AN INTRODUCTION TO APPLICATIONS OF ELEMENTARY SUBMODELS TO TOPOLOGY

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ABSTRACT. The purpose in writing this paper is to demonstrate that the notion of an elementary submodel is a simple but powerful tool which should be brought into the mainstream of set-theoretic topology. The hope is that this can be accomplished simply by supplying the reader with proofs of results, ranging from very trivial ZFC results to some quite difficult independence results, in which elementary submodels play a key role. The topological questions which are discussed concern, for the most part, metrizable and first countable spaces. However one new result in the paper is a proof from PFA that compact spaces with countable tightness have some points of first countability.

INTRODUCTION

This paper is an expanded version of the author's talk given at the Spring Topology Conference in Gainseville. The main purpose of both the talk and the paper is to give examples to demonstrate the usefulness of elementary submodels to set-theoretically oriented topologists. The author is not alone in believing that elementary submodels should become as familiar a part of the language of set-theoretic topology as is the pressing-down lemma for example. I believe that, for set-theoretic topologists, elementary submodels provide:

1. a convenient shorthand encompassing all standard closing-off arguments;
2. a powerful technical tool which can be avoided but often at great cost in both elegence and clarity; and
3. a powerful conceptual tool providing greater insight into the structure of the set-theoretic universe.

I hope to convince some readers of the validity of these points simply by (over-)using elementary submodels in proving some new and old familiar results. This paper is not a survey of their use nor an adequate (or even rigorous) introduction to the concept – it is intended solely as a demonstration of how useful they can be even in some rather unexpected applications. The author's primary reference is Kunen's text [K] and the reader is directed there for both an introduction and to discover what I probably should really have said in many of the proofs and discusssions.

There are two new results worth mentioning in the paper. The first is that it follows from the consistency of large cardinals that it is consistent that non-metrizability reflects in the class of locally-$\aleph_1$ spaces. This result is similar to Fleissner's results in [F] about left-separated spaces with point-countable bases. The second is that it follows from PFA that each compact space of countable tightness necessarily contains points of countable character. The second result

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is related to a question of Arhangel'skii [A2] and is just something that Fremlin, Nyikós and Balogh "missed" in the papers [Fr], [FrN] and [B].

In the first section we will introduce elementary submodels and establish some of the non-standard assumptions we will make in the remainder of the paper. In the three sections following we apply elementary submodels in increasingly difficult arguments. Most of the results in these sections concern metric spaces and the remainder are concerned with spaces of countable tightness. None of the results in these sections involve forcing or large cardinals (although their existence is acknowledged). Section five concerns applications of elementary submodels to forcing arguments. Not surprisingly this is an area in which elementary submodels are particularly useful – especially when proper forcing is involved. The last two sections discuss large cardinals and iterated forcing respectively.

1 Preliminaries

For a set or class M and a formula, φ , in the language of set theory, the formula φ^M is defined recursively (see IV of [K]). φ^M is just the formula you get when you "restrict all the quantifiers to M". However note that $(x \subset y)^M$ is really $\big((\forall a \in M)(a \in x \Rightarrow a \in y)\big)$, since $(x \subset y)$ is not in the language of set theory. However it does not take long for one to become accustomed to the meaning of φ^M especially when M is a "model" of most of ZF . We say that M is a model of φ (denoted $M \models \varphi$) if φ^M holds .

Definition: *If $\{a_1, \ldots, a_n\} \subset M \subset N$ then $\varphi(a_1, \ldots, a_n)$ is* <u>*absolute*</u> *for M, N if*

$$\varphi^M(a_1, \ldots, a_n) \quad \textit{holds} \qquad \textit{iff} \qquad \varphi^N(a_1, \ldots, a_n) \quad \textit{holds.}$$

Definition: *M is an elementary submodel of N , denoted $M \prec N$, if $M \subset N$ and for all $n < \omega$ and formulas φ with at most n free variables and all $\{a_1, \ldots, a_n\} \subset M$ the formula $\varphi(a_1, \ldots, a_n)$ is absolute for M, N.*

For a cardinal κ, the set $H(\kappa)$ is the set of all "hereditarily $< \kappa$ sized sets". That is, $H(\kappa)$ is the set of all sets whose transitive closure has size less than κ. These sets are useful because if κ is regular then

$$H(\kappa) \models \text{ ZF - P} \quad \text{(see IV in [K])}.$$

In practice, when one is investigating a property of some objects, say $\langle X, \mathcal{T}, \mathcal{C}\rangle$, one usually knows the largest possible size of any set at all relevant to the validity of the property. Therefore there is a cardinal θ large enough and a formula of set theory φ so that $\varphi(X, \mathcal{T}, \mathcal{C})$ expresses the property and such that $\varphi(X, \mathcal{T}, \mathcal{C})$ is absolute for $V, H(\theta)$. (For example, see the Levy Reflection Theorem, IV in [K]).

Throughout this paper we shall often choose such "large enough" θ or $H(\theta)$ with little or no discussion as to how large it needs to be.

Once we have shrunk our model to a set (namely $H(\theta)$), we then have the downward Lowenheim-Skolem theorem. The proof of this theorem makes very transparent the concept of elementary submodels.

Theorem 1.1: *For any set H and $X \subset H$, there is an elementary submodel M of H, such that $X \subset M$ and $|M| \leq |X| \cdot \omega$.*

Another very useful notion and resulting basic fact concerns elementary chains. $\mathcal{M}$ is called an elementary chain if it is a chain when ordered by $\prec$. It is worth noting that $\prec$ is a transitive order.

Theorem 1.2: *If $\mathcal{M}$ is an elementary chain then $M \prec \bigcup \mathcal{M}$ for all $M \in \mathcal{M}$.*

Corollary 1.3: *If $\mathcal{M}$ is a chain under inclusion of elementary submodels of H, then $\mathcal{M}$ is an elementary chain and $\bigcup \mathcal{M} \prec H$.*

Corollary 1.4: *For uncountable regular cardinals $\kappa \leq \theta$ and $X \in H(\theta)$ with $|X| < \kappa$,*

$$\{\lambda < \kappa : \exists M \prec H(\theta)\big(X \subset M \ , \ |M| < \kappa \ \text{ and } \ M \cap \kappa = \lambda\big)\}$$

is a closed and unbounded set (cub) in κ.

Proof. Inductively build an elementary chain of length κ, $\{M_\alpha : \alpha < \kappa\}$, so that for α a limit ordinal, $M_\alpha = \bigcup\{M_\beta : \beta < \alpha\}$. □

Note that, for regular cardinals θ, if $M \subset H(\theta)$ has cardinality less than θ then $M \in H(\theta)$. Therefore we could have built the elementary chain so that $M_\alpha \in M_{\alpha+1}$ – this will be called an elementary $\in$-chain. A <u>continuous</u> elementary chain or elementary $\in$-chain is one in which, for each limit α, we have that $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.

Another corollary to theorems 1.1 and 1.2 which we shall use frequently is the following.

Theorem 1.5: *For any regular $\theta \geq 2^\omega = \underline{c}$ and any $X \subset H(\theta)$ with $|X| \leq \underline{c}$, there is an $M \prec H(\theta)$ so that $X \subset M$, $|M| = \underline{c}$ AND $M^\omega \subset M$.*

A remark often made for its shock value is to suppose that M is a countable elementary submodel of $H(\ (2^{\underline{c}})^+\ )$ such that the reals $\mathbf{R}$ are in M . Then $M \models$ ($\mathbf{R}$ is uncountable) and yet $\mathbf{R} \cap M$ is only countable. There's no paradox here: M thinks $\mathbf{R}$ is uncountable not $\mathbf{R} \cap M$. Indeed the set $\mathbf{R} \cap M$ is not even in M so M can't think anything about it. The lesson here is that if $M \prec H$, then $\varphi^M(m) \iff \varphi^H(m)$ holds for <u>elements</u> of M, and that, in general, neither of the implications $X \in M \Rightarrow X \subset M$, $X \subset M \Rightarrow X \in M$ hold.

However in some case $X \in M$ does imply $X \subset M$.

Theorem 1.6: *If $M \prec H(\theta)$, θ regular, and $\kappa \in M$ is a cardinal such that $\kappa \subset M$, then for all $X \in M$ with $|X| \leq \kappa$, X is a subset of M. In particular, each countable element of M is a subset of M.*

Proof. If $|X| \leq \kappa$, then $H(\theta) \models \left(\exists f \; \kappa \overset{\text{onto}}{\to} X\right)$. Since κ , X are both in M, $M \models \left(\exists f \; \kappa \overset{\text{onto}}{\to} X\right)$. That is, $(\exists f \; \kappa \overset{\text{onto}}{\to} X)^M$ holds, hence there is an $f \in M$ such that $(f \text{ maps } \kappa \text{ onto } X)^M$. Now we are down to what is know as a Δ_0-sentence (see IV in [K]) - these formulas are absolute in many circumstances; that is f "really" is a function from κ onto X. Indeed, $M \models (f \subset \kappa \times X)$ so we show that $f \subset \kappa \times X$ as follows. $M \models (f \subset \kappa \times X)$ really means $M \models (\neg \exists x \in f \setminus \kappa \times X)$ – hence $H(\theta) \models (\neg \exists x \in f \setminus \kappa \times X)$. Similarly $H(\theta) \models (f$ is a function $)$ since $M \models (\forall \alpha \in \kappa)(\forall x, y \in X) (\{(\alpha, x), (\alpha, y)\} \subset f \Rightarrow x = y)$. Also, of course, $M \models (\forall x \in X)(\exists \alpha \in \kappa) \; (\alpha, x) \in f)$. Finally, we show that $x \in M$ for each $x \in X$ follows from $\kappa \subset M$. Indeed, let $x \in X$. Since f is "really" onto, we may choose $\alpha \in \kappa$ such that $(\alpha, x) \in f$. Now $M \models (\exists y)(\alpha, y) \in f$ so choose $y \in M$ such that $M \models (\alpha, y) \in f$. Clearly $H(\theta)$ "thinks" (realizes?) that $x = y$.

When one says "let $\langle X, \mathcal{T} \rangle$ be a topological space" it is usually meant that $\mathcal{T}$ is the topology on X. However we shall mean that $\mathcal{T}$ is a <u>base</u> for a topology on X. As we shall see below this is much more convenient.

Suppose $\langle X, \mathcal{T} \rangle$ is a topological space in some $H(\theta)$. Our general procedure is to take some kind of submodel, $M \subset H(\theta)$ (frequently an elementary submodel), such that $\langle X, \mathcal{T} \rangle \in M$. We then consider the (generally much smaller) subset $X_M = X \cap M$. At this point there are two natural topologies to consider on X_M. On the one hand we have the subspace topology generated by $\{U \cap X_M : U \in \mathcal{T}\}$. And on the other hand, if $M \models \mathcal{T}$ is a base for a topology on X plus some basic axioms, then we'd get the base $\mathcal{T}_M = \{U \cap X_M : U \in \mathcal{T} \cap M\}$. In general, these give very different topologies on X_M. For example, if $X = \beta\omega$ and M is countable then, of course, $\langle X_M, \mathcal{T}_M \rangle$ is a countable metric space.

However, it is by comparing these two topologies that we prove our reflection results. The game we play is to jump back and forth between M and $H(\theta)$, comparing what M "thinks" with what $H(\theta)$ "thinks".

Most of the results in this article are what are known as reflection results. A reflection question in topology usually has the form "if a space X has property P, then what is the size of the smallest subspace Y which also has property P ?". However it is usually the case that P is the *negation* of a nice property. So one might rephrase the question as "if κ is a cardinal and X is a space such that every subspace of X of cardinality at most κ has P, then does this guarantee that X has P ?".

We will adopt the following notation :
If P is a class of spaces or a property (which defines the class of spaces having that property) then for a space X

$$\kappa(X, P) = min\{|Y| : Y \subset X \text{ and } Y \text{ does } \underline{\text{not}} \text{ have property } P\} ,$$

(where we assume the minimum of the empty set is ∞).

There are not too many reflection results that hold for the class of all topological spaces but, for example, if we consider the separation property T_1 then for any space X we have $\kappa(X, T_1) \in \{2, \infty\}$. Another less trivial example is that $\kappa(X, \text{first countable}) \leq \omega_1$ for all X such that $\chi(X) = \omega_1$ (but not for all X such that $\chi(X) > \omega)$. We shall use such self-explanatory abbreviations for classes of spaces as '$\chi = \omega$', '$w = \omega$' and '$t \le \kappa$' for 'first countable', 'countable weight' and 'tightness at most κ' respectively. The reader is, of course, referred to the Handbook of Set-theoretic Topology for all topological definitions and basic facts.

II Some Elementary Applications.

In this section we prove a few simple theorems as an introduction to elementary submodel arguments.

Example 2.1 The Delta System Lemma. Let κ be a regular cardinal and let $\{F_\alpha \mid \alpha < \kappa\} \subset [\kappa]^{<\omega}$. Of course we want to show that there are

$$n < \omega \ , \ F \in [\kappa]^{<\omega} \text{ and } \ I \in [\kappa]^\kappa$$

so that $|F_\alpha| = n$ and $F_\alpha \cap F_\beta = F$ for all $\alpha, \beta \in I$ with $\alpha \ne \beta$. Let M be an elementary submodel of $H(\kappa^+)$ such that $\{F_\alpha \mid \alpha < \kappa\}$ is in M and $|M| < \kappa$. Let $\lambda = sup(M \cap \kappa)$ and choose any $\alpha \in \kappa - \lambda$. We have found our n and F; let $n = |F_\alpha|$ and $F = F_\alpha \cap M$. Then one notes that

$$M \models \forall\gamma < \kappa\exists\alpha(|F_\alpha| = n \wedge (F_\alpha \cap (\max(F \cup \{\gamma\}) + 1) = F).$$

To see this, note that the set $S = \{\gamma \mid (\exists\alpha \in \kappa)F_\alpha \cap \gamma = F\}$ is an element of M. Furthermore $\lambda \in S$ hence $M \models S$ is cofinal in κ.

It follows that we may pick, by induction on $\alpha < \kappa$, F_α so that $F_\alpha \cap \max F_\beta = F$ for all $\beta < \alpha$. Alternatively, we may choose an elementary chain $\{M_\alpha : \alpha < \kappa\}$ of elementary submodels of cardinality less than κ so that $M = M_0$ and choose $F_\alpha \in M_{\alpha+1}$ so that $F_\alpha \cap M_\alpha = F$. $\square$

In the next example we prove Arhangel'skii's famous result that the cardinality of a Lindelöf first countable space is at most c .

Example 2.2: *A Lindelöf space with countable pseudocharacter and countable tightness has cardinality at most c .*

Let $\mathcal{T}$ be a base for a Lindelöf topology on X which has countable tightness and pseudocharacter. Let $\langle X, \mathcal{T}\rangle \in M \prec H(\theta)$ such that $M^\omega \subset M$ and θ is "large enough".

Claim. $M \cap X = X$

Indeed, suppose not, and choose $z \in X \setminus M$.

Subclaim 1. For each $y \in X \cap M \exists U_y \in \mathcal{T} \cap M$ such that $y \in U_y$ and $z \notin U_y$.

Proof of Subclaim 1. $H(\theta) \models (\exists\{U_n : n < \omega\} \subset \mathcal{T}$ such that $\{y\} = \cap\{U_n : n < \omega\})$. Therefore M is a model of this , so let $\{U_n : n < \omega\} \in M$ be such that $M \models \{y\} = \cap_n U_n$. Now since $\{U_n\}_n \in M$ it follows that $V \models \{y\} = \cap_n U_n$, hence we may choose U_y as required.

Subclaim 2. $X \cap M$ is closed (hence Lindelöf)

Proof of Subclaim 2. Assume $x \in \overline{X \cap M}$. By countable tightness, choose a countable set $Y \subset X \cap M$ so that $x \in \overline{Y}$. Fix a set $\{U_n\}_{n\in\omega} \subset \mathcal{T}$ exhibiting that X has countable pseudocharacter at x . Next choose, for each $n \in \omega$ a collection $\{U_{n,m}\}_{m\in\omega} \subset \mathcal{T}$ such that $x \notin \cup\{\overline{U_{n,m}}\}_{m\in\omega} and X \setminus U_n \subset \cup_{m\in\omega}\{U_{n,m}\}$. It follows that $\overline{Y} \setminus \{x\} = \bigcup\{\overline{Y_{n,m}} : n, m \in \omega\}$ where, for each $n, m \in \omega$ $Y_{n,m} = Y \cap U_{n,m}$. But since $\{Y_{n,m} : n, m \in \omega\}$ is a countable collection of countable subsets of M , the collection and each member of it is an element of M. Now if x were not in M we would have

$$M \models \overline{Y} = \bigcup\{\overline{Y_{n,m}} : n, m \in \omega\}$$

whereas

$$H(\theta) \models x \in \overline{Y} \setminus \overline{Y_{n,m}} \text{ for each } n, m \in \omega .$$

Now by subclaim 1 $\mathcal{U} = \{U_y : y \in X \cap M\}$ forms an open cover of $X \cap M$ but not of X. By subclaim 2 , $\mathcal{U}$ has a countable subcollection $\mathcal{W}$ which still covers $X \cap M$. Now $\mathcal{W}$ is a countable subset of M and therefore is an element of M. But this is a contradiction since $M \models \mathcal{W}$ covers X. The result now follows from the fact that we may assume that $|M| = c$. □

Proposition 2.3: *If a space X with base $\mathcal{T}$ has a point-countable base and $\langle X, \mathcal{T}\rangle \in M \prec H(\theta)$ then $\mathcal{T} \cap M$ is a base for each point of $\overline{X \cap M}$.*

Proof. Let $M \prec H(\theta)$ with $\langle X, \mathcal{T}\rangle \in M$. Since

$$H(\theta) \models \langle X, \mathcal{T}\rangle \text{ has a point-countable base}$$

and M is an elementary submodel, there must be a set $\mathcal{B} \in M$ such that

$$M \models \mathcal{B} \text{ is a point-countable base for } \langle X, \mathcal{T}\rangle .$$

It is straightforward to check that absoluteness guarantees that $\mathcal{B}$ is a base for $\langle X, \mathcal{T}\rangle$ (in $H(\theta)$). Also $H(\theta) \models \mathcal{B}$ is point-countable since this follows from $M \models \forall x \in X \ \{B \in \mathcal{B} \mid x \in B\}$ is countable. Now let x be any point of $\overline{X \cap M}$ and suppose $B \in \mathcal{B}$ is a neighbourhood of x. Choose $U \in \mathcal{T}$ and $W \in \mathcal{B}$ so that $x \in W \subset U \subset B$. Now choose $y \in W \cap M$ which we may do since $x \in \overline{X \cap M}$. Since $\mathcal{B}$ is point-countable and $\{y, \mathcal{B}\} \in M$ it follows that $\{S \in \mathcal{B} : y \in S\} \subset M$; hence , in particular, $\{B, W\} \subset M$. Furthermore, since $U \in \mathcal{T}$ and $W \subset U \subset B$, it follows that

$$M \models \exists T \in \mathcal{T} \text{ such that } W \subset T \subset B .$$

Therefore there is a $T \in \mathcal{T} \cap M$ such that $x \in T \subset B$ which was to be shown. □

As we shall see later, the hidden strength of the previous result is that the base $\mathcal{T}$ is not assumed to be point-countable (recall our assumption that $\mathcal{T}$ denotes a base, not the whole topology). The next result uses some compactness in the topological sense to find when $\mathcal{T} \cap M$ is not a base at all points of $\overline{X \cap M}$.

Proposition 2.4: *Let $\langle X, \mathcal{T}\rangle$ be a countably compact space which is an element of a countable elementary submodel, M, of some sufficiently large $H(\theta)$.*
if $\mathcal{T} \cap M$ is not a base for $\langle X, \mathcal{T}\rangle$
then $\exists z \in \overline{X \cap M}$ such that $\mathcal{T} \cap M$ is not a base at z.

Proof. Clearly we may as well assume that $X \cap M$ is not dense in X , so choose any $z \in X \setminus \overline{X \cap M}$. Now if $\mathcal{T} \cap M$ does contain a base for all points of $\overline{X \cap M}$ then there is a cover $\mathcal{U} \subset \mathcal{T} \cap M$ of $\overline{X \cap M}$ whose union does not contain z. But now $\overline{X \cap M}$ is countably compact and $\mathcal{U}$ is a countable cover of it (since $\mathcal{T} \cap M$ is countable) . Therefore there is a finite subcover , say $\mathcal{W} \subset \mathcal{U}$, of $\overline{X \cap M}$ and hence of $X \cap M$. But now $\mathcal{W} \in M$ and $M \models \cup\mathcal{W} = X$ while $H(\theta) \models z \notin \cup\mathcal{W}$. □

The following non-trivial result is an immediate consequence of the previous two propositions.

Example 2.5 Misčenko's Lemma: *A countably compact space with a a point-countable base has a countable base.*

III Elementary Chains and the ω-covering property

As we saw in the proof of Arhangel'skii's theorem it is a very powerful assumption to have that your elementary submodel is "closed under ω -sequences". Also we cannot expect that countable elementary submodels can "trap" a great deal. Indeed a typical inductive construction usually carries through without much difficulty through the countable limit ordinals (discounting the problems of "trapping" the uncountable sets). On the other hand most constructions have considerable difficulty passing ω_1, so we can expect some non-trivial reflection by taking elementary submodels of cardinality ω_1 even in the absence of CH .

A useful property, which can to some extent replace "closed under ω-sequences", is the *ω-covering property.* We shall say that a set M has the ω-covering property if for each countable $A \subset M$ there is a countable $B \in M$ such that $A \subset B$. If $\{M_\alpha : \alpha \in \omega_1\}$ is an elementary $\in$-chain of countable elementary submodels of some $H(\theta)$ such that for each $\alpha \in \omega_1$ $M_\alpha \in M_{\alpha+1}$ then clearly the union of the M_α's is an ω-covering elementary submodel of $H(\theta)$ of cardinality ω_1.

In this section we shall present several proofs that use elementary submodels of cardinality ω_1 which satisfy the ω-covering property. It can be shown that such elementary submodels are exactly those which are uncountable and are the union of an elementary $\in$-chain of countable elementary submodels.

Theorem 3.1: *If every subspace of cardinality ω_1 of a countably compact space is metrizable, then the space itself is metrizable.*

It is convenient to make a few preliminary remarks before actually proving the theorem. To give a slick proof using elementary submodels it seems to be necessary to first prove that such a space is necessarily first countable, or at least that we may assume that if there is a counterexample then there is a first countable one. This can be done directly with relative ease — because of countable compactness a counterexample would have a subspace with density at most ω_1 which was also a counterexample. However it seems more appropriate to proceed by first proving the following surprising result of Hajnal and Juhasz (the result for regular spaces was proven by Tkacenko [Tk]). This was proven during their systematic study of cardinal functions on unions of chains of spaces which is very similar to investigating reflection properties of the cardinal functions. We state this result twice in order to recall our notation introduced in §I.

Proposition 3.2 [J]: *If every subspace of cardinality at most ω_1 has countable weight then the space itself has countable weight.*

Proposition 3.2 [J]: *For any space X, $\kappa(X, w = \omega) > \omega_1$ implies $w(X) = \omega$.*

Proof. Let $\langle X, \mathcal{T}\rangle \in M$ where M is an ω-covering elementary submodel of $H(\theta)$ of cardinality ω_1 . We must first show that $\mathcal{T} \cap M$ is a base for the subspace topology on $X \cap M$. Indeed suppose $x \in X \cap M$ and U is an open neighbourhood of x . Since $\kappa(X, w = \omega) > \omega_1$, $X \cap M \setminus U$ has a countable dense subset D. Since M has the ω-covering property we may choose a countable $D' \in M$ so that $D \subset D' \subset X$. Now $M \models w(D' \cup \{x\}) = \omega$ hence there is $T \in \mathcal{T} \cap M$ such that $x \in T$ and $T \cap D' \subset U$. So we now have $X \cap M \setminus U = \overline{D} \subset \overline{D' \setminus T} \subset X \setminus T$, hence $M \cap T \subset U$ as was to be shown.

It now follows that there is a countable subset $\mathcal{B}$ of $\mathcal{T} \cap M$ which is a base for $X \cap M$ since $w(X \cap M) = \omega$ and $\mathcal{T} \cap M$ forms a base. We may suppose $\mathcal{B} \in M$ by the ω-covering property . But now $M \models w(X) = \omega$, hence the result follows by elementarity.

Proof of 3.1. Let $\mathcal{T}$ be a base for the topology on X and assume that $\langle X, \mathcal{T}\rangle$ is not metrizable. Let $\langle X, \mathcal{T}\rangle \in M$ where M is an ω_1-sized, ω-covering elementary submodel of some $H(\theta)$. We shall show that $X \cap M$ with the subspace topology is not metrizable; hence $\kappa(X, metriz) = \omega_1$. By 3.2, we know that X has a subspace Z with $|Z| = \omega_1$ and $w(Z) > \omega$. By elementarity, there is such a set Z in M, so assume $Z \in M$. Since X is countably compact and $w(Z)$ is uncountable we know that $\overline{Z}$ is not metrizable — hence we may as well assume that $X = \overline{Z}$.

We may also assume that, for each $x \in X$, $Z \cup \{x\}$ is metrizable, hence first countable. Therefore $M \models Z \cup \{x\}$ is metrizable. If X is not regular at x then M will reflect this since $Z \in M$ is dense. Indeed, suppose $U \in \mathcal{T}$ is a neighborhood of x such that $\overline{V} \setminus U \neq \emptyset$ for each neighborhood of x. By elementarity, we may assume that $U \in M$. Assume though that x has a neighborhood such that $\overline{V} \cap M \subset U$ (i.e. $X \cap M$ is regular). Since $Z \cup \{x\}$ is first-countable and in M we may choose $W \in M$ such that $W \cap Z \subset V$. Therefore $\overline{W} \subset \overline{V} \subset U$. Since W and U are both members of M, this is a contradiction since $M \models \overline{V} \subset U$ while $H(\theta) \models \overline{V} \setminus U \neq \emptyset$.

So we may assume that X is regular at x and therefore it follows that X is first countable at x and $\mathcal{T} \cap M$ contains a local base at x. Therefore it suffices to show that $\langle X \cap M, \mathcal{T} \cap M\rangle$ is not metrizable.

Let $\{M_\alpha : \alpha < \omega_1\}$ be a continuous ϵ-chain of countable elementary submodels of M with $\langle X, \mathcal{T}\rangle \in M_0$ and whose union is all of M. For each $\alpha \in \omega_1$, we have that $\exists x \in \overline{X \cap M_\alpha}$ such that $\mathcal{T} \cap M_\alpha$ does not contain a base at x. But since $\{X, \mathcal{T}, M_\alpha\} \in M_{\alpha+1}$ there is in fact an $x \in M_{\alpha+1} \cap \overline{X \cap M_\alpha}$ such that $\mathcal{T} \cap M_\alpha$ does not contain a base at x.

Finally, let us suppose that $\langle X \cap M, \mathcal{T} \cap M\rangle$ has a point-countable base and obtain a contradiction to Proposition 2.3 . Let N be a countable elementary submodel of $H(\theta)$ such that each of $X, M, \mathcal{T}$ and $\{M_\alpha : \alpha \in \omega_1\}$ are in N. Let $\alpha = N \cap \omega_1$ and consider a point $x \in M \cap \overline{X \cap M_\alpha}$ such that $\mathcal{T} \cap M_\alpha$ does not contain a neighbourhood base at x as discussed in the previous paragraph. But now $N \models$

$M = \bigcup\{M_\alpha : \alpha \in \omega_1\}$ hence

$$(\mathcal{T} \cap M) \cap N = \bigcup\{\mathcal{T} \cap M_\beta : \beta \in \alpha\}.$$

Therefore $(\mathcal{T} \cap M) \cap N$ does not contain a local base for $x \in \overline{X \cap M \cap N}$, which is the contradiction we seek. □

A noteworthy aspect of the above proof is the double usage of elementary submodels. That is we developed some of the properties of the model M and then put M itself into a countable submodel.

Clearly one of the awkward things about the above proof is that we had to first show that the space would have to be first countable in order to deduce that $\mathcal{T} \cap M$ yielded the subspace topology on $X \cap M$. We shall now discuss the situation for reflecting countable character.

It is easy to see that

$$\kappa(X, \chi = \omega) > \omega_1 \not\Rightarrow \chi(X) = \omega.$$

Indeed remove the limit ordinals having cofinality ω_1 from $\omega_2 + 1$ and observe that this example shows that even

$$\kappa(X, \chi = \omega) > \omega_1 \& X \text{ is countably compact} \not\Rightarrow \chi(X) = \omega.$$

Therefore we could not have proceeded directly in 3.1 . But for which spaces does $\kappa(X, \chi = \omega) > \omega_1$ imply first countability?

Proposition [J]: *For compact spaces X,*

$$\kappa(X, \chi = \omega) > \omega_1 \Rightarrow \chi(X) = \omega.$$

It makes sense to ask how much compactness you need to obtain the above result. A space is called *initially $\omega_1 -$ compact* if every cover by ω_1 open sets has a finite subcover. This condition is, of course, equivalent to each of the conditions "there is no free closed filter base of size ω_1" and "each set of size at most ω_1 has a *complete* accumulation point". Let us first observe that this is how much compactness one needs to prove Arhangel'skii's result relating free sequences and countable tightness. Recall that a sequence $\{x_\alpha : \alpha < \kappa\}$ is called a *free sequence of length κ* if for each $\alpha < \kappa$ it is the case that $\overline{\{x_\beta : \beta < \alpha\}}$ is disjoint from $\overline{\{x_\beta : \beta \geq \alpha\}}$. When we say free sequence we shall assume the length is ω_1.

Proposition 3.3: *If a countably compact space does not have countable tightness then it contains free sequences. In addition, for an initially ω_1-compact space X , $t(X) = \omega$ iff X has no free sequences .*

Note that 3.3 is actually a reflection type result as well since it has as an immediate Corollary the fact that

$$\kappa(X, t = \omega) > \omega_1 \Rightarrow t(X) = \omega$$

for all initially ω_1-compact spaces.

Proposition 3.4: *For initially ω_1-compact regular spaces X ,*

$$\kappa(X, \chi = \omega) > \omega_1 \Rightarrow \chi(X) = \omega.$$

Proof. Let $\langle X, \mathcal{T}\rangle$ be a regular initially ω_1-compact space such that $\kappa(X, \chi = \omega) > \omega_1$. By the remark following 3.3 we have that $t(X) = \omega$. Let M be an ω-covering elementary submodel of some $H(\theta)$ so that $\langle X, \mathcal{T}\rangle \in M and |M| = \omega_1$. It suffices to show that $M \models \chi(X) = \omega$.

As in 3.2 it suffices to show that $\mathcal{T} \cap M$ induces the subspace topology on $X \cap M$. Let $x \in X \cap M$ and $\mathcal{T}_x = \{T \in \mathcal{T} \mid x \in T\}$. Let $U \in \mathcal{T}_x$ and suppose that $T \cap M \setminus U \neq \emptyset$ for all $T \in \mathcal{T}_x \cap M$. Using initial ω_1-compactness we may choose

$$z \in \bigcap\{\overline{T \cap M \setminus U} : T \in \mathcal{T}_x \cap M\}.$$

Using $t(X) = \omega$, choose a countable set $D \subset X \cap M \setminus U$ so that $z \in \overline{D}$. Again, by ω-covering of M we can find $T \in \mathcal{T}_x \cap M$ so that $x \in T and T \cap D = \emptyset$. Now, since we are assuming that X is regular and $T \in M$ we may choose $T' \in \mathcal{T}_x \cap M$ so that $\overline{T'} \subset T$, hence $\overline{T'} \cap \overline{D} = \emptyset$. This is a contradiction since z is supposed to be in $\overline{T'} \cap \overline{D}$.

I do not know if one needs to assume that X is regular in the previous result. If there is a non-compact first-countable initially ω_1-compact space then there is an example to show that the assumption of regularity in 3.4 is necessary. On the other hand, it easy to see that one does not need to assume regularity if CH holds. Indeed, this is because under CH (and it is consistent with $\neg$CH) that every initially ω_1-compact Hausdorff space of countable tightness is compact!

Proposition 3.5: *Let $\langle X, \mathcal{T}\rangle$ be an initially ω_1-compact Hausdorff space of countable tightness. Then every maximal free filter of closed sets has a base of separable sets. Furthermore if CH holds then the space is compact.*

Proof. Suppose that $\mathcal{F}$ is a maximal free filter of closed subsets of $\langle X, \mathcal{T}\rangle$. Let M be an ω-covering elementary submodel of some appropriate $H(\theta)$ such that $\{X, \mathcal{T}, \mathcal{F}\} \in M and |M| = \omega_1$. If CH holds we assume in addition that $M^\omega \subset M$. Choose any $z \in F_M = \bigcap\{\overline{F \cap M} : F \in \mathcal{F} \cap M\}$, which we may do since $|\mathcal{F} \cap M| = \omega_1$. Let $A \in \mathcal{F} \cap M$ be arbitrary and, by countable tightness, choose a countable set $D \subset A \cap M$ so that $z \in \overline{D}$. Since M has the ω-covering property and $A \in M$ we may assume that $D \in M$. Since $z \in \overline{F} \cap \overline{D} = F \cap \overline{D}$ for each $F \in \mathcal{F} \cap M$ it follows that

$$M \models \overline{D} \cap F \neq \emptyset \text{ for each } F \in \mathcal{F}.$$

Therefore, by elementarity and the maximality of $\mathcal{F}$, $\overline{D} \in \mathcal{F}$, $\mathcal{F}$ has a base of separable sets. It also shows that $\{\overline{F \cap M} \mid F \in \mathcal{F} \cap M\} \subset \mathcal{F}$, $F_M \in \mathcal{F}$ and furthermore that $|F_M| > \omega_1$ since $\mathcal{F}$ is a free filter and X is initially ω_1-compact.

Now suppose that M is closed under ω-sequences and that z' is any other point of F_M. Let $U_z and U_{z'}$ be disjoint neighbourhoods of z and z'. Let $D_z = D \cap U_z$ and $D_{z'} = D \cap U_{z'}$. Now just as we showed that $\overline{D}$ was in $\mathcal{F}$, the same proof shows that both $\overline{D_z}$ and $\overline{D_{z'}}$ are in $\mathcal{F}$ since they are both in M. However this contradicts that $z \in \overline{F \cap M}$ for all $F \in \mathcal{F} \cap M$ since $z \notin \overline{D_{z'}}$. $\square$

One can prove even a stronger result than the above one but the proof does not benefit by the use of elementary submodels and can be proven by a simple induction of length ω_1 .

Proposition 3.5A [Fremlin]: *If $\langle X, \mathcal{T}\rangle$ contains no free sequences then for each countably complete maximal filter $\mathcal{F}$ of closed sets and each set $H \in \mathcal{F}^+ = \{Z \subset X : Z \cap F \neq \emptyset$ for each $F \in \mathcal{F}\}$ there is a countable $H' \subset H$ so that $\overline{H'} \in \mathcal{F}$*

However an interesting feature of the proof of 3.5 is that it gives us a pretty good idea of how the consistency results in both directions must go. For example to show that it is consistent with $\neg$CH we can imagine that M is an inner model of CH and there are more reals to be added. It must be the case that new subsets of $X \cap M$ are added which can serve as the pair $U_z, U_{z'}$ mentioned above. There are a lot of properties that we can show the pair must have — for example they both meet every countable set in M whose closure is a member of $\mathcal{F} \cap M$ and that $M \models \mathcal{F}$ is a countably complete filter. We then investigate which kinds of forcings which add reals could not possibly add such a pair. It turns out that Cohen forcing is such a forcing but we shall not give the details here. In sections 5–7 we shall prove the result, due to , Fremlin and Nyikós that assuming the Proper Forcing Axiom, each initially ω_1-compact space of countable tightness is compact. As for the consistency of there being such spaces the above analysis indicates that we have to plan for those inner models of CH and be building a space in such a way that it is possible to add the necessary sets. This is still open.

Another question which suggests itself is whether or not we could replace 'compact' in the character reflection result with 'countably compact & countable tightness'. It turns out that if there are large cardinals then it is consistent that simply 'countable tightness' will suffice and no compactness is necessary at all. This will be proven in section 6. However it is consistent that these two properties do not suffice.

Example 3.6: *In the constructible universe, L, there is a countably compact space of countable tightness and uncountable character such that each subspace of cardinality ω_1 has countable character.*

It is shown in [DJW], that there is a family of functions $\{f_\alpha : \alpha < \omega_2\}$ in L so that

1. $f_\alpha : \alpha \longrightarrow \omega$ for each $\alpha < \omega_2$
2. $\alpha < \beta < \omega_2$ implies $\{\gamma < \alpha : f_\alpha(\gamma) \neq f_\beta(\gamma)\}$ is finite
3. $\forall f : \omega_2 \longrightarrow \omega$ $\exists \alpha < \omega_2$ such that $\{\gamma < \alpha : f(\gamma) \neq f_\alpha(\gamma)\}$ is infinite.

For each $\alpha < \omega_2$, let $A_{\alpha,0} = \{(\beta, m) \in \alpha \times \omega : m \leq f_\alpha(\beta)\}$ and for $n > 0$ let $A_{\alpha,n} = \{(\beta, n + f_\alpha(\beta)) : \beta < \alpha\}$. By a straightforward 'Ostaszewski-type' induction one can define a locally countable, locally compact topology on $\omega_2 \times \omega$ so that for each $\lambda < \omega_2$ with uncountable cofinality the subspace $\lambda \times \omega$ is countably compact and furthermore ensure that for each $n < \omega$ the set $A_{\lambda,n}$ is clopen.

Next one defines, just as in [DJW] , a topology on $X = \{p\} \cup \omega_2 \times \omega$ by declaring that $\omega_2 \times \omega$ is endowed with the above topology and U is a neighbourhood of $\{p\}$ providing $p \in U$ and $\forall \alpha \in \omega_2 \exists n \in \omega$ so that $U \supset \bigcup\{A_{\alpha,m} : n < m \in \omega\}$.

IV MORE ON METRIC SPACES – HAMBURGER'S QUESTION.

Peter Hamburger has asked a natural question about metric spaces which can be asked in our terminology as "Does there exist a first countable non-metrizable space, X, such that $\kappa(X, metriz) > \omega_1$?". If the existence of large cardinals is inconsistent, then the answer is known to be "yes". In fact the example would just be a special kind of subspace of the ordinal space ω_2 – called an *E-set* . An E- set is what is known as a *non-reflecting* stationary set. A set E of ordinals is called an E-set if E is stationary in its supremum, $(\forall \alpha \in E)\ cf(\alpha) < \omega_1$ and for each $\lambda < sup(E)$ with $cf(\lambda) > \omega \quad E \cap \lambda$ is not stationary in λ.

As mentioned above if there are no large cardinals then in fact there is an E-set contained in ω_2 (see [De2]). In section 6 we shall discuss the consistency, from a large cardinal, of there being no E-sets. Therefore Hamburger's question for ordinal spaces is resolved. We shall shall show that the situation is the same for locally-$\aleph_1$ spaces. Recall that X is locally-λ if every point has a neighbourhood of cardinality at most λ.

We proceed by analyzing the inductive step: "if X is a *locally small* space, does

$$\kappa(X, metriz) \geq \kappa \Rightarrow \kappa(X, metriz) > \kappa?"$$

The singular case holds in ZFC and the consistency of the regular case follows from (and implies) the consistency of large cardinals. The main tools will be Proposition 2.3 and elementary chains.

Theorem 4.1: *Suppose $\omega = cf(\kappa) \leq \lambda < \kappa$ and that X is a locally-λ space. Then $\kappa(X, metriz) \neq \kappa$.*

Proof. We may as well assume that X has cardinality κ. Fix a base $\mathcal{B}$ for X consisting of open sets of cardinality at most λ.

Choose a regular cardinal θ much larger than κ and an elementary $\in$-chain $\{M_n : n < \omega\}$ so that $\lambda \cup \{\kappa, \langle X, \mathcal{B}\rangle\} \subset M_0 \prec H(\theta)$, $|M_n| < \kappa$ for each $n \in \omega$, and $X \subset \bigcup M_n$. By assumption, $X_n = M_n \cap \kappa$ is metrizable for each $n \in \omega$. Furthermore, for each $B \in \mathcal{B} \cap M_n$, Theorem 1.6 implies that $B \subset X_n$ - hence X_n is open in X. Therefore X has a point-countable base. Furthermore, $B \in \mathcal{B} \cap M_n \Rightarrow B \subset X_n$, hence $\mathcal{B} \cap M_n$ does not contain a base for any point of $X \setminus X_n$. Also by 2.3, $\mathcal{B} \cap M$ contains a base for all points of $\overline{X_n}$. Therefore X_n is a clopen subset of X and $\{X_{n+1} \setminus X_n : n \in \omega\}$ is a partition of X into clopen metrizable pieces. $\square$

Theorem 4.2: *Suppose $cf(\kappa) \leq \lambda < \kappa$ and that X is a locally-λ space. Then $\kappa(X, metriz) \neq \kappa$.*

Proof.

The former (published) proof of Theorem 4.2 is very wrong. I would like to thank Paul Szepticky for bringing this to my attention. In addition I am very grateful to Stavros Christodoulou for supplying a correct proof.

The idea is to generalize the proof of Theorem 4.1. One must first prove that Proposition 2.3 can be generalized by replacing "point-countable" by "point-lambda" and demanding that λ be a subset of M.

Proposition 2.3(a). *If a space X with a base T has a point-λ base and $\langle X,T\rangle \in M \prec H(\theta)$ and $\lambda \subset M$, then $T\cap M$ is a base for each point of $\overline{X\cap M}$.*

Next one must show that an X as in 4.2 has a point-λ base. To see this, let $\{M_\alpha : \alpha \in \mathrm{cf}(\kappa\}$ be an as in the "proof" of 4.2. For each α, let T_α be a point-countable base for the open metrizable space $M_\alpha \cap X$. Then one can easily see that $T = \bigcup T_\alpha$ is a point-cf$(\kappa)\}$ base for X.

Now one finishes the proof by showing that each $M_\alpha \cap X$ is closed just as in 4.1.

'Locally-λ' can be replaced by 'locally-$<\kappa$' in 4.1, but I don't know if it can be in 4.2. Also we leave as an open question, the regular cardinal version of 4.2.

Question 4.3: *If κ is regular and there is a locally-$<\kappa$ topology on the set κ in which every subspace of cardinality less than κ is metrizable , does it follow that $\langle \kappa, \mathcal{T}\rangle$ is not metrizable iff $\{\lambda < \kappa \mid cf(\lambda) = \omega and\ \overline{\lambda} \neq \lambda\}$ is stationary ?*

The proof of the next result must be delayed until 6.1.

Theorem 4.4: *If it is consistent that there is a supercompact cardinal then it is consistent that, for the class of locally-$\aleph_1$ spaces ,*

$$\kappa(X, metriz) > \omega_1 \Rightarrow X \text{ is metrizable}$$

Recall that a space is said to be ($\aleph_1$-)CWH (for Collection-Wise Hausdorff) if every ($\aleph_1$-sized) discrete set can be separated by disjoint open sets. Shelah has also proven that it is consistent (subject to a large cardinal) that a locally-$\aleph_1$ first countable space which is $\aleph_1$-CWH is CWH. However when the local smallness condition in this and the above results on metric spaces are dropped no such reflection results are known to hold. It is known that the situation is different since an example in [F] shows that 4.4 does not hold if local smallness is dropped. To finish this section we will first formulate a combinatorial principal on ω_2 and then construct a space from it. I do not know whether or not this combinatorial principle is a consequence of GCH or even $2^{\omega_1} = \omega_2$ — it is consistent with these assumptions.

Let S^0_2 be the cofinality ω limits in ω_2 and let (†) denote the statement:
(†) $\exists\{A_\lambda \mid \lambda \in S^0_2\}$ and $\{g_\lambda \mid \lambda \in S^0_2\}$ so that:

1. A_λ is a cofinal increasing sequence in λ;
2. g_λ is a function from A_λ into ω;
3. $\forall \mu < \omega_2\ \exists h_\mu : \mu \to \omega$ such that $\forall \lambda \in \mu \cap S^0_2$
   $\{\alpha \in A_\lambda \mid h_\mu(\alpha) \leq g_\lambda(\alpha)\}$ is finite; and
4. $\forall g : \omega_2 \to \omega\ \exists \lambda \in S^0_2$ so that
   $\{\alpha \in A_\lambda \mid g(\alpha) \leq g_\lambda(\alpha)\}$ is infinite.

Example 4.5: *(†) implies there is a first countable space which is $\aleph_1$-CWH and for which subspaces of size $\aleph_1$ are metrizable but which is not CWH and not metrizable.*

Let $\{A_\lambda : \lambda \in S^0_2\}$ and $\{g_\lambda : \lambda \in S^0_2\}$ be as in $(\dagger)$. We shall define a topology on the set $\omega_2 \cup \omega_2 \times \omega_2 \times \omega$ so that ω_2 is closed discrete and unseparated and the rest of the space is open and discrete.

For each $\lambda \in S^0_2$ let $\{\alpha^\lambda_n : n \in \omega\}$ list A_λ in increasing order. For each point $\alpha \in \omega_2$ we define a countable neighbourhood base $U(\alpha, n)$ as follows:
for $\alpha \notin S^0_2$ $U(\alpha, n) = \{\alpha\} \cup \{\alpha\} \times \omega_2 \times (\omega \setminus n)$;
for $\lambda \in S^0_2$
$U(\lambda, n) = \{\lambda\} \cup \{\lambda\} \times \omega_2 \times (\omega \setminus n) \cup \{(\alpha^\lambda_m, \lambda, g_\lambda(\alpha^\lambda_m)) : n < m \in \omega\}$.

The simplicity of the space ensures that a subspace will be metrizable if and only if it is CWH. To see that the space is $\aleph_1$-CWH , let $\mu < \omega_2$ and choose h_μ as in $(\dagger)$. For each $\lambda \in \mu \cap S^0_2$, define $h'(\lambda) = h_\mu(\lambda) + j$ where j is such that $h_\mu(\alpha^\lambda_n) > g_\lambda(\alpha^\lambda_n)$ for all $n > j$. Otherwise define h' equal to h_μ. It is easy to check that this h' as a function from μ into the neighbourhood bases yields a separation of μ.

Let us now show that the space is not CWH. Indeed suppose that $g : \omega_2 \to \omega$ is such that $U(\alpha, g(\alpha))$ is disjoint from $U(\lambda, g(\lambda))$ for each $\alpha < \lambda < \omega_2$. Choose $\lambda \in S^0_2$ so that $A' = \{m \in \omega : g(\alpha^\lambda_n) \le g_\lambda(\alpha^\lambda_n)\}$ is infinite. Let $m = g(\lambda)$ and choose $m < n \in A'$. But now the point

$$(\alpha^\lambda_n , \lambda, g_\lambda(\alpha^\lambda_n)$$

is in both the sets $U(\alpha^\lambda_n , g(\alpha^\lambda_n))$ and $U(\lambda , g(\lambda))$. $\square$

V Elementary Submodels in Forcing Proofs

Forcing, of course, is the technique developed by Cohen which takes a (ground) model of set theory, together with a 'new' desired set, and canonically constructs a model of set theory (the extension) containing the new set and the ground model. The difficult part of most forcing arguments is to show what sets are not added. That is, one must prove some kind of preservation argument. For example, it is frequently important that the ordinal which is ω_1 in the ground model remains so in the extension — we would say that "ω_1 is preserved". Some other examples of properties which we may want preserved include: "being an ultrafilter over ω"; "a tree having no cofinal branches"; "a Souslin tree remaining Souslin".

If V is the ground model, and $P \in V$ is a poset then we assume the existence of $G \subset P$ – a **generic** filter (i.e. for each dense open $D \subset P$ with $D \in V$ $G \cap D \neq \emptyset$). $V[G]$ is just the model obtained by adding G to V and using the axiom of comprehension to interpret all the P-names from V. (*The fact that this works is remarkable and difficult to prove but to apply it is not as difficult as I suspect is commonly assumed*). Therefore we now have two models of set theory, V and $V[G]$. If $\mathcal{T} \in V$ is a base for a topology on $X \in V$, then we can still discuss $\langle X, \mathcal{T}\rangle$ in $V[G]$ — it will be the same topological space but it may have different *second order properties.* That is, we would want to discuss the preservation of topological properties such as: the countable compactness of $\langle X, \mathcal{T}\rangle$, the non- normality of $\langle X, \mathcal{T}\rangle$, etc. .

In this section we give some examples of how elementary submodels can be utilized in proving such preservation results. We begin with Cohen-real forcing. Recall that the poset $Fn(I, 2) = \{s : s \text{ is a function into } 2 , dom(s) \in [I]^{<\omega} \}$ and is ordered by $s < t \iff s \supset t$.

Lemma 5.1: *If G is $Fn(I,2)$-generic over V and $A \in V[G]$ is a subset of ω, then both*

$$\mathcal{F}_A = \{B \subset \omega : B \in V and A \subset B\} \text{ and } \mathcal{I}_A = \{B \subset A : B \in V\}$$

are countably generated.

Proof. Let $\dot{A}$ be a $Fn(I,2)$-name for A and let $\dot{A} \in M \prec H(\theta)$.

Claim: *If $B \in V$ and $p \Vdash \dot{A} \subset B$, then $\exists B' \in M$ with $p \Vdash \dot{A} \subset B' \subset B$.*

To prove the claim, let $p' = p \cap M$ and $B' = \{n : \exists q < p'$ so that $q \Vdash n \in \dot{A}\}$. Clearly $B' \subset B$ and $B' \in M$. Furthermore $p' \Vdash \dot{A} \subset B'$. This proves the claim and that $\mathcal{F}$ is countably generated. That $\mathcal{I}$ is countably generated is proven analogously. $\square$

Lemma 5.2: *Suppose $\langle X, \mathcal{T}\rangle$ is a space and $x \in X$ is such that $t(x,X) = \omega$ then $1 \Vdash_{Fn(I,2)} t(x,X) = \omega$.*

Proof. Suppose $1 \Vdash (x \in \overline{\dot{A}})$ and $\dot{A} \in M \prec H(\theta)$. We shall complete the proof by showing that

$$1 \Vdash x \in \overline{\dot{A} \cap M}.$$

Assume that $p \Vdash U \cap (\dot{A} \cap M) = \emptyset$ where $x \in U \in \mathcal{T}$. Let $p' = p \cap M$ and define $A_{p'} = \{y \in X : \exists q < p' \; q \Vdash y \in \dot{A}\}$ and note that $x \in \overline{A_{p'}}$ and $A_{p'} \in M$. Since $M \models t(x,X) = \omega$, x is in the closure of some countable subset B of $A_{p'}$ which is an element of M . Now choose $y \in U \cap B$ and, by elementarity, $p' > q \in M$ so that $q \Vdash y \in \dot{A}$. Finally, we have our desired contradiction since $dom(q) \cap dom(p) = dom(p')$, hence $p \cup q \in Fn(I,2)$, and $(p \cup q) \Vdash y \in U \cap \dot{A}$ $\square$

Although countably closed forcing does not preserve countable tightness in general, it is often the case that additional hypotheses on the space are required to prove the desired preservation result.

Lemma 5.3: *If X is a space of countable tightness which is first countable on countable subsets then the countable tightness of X is preserved by countably closed forcing.*

Proof. Let P be a countably closed forcing , $\dot{A}$ a P-name and assume $p \Vdash x \in \overline{\dot{A}}$ for some $x \in X$. Let M be a countable elementary submodel such that $\{p, x, \dot{A}, X\} \in M$. With A_q defined as above, we have that $x \in \overline{A_q \cap M}$ for each $q \in M \cap P$. By assumption $X \cap M$ is first countable hence choose $\{U_n : n \in \omega\}$ a neighbourhood base for x in the subspace $X \cap M$. Within M choose a descending sequence $\{p_n : n \in \omega\} \subset P$ with $p_0 = p$ and for each $n \in \omega$ there is an $x_n \in X \cap M$ so that $p_n \Vdash x_n \in U_n \cap \dot{A}$. Finally since P is countably closed there is a $q \in P$ with $q < p_n$ for each n, and $q \Vdash \{x_n : n \in \omega\} \subset \dot{A}$. This completes the proof since $x \in \overline{\{x_n : n \in \omega\}}$. $\square$

Another preservation result for Cohen forcing we'll need is 5.4. This result is proven in [DTW] and we shall not give a proof here. The proof uses a combinatorial structure on the Cohen poset called an endowment and elementary submodels do not play a role.

Proposition 5.4: *If $\langle X, \mathcal{T}\rangle$ is a space such that for some set I, $1 \Vdash_{Fn(I,2)} \langle X, \mathcal{T}\rangle$ has a σ-discrete base then X must already have one.*

A poset P is defined to be **proper** [S] if for each $\lambda > \omega$ the stationarity of each stationary $S \subset [\lambda]^\omega$ is preserved by forcing with P. Recall that $C \subset [\lambda]^\omega$ is closed if the union of each countable chain contained in C is again in C. The elementary submodel approach makes the concept of properness much easier to understand and to use. In fact properness can be viewed as a condition which guarantees that many elementary submodels in V will extend to elementary submodels in $V[G]$. If θ is a large enough cardinal and if $\lambda = |H(\theta)|$ we can identify $[\lambda]^\omega$ and $[H(\theta)]^\omega$. Furthermore the set of countable elementary submodels of $H(\theta)$ is closed and unbounded. Since P is proper , it can be shown that if G is P-generic over V, then in $V[G]$ the set $\{M \in H(\theta) : M \prec H(\theta) and M \cap V \in V\}$ is stationary in $[H(\theta)]^\omega$. Therefore there are "stationarily many" such M such that $P, G \in M$. Now we have

$$H(\theta) \models G \cap D \neq \emptyset \text{ for each dense open } D \subset P \text{ such that } D \in V$$

hence by elementarity $M \models G \cap D \neq \emptyset$ for each dense open subset D of P such that $D \in V$. It also follows that $M \cap V$ is an elementary submodel of the $H(\theta)$ of V. Any condition $q \in P$ which forces that $G \cap M$ meets each dense open subset from $M \cap V$ is called a $(P, M \cap V)$-generic condition. Combinatorially, in V, this translates to $q \in P$ is (P, M)-generic if for each $r < q$ and each dense open $D \in M$ there is a condition $p \in D \cap M$ such that r is compatible with p.

As a result there is an equivalent definition of proper which is the one we shall work with. P is proper iff for each regular $\theta > 2^{|P|}$ and each countable elementary submodel, M, of $H(\theta)$ which includes P, there is a (P, M)-generic condition below each $p \in P \cap M$ (see [S]).

Lemma 5.5: *If P is an ω_1-closed poset, then P is proper and furthermore, if $X \in V$ and G is P-generic over V then $[X]^\omega \subset V$.*

Proof. Let $\{D_n : n \in \omega\}$ list the dense open subsets of P which are in M - a countable elementary submodel. Let $p_0 = p$ be any element of $P \cap M$ and choose a descending sequence p_n , $n \in \omega$ so that $p_n \in D_n \cap M$. Since P is countably closed, there is a $q \in P$ so that $q < p_n$ for each $n \in \omega$. This q is clearly an (M, P)-generic condition. Furthermore, this q has the property that for each element of M which is a P-name of a function from ω into V, q forces it to equal a function in V This is how one proves $[X]^\omega \subset V$.

A useful generalization of countably closed forcing is the iteration of Cohen forcing followed by countably closed forcing. There are many preservation results for the iteration which do not hold for countably closed forcing itself.

Lemma 5.6: *Suppose $\dot{Q}$ is a $Fn(I,2)$-name of a countably closed poset and that I is uncountable. If $\langle X,\mathcal{T}\rangle$ has countable tightness at $x\in X$, then*

$$1 \Vdash_{Fn(I,2)*Q} \langle X,\mathcal{T}\rangle \textit{ has countable tightness at } \ x\ .$$

Proof. By 5.2 we may begin by assuming that G is $Fn(I,2)$-generic over V and $\langle X,\mathcal{T}\rangle\in V$. Let $\dot{A}$ be a Q-name of a subset of X , $q\in Q$, and assume that $q\Vdash x\in \dot{A}$. Let M be a countable elementary submodel containing $\{X,\mathcal{T},I,Q,q,A,x\}$. Now since I is uncountable and M is not, there are, in $V[G]$, filters on $Fn(\omega,2)$ which are generic over $V[G\cap M]$. That is, if P is any countable atomless poset which is an element of $V[G\cap M]$, then there is a filter $H\subset P$ so that $H\in V[G]$ and $H\cap D\neq\emptyset$ for all dense open $D\subset P$ with $D\in V[G\cap M]$. Well, such a P is $Q\cap M$ and so we choose such an $H\subset Q\cap M$. Since Q is countably closed, choose $q'\in Q$ so that $H\subset\{p\in Q: q'<p\}$.

Claim: $x\in\overline{\{y\in X\cap M: q'\Vdash y\in\dot{A}\}}$.

Proof of Claim. Let $U\in\mathcal{T}$ be an open neighbourhood of x and let $p\in Q\cap M$. Recall the definition of $A_p=\{y\in X:\exists p'<p\ \ p'\Vdash y\in\dot{A}\}$. Since $A_p\in M$ and $M\models t(x,X)=\omega$, there is a countable $B\subset A_p$ such that $x\in\dot{A}$ and $B\in M$. Therefore $U\cap B\neq\emptyset$ and furthermore, by elementarity for each $y\in B$ there is a $p'\in M$ so that $p'\Vdash y\in\dot{A}$. This shows that

$$D_U=\{p\in Q\cap M:\exists y\in U\cap M \text{ such that } p\Vdash y\in\dot{A}\}$$

is a dense open subset of $Q\cap M$. Furthermore $U\in V$, hence $U\in V[G\cap M]$, and so $H\cap D_U\neq\emptyset$. Since q' is below every member of H, this completes the proof of the claim and the Lemma. □

The condition $[X]^\omega\subset V$ in 5.5 gives us a kind of ω- absoluteness for V relative to $V[G]$ which is similar to what we had when we were taking elementary submodels closed under ω-sequences. For example we have the following result.

Lemma 5.7: *If G is generic over a countably closed poset P and $\langle X,\mathcal{T}\rangle\in V$ is a countably compact space having no free ω_1-sequences, then*

$$V[G]\models\langle X,\mathcal{T}\rangle \textit{ is countably compact, with no free } \ \omega_1\textit{-sequences}\ .$$

Proof. It is a trivial consequence of 5.5 that X is still countably compact since, for example, there are no new countable subsets of X and each countable subset from V still has all its limit points. Now suppose that $\{\dot{x}_\alpha:\alpha<\omega_1\}$ is a P-name so that

$$1\Vdash\{\dot{x}_\alpha:\alpha<\omega_1\} \text{ is a free sequence in } \ X.$$

Since P is countably closed we can choose, in V, a descending sequence $\{p_\alpha:\alpha<\omega_1\}\subset P$ and $\{y_\alpha:\alpha<\omega_1\}$ so that , for each α, $p_\alpha\Vdash y_\alpha=\dot{x}_\alpha$. It follows that $\overline{\{y_\beta:\beta<\gamma\}}\cap\overline{\{y_\beta:\gamma\le\beta<\alpha\}}=\emptyset$, for each $\alpha<\omega_1$, since $p_\alpha\Vdash y_\beta=\dot{x}_\beta$ for each $\beta<\alpha$. But now the sequence is $\{y_\alpha:\alpha\in\omega_1\}$ is a free sequence since, by 3.3, X has countable tightness in V. □

Todorčević pioneered the use of elementary submodels as "side conditions" in building proper posets. The following result is due to Fremlin for some special cases and the general result is due to Balogh.

Proposition 5.8: *If $\langle X, \mathcal{T}\rangle$ is a non-compact, countably compact space , then there is a proper poset Q so that*

$$\Vdash_Q \langle X, \mathcal{T}\rangle \textit{ contains a copy of the ordinal space } \omega_1 .$$

Proof. It turns out that the proof splits into two essentially different cases, depending on whether or not X contains free sequences. As we only plan to use the case when X does not, we shall only prove the result for this case and refer the reader to [B], or [D] for a proof of the other case. Since the iteration of proper posets is again proper, we may assume, by 5.7, that we have already forced with the countably closed collapse of the cardinal $|\mathcal{T}| + |X|$. Therefore we may assume that X has cardinality and character ω_1. Choose a free maximal closed filter $\mathcal{F}$ and define Q as follows. $q \in Q$ if $q = \langle g_q,\ H_q\ ,\ \mathcal{M}_q\rangle$ where:

1. $H_q \in [\mathcal{T}]^{<\omega}$;
2. $\mathcal{M}_q$ is a finite elementary $\in$-chain of countable elementary submodels of $H(\theta)$ such that $\{X, \mathcal{T}, \mathcal{F}\} \in M$ for each $M \in \mathcal{M}_q$;
3. g_q is a function whose domain, E_q, is a subset of $\{\lambda \in \omega_1 : \exists M \in \mathcal{M}_q\ M \cap \omega_1 = \lambda\}$;
4. for each $\lambda \in E_q$ and each $M, M' \in \mathcal{M}_q$, $\lambda \in M' \setminus M \Rightarrow g_q(\lambda) \in M' \cap \bigcap\{F \cap M : F \in \mathcal{F} \cap M\}$.

The actual definition of the conditions is designed to make the finding of (M, Q)-generic conditions a triviality. Indeed, if $\mu > 2^{|P|}$ and $M \prec H(\mu)$ with $P \in M$ and $p \in M \cap P$, then, we will show below that, $q = \langle g_p, H_p, \mathcal{M}_p \cup \{M \cap H(\theta)\}\rangle$ is (M, P)-generic.

We take care to ensure that the range of the union of the first coordinates over the generic filter will yield a copy of ω_1 by defining $q < p$ providing: $g_q \supset g_p$, $H_q \supset H_p$, $\mathcal{M}_q \supset \mathcal{M}_p$ and,

$$g_q(\lambda) \in \bigcap\{U \in H_p : g_p(\lambda^*) \in U\}$$

for each $\lambda \in E_q \cap \max E_p$, where

$$\lambda^* = \min(E_p \setminus \lambda) \quad .$$

It is not too difficult to show that, if P does not collapse ω_1 and G is P-generic then

$$\omega_1 \approx \bigcup\{\mathrm{range}(g_p) : p \in G\} \subset \langle X, \mathcal{T}\rangle.$$

The main difficulty to this claim is in showing that

$$\forall p \in G, \forall M \in \mathcal{M}_p, \exists q \in G, M \cap \omega_1 \in E_q \quad .$$

However anyone who reads the rest of the proof can easily do this. One may find it easier to slightly change the definition of the conditions by allowing $H_q \in [\mathcal{T} \cup \omega_1]^{<\omega}$ and for $q < p$ add the condition that
$\lambda > \max(H_q \cap \lambda^*)$ for each $\lambda \in E_q \cap \max(E_p)$ and $\lambda^* = \min(E_p \setminus \lambda)$. The result of this is that, if G is Q-generic, $E = \bigcup\{E_p \mid p \in G\}$ is a *cub* in ω_1 and $g = \bigcup\{g_p \mid p \in G\}$ is a homeomorphism. That is, if $\lambda^* \in E_p$, then $p \Vdash \lambda^* \in E$ and if $p \Vdash E \cap \lambda^*$ has no maximum then $p \Vdash (E \cap \lambda^*)$ cofinal in λ^* (keep adding things to $H_p \cap \lambda^*$) and

$p \Vdash \{g(\beta) \mid \beta \in E \cap \lambda^*\}$ converges to $g(\lambda^*)$ - again keep adding neighbourhoods of $g(\lambda^*)$ to $H_p \cap \mathcal{T}$.

However the hard part of the proof is to show that P is proper (hence preserves ω_1). Let $\mu > 2^{|P|}$ and let $P \in M \prec H(\mu)$ with $|M| = \omega$ and let $p \in P \cap M$. Define $p' = \langle g_p, H_p, \mathcal{M}_q \cup \{M \cap H(\theta)\}\rangle$. We must first show that $p' \in P$ and that $p' < p$. First of all $P \in M$ and $\theta = \sup\{\alpha : \exists q \in P, \exists M' \in \mathcal{M}_q\ \alpha \in M'\} \in M$, hence $M \cap H(\theta) \prec H(\theta)$. Furthermore, if $M' \in \mathcal{M}_p$, then $M' \in M$ and $M \models M' \prec H(\theta)$, hence $M' \prec M \cap H(\theta)$. It follows that $p' < p$.

Now consider $r < p'$ and $D \in M$ such that D is a dense open subset of P; without loss of generality we may assume $r \in D$. Let $r_0 = \langle g_r \cap M\ ,\ H_r \cap M\ ,\ \mathcal{M}_r \cap M\ \rangle$ and note that $r_0 \in P \cap M$ and that $r \leq r_0$. Let $\tilde{D} = \{\langle g_q, H_q\rangle : r_0 \leq q \in D\} \in H(\theta) \cap M$. Let us first note that it suffices to find a pair $\langle g, H\rangle \in \tilde{D} \cap M$ such that range$(g \setminus g_r) \subset U^*$ where $U^* = \bigcap\{U \in H_r : g_r(\lambda_0^*) \in U\}$ and $\lambda_0^* = \min E_r \setminus M$. Indeed, if $\langle g, H\rangle \in \tilde{D} \cap M$, then by elementarity there is $q \in D \cap M$ so that $\langle g_q, H_q\rangle = \langle g, H\rangle$ and $r_0 < q$. One easily checks that q and r are compatible.

Let $E_r \setminus M = \{\lambda_0\ , \ldots , \lambda_{n-1}\}$ listed in increasing order. For expository purposes, first suppose that $n = 1$. Then, by definition of $r \in P$ we know that

$$g_r(\lambda_0) \in \bigcap\{F : F \in \mathcal{F} \cap M\} \quad .$$

Now, by 3.5A , we may assume that $\mathcal{F}$ is just a base for the filter which consists of separable sets. Therefore $F = \overline{F \cap M}$ for each $F \in \mathcal{F} \cap M$ and, since $\mathcal{F}$ is countably complete $\bigcap\{F : F \in \mathcal{F} \cap M\} \neq \emptyset$. Since $\tilde{D} \in M \cap H(\theta)$, it follows that $Z = \{x \in X : \exists\langle g, H\rangle \in \tilde{D}\ \ \{x\} = \text{range}(g - g_{r_0})\} \in M \cap H(\theta)$. Therefore if $\overline{Z} \notin \mathcal{F}^+$, then there is some $F \in \mathcal{F} \cap M$ such that $F \cap \overline{Z} = \emptyset$. But this contradicts that $g_r(\lambda_0) \in F \cap Z$. Therefore it follows that $g_r(\lambda_0) \in \overline{Z}$ Hence we may choose such an x in U^* and a $\tilde{q} \in \tilde{D} \cap M$ such that $\{x\} = \text{range}(g_{\tilde{q}} - g_{r_0})$.

The idea of the elementary chains is that we can then handle the case $n > 1$. For $i = 0, 1, \ldots , n-1$, let $M_i \in \mathcal{M}_r$ be so that $M_i \cap \omega_1 = \lambda_i$ and let $g_r(\lambda_i) = x_i$. Also, for $i = 0, \ldots , n-1$,let $g_i = g_r \cap M_i$ and $H_i = H_r \cap M_i$ and let $g_n = g_r$ and $H_n = H_r$.

Just exactly as in the case $n = 1$, but using M_{n-1} in place of $M = M_0$, we obtain that

$$cl[\{x : \exists\langle g, H\rangle \in \tilde{D}\ \ s.t.\ g = g^\frown\langle\max(\text{dom}(g')), x\rangle \& H_{n-1} \subset H\}] \in \mathcal{F}.$$

Define, for $i = 0, \ldots , n-1$:

$$\tilde{D}_{n-(i+1)} = \{\langle g, H\rangle : cl\big[Z_{n-(i+1)}[\langle g, H\rangle]\big] \in \mathcal{F}\};$$

where for any $\langle g, H\rangle$ we let

$$(1)\quad Z_{n-(i+1)}[\langle g, H\rangle] \;=\; \{x : \exists\langle g', H'\rangle \in \tilde{D}_{n-i}\ \ s.t.\ \ g' = g^\frown\langle\max(\text{dom}(g')), x\rangle \ \text{ and } \ H \subset H'\}\ .$$

Note that $\tilde{D}_{n-i} \in M_0$ for all $i = 0, \ldots , n$, where $\tilde{D}_n = \tilde{D}$. Furthermore we have noted above that $\langle g_{n-1}, H_{n-1}\rangle \in \tilde{D}_{n-1}$. Assume that $i < n$ and that $\langle g_{n-i}, H_{n-i}\rangle \in \tilde{D}_{n-i}$. Now since $x_{n-(i+1)} \in Z_{n-(i+1)}[\langle g_{n-(i+1)}, H_{n-(i+1)}\rangle]$ and $Z_{n-(i+1)} \in M_{n-(i+1)}$, we again obtain that

$$clZ_{n-(i+1)} \in \mathcal{F} \quad .$$

Therefore $\langle g_0, H_0\rangle \in \tilde{D}_0 \in M_0$.

We may now pick, $\langle \mu_0, y_0\rangle, \ldots, \langle \mu_{n-1}, y_{n-1}\rangle$ and $H'_1, \ldots, H'_{n-1}$ all in M_0 , so that $\{y_0, \ldots, y_{n-1}\} \subset U^*$. These are picked recursively so that for each $i = n-1, \ldots, 0$, if

$$g'_{n-i} = g'_{n-(i+1)}{}^\frown\langle \mu_{n-(i+1)}, y_{n-(i+1)}\rangle \ ,$$

then $H'_{n-i} \supset H'_{n-(i+1)}$ and $\langle g'_{n-i}, H'_{n-i}\rangle \in \tilde{D}_{n-i}$. To carry out the inductive step we note that since $\langle g'_{n-i}, H'_{n-i}\rangle \in \tilde{D}_{n-i}$, we have that $clZ_{n-i+1}[\langle g'_{n-i}, H'_{n-i}\rangle] \in \mathcal{F}$. Therefore $U^* \cap Z_{n-i+1}[\langle g'_{n-i}, H'_{n-i}\rangle] \neq \emptyset$ and we may choose $y_{n-(i+1)} \in M_0 \cap U^* \cap Z_{n-i+1}[\langle g'_{n-i}, H'_{n-i}\rangle]$. Then by elementarity we can choose μ_{n-i+1} and H_{n-i+1} in M as required above. $\square$

VI. Large Cardinals and Reflection Axioms

We have seen lots of examples where we had a space $\langle X, \mathcal{T}\rangle \in H(\theta)$ and a property or formula $\varphi(v_1, v_2, v_3)$ so that when we took $M \prec H(\theta)$ and a parameter $A \in M$, we had

$$H(\theta) \models \varphi(X, \mathcal{T}, A) \ \ \text{(hence } M \models \varphi(X, \mathcal{T}, A)\ )$$

but

$$H(\theta) \models \neg\varphi(X \cap M, \mathcal{T} \cap M, A \cap M) \ \ .$$

In fact the whole point of reflection is to find conditions on M which are sufficient to guarantee that φ does reflect, as opposed to the situation described above.

If a cardinal κ is supercompact and $\langle X, \mathcal{T}\rangle, A \in H(\theta)$ and $\varphi(v_1, v_2, v_3)$ is any formula such that $H(\theta) \models \varphi(X, \mathcal{T}, A)$ then there is an $M \prec H(\theta)$ such that

$$|M| < \kappa \ \&\ \{X, \mathcal{T}, A\} \in M \ \ \&\ \ H(\theta) \models \varphi(X \cap M, \mathcal{T} \cap M, A \cap M)$$

(see [KaMa]). When we combine this with forcing we get reflection results at "small" cardinals which need large cardinals. To get the most out of this, one would want to master the techniques described in such articles as [De1], [KaMa] and [DTW]. We shall just take the results after the fact as Axioms.

PFA is, of course, the Proper Forcing Axiom: Given a proper poset P and a family $\{D_\alpha : \alpha < \omega_1\}$ of dense open subsets of P, there is a filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

Fleissner has an axiom called Axiom R: If $S \subset [X]^\omega$ is stationary and $C \subset [X]^{<\omega_2}$ is t.u.b. then $\exists Y \in C$ such that $S \cap [Y]^\omega$ is stationary in $[Y]^\omega$.
The set $C \subset [X]^{<\omega_2}$ is said to be t.u.b. if it is unbounded and if the union of every chain of length ω_1 from C is again a member of C.

Axiom R is a specific case of a scheme (see [DTW] for more details). Roughly speaking, if $\mathcal{P}$ is a nice class of forcing notions, then we could have Axiom $\mathcal{P}_\kappa$: If $\varphi(v_1, v_2, v_3)$ is a (*local + structural*) property which is preserved by forcings from $\mathcal{P}$ and if $(a, b, c) \in H(\theta)$ is such that $\varphi(a, b, c)$ holds — then $\exists Y \in [H(\theta)]^{<\kappa}$ such that $\varphi(a \cap Y, b \cap Y, c \cap Y)$ holds and $Y \cap \kappa \in \kappa$.

For example, for axiom R, take $X, S, C \in H(\theta)$ and $\mathcal{P}$ is the class of proper posets of cardinality $< \kappa = \omega_2$. Then
$\varphi(X, S, C, \kappa) \equiv$ "S is stationary in $[X]^\omega$, C is unbounded in $[X]^{<\kappa}$ and closed

under unions of chains of length ω_1 ". Now note that proper forcing preserves stationarity – hence φ.

PFA$^+$ is what you get when you combine PFA and axiom R. For what it's worth, the author finds it easiest to apply PFA by recalling how the consistency of PFA$^+$ is proven. That is, the model in which PFA holds is obtained by forcing with an iteration of length κ of proper posets. When you are considering a space $\langle X, \mathcal{T}\rangle$ in the extension you know, from the fact that κ is a large cardinal, that this space and any of its properties will reflect to an inner model – but there will be more forcing to be done. But now the difference between PFA and the above axiom scheme is that you get to choose the next forcing in the iteration. The idea then is to choose the next forcing so that the iteration of it with any other proper poset will preserve the properties of interest.

Other axioms which are frequently used (but not as axioms) are:

"the Cohen forcing Axiom" = Axiom $Cohen_{2^\omega}$;

"Mitchell forcing Axiom" = Axiom $(Cohen{*}\omega_1\text{-}closed)_{\omega_2}$; where $Cohen{*}\omega_1$-*closed* denotes the class of posets which are of the form $Fn(\omega_2, 2){*}P$ and P is forced to be a countably closed poset by the Cohen posets.

"Levy forcing Axiom" = Axiom $\omega_1\text{-}closed_{\omega_2}$.

Using these axioms together with a judicious choice of φ we can obtain ω_1-sized , ω-covering elementary submodels of some $H(\theta)$ together with some instances of Π^1_1-reflection. For example as we promised in §IV .

Proposition 6.1: *Axiom R* $\Rightarrow \kappa(X, metriz) < \omega_2$ *for* X *in the class of locally-*$\aleph_1$ *spaces. Hence, in particular, Axiom R implies there are no E-sets.*

Proof. Let X be a locally-$\aleph_1$ space and assume that X is not metrizable. By 4.1 and 4.2, we may assume that $\lambda = |X| = X$ is a regular cardinal. Let $\mathcal{B}$ be a base for X consisting of open sets of cardinality at most $\aleph_1$. For our application of Axiom R we define

$$S = \{s \in [X]^\omega \mid \ \overline{s} \setminus sup(s) \neq \emptyset\} \ .$$

Next, fix a sufficiently large cardinal θ and let $\mathcal{M}$ denote the set of ω_1-sized, ω-covering elementary submodels of the structure $(H(\theta), X, \mathcal{B}, \in)$. Let

$$C = \{Y \in [X]^{\omega_1} : (\exists\ M \in \mathcal{M})(M \cap X = Y)\} \ .$$

Since X is first-countable and locally-$\aleph_1$, it follows from an easy generalization of 1.4 that C is a t.u.b. subset of $[X]^{\omega_1}$. Before we show that S is stationary in $[X]^\omega$, let us suppose that it is and show how to deduce the result from Axiom R. By Axiom R, we may choose $Y \in C$ such that $S \cap [Y]^\omega$ is stationary in $[Y]^\omega$. Let $M \in \mathcal{M}$ be such that $M \cap X = Y$. Since M is ω_1-covering we may choose a continuous elementary $\in$-chain $\{M_\alpha : \alpha \in \omega_1\}$ whose union is M. Choose a large θ' and a continuous $\in$-chain $\{N_\alpha : \alpha \in \omega_1\}$ of elementary submodels of $H(\theta')$ so that $\{M_\alpha : \alpha \in \omega_1\} \in N_0$. Since $S \cap [Y]^\omega$ is stationary, we may choose a $\delta \in \omega_1$ such that $M_\delta \cap Y = N_\delta \cap Y \in S$. By the definition of S, there is an $x \in \overline{Y \cap N_\delta} - \sup(Y \cap N_\delta)$. We first observe that, by elementarity, there is such an x in Y. Indeed, first notice that, since M_δ and X are in $M_{\delta+1}$ and $M_\delta \cap X = M_\delta \cap Y = N_\delta \cap Y$, it follows that $N_{\delta+1} \models (\exists x)\ (x \in \overline{Y \cap M_\delta} - \sup(Y \cap M_\delta))$.

Finally we use 2.3 again to contradict that Y is supposed to be metrizable. For this we will show that $x \notin \bigcup(\mathcal{B} \cap N_\delta)$, where x is chosen as above. For each $B \in \mathcal{B} \cap N_\delta$, the element $\sup(B)$ is a member of N_δ, hence $B \subset \sup(N_\delta \cap X)$.

Therefore it suffices to show that $\sup(N_\delta \cap X) \leq \sup(M_\delta \cap X)$. To see this observe that for each $\beta \in X \cap N_\delta$, $\sup(\{\alpha < \omega_1 : \{\sup(M_\alpha \cap X) \leq \beta)\}) \in N_\delta$.

Now to prove that S is stationary. Suppose that A is a closed and unbounded subset of $[X]^\omega$ such that $A \cap S = \emptyset$. Let $\{M_\alpha \mid \alpha \in \lambda\}$ be a continuous elementary $\in$-chain of elementary submodels of $H(\theta)$ such that, for each $\alpha \in \lambda$, $\omega_1 \cup \{X, \mathcal{B}, A, S\} \subset M_\alpha$, $|M_\alpha| < \lambda$, and $M_\alpha \cap \lambda \in \lambda$. As in the proof of 4.2, each $X \cap M_\alpha = \lambda \cap M_\alpha$ is an open metrizable subspace of X. Since X is not metrizable, there is an α such that $X \cap M_\alpha$ is not closed. Since X is first-countable, we may choose a countable set $s \subset X \cap M_\alpha$ such that $\overline{s} \setminus M_\alpha \neq \emptyset$. Therefore $\overline{s} \setminus sup(M_\alpha \cap \lambda) \neq \emptyset$. Now let N be a countable elementary submodel of M_α which contains $s \cup \{A\}$; we claim that $a = N \cap X \in A \cap S$. First $a \in S$ since $\overline{a} \supset \overline{s}$ and $sup(a) \leq sup(M_\alpha)$. Since $N \models (A$ is an unbounded subset of $[X]^\omega)$, it follows that there is a countable chain $\mathcal{C} \subset N \cap A$ such that $a = \cup\mathcal{C}$ - hence $a \in A$. This contradicts that A was chosen to miss S □

Now we show that it is consistent that 3.4 can be improved.

Proposition 6.2: *It follows from the Mitchell Axiom (and the Lévy Axiom) that a space with countable tightness and uncountable character has a $\leq \omega_1$-sized subspace with uncountable character. That is, $\kappa(X, \chi \leq \omega_1) \leq \omega_1$ for any X with countable tightness.*

Proof. Let $\langle X, \mathcal{T}\rangle$ have countable tightness and assume that $x \in X$ has uncountable character. Suppose further that the character of every countable subspace of X is countable (for the Lévy Axiom we must assume that every $\aleph_1$-sized subspace has countable character). Choose a regular cardinal θ large enough to contain the power set of the power set of X. Define the formula φ so that

$$\varphi(x, X, \mathcal{T}, H(\theta)) \text{ iff } \begin{cases} t(x, \langle X, \mathcal{T}\rangle) = \omega \text{ and } \chi(x, \langle X, \mathcal{T}\rangle) > \omega \\ \\ \big(\forall a \in [H(\theta)]^\omega \exists \tilde{a} \in H(\theta) \text{ so that} \\ \quad |\tilde{a}| = \omega \ , \ a \subset \tilde{a} \ , \text{ and} \\ \quad H(\theta) \models \chi(x, \langle X \cap \tilde{a}, \mathcal{T}\rangle = \omega)\big). \end{cases}$$

Now we must check that forcing by *Cohen* $*$ ω_1*-closed* preserves that φ holds. Lemma 5.6 (and 5.3 for Lévy) proves that countable tightness is preserved. The second line in the definition of φ is also preserved by any proper forcing but it deserves more discussion. At first glance it seems a total triviality — but the important point is that we are talking about the set $H = H(\theta)$ as opposed to the defined notion. All the second line is really saying is that "H has the ω-covering property " and we are simply asserting that this is preserved by proper forcing. Of course if we had put $a = \tilde{a}$ in line two – this would not have been preserved by any forcing which adds a real. Since H has the ω-covering property, uncountable character is preserved.

Now either of the above Axioms gives us a set M with $|M| = \omega_1$ (which we may as well assume is a subset of $H(\theta)$) so that

$$\varphi(x, X \cap M, \mathcal{T} \cap M, M) \text{ holds.}$$

Therefore, in the subset $X' = X \cap M$ and with respect to the topology induced by $\mathcal{T}' = \mathcal{T} \cap M$, the point x has countable tightness and uncountable character.

Therefore to finish the proof we wish to show, just as we were doing in § III , that $\mathcal{T}'$ induces the subspace topology on X' – at least at x. So let $x \in U \in \mathcal{T}$ and assume that $x \in cl_{\mathcal{T}'}[X' \setminus U]$. Since we have $t(x, \langle X', \mathcal{T}' \rangle) = \omega$, we may choose a countable $a \subset X' \setminus U$ so that $x \in cl_{\mathcal{T}'} a$. But now M has the ω-covering property hence we may choose a countable $\tilde{a} \in M$ so that $a \subset \tilde{a} \in M$. This contradicts that $M \models \tilde{a}$ has countable character with respect to the topology induced by $\mathcal{T}$ since M would then contain a base for the subspace topology at x . □

We finish this section with the PFA results on initially ω_1-compact spaces of countable tightness. Fremlin and Nyikós proved (i) and (ii) is due to Balogh. In fact Balogh proved that, under PFA, compact spaces of countable tightness are sequential but we do not include this result since it depends on the case in 5.7 which we did not prove.

Theorem 6.3: *PFA implies that if X is an initially ω_1-compact space of countable tightness then :*
(i) X is compact;
(ii) X is sequentially compact; and
(iii) X is first countable at some of its points.

Proof. Let $\langle X, \mathcal{T} \rangle$ be an initially ω_1-compact space with countable tightness. Let P be the usual countably closed collapse of $|X|$. In the extension obtained by forcing with P, the space $\langle X, \mathcal{T} \rangle$ will not be compact if any of the conditions (i) - (iii) failed to hold. Indeed, if (ii) fails to hold then clearly X contains a closed subspace in which there are no points of first-countability – so we may as well assume that (iii) fails. For each $x \in X$, fix a closed G_δ, F_x such that $x \notin F_x$. Let

$$\tilde{P} = \{g \in {}^{<\omega_1}X \mid \bigcap_{\alpha \in \mathrm{dom}(g)} F_{g(\alpha)} \neq \emptyset\}.$$

If $\tilde{G}$ is a $\tilde{P}$-generic branch, then $\bigcap_{\alpha \in \omega_1} F_{\tilde{G}(\alpha)} = \emptyset$ since each non-empty G_δ subset of X must contain many points. One can now observe that forcing with P will add a generic branch through $\tilde{P}$; (or force with $\tilde{P}$ in the first place, or even that we may assume without loss of generality that each non-empty G_δ subset of X has the same cardinality as X hence $\tilde{P} \cong P$).

In the extension $\langle X, \mathcal{T} \rangle$ is still countably compact and contains no free sequences by 5.7 . Now use 5.8 to find a proper poset Q in the extension so that there is a $P * Q$-name $\dot{g}$ so that

$$1 \Vdash \dot{g} \text{ is a homeomorphism from } \omega_1 \text{ into } \langle X, \mathcal{T} \rangle \ .$$

Therefore there are also $P * Q$-names $\{\{\dot{W}_\alpha, \dot{U}_\alpha\} : \alpha \in \omega_1\}$ such that

$$(2) \quad 1 \Vdash_{P*Q} \forall \alpha \in \omega_1 \quad \{\dot{W}_\alpha, U_\alpha\} \subset \mathcal{T} \text{ and}$$
$$\dot{g}([0,\alpha]) \subset \dot{W}_\alpha \subset \overline{\dot{W}_\alpha} \subset \dot{V}_\alpha \quad \text{and} \quad \dot{V}_\alpha \cap \dot{g}((\alpha, \omega_1)) = \emptyset \ .$$

Finally, we define D_α for $\alpha \in \omega_1$ to be $D_\alpha = \{p \in P * Q : \exists x, W, U$ such that $p \Vdash \dot{g}(\alpha) = x$, $\dot{W}_\alpha = W$, and $\dot{U}_\alpha = U\}$. Since the above statements are forced by 1, it follows that D_α is dense for each $\alpha \in \omega_1$. Use PFA to find a filter $G \subset P$ which meets each D_α. Pick, for each $\alpha \in \omega_1$, $x_\alpha, W_\alpha, U_\alpha, and\ p_\alpha$ so that $p_\alpha \in G \cap D_\alpha$ and

$$p_\alpha \Vdash \dot{g}(\alpha) = x_\alpha, \dot{W}_\alpha = W_\alpha, \text{ and } \dot{U}_\alpha = U_\alpha.$$

Since G is a filter, it follows that for $\beta < \alpha, x_\beta \in W_\alpha$ and that $x_\alpha \notin V_\beta$. Therefore (back in V) $\{x_\alpha : \alpha \in \omega_1\}$ is a free sequence – since we have the same base for the topology in both models, $\overline{W_\alpha} \cap \overline{X \setminus V_\alpha}$ must be empty. This contradicts the fact that X contains no free sequences (in V) . □

Remark. The role of the pair $\{W_\alpha, V_\alpha\}$ in the above proof is critical. It is not true, in general, that if you introduce a free sequence with proper forcing then you must have had one to begin with. Perhaps the easiest way to see what is going on is to think of the above mentioned "author's-view" of PFA. When you meet ω_1-many dense sets from the poset $P * Q$, you are really forcing over some inner model. We can think of this forcing as introducing a sequence which is "free with respect to the inner model space". However there are still points to be added to the space which can destroy that freedom. Also there are still neighbourhoods to be added of the points you do have and this is why we do not, and can not, assert that we get a copy of ω_1 in X.

VII Submodels closed under ω-sequences and Forcing

In this last section we will prove a few results that show that the techniques involved when using large cardinals can be used even without the large cardinals. All that is going on in the results of the previous section is that a forcing statement is first reflected, then the forcing is factored and finally a preservation result is proven. When countable objects seem to determine all the reflection that you need then it is possible that a large cardinal is not needed. It may suffice to reflect the forcing statement (as in the above outline) by simply taking an elementary submodel closed under ω-sequences. The more difficult arguments (e.g. those using PFA) may require the assumption of $\diamondsuit(\omega_2)$ because it sometimes depends on the order in which you iterate your posets. If the forcing is simply an iteration of the same poset then you probably just need to assume CH in the ground model as we shall demonstrate below with Cohen forcing. Frequently these results are proven using the Δ-system lemma and other combinatorics .

The general procedure is to let, say, $\{\dot{A}_\alpha : \alpha < \omega_2\}$, $\{\dot{B}_\alpha : \alpha < \omega_2\}$ and $\{\dot{C}_\alpha : \alpha < \omega_2\}$ be $Fn(\omega_2, 2)$-names of subsets of ω_2 . Let $\mathcal{A} = \{\dot{A}_\alpha : \alpha < \omega_2\}$ and similarly define $\mathcal{B}$ and $\mathcal{C}$. Let $M \prec H(\omega_3)$ be such that

$$\{\mathcal{A}, \mathcal{B}, \mathcal{C}\} \in M \ , \quad M^\omega \subset M \quad , \quad \text{and} \quad M \cap \omega_2 = \lambda < \omega_2 \ .$$

Recall that a $Fn(\omega_2, 2)$-name, say $\dot{A}$, of a subset of ω_2 can be assumed to be a subset of $\omega_2 \times Fn(\omega_2, 2)$ where $p \Vdash \alpha \in \dot{A}$ iff $(\alpha, p) \in \dot{A}$.

Now we let $\dot{A}_{\alpha\lambda} = \dot{A}_\alpha \cap \big(\lambda \times Fn(\lambda, 2)\big)$ for each $\alpha < \lambda$. Similarly define $\dot{B}_{\alpha\lambda}$ and $\dot{C}_{\alpha\lambda}$. Using the facts that $M \prec H(\omega_3)$ and $M^\omega \subset M$ one can easily prove that $1 \Vdash_{Fn(\lambda,2)} \dot{A}_{\alpha\lambda} = \dot{A}_\alpha \cap \lambda$ and many other reflection results of the form

$$1 \Vdash_{Fn(\lambda,2)} \varphi\big(\dot{A}_{\alpha\lambda}, \ldots, \dot{C}_{\gamma\lambda}\big) \Leftrightarrow 1 \Vdash_{Fn(\omega_2,2)} \varphi\big(\dot{A}_\alpha, \ldots, \dot{C}_\gamma\big) \ .$$

The final and crucial step after having obtained the validity of the appropriate forcing reflection is to prove that further Cohen forcing preserves the property.

Let us begin with a well-know result of Kunen's.

Proposition 7.1: *In the model obtained by adding ω_2-Cohen reals to a model of CH, there are no ω_2-chains in $\mathcal{P}(\omega)$ mod fin.*

Proof. Suppose $\mathcal{A}, \{\dot{A}_\alpha \mid \alpha < \omega_2\}$, are $Fn(\omega_2, 2)$-names such that

$$1 \Vdash_{Fn(\omega_2,2)} \mathcal{A} = \{\dot{A}_\alpha \mid \alpha < \omega_2\} \subset \mathcal{P}(\omega) \text{ and } \dot{A}_\alpha \subset^* \dot{A}_\beta \ for\ \alpha < \beta < \omega_2.$$

Also fix a name $\mathcal{B}$ for $\{\dot{B}_\alpha : \alpha \in \omega_2\}$ so that

$$1 \Vdash \mathcal{B} = \{B \in \mathcal{P}(\omega) \mid \ |\dot{A}_\alpha \setminus B| < \omega \ for\ all\ \alpha < \omega_2\}.$$

Let $\mathcal{A}$, $\mathcal{B} \in M \prec H(\omega_3)$ with $M^\omega \subset M, |M| = \omega_1$ and $M \cap \omega_2 = \lambda$. For $\alpha < \lambda$, we may assume that, in fact, $\dot{A}_\alpha$ and $\dot{B}_\alpha$ are $Fn(\lambda, 2)$-names. Now

$$M \models 1 \Vdash \{\dot{B}_\beta : \beta < \omega_2\} = \{B \mid \ (\forall \alpha < \omega_2)\ |\dot{A}_\alpha \setminus B| < \omega\}.$$

Suppose $\dot{A}$ is such that $1 \Vdash_{Fn(\lambda,2)} (\forall \alpha \in \lambda)\ |\dot{A}_\alpha \setminus \dot{A}| < \omega$. Since $Fn(\lambda, 2)$ is *ccc* and ω is countable, there is a $\dot{B} \in M$ so that $1 \Vdash_{Fn(\lambda,2)} \dot{A} = \dot{B}$. Therefore,

$$1 \Vdash \exists \beta < \lambda \text{ such that } \dot{B} = \dot{B}_\beta.$$

This application of "ω-absoluteness" has shown that if G is $Fn(\omega_2, 2)$-generic and if $G_\lambda = G \cap Fn(\lambda, 2)$, then

$$V[G_\lambda] \models \{val(\dot{B}_\alpha, G_\lambda) \mid \alpha < \lambda\} = \{B \in \mathcal{P}(\omega) \mid (\forall \alpha < \lambda) val(\dot{A}_\alpha, G_\lambda) \subset^* B\}.$$

Now in $V[G]$, let $A_\alpha = val(\dot{A}_\alpha, G)$ for $\alpha < \omega_2$. By assumption, $|A_\alpha \setminus A_\lambda| < \omega$ for all $\alpha < \lambda$ and $|A_\lambda \setminus B_\alpha| < \omega$ for all $\alpha < \lambda$. Now refer to 5.1 and let $\mathcal{I} = \{B \subset \omega : B \in V[G_\lambda] \text{ and } B \subset A_\lambda\}$. Since $\mathcal{I}$ is countably generated and $cf(\lambda) = \omega_1$, there is an $I \in \mathcal{I}$ so that $A_\alpha \subset^* I$ for cofinally many $\alpha \in \lambda$. Therefore, $A_\alpha \subset^* I$ for all $\alpha \in \lambda$. But then, by the above, there is a $\beta \in \lambda$ such that $I = B_\beta$. But this implies that $A_\lambda =^* A_{\lambda+1}$ (a contradiction) since $B_\beta = I \subset A_\lambda \subset^* A_{\lambda+1} \subset^* B_\beta$ □

This technique is also useful in proving Malykin's interesting new result. Van Douwen and van Mill have shown that it is consistent that (e.g. under PFA) $\omega^* - \{x\}$ is C*-embedded in ω^* for any point $x \in \omega^*$. Malykin has shown that this is also true in the Cohen model. I feel that this result demonstrates that there are still interesting consistency results to be obtained in the Cohen model.

Proposition 7.2: *If G is $Fn(\omega_2, 2)$-generic over V, a model of CH, then, in $V[G]$, ω^*-$\{x\}$ is C*-embedded for each $x \in \omega^*$.*

Sketch of Proof. Assume that $f\ \omega^*\text{-}\{x\} \to [0,1]$ is continuous and that $r_0 < r_1$ are such that $x \in \overline{f^{\leftarrow}[0,r_0]} \cap \overline{f^{\leftarrow}[r_1,1]}$. It is well-known that $R_0 = \overline{f^{\leftarrow}[0,r_0]}$ and similarly R_1 are regular closed subsets of ω^*. Let $\{A_\alpha\}_{\alpha<\omega_2}$ and $\{B_\beta\}_{\beta<\omega_2}$ be the subsets of ω whose remainders are contained in R_0 and R_1 respectively. Fix $Fn(\omega_2,2)$-names for the A_α's and the B_β's and find $\lambda < \omega_2$ just as in 7.1 (you would also want to ensure that x was in M). Using 5.1 and the fact that ω^* is an F-space one can show that $R_0 \cap R_1 \supset \bigcap\{C^* : C \in x \cap V[G_\lambda]\}$. Indeed, suppose that $D \subset \omega$ is such that $D \cap X \neq^* \emptyset$ for all $X \in x \cap V[G_\lambda]$ and that $D^* \cap R_0 = \emptyset$. By 5.1, there are $\{X_n : n \in \omega\} \subset x \cap V[G_\lambda]$ which generate the filter $\{Y \in V[G_\lambda] : D \subset^* Y\}$. This set $\{X_n : n \in \omega\}$ need not be in $V[G_\lambda]$ in general but since x is a filter we can enlarge the set $\{X_n\}$ so that we may assume that it is in $V[G_\lambda]$. Let $Z = \bigcap\{X_n^* : n \in \omega\}$ and note that $x \in Z \cap R_0$ and that $Z \cap R_0$ is again regular closed. But now $Z \in V[G_\lambda]$, hence we may choose an $\alpha < \lambda$ so that $A_\alpha \in V[G_\lambda]$ and $A_\alpha^* \subset Z \cap R_0$. This contradicts that there should be an n so that $X_n \subset^* \omega \setminus A_\alpha$. To finish the proof then we just have to note that $x \cap V[G_\lambda]$ does not generate x. □

We finish with a new proof of a result from [DTW]. The original proof of this (and the PMEA analogue) involved rather more difficult filter combinatorics.

Proposition 7.3: *If G is $Fn(\omega_2,2)$-generic over V, a model of CH, then, in $V[G]$, a first countable space of weight ω_1 is metrizable if each of its $\aleph_1$-sized subspaces are metrizable.*

Proof. Let $\{\dot B_\alpha : \alpha \in \omega_1\}$ be $Fn(\omega_2,2)$-names of subsets of ω_2 so that $1 \Vdash \langle \omega_2, \{\dot B_\alpha \mid \alpha < \omega_1\}\rangle$ is a first countable space in which each subspace of size ω_1 is metrizable. Let $M \prec H(\omega_3)$ be so that $M^\omega \subset M$, $|M| = \omega_1$ and $\{\dot B_\alpha\}_{\alpha<\omega_1} \in M$. Let $\lambda = M \cap \omega_2$, $G_\lambda = G \cap Fn(\lambda,2)$ and let $\{\dot B_{\alpha\lambda}\}_{\alpha<\omega_1}$ be as above. Then $V[G_\lambda] \models \langle \lambda, \{\dot B_{\alpha\lambda}\}_{\alpha<\omega_1}\rangle$ is a first countable space. By 5.4, $1 \Vdash_{Fn(\lambda,2)} \langle \lambda, \{\dot B_{\alpha\lambda}\}_{\alpha<\lambda}\rangle$ has a σ-discrete base. Fix a $Fn(\lambda,2)$-name $\dot{\mathcal{U}}$ so that $1 \Vdash \dot{\mathcal{U}} \subset \omega_1 \times \omega_1 \times \omega$ and so that $\dot{\mathcal{U}}$ "codes" a σ-discrete base for λ. That is, the $(\alpha,n)^{th}$ member of the base will be the union of $\{\dot B_{\beta\lambda} : (\beta,\alpha,n) \in \dot{\mathcal{U}}\}$. We will show that the collection whose $(\alpha,n)^{th}$ member is the union of $\{\dot B_\beta : (\beta,\alpha,n) \in \dot{\mathcal{U}}\}$ will form a σ-discrete base for the whole space.

We would be done if the name $\dot{\mathcal{U}}$ were a member of M but there is no reason to suppose that this would be so. However, the trick is to isolate, for each remaining $x \in \omega_2$ a countable piece of the name $\dot{\mathcal{U}}$ which will do the job. This countable piece will be in M which will allow us to play the ω-absoluteness game.

Let $x \in \omega_2$ and let $\alpha_0 \in \omega_1$ be such that $1 \Vdash \{\dot B_\beta : \beta < \alpha_0\}$ contains a base at x. Let N be a countable elementary submodel of $H(\omega_3)$ which contains the set $\{x, \{\dot B_\beta : \beta < \alpha_0\}, \dot{\mathcal{U}}, M\}$. Now let $\mu = N \cap \omega_1$ and let $\dot{\mathcal{U}}_N = \dot{\mathcal{U}} \cap N$. Since M is closed under ω-sequences, $\dot{\mathcal{U}}_N \in M$.

Let $\varphi(\alpha, \dot U)$ denote the formula (with parameter $\{\dot B_\beta \mid \beta \in \omega_1\}$):
"$1 \Vdash \big($ If $y \in \omega_2$ is such that $\{\dot B_\beta \mid \beta \in \alpha\}$ contains a base for y, then,
(i) for each $n \in \omega$, y has a neighbouhood meeting at most one member of the family $\{\cup\{\dot B_\beta \mid (\beta,\gamma,n) \in \dot U\} \mid \gamma \in \omega_1\}$, and

(ii) for each $\xi \in \alpha$, such that $y \in \dot{B}_\xi$, there are $\gamma \in \omega$ and $n \in \omega$ such that $y \in \cup\{\dot{B}_\beta \mid (\beta,\gamma,n) \in \dot{U}\} \subset B_\xi$)."

Now we observe that :

$$M \models \varphi(\alpha_0, \dot{\mathcal{U}}_N).$$

Therefore,

$$H(\omega_3) \models \varphi(\alpha_0, \dot{\mathcal{U}}_N).$$

But now, since $\dot{\mathcal{U}}_N = \dot{\mathcal{U}} \cap N$, we have:

$$N \models \varphi(\alpha_0, \dot{\mathcal{U}}).$$

And, finally, since $N \prec H(\omega_3)$, we obtain that:

$$H(\omega_3) \models \varphi(\alpha_0, \dot{\mathcal{U}}).$$

This completes the proof since it shows that, at least with respect to x, $\dot{\mathcal{U}}$ codes a σ-discrete base. $\square$

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