point $x \in X$ a p-limit of the sequence $\{A_n : n \in N\}$ if x is a limit point of $\{A_n : n \in I\}$ for each $I \in p$. Equivalently, x is a p-limit point of $\{A_n : n \in N\}$ if, for each neighbourhood U of x, the set $\{n \in N : U \cap A_n \neq \emptyset\}$ is in p. We let $p\text{-}\lim\{A_n : n \in N\}$ denote the set of all p-limits of $\{A_n : n \in N\}$, hence*

$$p\text{-}\lim\{A_n : n \in N\} = \bigcap_{I \in p} cl_X \left[\bigcup_{n \in I} A_n \right] .$$

We also say that x is a p-limit of $\{x_n : n \in N\}$ if $x \in p\text{-}\lim\{\{x_n\} : n \in N\}$.

Proposition 1.4 *If p, q are in βN, then $p \leq_{RK} q$ iff q is a p-limit of some discrete sequence of subsets of βN.*

Also, $p \leq_{RF} q$ iff q is a p-limit of a discrete sequence of points in βN.

Proof. If $p \leq_{RK} q$, then there is a function f from N to N such that $f^{-1}(I) \in q$ for each I in p. This is easily seen to be a restatement of the fact that q is the p-limit of the sequence $\{f^{-1}(n) : n \in N\}$. Conversely, suppose that q is the p-limit of a discrete sequence $\{K_n : n \in N\}$. Since this sequence is discrete, there is a neighbourhood U_0 of K_0 whose closure misses $\bigcup_{n>0} K_n$. Let $A_0 = U_0 \cap N$ and notice that $\overline{A_0} \subset \overline{U_0}$. Clearly the sequence $\{\overline{A_0}\} \cup \{K_n : n > 0\}$ is again a discrete sequence. Hence we can inductively choose pairwise disjoint $A_n \subset N$ so that $K_n \subset \overline{A_n}$. Let $f \in {}^N N$ be any function such that $f[A_n] = \{n\}$ for each $n \in N$. To show f witnesses that $p \leq_{RK} q$, it suffices to show that $f(J)$ is a member of p for each $J \in q$. Since q is the p-limit of $\{K_n : n \in N\}$, it follows that $f(J) = \{n : J \cap A_n \neq \emptyset\} \supset \{n : \overline{J} \cap K_n \neq \emptyset\}$ is a member of p . $\Box$

One of the fundamental and remarkable properties of $\leq_{RF}$ is that it is tree-like, i.e. for each $q \in \beta N$, the set $\{p \in \beta N : p \leq_{RF} q\}$ is linearly ordered.

This is basically a consequence of the following.

Proposition 1.5 *If A, B are σ-compact subsets of βN and q is a common limit point, then there is an $I \in q$ such that either $(A \cap \overline{I}) \subset \overline{B}$ or $(B \cap \overline{I}) \subset \overline{A}$.*

A pair of sets are said to be *separated* if each one is disjoint from the closure of the other; i.e. separated σ-compact subsets of βN have disjoint closures. Now to prove that $\{p \in \beta N : p \leq_{RF} q\}$ is linearly ordered by $\leq_{RF}$, assume that q is a p-limit of a discrete set A and an r-limit of a discrete set B and (wlog) that $B \subset \overline{A}$. Now the canonical homeomorphism from βN to $\overline{A}$ will take p to the p-limit of A, namely q. Since q is an r-limit of a discrete subset of $\overline{A}$, it follows that p is an r-limit of a discrete subset of βN – hence $r \leq_{RF} p$.

By reversing our perspective in the above paragraph we come to another extremely important result of Frolík's: (in Frolík's notation) a type cannot *produce* itself. Let us choose a discrete B contained in $\overline{A} - A$, for some discrete $A \subset \beta N$ and fix q, r, p as above. The result is that p is not equal to r. That is, p is not the p-limit of any discrete subset of $\beta N - N$. We will deduce this from the following result of Rudin.

Proposition 1.6 *q is $\leq_{RK}$-minimal in $\{q : p \leq_{RK} q$ and $r \leq_{RK} q\}$ if and only if whenever there are pairwise disjoint $\{A_n : n \in N\}$ and pairwise disjoint $\{B_n : n \in N\}$ such that q is a p-limit of $\{A_n : n \in N\}$ and an r-limit of $\{B_n : n \in N\}$, there is an I in q such that $I \cap A_n \cap B_m$ is at most a singleton for each pair n and m.*

We can deduce Frolík's result from the above as follows. Suppose that p is a p-limit of a discrete set $\{x_n : n \in N\} \subset N^*$. Let $A_n \in x_n$ be chosen pairwise disjoint as in Proposition 1.2. Now let $B_n = A_n$ and apply the above Proposition with $q = r = p$ to find $I \in p$ such that $I \cap A_n$ is at most a singleton for each n. We now have a contradiction since it follows that $\overline{I}$ is a neighbourhood of p which does not contain any of the x_n.

The $\leq_{RK}$ ordering is most definitely not tree-like. In fact it is as upwards directed as it can get. It follows immediately from the definition that no point can have more than $\mathfrak{c}$ predecessors. Rudin showed that the $\leq_{RK}$ order is directed, Blass [3] showed that it was ω_1-directed and Comfort and Negrepontis (and independently Kunen, see [11, 10.10]) showed that any set of at most $\mathfrak{c}$ types has an upper bound (this is proven in Proposition 2.7 below). Kunen [17] showed that the $\leq_{RK}$ ordering is not a chain (hence neither is the $\leq_{RF}$ ordering). The fact that the $\leq_{RF}$ ordering is tree-like but is not a chain has many applications, especially in the class of those spaces which share the property of Proposition 1.5.

Proposition 1.7 *No infinite homogenous compact space has the property that all pairs of countable separated subsets have disjoint closures.*

Proof. Assume that X is an infinite compact space has the property and choose an infinite discrete set $A \subset X$. Fix $\leq_{RF}$-incomparable p, q in N^*. Let x be a (unique) p-limit of A and let y be a q-limit. Assume that f is a homeomorphism of X into itself which takes x to y. Let D be the dense set of isolated points in $A \cup f[A]$ and indentify $\overline{D}$ with βN. In this copy of βN, the point y is both a p-limit (of $f[A]$) and a q-limit of A. It then follows that p and q are $\leq_{RF}$-comparable. Since there are $\leq_{RF}$-incomparable points, X is not homogeneous. $\square$

Kunen has found substantial strengthenings of the above result using *weak P-points* in [20] (also see §2). Kunen defined weak P-points in [19] as a point of N^* which is not the limit point of any countable subset of N^* and showed that there are $2^{\mathfrak{c}}$ such points (see Proposition 2.6). We can use this to answer a recent question of Pelant and Weiss (answered independently by van Mill).

Proposition 1.8 *If $\mathcal{D}$ is any family of at most $\mathfrak{c}$ separable subsets of N^*, then there is an infinite closed set contained in $N^* - \bigcup \mathcal{D}$.*

Proof. Fix a set $D \subset \bigcup \mathcal{D}$ of cardinality $\mathfrak{c}$ such that D contains a dense subset of each member of $\mathcal{D}$. We may assume that D has the property that for each $d \in D$, if there is a countable set of weak P-points with d as a limit, then there is such a subset of D. Since there are $2^{\mathfrak{c}}$ weak P-points, fix a countable set $E \subset N^* - D$ of weak P-points. We check that $\overline{E}$ is disjoint from the closure of every countable subset of D - hence $\overline{E} \subset N^* - \bigcup \mathcal{D}$ is empty. Assume that $A \subset D$ is countable and that $\overline{E} \cap \overline{A}$ is not empty. Then, by Proposition 1.5 and the fact that the members of E are weak P-points, it follows that $A \cap \overline{E}$ is not empty. Therefore each member of $A \cap \overline{E}$ is a limit of a countable set of weak P-points, hence there is a countable set of weak P-points in D, say B, such that $\overline{B} \cap \overline{E}$ is not empty. However, this contradicts Proposition 1.5, because every point of $B \cup E$ is a weak P-point. $\square$

Much of the paper [28] is devoted to showing that we cannot expect (unfortunately) that the structure of the partial orders on βN will lead to a characterization of the types. This is done primarily by constructing several different ultrafilters (assuming CH). We will focus on the constructions and the information gained about the orders and not discuss the possibility of characterizing the types. Rudin actually also defines a third order which we will denote by $\sqsubseteq$. The definition of $\sqsubseteq$ is the condition 3 below.

Proposition 1.9 *If $p, q \in \beta N$ are not the same type then the following are equivalent.*

1. *$p \sqsubseteq q$,*
2. *q is a p-limit of a pairwise disjoint sequence of clopen subsets of N^**
3. *there is a function $f: N \mapsto N$ such that $\beta f(q) = p$ and, for each $I \in q$, the set $\{n : I \cap f^{-1}(n)$ is infinite $\}$ is a member of p,*
4. *there is a sequence $\{A_n : n \in N\}$ of disjoint subsets of N, such that for each $I \in q$, the set $\{n : I \cap A_n$ is infinite $\}$ is a member of p.*
5. *q is a p-limit of a sequence $\{A_n : n \in N\}$ of disjoint subsets of N, such that for each sequence $A'_n \in [A_n]^{<\omega}$, q is not a p-limit of $\{A'_n : n \in N\}$.*
6. *q is a p-limit of a discrete sequence $\{K_n : n \in N\}$ of subsets of N^**

It is not immediately clear that the three orders are in fact distinct. It is clear that $\leq_{RF} \subseteq \sqsubseteq \subseteq \leq_{RK}$. Indeed, if p and q are not the same type then, using p-limits:

1. $p \leq_{RF} q$ if q is a p-limit of a discrete sequence of points in N^*.
2. $p \sqsubseteq q$ if q is a p-limit of a discrete sequence of subsets of N^*.
3. $p \leq_{RK} q$ if q is a p-limit of a discrete sequence of subsets of βN.

One can establish that $\leq_{RK} \not\subseteq \leq_{RF}$ is several distinct ways.

Proposition 1.10 1. *If q is a weak P-point which is not a P-point, then there is a $p \leq_{RK} q$ but q is $\leq_{RF}$-minimal.*

2. *(CH) For each $p \in \beta N$, there are $\leq_{RK}$-incomparable q, r such that $p \leq_{RK} q$, $p \leq_{RK} r$, and q is $\leq_{RF}$ minimal. In fact q can be chosen so that there is no r which is strictly $\leq_{RK}$ between p and q.*
3. *If p and r are $\leq_{RK}$-incomparable, then any q which is $\leq_{RK}$ above them both is $\leq_{RF}$ above at most one.*

Rudin established (CH) that all three orderings are distinct. The $\sqsubseteq$ relation can be substituted in for $\leq_{RK}$ in item 1 above. Interestingly, it seems to have been open if it can be shown in ZFC that $\sqsubseteq$ and $\leq_{RK}$ are distinct. This fact was brought to my attention by S. Garcia-Ferreira. Even the statement that they are distinct can be reformulated to sound like a much more interesting question.

Recall that a point q is a P-point if and only if whenever q is a limit point of a pairwise disjoint sequence $\{A_n : n \in N\}$ of subsets of N, there is an $I \in q$ such that $I \cap A_n$ is finite for each n.

Call q a p-P-point if whenever q is a p-limit point of a pairwise disjoint sequence $\{A_n : n \in N\}$ of subsets of N, there is an $I \in q$ such that $I \cap A_n$ is finite for each n. Since it may happen that q is not a p-limit of any sequence of sets (i.e. in case $p \not\leq_{RK} q$) and we already know that p is a p-P-point, we will say that q is a strict p-P-point if q is strictly above p in the $\leq_{RK}$ ordering.

Therefore the question of whether the two orders, $\sqsubseteq$ and $\leq_{RK}$, are equal can be restated as: 'Is there a strict p-P-point for some $p \in N^*$?'. Andreas Blass [7] observed that there are such points (in fact our earlier construction from a P-point only required a weak P-point). Recall that Kunen showed that there are many $\leq_{RK}$-incomparable weak P-points.

Proposition 1.11 *Suppose that p is a weak P-point which is not the* <u>*minimum*</u> *in the $\leq_{RK}$ order. Then there is a strict p-P-point (i.e. $\leq_{RK}$ is not contained in $\sqsubseteq$).*

Proof. Let $\{A_n : n \in N\}$ be any sequence of pairwise disjoint infinite subsets of N. For each n, let p_n be the unique p-limit of the sequence obtained by enumerating A_n in increasing order. Let r be any point which is not $\leq_{RK}$ above p. Let q be the r-limit of the sequence $\{p_n : n \in N\}$ (q is equal to $r \otimes p$ according to the terminology of Proposition 3.10). To see that $p \leq_{RK} q$ note that q is the p-limit of the sequence $\{I_m : m \in N\}$ where I_m is the set which selects the m-th element from each A_n.

Now suppose that q is the p-limit of a disjoint sequence $\{B_n : n \in N\} \subset \mathcal{P}(N)$. Without loss of generality $\bigcup_n B_n = N$. Define $g \in {}^N N$ by $g(B_n) = n$ for each n and let βg denote the continuous extension from βN onto βN. It follows that $\beta g(q) = p$ and that p is the r-limit of the sequence $\{\beta g(p_n) : n \in N\}$. Clearly $\{\beta g(p_n) : n \in N\} \cap N$ is a discrete set, hence p cannot be the r-limit of it (since $p \not\leq_{RK} r$). However neither is p a limit of any countable subset of $N^* - \{p\}$. Therefore we have that $J = \{n : \beta g(p_n) = p\}$ is a member of r.

Now fix an n in J and let f_n denote the unique order preserving map from N onto A_n, hence $p_n = \beta f_n(p)$. Since $\beta(g \circ f_n)(p) = p$, it follows from Proposition 1.2 that $\tilde{A}_n = \{m : g \circ f_n(m) = m\}$ is a member of p. Therefore $I = \bigcup_{n \in J} f_n[\tilde{A}_n - n]$ is a member of q. Finally we finish by observing that

$|I \cap B_k| \leq k$ for each k; indeed, by the definition of g, we have that, for each $n \in J$, $f_n[\tilde{A}_n] \cap B_k \subset \{f_n(k)\}$. $\square$

It appears very unlikely but it is not known if it is consistent that the $\leq_{RK}$ order has a minimum. A related question raised by Shelah is if there is some ultrafilter p which is $\leq_{RK}$ comparable with all others. It has been shown that there is no such p if $\mathfrak{c}$ is regular (Hindman [15], see also Butkovičová [10]). Keisler proved that it follows from CH that there are minimals in $\leq_{RK}$. An ultrafilter p is $\leq_{RK}$-minimal if and only if it is *selective*. The definition "follows" from Proposition 1.6: whenever p is a limit of pairwise disjoint $\{A_n : n \in N\}$, there is an $A \in p$, such that $|A \cap A_n| \leq 1$ for all n. Kunen [18] proved that it is consistent that there are no $\leq_{RK}$-minimal elements. The next result is probably the main result of Rudin's paper [28]. The construction of the point which is a weak P-point but not a P-point is quite ingenious.

Theorem 1.12 (CH) *There are p, q, r such that p is $\leq_{RK}$-minimal and, for each of q and r, p is the only type which is $\sqsubseteq$ below. But, in addition and by contrast, $p \leq_{RF} r$ while q is a weak P-point and $p \sqsubseteq q$ (hence not a P-point).*

Proof. Rather than repeating the proof from [28] of this theorem we will give a sketch by isolating the different aspects of the proof which are designed to accomplish the separate goals of the construction.

It is actually rather straightforward to construct r. Suppose we have found p - an $\leq_{RK}$-minimal type. Take r to be the p-limit of any countable discrete set of points, $\{p_n : n \in N\}$, each of the same type as p - clearly $p \leq_{RF} r$. Now we suppose that r is an s-limit of a sequence $\{B_m\}_m$, and define $g[B_m] = m$ as in Proposition 1.11. Hence $\beta g(r) = s$ and consider the set $\{\beta g(p_n) : n \in N\}$. By the minimality of p we may as well assume that $\{n : \beta g(p_n) = s\}$ is empty. Therefore s is the p-limit of the sequence $\{\beta g(p_n) : n \in N\}$, which we may assume by the minimality of p, are all distinct. Furthermore, each $\beta g(p_n)$ is either in N or of the same type as p. If "p-many" are in N, then s is equal to p. Otherwise, we may assume that each $\beta g(p_n)$ is the same type as p (hence a P-point), and it follows that s is the same type as r.

Now we analyse (topologically) what we need to construct q. Of course, q will be constructed in an induction of length ω_1 such that at each stage of the induction the approximation to q will be countably generated. The trick is to find ω_1 "do-able" tasks which will successfully construct q.

Since p is to be $\sqsubseteq$ below q, we must fix infinite A_n and guarantee that q will be a p-limit of $\{A_n^* : n \in N\}$. Let $X_n = A_n^*$ and let $X = \bigcup_n X_n$. Since $\overline{X}$ is homeomorphic to βX, let us, at the risk of confusion, use X^* to denote $\overline{X} - X$.

It is a routine part of the induction to ensure that the filter p will be $\leq_{RK}$-

minimal.

It is equally routine to ensure that q will be a P-point of the boundary X^* – i.e. q will not be a limit point of any σ-compact subset of X^*. This goes a long way towards ensuring both that q will be a weak P-point and that p will be the only $\sqsubseteq$ predecessor to q. Furthermore, for these two properties, we really only have to worry about subsets of $\overline{X}$ since q will be a limit of a σ-compact set C if and only if q is a limit of $C \cap \overline{X}$. More is needed to guarantee that p is the only $\sqsubseteq$ predecessor to q. It is easily checked that the following suffices and it is clearly "do-able". For every sequence $\{B_m : m \in N\}$ of pairwise disjoint subsets of N, there is a member I of q such that, for each n there is at most one m such that $I^* \cap X_n \cap B_m^* \neq \emptyset$.

Now all that remains is to make q a weak P-point. Kunen had a different CH construction of weak P-points at the time and later, of course, a ZFC construction. This step is the least obvious because as Kunen remarks, one needs a base-type property because one cannot simply list all the countable sets and avoid them one at a time.

Rudin uses an extremely novel idea. As we carry out the induction, we pick P-points p_α in X (say $p_{\beta+n} \in X_n$ for each limit β) together with enumerations $\{A_{\alpha,\beta} : \alpha \leq \beta < \omega_1\}$ of a descending (mod finite) base for p_α such that

$$\bigcup_n A_{\beta+n,\beta+n} \in q \text{ for each limit } \quad \beta < \omega_1 \ .$$

We may obtain the following two properties, for each limit β:

for each $n \in N$, there is an $\alpha_n < \beta$ such that $A_{\beta+n,\beta+n} \subset A_{\alpha_n,\beta}$

and, $\{A_{\alpha,\beta} : \alpha < \beta\}$ are pairwise disjoint.

Amazingly, this guarantees that q is not the limit point of any countable subset of X. Indeed, suppose that D is a countable subset of X. Since each p_α is a P-point, there is an $f(\alpha) \in \omega_1$ such that $A^*_{\alpha,f(\alpha)}$ is disjoint from $D - \{p_\alpha\}$. Since D is countable, there is a $\gamma < \omega_1$ such that $D \cap \{p_\alpha : \gamma \leq \alpha < \omega_1\}$ is empty. Let β be a limit greater than $f(\alpha)$ for all $\alpha < \gamma$. Then $\bigcup_n A_{\beta+n,\beta+n}$ is a member of q which misses D since

$$A^*_{\beta+n,\beta+n} \cap D \subset \left[A^*_{\alpha_n^\beta,\beta} - A^*_{\alpha_n^\beta,\beta+n}\right] \cap D \subset \left[A^*_{f(\alpha_n^\beta)} - \{p_{\alpha_n^\beta}\}\right] \cap D = \emptyset \ . \ \square$$

The fact that the proof above could be broken into separate parts reflects a more general situation in how the constructions of special filters is often carried out. Given a collection $\mathcal{Q}$ of subsets of a space X, a collection $\mathcal{F}$ (usually a filter) is $\mathcal{Q}$-remote if for each $Q \in \mathcal{Q}$, there is a member of $\mathcal{F}$ whose closure in βX is disjoint from the closure of Q. Therefore a weak P-point is $\mathcal{Q}$-remote where $\mathcal{Q}$ is the collection of countable sets. Other interesting classes for $\mathcal{Q}$ are "countable and nowhere dense", "countable and discrete" and, for the original remote points, simply "nowhere dense".

Let X be a space which can be partitioned into a countable locally finite collection of closed sets X_n. Call a filter $\mathcal{F}$ of subsets of X *nice*, if each member of $\mathcal{F}$ is disjoint from at most finitely many of the X_n. As usual, let us denote $\beta X - X$ by X^*. J. van Mill has adapted Kunen construction of weak P-points and shown how to extend any nice filter $\mathcal{F}$ to a filter $\mathcal{F}^*$ such that the unique point in the intersection of the βX closure of members of $\mathcal{F}^*$ is not the limit of any countable subset of X^* (the adaptation of Kunen's technique for those X^* with weight $\mathfrak{c}$ is relatively straightforward but van Mill has a very clever innovation for larger spaces).

Therefore, the current situation in the problems of constructing points in βX which are $\mathcal{Q}$-remote, where $\mathcal{Q}$ is some collection of countable sets, is that it is only necessary to construct nice $(\mathcal{Q} \cap \mathcal{P}(X))$-remote filters on X. The major remaining open problem is to determine if every such βX will have a point which is not the limit of any countable discrete set.

2 Other constructions

Some obvious questions we can ask about a partial order:

Are there minimal/maximal elements? If so what are their properties? Can an element have an immediate successor? Does every element have an immediate predecessor?

We already discussed the problem of $\leq_{RK}$-minimals. Their existence is not provable in ZFC and if they exist they are selective. One of the basic properties of all the orders is that they do not have maximal elements.

The minimals in the $\leq_{RF}$ order are points that are not the limit of any countable discrete set. They exist in ZFC. There are even $\leq_{RF}$-minimal points which are not weak P-points.

It is not known to me if, in ZFC, there is an element which has no immediate $\leq_{RK}$-predecessor. However Butkovacova [9] showed that there is such a point for the $\leq_{RF}$ order. Consider the space ${}^{<\omega}N$ with the following topology. Let p be a weak P-point ultrafilter and take the largest topology such that each $t \in {}^{<\omega}N$ is the p-limit of $\{t^\frown n : n \in N\}$. It turns out (see [13]) that this space is regular, Hausdorff and extremally disconnected (i.e. the closure of every open set is open). Therefore it can be embedded into N^* as $\{x_t : t \in {}^{<\omega}N\}$. Consider the $\leq_{RF}$-predecessors of $x_\emptyset$. Certainly $p \leq_{RF} x_\emptyset$ and so is p^2, where $p^{k+1} = p^k$-$\lim\{p_n : n \in N\}$ and $\{p_n : n \in N\}$ is a discrete set of points each of type p. Indeed, $x_\emptyset$ is the p^2-limit of the discrete set $\{x_t : t \in {}^2N\}$. More generally, if $x_\emptyset$ is a q-limit of a discrete subset D of $\{x_t : t \in {}^{<\omega}N\}$, then each $x \in D$ is a q-limit of a discrete subset of $\{x_t : t \in {}^{<\omega}N\}$. It follows easily then, that $x_\emptyset$ is a q^2-limit of a discrete subset of $\{x_t : t \in {}^{<\omega}N\}$. Therefore if we are able to ensure that $x_\emptyset$ is not a limit of any discrete set which is disjoint from $\{x_t : t \in {}^{<\omega}N\}$, then no q is an immediate predecessor of $x_\emptyset$, since $q \leq_{RK} x_\emptyset$ implies $q \leq_{RK} q^2 \leq_{RK} x_\emptyset$.

It is shown in [9] and in [13] that no point of ${}^{<\omega}N$ is a limit point (in this topology) of any countable subset of of $\beta({}^{<\omega}N) - {}^{<\omega}N$. Hence if we embed ${}^{<\omega}N$ (with the topology as above) into N^* so that its closure is a weak P-set, then $x_\emptyset$ (indeed, x_t for each $t \in {}^{<\omega}N$) will have no immediate $\leq_{RF}$-predecessor.

This is what Butkovacova did but we will see that we can do this by Theorem 2.6 – a special case of a result of van Mill[24]. In the previous proof we saw that p^2 is an immediate $\leq_{RF}$-successor of p.

Corollary 2.1 *There are points p, q such that q is an immediate $\leq_{RF}$-successor of p.*

To prove the theorem of van Mill (Theorem 2.6) we first obtain a simple generalization of Kunen's result that there is a $\mathfrak{c}$ by $\mathfrak{c}$ independent matrix.

Lemma 2.2 *If N^* maps onto a space X, then N^* also maps onto $X \times (\mathfrak{c}+1)^{\mathfrak{c}}$*

Proof. We prove this algebraically using Simon's proof that N^* maps onto $(\mathfrak{c}+1)^{\mathfrak{c}}$ (which was originally proved by Kunen). We may assume that X is zero-dimensional and fix a subalgebra $\mathcal{X} \subset \mathcal{P}(N)$ such that $\mathcal{X}/fin$ is isomorphic to $CO(X)$. We will produce an appropriate subalgebra of the power set of

$$\bigcup_{n \in N} {}^{\mathcal{P}(n)}\mathcal{P}(n) \quad .$$

For each pair $S, T \subset N$, let

$$A_{S,T} = \bigcup_{n \in N} \{s \in {}^{\mathcal{P}(n)}\mathcal{P}(n) : s(S \cap n) = T \cap n\}$$

and for each $X \in \mathcal{P}(N)$, let

$$A_X = \bigcup_{n \in X} {}^{\mathcal{P}(n)}\mathcal{P}(n) \quad .$$

It should be clear that $\{A_X : X \in \mathcal{X}\}$ is isomorphic to $\mathcal{X}$. We can repeat the proof from [19] that for each $S \subset N$, the family

$$\{A_{S,T} : T \subset N\}$$

is pairwise almost disjoint and if $\mathcal{S}$ is any finite subset of $\mathcal{P}(N)$ and, for each $S \in \mathcal{S}$, a set $T(S) \subset N$ is chosen then, for each n such that $\{S \cap n : S \in \mathcal{S}\}$ are pairwise distinct (i.e. all but finitely many n), there is an $s_n \in {}^{\mathcal{P}(n)}\mathcal{P}(n)$ such that $s_n \in A_{S,T(S)}$ for each $S \in \mathcal{S}$. Therefore, it follows that, for each infinite $X \in \mathcal{X}$, $A_X \cap \bigcap_{S \in \mathcal{S}} A_{S,T(S)}$ is infinite. $\square$

Next it is useful to isolate a couple more of Kunen's wonderful ideas for constructing weak P-points.

Definition 2.3 *For a cardinal κ and a space X, call a set $K \subset X$, a κ-OK set if for every increasing sequence $\{C_n : n \in \omega\}$ of regular-closed sets disjoint from K, there is a set $\mathcal{U}$, of neighbourhoods of K, such that $|\mathcal{U}| = \kappa$ and, for each $n \in N$, the intersection of any n-many members of $\mathcal{U}$ is disjoint from C_n. When there is such a family $\mathcal{U}$, let us say that K is κ-OK with respect to $\{C_n : n \in N\}$*

Lemma 2.4 *If K is κ-OK for some uncountable κ, then no point of K is a limit point of any countable subset of $X - \overline{K}$.*

Proof. Let $\{x_n : n \in N\}$ be any countable subset $X - \overline{K}$ and fix, for each $n \in N$, a regular-open neighbourhood, W_n, of x_n whose closure misses $\overline{K}$. Let $C_n = \bigcup_{k \leq n} \overline{W_k}$ and let $\mathcal{U}$ witness that K is κ-OK with respect to $\langle C_n : n \in N\rangle$. Then some $U \in \mathcal{U}$ must be disjoint from $\{x_n : n \in N\}$ since otherwise a single x_n will be a member of infinitely many (in fact, uncountably many) members of $\mathcal{U}$. Take any n of these to obtain an intersection which is not disjoint from C_n (i.e. it contains x_n). $\square$

The next lemma is the main idea behind the construction.

Lemma 2.5 *Suppose that φ is a mapping from N^* onto $(\kappa+1)^I$ (with I infinite) and that $K \subset N^*$ is a closed subspace such that φ also maps K onto $(\kappa+1)^I$. Finally suppose that $\{C_n : n \in N\}$ is an increasing sequence of closed sets which are disjoint from K. Then there is a countable subset I' of I and a closed set $K' \subset K$ such that φ composed with the projection map, maps K' onto $(\kappa+1)^{I-I'}$ and such that K' is κ-OK with respect to $\{C_n : n \in N\}$*

Proof. Fix $I' = \{i_n : n \in N\} \subset I$ arbitrarily. We may suppose that each C_n is clopen. For each $\xi \in \kappa$, define $Y(\xi, n) = \varphi^{-1}([(i_n, \xi)]) - C_{n+1}$. Now, $\bigcup_n Y(\xi, n)$ is a cozero subset of N^* which meets each C_n in a clopen set. It follows that there is a clopen set Y_ξ such that, for each n,

$$Y_\xi \cap C_n = C_n \cap \bigcup_n Y(\xi, n) \ .$$

K' is simply $K \cap \bigcap_{\xi \in \kappa} Y_\xi$ and the Y_ξ witness that K' is κ-OK with respect to $\{C_n : n \in N\}$. To check this, fix any $\xi_0 < \xi_1 < \cdots \xi_{n-1}$. Now $C_n \cap \bigcap_{i<n} Y_{\xi_i}$ is not empty if and only if there is a sequence j_i $(i < n)$ of integers less than $n-1$ such that $C_n \cap \bigcap_{i<n} Y(\xi_i, j_i)$ is not empty. But clearly, there must be $i < i' < n$ such that $j = j_i = j_{i'}$, which means that $Y(\xi_i, j) \subset \varphi^{-1}([(j, \xi_i)])$ and $Y(\xi_{i'}, j) \subset \varphi^{-1}([(j, \xi_{i'})])$ are disjoint.

It remains to show that K' maps onto $(\kappa+1)^{I-I'}$. It suffices to show that for any basic clopen set $B \subset (\kappa+1)^{I-I'}$ and any increasing finite sequence $(\xi_k : k < n)$,

$$\varphi^{-1}(B) \cap \bigcap_{k<n} Y(\xi_k) \cap K$$

is not empty. This follows from the fact that

$$\varphi^{-1}(B) \cap \bigcap_{k<n} Y(\xi_k) \supset \varphi^{-1}\big(B \cap \bigcap_{k<n} [(i_k, \xi_k)]\big) - C_{n+1}$$

which meets K since the preimage of every non-empty basic clopen subset of $(\kappa+1)^I$ meets K while C_{n+1} is disjoint from K. $\square$

Before proving the next lemma we recall the notion of an irreducible map. A map f from X to Y is *irreducible* if it is surjective and no proper closed subset of X maps onto Y. A simple Zorn's Lemma argument can be used to establish that for every continuous function f with compact domain X there is a closed subspace K of X such that $f \restriction K$ is irreducible and $f[K] = f[X]$.

Theorem 2.6 *If N^* maps onto an extremally disconnected space X, then X can be embedded in N^* as a weak P-set.*

Proof. Let φ be a mapping onto $X \times (\mathfrak{c}+1)^{\mathfrak{c}}$ as given as in Lemma 2.2. For a subset I of $\mathfrak{c}$ let π_I denote the projection from $X \times (\mathfrak{c}+1)^{\mathfrak{c}}$ onto $X \times (\mathfrak{c}+1)^I$. Now fix an enumeration $\{\langle C(\alpha, n) : n \in N\rangle : \alpha < \mathfrak{c}\}$ of all increasing sequences of subsets N with each listed $\mathfrak{c}$ times.

We will inductively construct a sequence K_α for $\alpha < \mathfrak{c}$ of closed subsets of N^* together with sets $I_\alpha \subset \mathfrak{c}$ such that $\pi_{I_\alpha} \circ \varphi$ maps K onto $X \times (\mathfrak{c}+1)^I$ (this is just dual to Kunen's notion of an independent matrix modulo a filter). In order to be sure the induction will go $\mathfrak{c}$ steps, we ensure that $\mathfrak{c} - I_\alpha$ has cardinality at most $|\alpha| \cdot \aleph_0$. Since Gleason showed that compact extremally disconnected spaces are projective, it follows that any irreducible map onto a compact extremally disconnected space is a homeomorphism. Therefore, to ensure that $K_{\mathfrak{c}}$ will be homeomorphic to X, we will guarantee that $\pi_\emptyset \circ \varphi$ is an irreducible mapping onto X. To do this, at each stage we will have that either $C(\alpha, 0)^*$ contains

$K_{\alpha+1}$, or that $\pi_\emptyset \circ \varphi$ does not map $K_{\alpha+1} \cap C(\alpha, 0)^*$ onto X. That is, we can take $K_{\alpha+1}$ to be $K_\alpha \cap C(\alpha, 0)^*$ if $\pi_{I_\alpha} \circ \varphi$ maps it onto $X \times (\mathfrak{c}+1)^{I_\alpha}$. If it does not then choose non-empty clopen subsets $B \subset (\mathfrak{c}+1)^{I_\alpha}$ and $A \subset X$ such that $\varphi^{-1}(A \times B) \cap K_\alpha$ is disjoint from $C(\alpha, 0)^*$. Let $K_{\alpha+1} = \varphi^{-1}(B) \cap K$ and let $I_{\alpha+1}$ be I_α minus the support of B. It is straightforward to check that the induction hypotheses hold.

Finally, to ensure that $K_\mathfrak{c}$ is a weak P-set, we will just apply the construction of Lemma 2.5 if each $C(\alpha, n)^*$ is disjoint from K_α – hence ensuring that $K_{\alpha+1}$ is $\mathfrak{c}$-OK with respect to $\langle C(\alpha, n) : n \in N\rangle$. It is enough to only worry about clopen sets since N^* is compact and zero-dimensional. It should be clear that in Lemma 2.5, we can add that K' will map onto $X \times (\mathfrak{c}+1)^{I'}$. $\square$

One of the difficulties with the $\leq_{RK}$ ordering in controlling the set of predecessors is that it is not tree-like. Laflamme [21] has shown that the predecessors of a type can form a **decreasing** chain of any countable well-order type. On the other hand there does not seem to be an easily stated property that would guarantee that a (non-minimal) point does not have an immediate predecessor. For example, a point which is simultaneously a q-point and a P-point is minimal but there are points with any other combination of these two properties which have immediate predecessors. There is a fairly routine CH construction of a point with no immediate $\leq_{RK}$ predecessors. Are either of the following consistent for the $\leq_{RK}$ order? No point has an immediate predecessor. Every point has an immediate predecessor.

We finish this section with the proof that the $\leq_{RK}$ order is upwards directed.

Proposition 2.7 *Any set of at most $\mathfrak{c}$ types has an upper bound in the $\leq_{RK}$ order.*

Proof. Fix any function φ from N onto a dense subset of the separable space $N^\mathfrak{c}$. Extend φ to a continuous function from βN onto $(\beta N)^\mathfrak{c}$ (still call it φ). Let $\{p_\alpha : \alpha < \mathfrak{c}\}$ be a set of free ultrafilters on N and let $\vec{p} \in (\beta N)^\mathfrak{c}$ denote the point $\langle p_\alpha : \alpha \in \mathfrak{c}\rangle$. Fix a closed subspace $K \subset N^*$ such that φ maps K irreducibly onto $(\beta N)^\mathfrak{c}$. We claim that each point p of $K \cap \varphi^{-1}(\vec{p})$ is $\leq_{RK}$ above each p_α. Indeed, $\vec{p}$ is a p-limit of the sequence $\{[(\alpha, n)] : n \in N\}$, hence, by the irreducibility of $\varphi \restriction K$, p is a p_α-limit of $\varphi^{-1}([(\alpha, n)])$. $\square$

3 Independent families and many types

Kunen constructed two $\leq_{RK}$-incomparable ultrafilters by using an independent family to *guide* an induction through $\mathfrak{c}$ steps. Rudin and Shelah [32] refined this technique to construct $2^\mathfrak{c}$ pairwise incomparable types. Simon[33] has a nice construction which combines the Rudin-Shelah construction with Kunen's weak P-point construction to prove a result which has frequent applications: there are $2^{\mathfrak{c}}$ pairwise $\leq_{RK}$-incomparable weak P-points.

Definition 3.1 *A family $\mathcal{I} \subset \mathcal{P}(N)$ is* **independent** *if for each disjoint pair of finite subfamilies $\mathcal{D}$ and $\mathcal{E}$, the set $\bigcap \mathcal{D} - \bigcup \mathcal{E} = \bigcap \mathcal{D} \cap \bigcap_{E \in \mathcal{E}} N - E$ is not empty. Given a filter $\mathcal{F}$, the family $\mathcal{I}$ is said to be independent mod $\mathcal{F}$, if*

$$F \cap \bigcap \mathcal{D} \cap \bigcap_{E \in \mathcal{E}} N - E$$

is not empty for each $F \in \mathcal{F}$ and $\mathcal{D}, \mathcal{E}$ as above.

We will formulate and prove the Rudin-Shelah result in the language of topological dynamics. For example, as mentioned above, the structure of an independent family mod a filter $\mathcal{F}$ is dual to that of a closed subspace K of $\bigcap_{F \in \mathcal{F}} F^*$ which maps onto $2^{\mathcal{I}}$. The $\leq_{RK}$ ordering on N^* can be just thought of as the ordering induced by the inclusion ordering on the *orbits* of points under the family of **surjective** continuous self-maps of βN.

If S is a collection of continuous partial functions from X to itself, let, for $x \in X$,

$$O_S(x) = \{f(x) : f \in \mathcal{S} \text{ and } x \in \operatorname{dom}(f)\} \ .$$

A set Y is pairwise S-incomparable if, for distinct $x, y \in Y$, $x \notin O_S(y)$.

First we shall prove the following two theorems. (Actually we will be lazy and always assume, with no more apologies, that our spaces are all zero-dimensional).

Theorem 3.2 *If X is any compact space of weight equal to a regular cardinal κ which has no points of character less than κ and if S is any family of at most κ continuous partial functions from X to X, then X contains a pairwise S-incomparable set of cardinality 2^κ.*

Theorem 3.3 *If X is any compact space of weight κ which has an irreducible mapping onto $[0,1]^\kappa$ (hence no points of character less than κ) and if S is any family of at most κ continuous partial functions from X to X, then X contains a pairwise S-incomparable set of cardinality 2^κ.*

Shelah observed that it suffices to find a set of size κ^+ because of the free set lemma. That is, each set $O_S(x)$ has cardinality at most κ and, by the Čech-Pospisil theorem, X has cardinality 2^κ, hence if $2^\kappa > \kappa^+$, there is a $O_S(\cdot)$-free set of cardinality 2^κ.

The idea is standard. We have a compact space X. A subset of X is called a $G_{<\kappa}$-set if it is equal to the intersection of fewer than κ open subsets of X. By an obvious generalization of the Baire category theorem, when κ is regular the intersection of any family $\mathcal{U}$ consisting of at most κ $G_{<\kappa}$-dense and $G_{<\kappa}$-open sets is $G_{<\kappa}$-dense. Most of the time, an inductive construction can be reformulated as the statement that in some compact space X, every member of a certain κ-sized family of sets are $G_{<\kappa}$-dense and $G_{<\kappa}$-open.

Let S be any collection of at most κ continuous functions from a compact subset of X into X. Suppose we have already chosen some collection Y of at most κ points in X. We want to find another point x in X so that

1. for each $f \in S$ and $y \in Y \cap \mathrm{dom}(f)$, $f(y) \neq x$.
2. for each $f \in S$ and $y \in Y \cap \mathrm{ran}(f)$, $x \notin f^{-1}(y)$.

Obviously, $X - \{f(y)\}$ is an open subset of X which is $G_{<\kappa}$-dense just as long as the character of $f(y)$ is not less than κ. Just as obviously, the set $X - f^{-1}(y)$ is $G_{<\kappa}$-dense just as long as $f^{-1}(y)$ does not contain a non-empty $G_{<\kappa}$-set – which in all likelihood it will. Therefore we must take care not to choose any such points; in particular, we must find a collection of no more than κ $G_{<\kappa}$-dense, $G_{<\kappa}$-open sets such that any point in the intersection will not be such a point. This is really the only hard part of the proof.

Definition 3.4 *Given a space X and a collection S of functions, let $B(S,X)$ denote the collection of points $x \in X$ such that $f^{-1}(x)$ contains a non-empty $G_{<w(X)}$-subset of X for some $f \in S$.*

By adopting the topological approach, most of the difficulty is taken care of by the following two lemmas. We will show that $X - B(S,X)$ contains the intersection of κ many $G_{<w(X)}$-dense, $G_{<w(X)}$-open sets.

Lemma 3.5 *Assume that X is a compact space in which every point has character at least κ, then no space of weight less than κ maps onto a $G_{<\kappa}$-dense subset of X.*

Proof. Suppose that $g\colon Y \mapsto X$ where Y has weight less than κ. Fix any x in the range of g and fix a base $\mathcal{A}$ for Y with cardinality less than κ. For each $a \in \mathcal{A}$, choose a neighbourhood U_a of x such that, if possible, $U_a \cap g(a) = \emptyset$. Let $F = \bigcap\{U_a : a \in \mathcal{A}\}$. Now if C is any clopen subset of X which meets F but does not contain x, then $g^{-1}(C \cap F)$ is empty. Indeed, fix any $a \in \mathcal{A}$ which is contained in $g^{-1}(C)$ and which meets $g^{-1}(C \cap F)$. Therefore, U_a was chosen to miss $g(a)$ which contradicts that $\emptyset \neq g(a) \cap F \subset C \cap F \subset U_a$. $\square$

Lemma 3.6 *Suppose that X is a compact space in which no point has character less than κ and suppose that f is a continuous function from some compact subset of X into X. Let B be any collection of fewer than κ regular-open subsets of X and define $U_{f,B}$ to be the set of points x such that $f^{-1}(x)$ does not contain any non-empty intersection of members of B. Then $U_{f,B}$ contains a $G_{<\kappa}$-dense, $G_{<\kappa}$-open subset.*

Proof. To show that $U_{f,B}$ is as above, it suffices to show that for any $G_{<\kappa}$-set K, $K - U_{f,B}$ is not $G_{<\kappa}$-dense in K. Since we may assume B is a Boolean subalgebra of $CO(X)$ (recall that we are being lazy), there is a canonical continuous function g from X onto $S(B)$. Let Y be the set of all points $y \in S(B)$ such that $f(g^{-1}(y))$ is a singleton in K, call it $h(y)$. Since no point of K has character less than κ and the weight of Y is less than κ, we are finished, by Lemma 3.5, if we show that h is a continuous function. Let C be a clopen subset of X and fix any $y \in Y$ such that $h(y) \in C$. Therefore, $g^{-1}(y) \subset f^{-1}(C)$. Since $\mathrm{dom}(f) - f^{-1}(C)$ is compact and $g^{-1}(y)$ is the intersection of members of B, there is a $b \in B$ such that $g^{-1}(y) \subset b$ and $b \cap \mathrm{dom}(f) \subset f^{-1}(C)$. Therefore $b \in y$ (in the sense of $S(B)$) and $h(b) \subset C$. It follows that h is continuous. $\square$

Corollary 3.7 *Let κ be a regular cardinal and let X be a compact space of weight κ such that no point of X has character less than κ. Then if S is any collection of no more than κ continuous partial functions on X and if $Y \subset X - B(S,X)$ has cardinality at most κ, there is a $G_{<\kappa}$-dense set of points $x \in X - B(S,X)$ such that $Y \cup \{x\}$ is pairwise S-incomparable.*

It should be clear how to prove Theorem 3.3 for regular cardinals κ. The singular case is handled by expressing the system as an inverse limit. This decomposition is much easier algebraically. For example the next lemma is trivial and is the only place we use the irreducible map onto $[0,1]^\kappa$.

Lemma 3.8 *Suppose that $S_0 \subset S$ and $B_0 \subset CO(X)$ both have cardinality less than κ. Then there is a $B_1 \subset CO(X)$ containing B_0 of cardinality equal to $|B_0|$, such that for each $f \in S_0$ and $b \in B_1$, $f^{-1}(b) \in B_1$. In addition, if there is an irreducible mapping of X onto 2^κ, then B_1 can be chosen so that no point of $S(B_1)$ has character less than $|B_1|$*

Therefore, by applying the lemma repeatedly, we can express X as an inverse limit system, $\varprojlim \{X_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ with bonding maps $\varphi_{\alpha,\beta}$ such that the weight of each X_α is a regular cardinal greater than $\Sigma_{\gamma<\alpha} w(X_\gamma)$ and such that for each $f \in S$ there is an $\alpha_f < \mathrm{cf}(\kappa)$ such that

$$f_\beta = \varphi_\beta \circ f \circ \varphi_\beta^{-1}$$

is a continuous function on X_β for each $\beta \geq \alpha$, where φ_β is the function from X onto X_β. We can then express S as an increasing union of S_α with $|S_\alpha|$ at most the weight of X_α such that $\alpha_f \leq \alpha$ for each $f \in S_\alpha$.

Now let us complete the proof of Theorem 3.3. Suppose that we have chosen $Y \subset X$ which is pairwise S-incomparable, has cardinality at most κ and, for each $y \in Y$ and $\alpha < \mathrm{cf}(\kappa)$, $\varphi(y) \notin B(S_\alpha, X_\alpha)$. Let $\{Y_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ be an increasing chain whose union is Y such that $|Y_\alpha| \leq w(X_\alpha)$ for each α. Now inductively choose x_α in the $G_{<w(X_\alpha)}$-set $\bigcap \varphi_{\alpha,\gamma}^{-1}(x_\gamma)$ so that $\{x_\alpha\} \cup \{\varphi_\alpha(y) : y \in Y_\alpha\}$ is pairwise $\{f_\alpha : f \in S_\alpha\}$-incomparable. To see that this can be done, we notice that we just have to meet at most $w(X_\alpha)$ many $G_{<w(X_\alpha)}$-dense, $G_{<w(X_\alpha)}$-open subsets of X_α.

The referee observes that it is worthwhile making the following deduction from Theorem 3.3.

Corollary 3.9 *Every infinite closed subset of N^* contains a family of $2^{\mathfrak{c}}$ Rudin-Keisler incomparable points.*

Proof. Let Y be any closed subset of N^* and let D be any countable infinite discrete subset. Let φ be any function from D onto a dense subset of $2^{\mathfrak{c}}$. It follows from Proposition 1.2 that $\overline{D}$ is homemorphic to βN and that φ extends to a continuous map from $\overline{D}$ onto $2^{\mathfrak{c}}$. As mentioned just before Theorem 2.6, there is a closed subset X of $\overline{D}$ such that $\varphi \restriction X$ is irreducible. For each $f \in N^N$ there is an extension to a function, βf, from βN to βN. For each such f, let f_X denote the restriction $\beta f \restriction (X \cap \beta f^{-1}(X))$. Now apply Theorem 3.3 using the collection $\mathcal{S} = \{f_X : f \in N^N\}$ $\square$

We are also grateful to Andreas Blass who supplied us with the following example. To motivate the example we remark that one is tempted to prove that every embedded copy of βN contains a large family of Rudin-Keisler incomparable points because the embedding should preserve the incomparability from the original Rudin and Shelah constructed family. However this next result shows that, at least consistently, this is not the case.

Proposition 3.10 (Blass) *If there are at least three $\leq_{RK}$-distinct $\leq_{RK}$-minimal free ultrafilters, then there are two $\leq_{RK}$-incomparable p, q and an embedding f of βN into itself such that $f(p) \leq_{RK} f(q)$.*

Proof. Given two ultrafilters p, q on N, let $p \otimes q$ be any ultrafilter which is a p-limit of a discrete collection $\{q_n : n \in N\}$ such that each q_n is $\leq_{RK}$-equivalent to q. Another way to think of this is to first construct the following ultrafilter

$\mathcal{U}$ on $N \times N$: a set $U \in \mathcal{U}$ if and only if

$$\{m \in N : \{n \in N : (m,n) \in U\} \in q\} \in p\ .$$

Then any embedding of $N \times N$ into N, sends $\mathcal{U}$ to a suitable copy of $p \otimes q$. It is shown in [8] that this operation is associative.

If p and q are $\leq_{RK}$-minimal, then $\{p, q, p \otimes q\}$ is equal (up to equivalence) to $\{r \in N^* : r \leq_{RK} p \otimes q\}$. This is essentially proven in Theorem 3.3 when we proved there that p was the only $\sqsubseteq$-predecessor to r.

Now suppose that p, q, r are all distinct $\leq_{RK}$-minimals. By the above it follows that $p \otimes q$ and $p \otimes r$ are $\leq_{RK}$-incomparable. We will find an embedding which sends $p \otimes q$ to an ultrafilter which is $\leq_{RK}$ above that to which $p \otimes r$ is sent. Indeed, fix any ultrafilter z and send $p \otimes q$ to $p \otimes q \otimes r \otimes z$ and $p \otimes r$ to $p \otimes r \otimes z$. We can easily do this as follows: fix complementary $A, B \subset N$ such that $A \in p \otimes q$ and $B \in p \otimes r$. Fix a discrete sequence $\{d_n : n \in A\}$ of "copies" of $r \otimes z$ and another discrete sequence $\{d_n : n \in B\}$ of "copies" of z such that $\{d_n : n \in N\}$ is discrete. It should be clear that by extending this embedding, $p \otimes q$ is sent to $(p \otimes q) \otimes (r \otimes z) = p \otimes q \otimes r \otimes z$ while $p \otimes r$ is sent to $(p \otimes r) \otimes z = p \otimes r \otimes z$. Consider $p \otimes q \otimes r \otimes z$ as an ultrafilter on N^4. For each $(\ell, n, o) \in N^3$, let $A_{(\ell,n,o)} = \{\ell\} \times N \times \{(n, o)\}$. It is routine to check that $p \otimes q \otimes r \otimes z$ is the $p \otimes r \otimes z$-limit of the sequence $\{A_{(\ell,n,o)} : (\ell, n, o) \in N^3\}$. The result now follows by Proposition 1.4. $\square$

4 NCF and composants

A very important and exciting development in this subject in recent years is the formulation of NCF by Blass and the proof, by Shelah, that it is consistent. Mary Ellen's involvement in this comes from her paper in which she proves that it is consistent that there are $2^{\mathfrak{c}}$ distinct composants in the Stone-Čech remainder of the half-line $\mathbb{H}$ (i.e. $\mathbb{H} = [0, \infty)$). NCF is equivalent to the statement that there is only one composant. NCF has a great deal of equally (or more) important applications (see [4, 5]). In addition, the study of the remainder of the real line has undergone something of a revival in recent years with the fine work of people such as Baldwin, Smith, Yu, and most of all Zhu. K.P. Hart has just finished a survey on the topic [14].

A pair x, y of a space X are in the same **composant** if there is a proper closed connected set C containing them both. A **continuum** is a compact connected space and a continuum is called an **indecomposable continuum** if it cannot be written as the union of two proper subcontinua. If X is an indecomposable continuum then the relation of being in the same composant is a transitive, hence an equivalence, relation. Mazurkiewicz proved that every metric indecomposable continuum contains a perfect set of points, each in a distinct composant. So after Woods and Bellamy had, independently, shown that $\mathbb{H}^*$ was an indecomposable continuum it was natural to want to estimate the number of composants.

Not surprisingly the structure of $\mathbb{H}^*$ is very tied up with the structure of N^* (although we still do not fully understand the relationship). For example, from [14], if we fix any sequence $[a_n, b_n]$ of disjoint intervals such that $b_n < a_{n+1}$ for each n, then, for any $p \in N^*$, the set of p-limits of the sequence $\{[a_n, b_n] : n \in N\}$ is a proper subcontinuum. In fact, it is not too difficult to see that any proper subcontinuum is contained in such a *standard* subcontinuum (any proper regular closed set containing the subcontinuum will induce the partition of $\mathbb{H}$ into intervals $\{[a_n, b_n] : n \in N\}$ and then each subset of N will correspond to a clopen subset of the remainder of the union of the intervals). Hence if two standard subcontinuum are contained in a single proper subcontinuum there is a natural finite-to-one map induced on the intervals.

Definition 4.1 *Two filters on N are said to be nearly coherent if there is a finite-to-one function $f\colon N \mapsto N$ which takes both the filters into a single filter.* **NCF** *is the assertion that all ultrafilters are nearly coherent.*

Theorem 4.2 $\mathbb{H}^*$ *has only one composant if and only NCF holds.*

Mary Ellen showed that if two ultrafilters on N when considered as points in $\mathbb{H}^*$ are in the same composant, then they are nearly coherent. She also showed that, assuming the continuum hypothesis, there is a set of $2^{\mathfrak{c}}$ ultrafilters on N, no two of which are nearly coherent (e.g. any family of $\leq_{RK}$-minimals).

Zhu has proven the remarkable fact that NCF is equivalent to the assertion that $\mathbb{H}^*$ can be covered by an increasing chain of nowhere dense P-sets. It is not immediately clear that it follows that N^* itself can then be covered by an increasing chain of such sets. However Zhu has established this fact (it can also be easily deduced from [4, Thm. 19]). Blass had already established that NCF implies that N^* can be covered by nowhere dense P-sets (even of a special kind). The question of whether a space can be covered by such sets arises in a variety of settings. One such setting is the question of whether the product of (κ-many) sequentially compact spaces is countably compact – the Scarborough-Stone problem. To give an example of how NCF can be applied we will give a direct proof that it implies that N^* can be covered by a chain of nowhere dense P-sets (but we do not know if this is equivalent to NCF). Also, following a suggestion of J. Vaughan, we will modify one of the forcings that Shelah showed will produce a model of NCF to produce a model in which there is a non-countably compact product of $\mathfrak{h}$ (defined below) sequentially compact spaces (this answers question 343 of [23]).

Lemma 4.3 ([4]) *NCF implies there is a P-point of character less than $\mathfrak{d}$. Hence every point is nearly coherent with a P-point, which in turn implies that N^* can be covered by nowhere dense P-sets*

Proof. By the definition of $\mathfrak{d}$ we may fix a family of strictly increasing functions $\{f_\alpha : \alpha < \mathfrak{d}\} \subset {}^N N$ of minimum cardinality such that for every function $g \in {}^N N$ there is an α such that $g(n) < f_\alpha(n)$ for all $n \in N$. Since every ultrafilter of character less than $\mathfrak{d}$ is a P-point [16], we need only show that there is some ultrafilter of character less than $\mathfrak{d}$. Assume there is not, fix any $p \in N^*$ and construct $q \in N^*$ which is not nearly coherent. For each $\alpha < \mathfrak{d}$ and $i < 4$, define $h_{i,\alpha} \in {}^N N$ by $h_{i,\alpha}([f_\alpha^{(4n+i)}(0), f_\alpha^{(4(n+1)+i)}(0))) = n$ and $h_{i,\alpha}([0, f_\alpha^i(0)\,)) = 0$.We first show that an ultrafilter q is nearly coherent with p if and only if there is an $\alpha < \mathfrak{d}$ and an $i < 4$ such that $h_{i,\alpha}$ takes both to the same (ultra)filter. Fix any finite-to-one g which takes p, q to the same filter and fix an α such that for each k, $g^{-1}(g(k)) \subset [0, f_\alpha(k))$. Choose $i < 4$ so that $I' = \bigcup_n [f_\alpha^{(4n+i+1)}(0), f_\alpha^{(4n+i+2)}(0))$ is in p. Let $I \in p$ be arbitrary and $J \in q$. We must show that $h_{i,\alpha}(I) \cap h_{i,\alpha}(J) \neq \emptyset$. Indeed choose any $k \in g(I \cap I') \cap g(J)$. Fix $\ell \in I \cap I'$ and $j \in J$ such that $g(\ell) = g(j) = k$. Since $\ell \in I'$ there is an n such that $f_\alpha^{(4n+i+1)}(0) \leq \ell < f_\alpha^{(4n+i+2)}(0)$. It follows that $j < f_\alpha^{(4n+i+3)}(0)$ since $g^{-1}(g(j)) = g^{-1}(g(i)) \subset f_\alpha(i) \leq f_\alpha(f_\alpha^{(4n+i+2)}(0))$. Similarly, $f_\alpha^{(4n+i)}(0) \leq j$ since $g^{-1}(g(j)) \subset f_\alpha(j)$ while $i \in g^{-1}(g(j)) - f_\alpha(f_\alpha^{(4n+i)}(0))$. Therefore, $h_{i,\alpha}(j) = h_{i,\alpha}(\ell) = n \in h_{i,\alpha}(I) \cap h_{i,\alpha}(J)$.

Re-enumerate the set $\{h_{i,\alpha} : i < 4 \text{ and } \alpha < \mathfrak{d}\}$ as $\{h_\alpha : \alpha < \mathfrak{d}\}$. Now inductively build an increasing sequence of filter bases $(q_\alpha \quad \alpha < \mathfrak{d})$, each of cardinality at most $|\alpha| \cdot \omega$ as follows. At stage α, $h_\alpha(p)$ is an ultrafilter and $h_\alpha(q_\alpha)$ is not (since it has character less than $\mathfrak{d}$). Therefore there is some set Y_α such that $N - Y_\alpha \in h_\alpha(p)$ and Y_α meets each member of $h_\alpha(q_\alpha)$. Let $q_{\alpha+1}$ be generated by $q_\alpha \cup \{h_\alpha^{-1}(Y_\alpha)\}$. Any ultrafilter q which contains $\bigcup_{\alpha<\mathfrak{d}} q_\alpha$ will not be nearly coherent with p. $\square$

Proposition 4.4 (Zhu) *NCF implies that N^* can be covered by an increasing chain of nowhere dense P-sets.*

Proof. Fix a P-point p_0 with character less than $\mathfrak{d}$ and let f_0 be the identity function on N. We will inductively construct P-point ultrafilters p_α and finite to one functions f_α for $\alpha < \mathfrak{d}$ so that $P_\alpha = (\beta f_\alpha)^{-1}(p_\alpha)$ is the desired chain. In addition, for each α there will be a function h_α such that $f_\alpha(n) = [h_\alpha^n(0), h_\alpha^{n+1}(0))$.

Let us first show that if $\{h_\alpha : \alpha < \mathfrak{d}\}$ is a dominating family of increasing functions then each ultrafilter is in some P_α. Let q be any ultrafilter and let g

be any finite to one function which witnesses that q and p_0 are nearly coherent. For each $X \in p_0$, define $h_X(n)$ to be large enough so that there are $j_0, j_1 \in X$ such that $\max(j_0 \cup g^{-1}(g(j_0))) < \min(j_1 \cup g^{-1}(j_1))$ and the interval $[n, h_X(n)]$ contains $\{j_0, j_1\} \cup g^{-1}(\{g(j_0), g(j_1)\})$. Find an α such that, for each X from a base for p_0, $h_\alpha(n) > h_X(n)$ for infinitely many n. We want to show that $q \in P_\alpha$. Let $Y = \{j : (\exists n) \;\; g^{-1}(g(j)) \subset f_\alpha^{-1}(n)\}$. It is sufficient to show that $Y \in p_0$, i.e. that it meets every member of the base for p_0. To see this, suppose that it is true. Then, $g^{-1}(g[Y])$ is in q and, for any $Z \in q$ with $g[Z] \subset g[Y]$, $f_\alpha(Z)$ is equal to $f_\alpha(g^{-1}(g[Z]))$. But now this is a member of p_α since $g^{-1}(g[Z]) \in p_0$ and f_α takes every member of p_0 to a member of p_α.

To show that $Y \in p_0$ let X be any member of the base for p_0. Fix any m such that $h_\alpha(m) > h_X(m)$ and fix $j_0, j_1 \in X$ as in the definition of $h_X(m)$. Let n be such that $m \in [h_\alpha^n(0), h_\alpha^{n+1}(0))$. Now either $g^{-1}(g(j_0)) \subset [h_\alpha^n(0), h_\alpha^{n+1}(0))$ (hence $j_0 \in X \cap Y$) or, if this is not the case, then $h_\alpha^{n+1}(0) \le \max(g^{-1}(j_0)) < \min(g^{-1}(j_1))$. Since we also have that $h_\alpha^{n+2}(0) = h_\alpha(h^{n+1}(0)) \ge h_\alpha(m) > h_X(m)$, it follows that $j_1 \cup f^{-1}(j_1) \subset [h_\alpha^{n+1}(0), h_\alpha^{n+2}(0))$, hence $j_1 \in X \cap Y$.

Now suppose we have chosen p_α and h_α for each $\alpha < \beta < \mathfrak{d}$. Then in place of g and p_0 above, do the same thing using f_α (hence the image of p_0 under f_α will be p_α). We can find an increasing function h (bigger than any given member of a dominating family) such that h "works" for each f_α, i.e. h is sufficiently large in the sense that the h_α above was sufficiently large. This will be our h_β, and p_β is simply the image of p_0 under h_β. In the above paragraph, we can observe that we showed that any q which was taken by g to the same thing as p_0 was sent by f_α to the same thing as p_0. Therefore, in this case, $P_\alpha \subset P_\beta$ for all $\alpha < \beta$. $\square$

Now we answer question 343 in [23], by showing that there is a model in which there is a non-countably compact product of $\mathfrak{h}$ sequentially compact spaces. First some definitions. A *tower* is a maximal descending mod finite sequence of infinite subsets of N. The cardinal $\mathfrak{t}$ is the minimum length of a tower. The cardinal $\mathfrak{h}$ can be defined as the minimum cardinal such that there is a chain of that length consisting of dense open subsets of N^* whose intersection has empty interior. Of course, $\mathfrak{t}$ is just the minimum length of a chain of clopen subsets of N^* which has empty interior. It is easy to see that if "chain" is removed from the definition of $\mathfrak{h}$ then the value is unchanged but it is not known if this is the case for $\mathfrak{t}$. The study of the cardinal $\mathfrak{h}$ (called the Baire number of N^*) was begun in [1].

It is straightforward to show that the product of $\mathfrak{t}$ sequentially compact spaces is countably compact and it was shown by Simon that the product of fewer than $\mathfrak{h}$ such spaces is countably compact. It is known (see [26]) that if N^* can be covered by a family of κ nowhere dense sets which are each equal to the intersection of a decreasing chain of clopen sets, then there is a family of κ many sequentially compact spaces whose product is not countably compact. We produce a model in which N^* is covered by $\mathfrak{h}$-many such nowhere dense sets.

Let $\mathbb{Q}$ be the rational perfect poset as defined in [25] and let $\mathbb{M}$ be the Mathias poset as defined in [22]. Start with a model of CH and fix disjoint stationary subsets of ω_2, S_0 and S_1, consisting of ordinals of cofinality ω_1. Now perform an ω_2-length countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \omega_2\rangle$ in which, at limits $\lambda \in S_0$ we use $Q_\lambda = \mathbb{M}$ and at all other α, we use $Q_\alpha = \mathbb{Q}$.

There is really nothing new to prove. Just collect together some published lemmas.

Lemma 4.5 *If $\langle P_\alpha, Q_\alpha : \alpha < \omega_2\rangle$ is a countable support proper iteration of proper posets of cardinality $\aleph_1$ such that , Q_λ is $\mathbb{M}$, for a stationary set of λ in ω_2, each with cofinality ω_1, then $\mathfrak{h}$ is ω_2.*

Proof. This lemma is essentially proven in [12] but we sketch the main idea. Recall that $\mathfrak{h}$ is equal to the minimum number of maximal antichains of $\mathcal{P}(N)/fin$ for which there is no subset of N which is almost contained in a member from each antichain. Suppose that we have a collection of fewer than ω_2 maximal antichains in the model, $V[G]$, described in the statement of the lemma. We apply a straightforward reflection argument to observe that there is a cub subset C of ω_2 such that for every $\lambda \in C$ the intersection of each of the antichains will be maximal in

$$\mathcal{P}(N) \cap \bigcup_{\beta<\lambda} V[G_\beta] \quad .$$

This is only useful for λ with cofinality ω_1 since in this case $\bigcup_{\beta<\lambda} V[G_\beta] \cap \mathcal{P}(N) = V[G_\lambda] \cap \mathcal{P}(N)$. Now we may choose a $\lambda \in C$, such that Q_λ is Mathias forcing. It is well-known that $\mathbb{M}$ will introduce a set which is almost included in some member of the restriction of each of the maximal antichains. $\square$

The following lemma is one of the key steps in showing that NCF holds in the model obtained by iterating $\mathbb{Q}$. It can be proven, for example, by combining Lemma 1 and Theorem 3 of [6].

Lemma 4.6 (CH) *Forcing with $\mathbb{Q}$ introduces a finite to one function f and a P-point p such that the filter $\{f^{-1}(X) : X \in p\}$ is contained in the filter generated by every ground model ultrafilter.*

The general iteration lemmas of Shelah in [31] imply that a proper iteration of posets which preserves all towers (i.e. all towers in the ground model are again maximal in the extension) will itself preserve all towers. Baumgartner and

Dordal showed that Mathias forcing preserves all towers. By a straightforward generalization one can show that rational perfect forcing also does but we will not prove it here. Hence the next lemma follows.

Lemma 4.7 *If $\mathcal{T}$ is a maximal descending mod finite family of subsets of N, then it remains so after any countable support iteration with iterands from $\{\mathbb{Q}, \mathbb{M}\}$.*

We outline how to verify that the model, $V[G]$ described above has the desired properties. Clearly $\mathfrak{h}$ will be ω_2 by Lemma 4.5. For each $\alpha \notin S_0$, consider the inner model $V[G \cap P_\alpha]$. By Lemma 4.6, we obtain the finite to one function f_α and a P-point p_α with the properties as stated. Hence, in the inner model $V[G \cap P_{\alpha+1}]$, $f_\alpha^{-1}(p_\alpha)$ will contain a maximal descending mod finite tower. This tower will remain maximal in $V[G]$ by Lemma 4.7. To finish the proof we have to show that every ultrafilter q in $V[G]$ contains one of these towers. By a standard reflection argument, we can fix a name for q and obtain a cub C in ω_2 such that for every $\lambda \in C \cap S_1$, $q_\lambda = q \cap V[G \cap P_\lambda]$ is a member of βN in the sense of $V[G \cap P_\lambda]$. Finally, we have, by Lemma 4.6, that $q \supset q_\lambda \supset f_\lambda^{-1}(p_\lambda)$.

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