TWO REAL ULTRAFILTERS ON ω

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Abstract. We prove the following results concerning the space $\omega\cup\{p\}$, where p is an ultrafilter on ω:

(i) There are ultrafilters p on ω with the property that every compactification of $\omega \cup \{p\}$ contains a copy of $\beta\omega$.

(ii) There is an ultrafilter p on ω with a strong Noetherian base, i.e. every member of the base is contained, mod finite, in only finitely many others.

1. Introduction

We prove several results which relate to the space $\omega\cup\{p\}$, where p is an ultrafilter on ω. In section 2, we prove, in ZFC, that there are ultrafilters p on ω with the property that every compactification of $\omega \cup \{p\}$ contains a copy of $\beta\omega$, thus answering a question raised by van Mill in [4]

In section 3, we construct an ultrafilter p for which there exists an almost disjoint refinement $\mathcal{I}$ of p and a base $\mathcal{B}$ for p such that for each subset $\mathcal{I}^{'} \in [\mathcal{I}]^{<\mathfrak{c}}$, there is a member of p which is almost disjoint with each member of $\mathcal{I}^{'}$. Additionally, we have that each $I \in \mathcal{I}$ is almost disjoint from all but finitely many members of $\mathcal{B}$. The last property implies that $\mathcal{B}$ is a strong Noetherian base.

2. Large Ultrafilter

It is asked in van Mill [4] if there is an ultrafilter p on the countable discrete space ω such that the space $\omega \cup \{p\}$ (as a subspace of $\beta\omega$) has the property that every compactification contains a copy of $\beta\omega$. Let us temporarily call such ultrafilter a *large ultrafilter*. Ryll-Nardzewski and Telgársky [5] showed that if p is a simple P-point then the space $\omega \cup \{p\}$ has a scattered compactification. It was also shown by Rajagopolan that p is not large if it contains a tower since space $\omega \cup \{p\}$ has a compactification whose remainder is an ordinal (and there exist such p in any model of Martin's Axiom) while Malyhin has shown that, in the Cohen model, every ultrafilter is large. In this section we show (in ZFC) that there are many large ultrafilters.

First of all, we give the following combinatorial lemma. It is certainly straightforward but for the reader's convenience, we provide a standard Δ-system proof. Note, however, that it is not totally trivial since $\mathfrak{c}$ may be singular.

Lemma 2.0.1 (Folklore). *Suppose $\{\varphi_\alpha : \alpha \in \mathfrak{c}\}$ is a sequence such that φ_α is a function into 2 with finite domain $F_\alpha \subseteq \mathfrak{c}$ and $\alpha \in F_\alpha$ for all $\alpha < \mathfrak{c}$. Then there exists $J \in [\mathfrak{c}]^{\mathfrak{c}}$ such that $\{\varphi_\alpha, \{\langle\beta, 1\rangle\} : \alpha \in G, \beta \in H\}$ is a set of mutually compatible functions, for any disjoint subsets G, H of J,*

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Proof. If $\mathfrak{c}$ is a regular cardinal, let $I \in [\mathfrak{c}]^{\mathfrak{c}}$ be such that $\{F_\alpha : \alpha \in I\}$ is a Δ-system with root R and such that $\varphi_\alpha \restriction R$ all agree.

If $\mathfrak{c}$ is singular apply the Δ-system lemma to $\{F_\alpha : \alpha < \mathfrak{c}\}$ as follows. Choose an increasing sequence κ_ξ ($\xi < \mathrm{cf}(\mathfrak{c})$) of regular cardinals which are cofinal in $\mathfrak{c}$ (each of which is greater than $\mathrm{cf}(\mathfrak{c})$). Let, for each $\xi < \mathrm{cf}(\mathfrak{c})$, $I_\xi \in [\kappa_\xi - \sup\{\kappa_\zeta : \zeta < \xi\}]^{\kappa_\xi}$ be such that $\{F_\alpha : \alpha \in I_\xi\}$ is a Δ-system with root R_ξ and for $\alpha, \beta \in I_\xi$, φ_α and φ_β agree on R_ξ. We can apply the Δ-system lemma to the R_ξ and by re-indexing a subset we may assume that $\{R_\xi : \xi \in \mathrm{cf}(\mathfrak{c})\}$ forms a Δ-system with root R and, again, that all the φ_α agree on R for $\alpha \in \bigcup_{\xi<\mathrm{cf}(\mathfrak{c})} I_\xi$. For each ξ, choose a subset I'_ξ of I_ξ with cardinality κ_ξ such that

$$(F_\alpha - R_\xi) \cap \left[\left(\bigcup_{\zeta<\mathrm{cf}(\mathfrak{c})} R_\zeta \right) \cup \left(\bigcup \{F_\beta : \beta \in \bigcup_{\zeta<\xi} I_\zeta\} \right) \right]$$

is empty. Let $I = \cup_{\xi<\mathrm{cf}(\mathfrak{c})} I'_\xi$.

It is easy to see that for each finite subset G of I, $\bigcup_{\alpha\in G} \varphi_\alpha$ is a function. Now, take a subset J of I with size $\mathfrak{c}$ such that, for each $\alpha \in J$, $J \cap \mathrm{dom}\,\varphi_\alpha = \{\alpha\}$. J has the required properties. $\square$

Fix any function f from ω onto a dense subset of $2^{\mathfrak{c}}$. Extend f to a map from $\beta\omega$ onto $2^{\mathfrak{c}}$. It is well known that we can find a closed subset K of $\beta\omega$ for which $f \restriction K$ maps K irreducibly onto $2^{\mathfrak{c}}$, i.e. $F[K] = 2^{\mathfrak{c}}$ but for any proper closed subset $J \subset K$, $f[J] \neq 2^{\mathfrak{c}}$. See 3.1.C.(a) of [2]

Theorem 2.0.2. *Let K be as defined above, then any point p in K has the property that every compactification of $\omega \cup \{p\}$ contains a copy of $\beta\omega$.*

Proof. Let X be any compactification of $\omega \cup \{p\}$. We have to show that X contains a copy of $\beta\omega$. By the maximality of $\beta\omega$, it suffices to show that some continuous image of X contains a copy of $\beta\omega$. We will construct a continuous mapping $h : X \to \mathbb{R}^{\mathfrak{c}}$ such that $\mathrm{ran}(h)$ contains $2^{\mathfrak{c}}$. Since $2^{\mathfrak{c}}$ contains a copy of $\beta\omega$, we will be done.

Let g be the map from $\beta\omega$ onto X which is the identity map on $\omega \cup \{p\}$. Without loss of generality, we can assume $f(p)$ is the constantly 0 function in $2^{\mathfrak{c}}$. For each $\alpha \in \mathfrak{c}$, let $A_\alpha \in p$ be chosen so that $A_\alpha = \omega \cap f^{\leftarrow}([\langle\alpha, 0\rangle])$ where $[\langle\alpha, 0\rangle]$ denotes the basic clopen subset of $2^{\mathfrak{c}}$ of all functions which are 0 at α.

Since $\omega \cup \{p\}$ is a subspace of X, there is a neighbourhood U_α of p in X such that $U_\alpha \cap \omega \subset A_\alpha$ – i.e. p is not in the closure of $\omega - A_\alpha$. Using the continuity of g, there is a $B_\alpha \in p$ such that the closure of $g[B_\alpha]$ is contained in U_α. Then $B^*_\alpha \cap K$ is a clopen subset of K which, by irreducibility of $f \restriction K$, will map to a subset of $2^{\mathfrak{c}}$ with non-empty interior. Fix a finite set $F_\alpha \subset \mathfrak{c}$ and a function $\varphi_\alpha : F_\alpha \mapsto 2$ such that $[\varphi_\alpha] \subset f[B^*_\alpha \cap K]$. It follows that $\alpha \in F_\alpha$ because

$$f(B^*_\alpha \cap K) \subset f(A^*_\alpha \cap K) = [\langle\alpha, 0\rangle] \ .$$

Applying Lemma 2.0.1 to $\{\varphi_\alpha, \alpha \in \mathfrak{c}\}$, we get a set J of size $\mathfrak{c}$ as in the Lemma.

Claim 2.1. The family $\{\langle B_\alpha, \omega - A_\alpha\rangle : \alpha \in J\}$ is a $J \times 2$-independent matrix, i.e. if G, H are disjoint finite subsets of J, then

$$\bigcap_{\alpha\in G} B_\alpha \cap \bigcap_{\beta\in H} \omega - A_\alpha$$

is not empty.

Proof of the Claim 2.1: Let G, H be disjoint finite subsets of J. It suffices to show that $\bigcap_{\alpha\in G} B^*_\alpha \cap K \ - \ \bigcup_{\beta\in H} A^*_\beta \cap K$ is not empty. We will instead show that $f(\bigcap_{\alpha\in G} B^*_\alpha \cap K \ - \ \bigcup_{\beta\in H} A^*_\beta \cap K)$ is not empty. Since $f \restriction K$ is irreducible onto $2^{\mathfrak{c}}$,

$$f(\bigcap_{\alpha\in G} B^*_\alpha \cap K \ - \ \bigcup_{\beta\in H} A^*_\beta \cap K) = \bigcap_{\alpha\in G} f(B^*_\alpha \cap K) \ - \ \bigcup_{\beta\in H} f(A^*_\beta \cap K) \ .$$

Recall that $f[B^*_\alpha \cap K]$ contains $[\varphi_\alpha]$ and that $f[A^*_\alpha \cap K]$ is equal to $[\langle\alpha, 0\rangle]$. Therefore we have

$$\begin{aligned}\bigcap_{\alpha\in G} f[B^*_\alpha \cap K] - \bigcup_{\beta\in H} f[A^*_\beta \cap K] \supseteq& \\ &\supseteq \bigcap_{\alpha\in G} [\varphi_\alpha] - \bigcup_{\beta\in H} [\langle\beta, 0\rangle] \supseteq \\ &\supseteq [\bigcup_{\alpha\in G} \varphi_\alpha \cup \{\langle\beta, 1\rangle : \beta \in H\}]\end{aligned}$$

Since $\varphi = \bigcap_{\alpha\in G} \varphi_\alpha \cup \{\langle\beta, 1\rangle : \beta \in H\}$ is a function, the last set in the above formula is not empty, which finishes the proof of the claim.

To finish the proof, fix, for each $\alpha \in J$, a function $h_\alpha \in C^*(X)$ such that $h_\alpha[g(B_\alpha)] = \{0\}$ and $h_\alpha[X - U_\alpha] = \{1\}$. Let $h = \Pi_{\alpha\in J} h_\alpha$. Since $X - U_\alpha$ contains $\omega - A_\alpha$, it follows immediately from Claim 2.1 and the compactness of $2^{\mathfrak{c}}$ that h is a mapping of X into $\mathbb{R}^J$ whose image contains $2^{\mathfrak{c}}$. $\square$

3. A Special Ultrafilter

Definition 3.0.3. Given a family $\mathcal{F}$ of subsets of some set S. A family $\mathcal{I}$ is called a refinement of $\mathcal{F}$ if for each F in $\mathcal{F}$, there is an I in $\mathcal{I}$ such that $I \subseteq F$. If $\mathcal{I}$ is an almost disjoint family we say $\mathcal{I}$ is an almost disjoint refinement of $\mathcal{F}$.

There has been many interesting results about almost disjoint refinements of filters on ω, see section 4 of [1] for a survey. The following Proposition gives an ultrafilter which has a quite unexpected almost disjoint refinement that does not follow from the constructions in [1]. A Noetherian space is a space with a base $\mathcal{B}$ such that for each $U \in \mathcal{B}$, the set $\{V \in \mathcal{B} \colon U \subseteq V\}$ is finite (see [3]). A base $\mathcal{B}$ for an ultrafilter p on ω is a *Noetherian base (strong Noetherian base)* if for each $A \in \mathcal{B}$, the set $\{B \in \mathcal{B} \colon A \subseteq B\}$ ($\{B \in \mathcal{B} \colon A \subseteq^* B\}$) is finite. Malykin has asked if there exists an ultrafilter with a Noetherian base. The base $\mathcal{B}$ in the ultrafilter p in the following proposition is actually strong Noetherian. Thus we give a positive answer to Malykin's question. In addition the property (i) of the filter p is quite unexpected.

Proposition 3.0.4. *There is an ultrafilter p on ω which has an almost disjoint refinement $\mathcal{I}$ and a base $\mathcal{B}$ such that*

- (i) *for each subset $\mathcal{I}' \in [\mathcal{I}]^{<\mathfrak{c}}$ there is a member of p which is almost disjoint with each member of $\mathcal{I}'$,* and
- (ii) *for each $B \in \mathcal{B}$* and *$I \in \mathcal{I}$ either $B \cap I$ is finite or B almost contains I and I is almost disjoint with all but finitely many members of $\mathcal{B}$.*

Remark 3.0.5. It is straightforward to see that the property (ii) in Proposition 3.0.4 implies the base $\mathcal{B}$ is a strong Noetherian base. A careful analysis of the construction in [1] will reveal that the almost disjoint refinements constructed by that method will fail to satisfy property (ii) for any base, i.e. for any base, there will be some member of the refinement almost contained in infinitely many members of the base.

The following Lemmas give the proof of the Proposition 3.0.4.

Lemma 3.0.6. *If $\mathcal{F}$ is a filter base on ω such that, for each $F \in \mathcal{F}$, there is an infinite $I \subseteq F$ such that $\{F' \in \mathcal{F} : I \cap F' is\ infinite\}$ is finite, then there is an ultrafilter p on ω which extends $\mathcal{F}$ and satisfies the following property*

$$(\dagger) \qquad \begin{gathered} For\ each\ F \in p\ , there\ exists\ I \in [F]^\omega \\ such\ that\ \ \{F' \in \mathcal{F} : I \cap F'\ \ is\ \ infinite\ \}\ is\ finite \end{gathered}$$

Proof. Let $[\omega]^\omega = \{Y_\alpha : \alpha \leq \mathfrak{c}\}$. We will build inductively an increasing sequence $\{\mathcal{F}_\alpha : \alpha \in \mathfrak{c}\}$ of filters on ω such that

(i) $\mathcal{F} \subseteq \mathcal{F}_0$

(ii) For each $\alpha \leq \mathfrak{c}, \mathcal{F}_\alpha$ satisfies $(\dagger)$(with p taken to be $\mathcal{F}_\alpha$).

(iii) For each $\alpha \in \mathfrak{c}$, either $Y_\alpha \in \mathcal{F}_{\alpha+1}$ or $\omega \setminus Y_\alpha \in \mathcal{F}_{\alpha+1}$,

Let $\mathcal{F}_0$ be the filter generated by $\mathcal{F}$. Suppose we have constructed $\{\mathcal{F}_\beta : \beta < \alpha\}$ which satisfies (i)-(iii). If α is a limit ordinal, let $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. If α is the successor of β, we have to build $\mathcal{F}_{\beta+1}$ which extends $\mathcal{F}_\beta$ and satisfies (ii) and (iii).

We say a set $Y \subseteq \omega$ is $\mathcal{F}_\beta$-acceptable if for each $F \in \mathcal{F}_\beta$ there exists an $I \in [F \cap Y]^\omega$ such that $\{F' \in \mathcal{F} : F' \cap I$ is infinite$\}$ is finite.

Claim 3.1. There exists a $Y \in \{Y_\beta,\ \omega \setminus Y_\beta\}$ which is $\mathcal{F}_\beta$-acceptable.

Proof of Claim 3.1: Suppose not. Then there exist F_1 and F_2 in $\mathcal{F}_\beta$ such that for each $I \in [\omega]^\omega$, if $I \subseteq F_1 \cap Y_\beta$ or $I \subseteq F_2 \setminus Y_\beta$, then $\{F' \in \mathcal{F} : F' \cap I$ is infinite$\}$ is infinite. Since $\mathcal{F}_\beta$ satisfies $(\dagger)$, there exists an $I_0 \subseteq F_1 \cap F_2$ such that $\{F' \in \mathcal{F} : F' \cap I_0$ is infinite$\}$ is finite. Without loss of generality, we can assume $I_0 \cap Y_\beta$ is infinite. Then $I_0 \cap Y_\beta \in [\omega]^\omega$ and $I_0 \cap Y_\beta \subseteq F_1 \cap Y_\beta$ and $\{F' \in \mathcal{F} : F' \cap I \cap Y_\beta$ is infinite$\}$ is finite. This contradicts the choice of F_1.

Now take a $Y \in \{Y_\beta,\ \omega \setminus Y_\beta\}$ which is $\mathcal{F}_\beta$-acceptable. Let $\mathcal{F}_{\beta+1}$ be the filter generated by $\mathcal{F}_\beta \cup \{Y\}$. Obviously, $\mathcal{F}_{\beta+1}$ satisfies (ii) and (iii). Finally, we let $p = \mathcal{F}_\mathfrak{c}$. It is easy see that p works. $\square$

Lemma 3.0.7. *Under the condition of the Lemma 3.0.6, if $|\mathcal{F}| = \mathfrak{c}$, and p is obtained from Lemma 3.0.6, then Proposition 3.0.4 holds for p.*

Proof. Let $p = \{a_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{F} = \{d_\alpha : \alpha < \mathfrak{c}\}$, such that if $\alpha \neq \beta$, then $d_\alpha \neq d_\beta$. For each $F \in [\omega]^\omega$, denote by $\mathcal{I}_F$ the set $\{I \in [F]^\omega : \{F' \in \mathcal{F} : F' \cap I$ is infinite$\}$ is finite$\}$. Thus Lemma 3.0.6 says that for each member F of p, $\mathcal{I}_F$ is not empty. We will construct, by induction, $\{b_\alpha : \alpha < \mathfrak{c}\} \subseteq p$ and $\{I_\alpha : \alpha < \mathfrak{c}\}$ such that

(i) $I_\alpha \subseteq b_\alpha \subseteq a_\alpha \cap d_\alpha$,

(ii) $\forall \xi < \alpha, I_\xi \cap b_\alpha$ is finite,

(*iii*) $\forall \alpha < \mathfrak{c}\{\xi < \alpha : I_\alpha \subseteq b_\xi\} = \{\xi < \alpha : I_\alpha \cap b_\xi$ is infinite $\}$ is finite,
(*iv*) $\{F \in \mathcal{F} : I_\alpha \cap F$ is infinite $\}$ is finite.

Suppose first that we have constructed the above sequences, then it is straightforward to see that $\mathcal{I}$ and $\mathcal{B}$ satisfies the Proposition 3.0.4, where $\mathcal{I} = \{I_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{B} = \{b_\alpha : \alpha < \mathfrak{c}\}$. Let b_0 be $a_0 \cap d_0$ and I_0 be any element of $\mathcal{I}_{b_0}$. At stage α, we have to find b_α and I_α which satisfies (i) to (iv). By condition (iv) and the fact that $|\mathcal{F}| = \mathfrak{c}$, we can find $F \in \mathcal{F}$ such that $F \cap I_\xi$ is finite for each $\xi < \alpha$. Let $b_\alpha = a_\alpha \cap d_\alpha \cap F$. Take any $I \in \mathcal{I}_{b_\alpha}$. Since for each $\xi \in \alpha$, $b_\xi \cap I \subseteq d_\xi \cap I$, there are only finitely many ξ such that $b_\xi \cap I$ is infinite. It is easy to find a subset I_α of I such that, for each of those finitely many b_ξ's which has infinite intersection with I, either it contains I_α or it is disjoint from I_α. $\square$

Lemma 3.0.8. *There exists a filter base, $\mathcal{F}$, on ω such that $|\mathcal{F}| = \mathfrak{c}$ and for each $F \in \mathcal{F}$ there exists an infinite set $I \subseteq F$ such that $\{F' \in \mathcal{F} : F' \cap I$ is infinite$\}$ is finite.*

Proof. We will identify ω with the set $P = \bigcup_{n\in\omega} \mathcal{P}({}^n2)$. For each $E \in [{}^\omega 2]^{<\omega}$, let $X_E = \bigcup_{n\in\omega}\{A \subseteq {}^n2 \colon E\restriction n \subseteq A\}$ and $I_E = \{E\restriction n \colon n \in \omega\}$.

For finite subsets E and E', it is easy to see that $I_E \subseteq X_E$ and $X_{E\cup E'} = X_E \cap X_{E'}$. Moreover, if $E' \not\subseteq E$, then, there is an m such that for all $n > m$, $E'\restriction n \not\subseteq E\restriction n$. Therefore, $I_E \cap X_{E'} \subseteq \{E\restriction n \colon n \leq m\}$ is finite. Let $\mathcal{F} = \{X_E \subseteq P : E \in [{}^\omega 2]^{<\omega}\}$. Clearly $\mathcal{F}$ is a filter base on P with cardinality $\mathfrak{c}$ and for each X_E in $\mathcal{F}$, I_E is an infinite subset of X_E which is almost disjoint from all but finitely many members of $\mathcal{F}$. $\square$

Remark 3.0.9. If we take p as in Proposition 3.0.4, the space $\omega \cup \mathcal{I} \cup \{p\}$ with the usual topology is quite interesting: the first property ensures that it has tightness $\mathfrak{c}$ while the second property says that it is a 0-dimensional scattered space.

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