PARTITION SUBALGEBRAS FOR MAXIMAL ALMOST DISJOINT FAMILIES

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ABSTRACT. Partitioner Algebras are defined in [BW82] as a natural tool for studying the properties of maximal almost disjoint families of subsets of ω. We prove from PFA$^+$ and $\diamondsuit(S^2_0)$ that there exists a representable Boolean Algebra which contains a non representable subalgebra.

1. INTRODUCTION

For any maximal almost disjoint family $\mathcal{M}$ on ω, we use $\mathcal{I}_{\mathcal{M}}$ to denote the ideal generated by the mad family $\mathcal{M}$ and the finite subsets of ω. A set A is a *partitioner* of $\mathcal{M}$ if, for all $M \in \mathcal{M}$, either $M \setminus A$ is finite or $M \cap A$ is finite. A partitioner A of $\mathcal{M}$ is called a non-trivial partitioner of $\mathcal{M}$ if A is not in $I_{\mathcal{M}}$. The set of partitioners $\mathfrak{A}$ of $\mathcal{M}$ forms a subalgebra of $\mathcal{P}(\omega)$. We some times use $\mathfrak{A}_{\mathcal{M}}$ to denote this subalgebra. The *partitioner algebra* $P_{\mathcal{M}}$ of $\mathcal{M}$ is the quotient algebra $\mathfrak{A}/\mathcal{I}_{\mathcal{M}}$. A Boolean Algebra is said to be *representable* if it is isomorphic to a partitioner algebra of some mad family on ω.

Baumgartner and Weese in [BW82] introduced the notions of partitioner algebra and representable algebra. In actual fact this study began with Mrówka's ground breaking result that the partitioner algebra could be trivial (See [Mro77]). It has been an interesting and useful topic for more than twenty years to determine if a given Boolean Algebra is representable (eg. see [Teres80]) . Mrówka's method was expanded in [BW82] and the authors gave the first fundamental theorem which says that, under CH, every Boolean Algebra with cardinality less or equal to $\mathfrak{c}$ is representable. It is also shown that in Cohen real extension any Boolean algebra containing a free Boolean Algebra with size $\mathfrak{c}$ is not representable. In the same paper, there was a number of open problems regarding the representability of a Boolean Algebra.

In answering question 7 of [BW82], Zbierski and Frankiewicz proved that it is consistent with MA (σ-linked) that $\mathfrak{c}$ is arbitrarily large and that each Boolean algebra of cardinality $\leq \mathfrak{c}$ is indeed representable. Dow and Nyikos proved in [DN92] that the free algebra with $\aleph_1$ many generators is representable. This answers question 5 of [BW82]. Dow and Frankiewicz in [DF91] proved that it is consistent that the class of representable algebras is not closed under the quotients and that there is an algebra of cardinality $\aleph_1$ which is not representable.

The first problem in [BW82] is to decide if every subalgebra of a representable algebra is also representable. By Baumgartner and Weese's result, the answer is

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yes under CH. In this paper, we prove that, with the presence of a supercompact cardinal, it is consistent there exists a nonrepresentable algebra which is a subalgebra of a representable algebra. We will actually prove that PFA^+ and the existence of a $\diamondsuit$-sequence on $\{\alpha \in \omega_2 \colon \operatorname{cf} \alpha = \omega\}$ imply the existence of such algebras. In section 3, we will contruct a partitioner algebra with a nonrepresentable subalgebra $\mathcal{B}$; the proof that $\mathcal{B}$ is not representable will occupy sections 4 through 6 .

1.1. **Definitions and Notations.** Our notation and definitions are standard. For any set S, we use $|S|$ to denote the cardinality of S. For any cardinal number κ, $[S]^\kappa$ is the set of subsets of S with cardinality κ. When we write $f \colon A \to B$, we mean f is a function or a partial function from the set A to the set B. We will use Fin to denote the ideal of finite subsets of ω. For a family $\mathcal{A}$ of subsets of S, we use $\mathcal{I}_{\mathcal{A}}$ to denote the minimal ideal on S containing $\mathcal{A}$. For a family $\mathcal{A}$ of subsets of S and two subsets A and B of S, we write $A \subseteq_{\mathcal{A}} B$ if $A \setminus B \in \mathcal{I}_{\mathcal{A}}$. Naturally, $A =_{\mathcal{A}} B$ means that $A \triangle B \in \mathcal{I}_{\mathcal{A}}$. When S is ω and $\mathcal{I}$ is Fin, we use the standard notation, for example, $A =^* \emptyset$ means that A is a finite set.

We call an almost disjoint family an *ad* family and a maximal almost disjoint family a *mad* family. A sequence $\{a_\alpha \colon \alpha < \lambda\}$ of sets is *increasing mod finite* if for $\alpha < \beta < \lambda$, $a_\alpha \subseteq^* a_\beta$

If an infinite set a is almost contained in every member of a family $\mathcal{B}$ of sets then we say that a is a pseudo-intersection of $\mathcal{B}$. A set a *splits* a set b if both $b \cap a$ and $b \setminus a$ are infinite. So a is a partitioner of a mad family $\mathcal{M}$ *iff* a does not split any member of $\mathcal{M}$.

Our standard references for forcing are [Kun80] and [She82]. We always assume a poset has a largest element $\mathbf{1}$. We use $\check{x}$ (and occasionally just x) to denote a canonical name for x in the ground model. Consequently, for a poset $\mathbb{P}$ and a set X in the ground model, if τ is a *nice* $\mathbb{P}$-name for a subset of X, we will write τ in the following form, $\bigcup\{\{x\} \times \pi_x : x \in X\}$ instead of the formally correct $\bigcup\{\{\check{x}\} \times \pi_x : x \in X\}$, where each π_x is an antichain of $\mathbb{P}$.

Definition 1.1. A sequence $\{(a_\alpha, b_\alpha) : \alpha \in \lambda\}$ is called a *coherent sequence* on ω provided that both $\{a_\alpha : \alpha \in \lambda\}$ and $\{b_\alpha : \alpha \in \lambda\}$ are increasing mod finite sequences on ω and for each $\alpha < \beta < \lambda$,

(i) $a_\alpha \subseteq b_\alpha$,
(ii) $a_\alpha =^* a_\beta \cap b_\alpha$.

If $\mathcal{A} = \{(a_\alpha, b_\alpha) : \alpha \in \lambda\}$ and $\mathcal{B} = \{(c_\alpha, d_\alpha) : \alpha \in \rho\}$ are coherent sequences, then $\mathcal{A}^\frown\mathcal{B}$ is the sequence $\{(u_\alpha, v_\alpha) : \alpha < \lambda + \rho\}$ where $(u_\alpha, v_\alpha) = (a_\alpha, b_\alpha)$ when $\alpha < \lambda$ and $(u_\alpha, v_\alpha) = (c_\xi, d_\xi)$ when $\alpha = \lambda + \xi$ for some $\xi < \rho$. It is easy to see that $\mathcal{A}^\frown\mathcal{B}$ will also be a coherent sequence if for all $\alpha < \lambda$, $a_\alpha =^* c_0 \cap b_\alpha$ and $b_\alpha \subseteq^* d_0$. In the case $\mathcal{B}$ is the singleton $\{(c_0, d_0)\}$, we write $\mathcal{A}^\frown(c_0, d_0)$ instead of $\mathcal{A}^\frown\{(c_0, d_0)\}$. A set X is said to *extend* $\mathcal{A}$, if for all $\alpha < \lambda$, $X \cap b_\alpha =^* a_\alpha$.

If $a \subseteq b$ are two sets, we adopt Shelah's notation in [She82] using Ch^a_b to denote the function r from b to $\{0,1\}$ such that $r^{-1}(1) = a$.

Definition 1.2. Suppose $\mathcal{A} = \{(a_\alpha, b_\alpha) : \alpha \in \lambda\}$ is a coherent sequence on ω. $\mathbb{P}_{\mathcal{A}}$ is the poset consisting of the partial functions p from ω to $\{0,1\}$ such that there is an $\alpha < \lambda$ such that $p =^* \mathrm{Ch}^{a_\alpha}_{b_\alpha}$, i.e the following holds,

1. $\operatorname{dom} p =^* b_\alpha$ and
2. $p^{-1}(1) =^* a_\alpha$

For p and q in $\mathbb{P}_{\mathcal{A}}$, $p \leq q$ if $p \supseteq q$.

Obviously, $\mathbb{P}_{\mathcal{A}}$ adds a subset X of ω which extends $\mathcal{A}$.

For sets I and J and an infinite cardinal λ, the set $\mathrm{Fn}(I, J, \lambda)$ is the poset, ordered by reverse inclusion, of partial functions from I into J with cardinality less than λ .

When we write $\exists^{\infty}$ we mean "there are infinitely many" and by $\forall^{\infty}$ we mean "for all but only finitely many". More generally, suppose λ is a cardinal number, then $\exists^{\lambda}$ means "there exists λ many"; $\forall^{\lambda}$ means "for all but fewer than λ many". We give one more notation to finish this section. If f and g are any two functions, we use $f \overline{\cup} g$ to denote the function $f \cup g \restriction (\mathrm{dom}\, g \setminus \mathrm{dom}\, f)$.

2. Preliminaries

Suppose λ is a cardinal. A poset $\mathbb{P}$ is *λ-directed closed* if whenever $D \subseteq P$ is directed and $|D| < \lambda$, then there is a $p \in \mathbb{P}$ with $p \leq q$ for all $q \in D$. Obviously $\mathbb{P}$ is *λ-directed closed* iff whenever $\gamma < \lambda$ and $\{p_\xi : \xi < \gamma\}$ is a decreasing sequence of elements of $\mathbb{P}$, then there is a $p \in \mathbb{P}$ such that $p \leq p_\xi$ for all $\xi < \gamma$. In case λ is ω_1, $\mathbb{P}$ is said to be countably closed.

Definition 2.1. [She82] A poset $\mathbb{P}$ is called *proper* if, for any cardinal number λ and any stationary subset S of $[\lambda]^{\leq\omega}$, S remains stationary in the forcing extension $V[G]$ for any $\mathbb{P}$-generic G over V.

We use S^2_0 to denote the stationary set of all ordinals in ω_2 with countable cofinality. For any ordinal α, $\lim \alpha$ is the set of all limit ordinals below α.

The following theorem is well-known, but does not seem to be in the literature, so we only give an outline of its proof.

Theorem 2.2. *If there is a supercompact cardinal, then there is a model V of ZFC such that*

$$V \models \text{“PFA}^+ + \diamondsuit(S^2_0)\text{”}.$$

Proof. Suppose there exists a supercompact cardinal κ. By the theorem in [Lav78], we can assume κ is indestructible, i.e. for any forcing notion Q which is κ-directed closed, κ remains supercompact in V^Q. In the forcing extension by κ-directed closed forcing $P = \mathrm{Fn}(\kappa, 2, \kappa)$, κ is still supercompact and there is a diamond sequence on $\{\alpha \in \kappa : \mathrm{cf}(\alpha) = \omega\}$. In V^P, let $\dot{Q}$ be the usual forcing to get PFA. It is well-known that $\dot{Q}$ forces PFA$^+$. Since, $\dot{Q}$ is κ-cc, in $V^{P*\dot{Q}}$, there is still a diamond sequence on $\{\alpha \in \kappa : \mathrm{cf}(\alpha) = \omega\}$. But $\kappa = \omega_2$ in the final model. $\square$

It is well-known that Martin's Maximum implies Chang's Conjecture. In fact, PFA$^+$ also implies Chang's Conjecture (See Todorčević's Theorem 3.25 and Theorem 3.27 in [Bek91] and the discussion after Corollary 3.6 on page 272 in [Vel92]). P. Koszmider proved in [Kos] that Chang's Conjecture implies there is no strong chain of length ω_2 in $[\omega_1]^{\omega_1}$ (see the definition below). Therefore PFA$^+$ implies there is no strong chain of length ω_2.

Definition 2.3. A sequence $\{X_\alpha : \alpha \in \lambda\}$ in $[\omega_1]^{\omega_1}$ is called a *strong chain* if for any $\alpha < \beta \in \lambda$, $X_\alpha \setminus X_\beta$ is finite and $X_\beta \setminus X_\alpha$ is uncountable.

A *weak Kurepa tree* is a tree of height ω_1 and cardinality $\aleph_1$ with more than $\aleph_1$ many $\aleph_1$-branches. The weak Kurepa Hypothesis (wKH) asserts that there is a weak Kurepa tree.

We will need to use the Open Coloring Axiom (OCA):

If X is a separable metric space and

$$[X]^2 = K_0 \cup K_1$$

is a partition of $[X]^2$ with K_0 open, i.e. for any $\{x, y\} \in K_0$, there are open neighbourhoods U_0 and U_1 of x and y respectively, such that, for any $x' \in U_0$ and $y' \in U_1$, $\{x', y'\} \in K_0$.

Then *either*

1. there exists an uncountable 0-homogeneous subset A of X, i.e. $[A]^2 \subseteq K_0$, *or*
2. X can be covered by countably many 1-homogeneous subsets, $\{A_n : n \in \omega\}$, i.e. for each n, $[A_n]^2 \subseteq K_1$.

We now list several of the well known consequences PFA in the following Theorem.

Theorem 2.4. *Suppose PFA holds. Then the following are true:*

1. OCA *(Todorčević,*[Tod89]*),*
2. *wKH is false (Todorčević,* [Tod81]*, see also* [Bau84]*)*
3. $2^{\aleph_0} = \omega_2$ *(Todorčević,* [Bek91]*).*

Definition 2.5. A family $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$ is a *gap* if for each $\alpha < \omega_1, \beta < \omega_1$, $a_\alpha \cap b_\beta$ is finite and $a_\alpha \cap b_\alpha$ is empty. For a set a, we say *a fills* the gap if for all α, $a_\alpha \subseteq^* a$ and $b_\alpha \cap a$ is finite. In this case we say that the gap can be filled.

Definition 2.6 ([Dow97, 2.6])**.** A family $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$ is a Hausdorff-Luzin type family if for each $\alpha < \omega_1$ and $m \in \omega$,

$$\{\beta < \alpha : |a_\alpha \cap b_\beta| < m\} \text{ is finite.}$$

If, in addition, $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$ is a gap, then we call it a Hausdorff-Luzin type gap.

It is easy to see that the property of being a Hausdorff-Luzin type family is absolute for models preserving ω_1. It is also easy to see that any Hausdorff-Luzin type gap can not be filled.

Definition 2.7 ([Dow97, 2.1])**.** Two ideals $\mathcal{I}$ and $\mathcal{J}$ can be *countably separated* if there are countably many sets X_n such that for each pair $(I, J) \in \mathcal{I} \times \mathcal{J}$, there is an n such that I is almost disjoint from X_n and J is almost contained in X_n. We will say that the ideals are *strongly countably separated* if the X_n's have the additional property that $X_n \cap I$ is finite for each $I \in \mathcal{I}$ and each $n \in \omega$ (i.e. $[X_n]^\omega \cap \mathcal{I} = \emptyset$ for each n).

If we say that a set of pairs $\{(I_\alpha, J_\alpha) : \alpha \in \omega_1\}$ is (strongly) countably separated then we mean that the ideal generated by family $\{I_\alpha : \alpha \in \omega_1\}$ is (strongly) countably separated from the one generated by $\{J_\alpha : \alpha \in \omega_1\}$. We recall the following result.

Theorem 2.8 ([Dow97])**.** *Suppose $\mathcal{I}$ and $\mathcal{J}$ are ideals of countable sets with $\mathcal{I}$ a P-ideal. If $\mathcal{I}$ and $\mathcal{J}$ are countably separated, then they are strongly countably separated.*

Moreover, if PFA holds. then either

1. *$\mathcal{I}$ and $\mathcal{J}$ can be countably separated,*

   or

2. *there are $\aleph_1$-generated subideals $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{J}' \subset \mathcal{J}$ that cannot be countably separated.*

3. Main Results

In this section, we will first construct a mad family and a subalgebra of its partitioner algebra. Then we list some basic facts about the mad family and the subalgebra. In the last subsection we will prove from Lemma 3.8 that the subalgebra is not representable and the proof of Lemma 3.8 will occupy the rest of the paper.

By Theorem 2.2, if there is a supercompact cardinal, then there is a model in which PFA$^+$ and $\diamondsuit(S^2_0)$ hold. Hence, for the rest of the paper, we will assume that PFA$^+$ and $\diamondsuit(S^2_0)$ hold in V. Therefore the following are true in our universe V:

1. OCA.
2. $\mathfrak{c} = \omega_2$.
3. there is no weak Kurepa tree.
4. There is no strong chain in $[\omega_1]^{\omega_1}$ of length ω_2.

3.1. **Constructions.** The actual construction of $\mathfrak{B}$ is relatively short but we will spend considerable effort to prove that it works. We will first construct a mad family $\mathcal{M}$ and simultaneously an increasing mod finite sequence $\{a_\alpha \colon \alpha \in \omega_2\}$ of non trivial partitioners of $\mathcal{M}$. The sequence $\{a_\alpha \colon \alpha \in \omega_2\}$ will generate a maximal proper ideal on ω. Using a different underlying set, it is easy to see that $\mathcal{P}_{\mathcal{M}} \times \mathcal{P}_{\mathcal{M}}$ is isomorphic to the partitioner algebra of the mad family $\widetilde{\mathcal{M}}$ defined as follows: $\widetilde{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}'$, where $\mathcal{M}' = \{M \times \{0\} \colon M \in \mathcal{M}\}$. We will use $\mathcal{P}_{\widetilde{\mathcal{M}}}$ instead of $\mathcal{P}_{\mathcal{M}} \times \mathcal{P}_{\mathcal{M}}$. $\mathfrak{B}$ will be generated by the union of the ideals generated by the two chains $\{a_\alpha : \alpha \in \omega_2\}$ and $\{a_\alpha \times \{0\} \colon \alpha \in \omega_2\}$. Formally we let $\mathfrak{B}$ be the set of equivalence classes in $\mathcal{P}_{\widetilde{\mathcal{M}}}$ of those partitioners x for which there is an $\alpha < \omega_2$ such that either x or its complement is contained in $a_\alpha \cup a_\alpha \times \{0\}$. Obviously ω is a nontrivial partitioner of $\widetilde{\mathcal{M}}$ which is not in $\mathfrak{B}$.

We were initially motivated by the observation that $\mathfrak{B}$ itself cannot be the actual partitioner algebra of some mad family on $\omega \cup \omega \times \{0\}$ because the gap structure from the a_α's will ensure that ω itself will be an unwanted partitioner. This theme will be carried through the proof as our method will be to show that in any representation there will be a partitioner analogous to $[\omega]$. Indeed if a potential representation does not have a partitioner analogous to $[\omega]$ then it is too wild and we will use Shelah's no non-trivial automorphism techniques to kill it.

Let us first fix an enumeration, $\{C_\alpha : \alpha \in \lim \omega_2\}$, of $[\omega]^\omega$ and choose a sufficiently large regular cardinal θ. For the elementary submodels $\mathfrak{M}_\alpha$ appearing below, we use T_α to denote the tree consisting of all countable coherent sequences in $\mathfrak{M}_\alpha$. The ordering on T_α is end extension. In other words, for $\mathcal{B}$ and $\mathcal{A}$ in T_α, $\mathcal{B} \prec \mathcal{A}$ iff $\mathcal{A} = \{(c_\xi, d_\xi) : \xi < \beta\}$ and, for some $\gamma \leq \beta$, $\mathcal{B} = \{(c_\xi, d_\xi) : \xi < \gamma\}$. $[T_\alpha]$ is the set of all $\aleph_1$-branches of T_α.

We will define for each $\alpha < \omega_2$, $\mathfrak{M}_\alpha$, $\mathcal{M}_\alpha$ and a_α such that $\{\mathfrak{M}_\alpha : \alpha < \omega_2\}$ is a $\in$-chain of elementary submodels of H_θ each with cardinality $\aleph_1$, $\{\mathcal{M}_\alpha : \alpha \in \omega_2\}$ is an increasing sequence of almost disjoint families on ω and $\{a_\alpha : \alpha \in \omega_2\}$ is an increasing, mod finite, sequence of infinite subsets of ω. Moreover, the following properties will hold.

1. $\mathfrak{M}_0$ contains a $\diamondsuit(S^2_0)$ sequence and $\{C_\alpha : \alpha \in \lim \omega_2\}$ as elements, thus if $\alpha \in \mathfrak{M}_\alpha$, then $C_\alpha \in \mathfrak{M}_\alpha$.

2. $[T_\alpha] \subseteq \mathfrak{M}_{\alpha+1}$.
3. $\forall \alpha \in \omega_2$, a_α is co-infinite and $\mathcal{M}_\alpha < a_\alpha$, i.e. $\forall M \in \mathcal{M}_\alpha$, $M \subseteq^* a_\alpha$.
4. $\forall \alpha \in \omega_2$, $\{\alpha, \mathcal{M}_\alpha, a_\alpha\} \in \mathfrak{M}_\alpha$, and if $\alpha < \beta$, then $\mathcal{M}_\beta \cap \mathfrak{M}_\alpha = \mathcal{M}_\alpha$.
5. $\forall \alpha \in \lim \omega_2$,
   (i) $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$;
   (ii) $\mathcal{M}_\alpha = \mathcal{M}_{\alpha+1} = \mathcal{M}_{\alpha+2}$ and $\mathcal{M}_{\alpha+n} = \mathcal{M}_{\alpha+3} (n \geq 3)$;
   (iii) for $a \subseteq \omega$, if $a \in \mathfrak{M}_{\alpha+2}$, then a is a partitioner of $\mathcal{M}_{\alpha+3} \setminus \mathcal{M}_\alpha$.
6. $\forall \alpha \in \lim \omega_2$, there exists $M \in \mathcal{M}_{\alpha+3}$ such that $C_\alpha \cap M \neq^* \emptyset$ and either $\omega \setminus C_\alpha \subseteq^* a_{\alpha+1}$ or $C_\alpha \subseteq^* a_{\alpha+1}$.
7. For all $\alpha \in \omega_2$, $\omega \setminus a_\alpha$ is infinite and there exist, in $\mathfrak{M}_{\alpha+1}$, Cohen generic reals g_0^α and g_1^α over $\mathfrak{M}_\alpha$ such that $(g_0^\alpha)^{-1}(0) \cup (g_1^\alpha)^{-1}(1) \subseteq a_{\alpha+1}$.

It will be clear during the construction that most of the conditions will be trivially satisfied. For example, to get condition (1) and condition (2), we just put the necessary sets into $\mathfrak{M}_\alpha$. Condition (6) ensures that $\{a_\alpha : \alpha < \omega_2\}$ will generate a maximal ideal and that $\bigcup_{\alpha<\omega_2} \mathcal{M}_\alpha$ is a mad family. The Cohen reals in condition (7), in some sense, guarantee that the mad family $\bigcup_{\alpha<\omega_2} \mathcal{M}_\alpha$ and its partitioner algebra are complicated enough to allow us to force that $\mathfrak{B}$ is not representable. Conditions (1) - (5) are mainly technical which will be used in the next three sections.

We will actually perform the induction along the non successor ordinals, i.e. for each limit α, we describe how to get $\{\mathfrak{M}_{\alpha+i} : i \in \omega\}$, $\{\mathcal{M}_{\alpha+i} : i \in \omega\}$ and $\{a_{\alpha+i} : i \in \omega\}$. Suppose we are at now at some non-successor stage α.

Case 1: $\alpha = 0$ Take any elementary submodel $\mathfrak{M}$ of H_θ as $\mathfrak{M}_0$ such that $\mathfrak{M}$ has cardinality $\aleph_1$ and it contains a $\diamondsuit(S_0^2)$ sequence and $\{C_\alpha : \alpha \in \lim \omega_2\}$ as elements. Let $\mathcal{M}_0 = \emptyset$. Let $a_0 \in \mathfrak{M}_0$ be any infinite and co-infinite subset of ω. For $\alpha \in \omega \setminus \{0\}$, proceed as in subcase 4 below.

Case 2: $\alpha \in \lim \omega_2$ Let $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$. Since $|\mathcal{M}_\alpha| \leq \aleph_1$ and PFA holds, there exists $a_\alpha \subseteq \omega$ such that $\omega \setminus a_\alpha$ is infinite and

$$\forall \beta < \alpha, a_\beta \subseteq^* a_\alpha \ .$$

Let $\mathfrak{M}_\alpha$ be an elementary submodel of H_θ such that $|\mathfrak{M}_\alpha| \leq \aleph_1$ and $\{\mathfrak{M}_\beta, \beta < \alpha\}, \{\mathcal{M}_\beta, \beta < \alpha\}$ and $\{a_\beta : \beta < \alpha\}$ and $\{C_\alpha, a_\alpha\}$ are elements of $\mathfrak{M}_\alpha$. It is easy to see that conditions (1)-(7) are satisfied. Next we define, by induction on i, $\{\mathfrak{M}_{\alpha+i}, \mathcal{M}_{\alpha+i}, a_{\alpha+i} : 0 < i \in \omega\}$.

Subcase 1 : $i = 1$ We have to find $\mathfrak{M}_{\alpha+1}$, $\mathcal{M}_{\alpha+1}$, $a_{\alpha+1}$ and g_0^α and g_1^α. Let

$$T_\alpha = \{\mathcal{A} \in \mathfrak{M}_\alpha : \mathcal{A} \text{ is a coherent sequence of countable length}\} \ .$$

Then T_α is a tree under the partial order $\prec$ defined above. Obviously T_α has cardinality not greater than $\aleph_1$ and height $\aleph_1$. By Theorem 2, the weak Kurepa Hypothesis fails, thus T_α has at most $\aleph_1$ many $\aleph_1$-branches, i.e. $|[T_\alpha]| \leq \aleph_1$. Since $\mathfrak{M}_\alpha$ has size $\aleph_1$, by $\mathrm{MA}_{\aleph_1}$, there are countably many Cohen reals over $\mathfrak{M}_\alpha$, say, $\{r_n : n \in \omega\}$ such that for any real $a \subseteq \omega$ in $\mathfrak{M}_\alpha$, $\{r_n^{-1}(0) : n \in \omega\}$ is independent on a.

Now let $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha$ and $\mathfrak{M}_{\alpha+1}$ be an elementary submodel of H_θ with cardinality $\aleph_1$ and $\{[T_\alpha], \{r_n : n \in \omega\}, \mathfrak{M}_\alpha\} \subseteq \mathfrak{M}_{\alpha+1}$. Note that since ω_1 is both a subset of $\mathfrak{M}_{\alpha+1}$ and an element of $\mathfrak{M}_{\alpha+1}$, the sets $[T_\alpha], \{r_n : n \in \omega\}$ and $\mathfrak{M}_\alpha$ are also subsets of $\mathfrak{M}_{\alpha+1}$. Since $a_\alpha \in \mathfrak{M}_\alpha$, the set $r_0^{-1}(0) \cup r_1^{-1}(1) \cup a_\alpha$ is co-infinite. If $\omega \setminus C_\alpha$ is almost contained in a_α, then we let $a_{\alpha+1}$ be $r_0^{-1}(0) \cup r_1^{-1}(1) \cup a_\alpha$. Otherwise, $a_\alpha \cup C_\alpha$ is co-infinite, therefore, $r_0^{-1}(0) \cup r_1^{-1}(1) \cup a_\alpha \cup C_\alpha$ is also co-infinite. Let

$a_{\alpha+1} = r_0^{-1}(0) \cup r_1^{-1}(1) \cup a_\alpha \cup C_\alpha$. Let $g_0^\alpha = r_0$ and $g_1^\alpha = r_1$. Obviously, conditions (1)-(7) are satisfied.

Subcase 2 : $i = 2$

We have to find $\mathfrak{M}_{\alpha+2}, \mathcal{M}_{\alpha+2}$, $a_{\alpha+2}$, $g_0^{\alpha+1}$ and $g_1^{\alpha+1}$. Let $\mathcal{M}_{\alpha+2} = \mathcal{M}_\alpha$. Take any elementary submodel $\mathfrak{M}$ of H_θ containing a sequence of reals $\{r_{1,n} : n \in \omega\}$ which are mutually Cohen generic over $\mathfrak{M}_{\alpha+1}$. Let $\mathfrak{M}$ also contain all the $\aleph_1$ branches $[T_{\alpha+1}]$ of $T_{\alpha+1}$ and $\mathfrak{M}_{\alpha+1}$. Let $\mathfrak{M}_{\alpha+2}$ be $\mathfrak{M}$. Let $a_{\alpha+2} = r_{1,0}^{-1}(0) \cup r_{1,1}^{-1}(1) \cup a_{\alpha+1}$. Let $g_0^{\alpha+1} = r_{1,0}$ and $g_1^{\alpha+1} = r_{1,1}$

Subcase 3: $i = 3$

We have to find $\mathfrak{M}_{\alpha+3}$, $\mathcal{M}_{\alpha+3}$, $a_{\alpha+3}$, $g_0^{\alpha+2}$, and $g_1^{\alpha+2}$.

Take an ultrafilter $\mathcal{V}$ on ω such that $\mathcal{V} \in \mathfrak{M}_{\alpha+2}$ and $\{\omega \setminus M : M \in \mathcal{M}_\alpha\} \subseteq \mathcal{V}$ and $C_\alpha \in \mathcal{V}$ in the case that, for all $M \in \mathcal{M}_\alpha$, $C_\alpha \cap M =^* \emptyset$. Again PFA implies there exists a pseudo-intersection M^* of $\mathcal{V} \cap \mathfrak{M}_{\alpha+2}$. Let

$$\mathcal{M}_{\alpha+3} = \mathcal{M}_{\alpha+2} \cup \{M^*\} .$$

Since M^* is a pseudo-intersection of $\mathcal{V} \cap \mathcal{M}_{\alpha+2}$, M^* is not in $\mathfrak{M}_{\alpha+2}$ and no subset of ω in $\mathfrak{M}_{\alpha+2}$ splits M^*.

Let $\{r_{2,n} : n \in \omega\}$ be mutually Cohen reals over $\mathfrak{M}_{\alpha+2}$. Let $\mathfrak{M}_{\alpha+3}$ be an elementary submodel of H_θ with cardinality $\aleph_1$ such that $\{\mathfrak{M}_{\alpha+2}, M^*, \{r_{2,n} : n \in \omega\}, [T_{\alpha+2}]\} \subseteq \mathfrak{M}_{\alpha+3}$

By the choice of M^*, it is easy to see that $a_{\alpha+2} \cup M^*$ is contained in some co-infinite subset of a in $\mathfrak{M}_{\alpha+2}$. Therefore, $a_{\alpha+2} \cup M^* \cup r_{2,0}^{-1}(0) \cup r_{2,1}^{-1}(1)$ is co-infinite. Let $a_{\alpha+3} = a_{\alpha+2} \cup M^* \cup r_{2,0}^{-1}(0) \cup r_{2,1}^{-1}(1)$ and $g_0^{\alpha+2} = r_{2,0}$ and $g_1^{\alpha+2} = r_{2,1}$.

Subcase 4: $i > 3$

In this case we let $\mathcal{M}_{\alpha+i} = \mathcal{M}_{\alpha+3}$ and simply choose $\mathfrak{M}_{\alpha+i} \prec H_\theta$ such that $\{\mathfrak{M}_{\alpha+i-1}, [T_{\alpha+i-1}], \{r_{i-1,j} : j \in \omega\}\} \subseteq \mathfrak{M}_{\alpha+i}$, where $\{r_{i-1,j} : j \in \omega\}$ are reals which are mutually Cohen over $\mathfrak{M}_{\alpha+i-1}$. Let $a_{\alpha+i} = a_{\alpha+i-1} \cup r_{i-1,0}^{-1}(0) \cup r_{i-1,1}^{-1}(1)$ and $g_0^{\alpha+i-1} = r_{i-1,0}$ and $g_1^{\alpha+i-1} = r_{i-1,1}$.

Finally let $\mathcal{M} = \bigcup_{\alpha<\omega_2} \mathcal{M}_\alpha$.

3.2. **Basic Facts and finally $\mathfrak{B}$.** We first prove some basic facts about $\{\mathfrak{M}_\alpha, \mathcal{M}_\alpha, a_\alpha : \alpha \in \omega_2\}$.

Fact 3.1. *$\mathcal{M}$ is a mad family.*

Proof. This is obvious by (6). □

Fact 3.2. *$\{a_\alpha : \alpha \in \omega_2\}$ generates a maximal ideal on in $\mathcal{P}(\omega)$.*

Proof. By (6), for each subset a of ω, either a or $\omega \setminus a$ is contained in some a_β. □

Fact 3.3.
1. *If $a \in \mathfrak{M}_\alpha$, then a is a partitioner of $\mathcal{M} \setminus \mathcal{M}_\alpha$. Moreover, if α is a limit and, for each $\beta < \alpha$, $a \cap a_\beta$ is a partitioner of $\mathcal{M}_\beta$, then, a is a partitioner of $\mathcal{M}$.*
2. *each a_α is a non trivial partitioner of $\mathcal{M}$.*

Proof. Since $\mathcal{M}_\alpha < a_\alpha$, (1) implies (2), so we prove (1). Let $M \in \mathcal{M} \setminus \mathcal{M}_\alpha$ and set β to be the minimal ordinal such that $M \in \mathcal{M}_\beta$. Then by (4), β must be a successor ordinal and β must be of the form $\xi + 3$ with ξ a limit. Since $\alpha < \beta = \xi + 3$, $a \in \mathfrak{M}_{\xi+2}$. By (4), a is a partitioner of $\mathcal{M}_{\xi+3} \setminus \mathcal{M}_\xi$ which contains M according to the choice of ξ. Therefore a does not split M. Suppose, in addition, α is a limit, and $a \cap a_\beta$ is a partitioner of $\mathcal{M}_\beta$ for each $\beta < \alpha$. Let $M \in \mathcal{M}_\alpha$, then there is $\beta < \alpha$ such that $M \in \mathcal{M}_\beta$. Since $M \subseteq^* a_\beta$, a does not split M. □

Recall that if $\mathcal{M}$ is an ad family on a countable subset S , we use $\mathcal{I}_{\mathcal{M}}$ to denote the ideal on $\mathcal{P}(S)$ generated by $\mathcal{M}$.

We need some additional notation to define $\mathfrak{B}$ which will actually be defined using a partitioner algebra on $\omega \cup \omega \times \{0\}$.

We use a'_α to denote $a_\alpha \times \{0\}$. Let $\mathcal{M}' = \{M \times \{0\} : M \in \mathcal{M}\}$ and $\widetilde{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}'$. Obviously, $\widetilde{\mathcal{M}}$ is a mad family on $\omega \cup \omega \times \{0\}$ and $P_{\mathcal{M}} \times P_{\mathcal{M}}$ is isomorphic to the partitioner algebra of $\widetilde{\mathcal{M}}$. Let $\mathfrak{A}$ denote the partitioner algebra of $\widetilde{\mathcal{M}}$.

Let $\widetilde{\mathfrak{B}} = \{a \subseteq \omega \cup \omega \times \{0\} \colon a$ is a partitioner of $\widetilde{\mathcal{M}}$ and $\exists \beta \; a \subseteq^* a_\beta \cup a'_\beta$ or $(\omega \cup \omega \times \{0\}) \setminus a \subseteq^* a_\beta \cup a'_\beta\}$. It is obvious that $\widetilde{\mathfrak{B}}$ forms a subalgebra of $\mathcal{P}(\omega \cup \omega \times \{0\})$.

Finally let $\mathfrak{B} = \widetilde{\mathfrak{B}}/\mathcal{I}_{\widetilde{\mathcal{M}}}$. Obviously, $\mathfrak{B}$ is the subalgebra of the partitioner algebra $\mathfrak{A}$. The following obvious fact will play a significant role in determining that $\mathcal{B}$ is not representable.

Fact 3.4. *There does not exist $a \in \widetilde{\mathfrak{B}}$ such that, for all $\alpha \in \omega_2$, $a_\alpha \setminus a \in \mathcal{I}_{\widetilde{\mathcal{M}}}$ and $a'_\alpha \cap a \in \mathcal{I}_{\widetilde{\mathcal{M}}}$* □

Since $\{a_\alpha : \alpha \in \omega_2\}$ generates a maximal ideal on ω, $\{\omega \setminus a_\alpha : \alpha \in \omega_2\}$ is a base for a P_{ω_2}-ultrafilter on ω. Let us call this ultrafilter z.

Let X, X_0 and X_1 denote the Stone spaces of $P_{\mathcal{M}} \times P_{\mathcal{M}}$, $P_{\mathcal{M}}$ and $P_{\mathcal{M}'}$ respectively. Then X is the topological union of X_0 and X_1. Let z_0 and z_1 be the copies of the P_{ω_2} point z in X_0 and X_1 respectively. In X , we identify z_0 and z_1 to get a quotient space Y. Hence another way to describe $\mathfrak{B}$ is that it is the clopen algebra of the topological space Y.

We need more notation.

Definition 3.5. Let $\mathcal{A}$ be a coherent sequence. We call $\mathcal{A}$ a *good coherent sequence* if $\mathcal{A}$ is of the form: $\{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \rho\}$, where $a_{\mu(\alpha)}$ are the non trivial partitioners of $\mathcal{M}$ in the above construction and b_α is also a nontrivial partitioner of $\mathcal{M}$ for all $\alpha \in \rho$. A good coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha < \rho\}$ is called a *reasonable sequence* if for all $\alpha < \rho$,

$$\mathcal{A}_\alpha = \{(b_\xi, a_{\mu(\xi)}) : \xi < \alpha\} \in \mathfrak{M}_{\mu(\alpha)} \; .$$

We will call a good coherent sequence with countable length a *g-sequence* and a reasonable coherent sequence with countable length an *r-sequence*.

3.3. **Main Theorem.** In this section we reduce the proof that $\mathcal{B}$ is not representable to a technical Lemma.

Theorem 3.6. *The Boolean Algebra $\mathfrak{B}$ is not representable.*

Definition 3.7. Let F be a function from $[\omega \cup \omega \times \{0\}]^\omega$ to $[\omega]^\omega$ and let $\mathcal{N}$ be a mad family on ω. We say that F induces an isomorphism from $\mathfrak{B}$ to $P_{\mathcal{N}}$ if the map $\varphi : [A] \mapsto [F(A)]$ is an isomorphism from $\mathfrak{B}$ onto $P_{\mathcal{N}}$. That is:

$$\begin{array}{ccc} A & \xrightarrow{F} & F(A) \\ {\scriptstyle\varphi_0}\downarrow & & {\scriptstyle\varphi_1}\downarrow \\ [A] & \xrightarrow{\varphi} & \varphi([A]) \end{array}$$

Where $\varphi_0 \colon \widetilde{\mathfrak{B}} \to \mathfrak{B}$ and $\varphi_1 \colon \mathfrak{A}_{\mathcal{N}} \to P_{\mathcal{N}}$ are the natural maps.

It will be clear from the proof of Theorem 3.6 that the following lemma essentially says that there is a coherent sequence $\mathcal{A}$ extended by some set X while the image of $\mathcal{A}$ under F can not be extended modulo $\mathcal{I}_{\mathcal{N}}$ by any subset of ω. The c_α's and e_α's are used to guarantee that no partitioner of $\mathcal{N}$ can exist as the image of X.

Lemma 3.8. *If $\mathcal{N}$ is a mad family whose partitioner algebra is isomorphic to $\mathfrak{B}$, then there are a function $F\colon [\omega \cup (\omega \times \{0\})]^\omega \to [\omega]^\omega$ and a reasonable coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \omega_1\}$ such that F induces an isomorphism from $\mathfrak{B}$ to $P_{\mathcal{N}}$ and $\mathcal{A}$ is extended by some set $X \subseteq \omega$. Moreover, there exist a sequence $C = \{(c_{\alpha+k}, e_{\alpha+k})\colon \alpha \in \lim\omega_1, 0 < k \leq \omega\}$ and an increasing sequence $\Sigma = \{\sigma_\alpha : \alpha \in \omega_1\}$ in $\lim\omega_1$ such that c_ξ and e_ξ are disjoint non trivial partitioners of $\mathcal{N}$ and the following hold*

1. $\forall \alpha \in \lim\omega_1, k,\ l \in \mathbb{N}$, *$c_{\alpha+k} \subseteq c_{\alpha+\omega}$ and $e_{\alpha+k} \subseteq e_{\alpha+\omega}$ and $k < l$ implies $c_{\alpha+k}$, $e_{\alpha+k}$, $c_{\alpha+l}$ and $e_{\alpha+l}$ are mutually disjoint and*
$$\begin{aligned} c_{\alpha+\omega} &=_{\mathcal{N}} F(b_{\alpha+\omega}) \\ e_{\alpha+\omega} &=_{\mathcal{N}} F(a_{\mu(\alpha+\omega)}) \setminus F(b_{\alpha+\omega}) \\ c_{\alpha+k} &=_{\mathcal{N}} F(b_{\alpha+k}) \setminus F(b_{\alpha+k-1}) \\ e_{\alpha+k} &=_{\mathcal{N}} F(a_{\mu(\alpha+k)}) \setminus (F(a_{\mu(\alpha+k-1)}) \cup F(b_{\alpha+k}))\ . \end{aligned}$$
2. *For evey $k \in N$ the family $\{(c_{\sigma_\alpha+k}, e_{\sigma_\alpha+k})\colon \alpha \in \omega_1\}$ is a Hausdorff-Luzin type family.*

Note that the role of the sequence C is simply to ensure that there is no suitable real which extends the image of $\mathcal{A}$ modulo $\mathcal{I}_{\mathcal{N}}$.

We introduce more notation. Let $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \lambda\}$ be a good coherent sequence with λ a limit ordinal such that $\lambda \neq \beta + \omega$ for any β. Suppose F induces an isomorphism from $\mathfrak{B}$ to some partitioner algebra $P_{\mathcal{N}}$. Then the *derived sequence* $C(\mathcal{A}, F) = \{(c_{\alpha+k}, e_{\alpha+k})\colon \alpha \in \lim\lambda \text{ and } 0 < k \leq \omega\}$ of $\mathcal{A}$ and F is defined by the following formulas,
$$\begin{aligned} d_{\alpha+\omega} &= F(a_{\mu(\alpha+\omega)}) \\ c_{\alpha+\omega} &= F(b_{\alpha+\omega}) \cap d_{\alpha+\omega}, \\ e_{\alpha+\omega} &= d_{\alpha+\omega} \setminus c_{\alpha+\omega} \\ d_{\alpha+k} &= (F(a_{\mu(\alpha+k)}) \setminus \bigcup_{i<k} F(a_{\mu(\alpha+i)})) \cap d_{\alpha+\omega}\ , \\ c_{\alpha+k} &= c_{\alpha+\omega} \cap d_{\alpha+k} \\ e_{\alpha+k} &= d_{\alpha+k} \setminus c_{\alpha+k} \end{aligned}$$

It is obvious that the derived sequence satisfies (1) in Lemma 3.8.

We first complete the proof of Theorem 3.6 from Lemma 3.8.

Proof of Theorem 3.6: Suppose that $\mathfrak{B}$ is representable. We will derive a contradiction. Let $\mathcal{N}$ be a mad family which represents $\mathfrak{B}$. Let F, $\mathcal{A}$, C, Σ be as in Lemma 3.8. Let $\mu = \sup\{\mu(\beta)\colon \beta < \omega_1\}$. Observe that $\{\mathcal{A}_\alpha\colon \alpha < \omega_1\} \in [T_\mu]$. so $\mathcal{A} \in \mathfrak{M}_{\mu+1}$. By elementarity we can assume that $X \in \mathfrak{M}_{\mu+1}$. We can also assume that $X \subseteq a_\mu$. Otherwise we use $X \cap a_\mu$ to replace X. Let $Y = F(X)$.

Since $X \in \mathfrak{M}_{\mu+1}$, it follows from Fact 3.3 that X does not split any $M \in \mathcal{M} \setminus \mathcal{M}_{\mu+1}$. However for any $M \in \mathcal{M}_{\mu+1}$, there exists $\alpha < \omega_1$ such that $M \subseteq^* a_{\mu(\alpha)}$.

Since X extends $\mathcal{A}$, $X \cap a_{\mu(\alpha)} =^* b_\alpha$ and recall that b_α is a partitioner of $\mathcal{M}$. Therefore X does not split M. We have proven that X is a partitioner of $\mathcal{M}$. Furthermore, since $X \subseteq a_{\mu+1}$, $X \in \widetilde{\mathfrak{B}}$, i.e $[X] \in \mathfrak{B}$.

Therefore Y must be a non-trivial partitioner of $\mathcal{N}$.

For $\alpha \in \omega_1$, since X extends $\mathcal{A}$ and F induces an isomorphism from $\mathfrak{B}$ to $P_\mathcal{N}$,

$$F(X) \cap F(a_{\mu(\alpha)}) =_\mathcal{N} F(b_\alpha) \ .$$

Therefore, $c_{\alpha+\omega} \subseteq_\mathcal{N} Y$ and $e_{\alpha+\omega} \cap Y =_\mathcal{N} \emptyset$.

For each $k \in \mathbb{N}$, if $c_{\alpha+k} \setminus Y$ is infinite, then there exists $N_k \in \mathcal{N}$ such that $N_k \subseteq^* c_{\alpha+k} \setminus Y \subseteq c_{\alpha+\omega} \setminus Y$. Since $c_{\alpha+k}$ $(k \in \omega)$ are mutually disjoint, the N_k's are distinct. Since $c_{\alpha+\omega} \subseteq_\mathcal{N} Y$, there are at most finitely many such N_k. Therefore, there are at most finitely many k such that $c_{\alpha+k} \not\subseteq^* Y$. Similarly there are only finitely many k such that $e_{\alpha+k} \cap Y \neq^* \emptyset$. Therefore we can find an uncountable subset Γ of ω_1 and $k \in \mathbb{N}$ and finite sets H, K such that for all $\alpha \in \Gamma$,

$$c_{\sigma_\alpha+k} \setminus Y = H \text{ and } e_{\sigma_\alpha+k} \cap Y = K \ .$$

Let $m > \max(H \cup K)$, and $\alpha \in \Gamma$ such that $\alpha \cap \Gamma$ is infinite. Then, for all $\beta \in \alpha \cap \Gamma$, $c_{\sigma_\alpha+k} \cap e_{\sigma_\beta+k} \subseteq m$. Therefore $\{\beta < \alpha \colon c_{\sigma_\alpha+k} \cap e_{\sigma_\beta+k} \subseteq m\}$ is infinite. But according to (2) of Lemma 3.8, $\{(c_{\sigma_\alpha+k}, e_{\sigma_\alpha+k}) \colon \alpha \in \omega_1\}$ is a Hausdorff-Luzin type family which is the contradiction we seek. □

Our strategy to prove Lemma 3.8 is first to get such a coherent sequence in some proper forcing extension. Then we use PFA to ensure that such a coherent sequence exists in the ground model. Then, since we are using PFA to produce such a coherent sequence in the ground model, we are only allowed to use ω_1 many dense subsets to determine the fact that no suitable subsets of ω extends the image of $\mathcal{A}$ modulo $\mathcal{I}_\mathcal{N}$. However, a Hausdorff-Luzin type family can be determined via ω_1 many dense subsets. The next three sections are actually to construct such a proper forcing and such a coherent sequence in the forcing extension. In other words, we will use the next three sections to prove the following Theorem 3.9 . We omit the standard dense set argument which shows how Lemma 3.8 follows from Theorem 3.9. It turns out that it is very difficult to get such a proper forcing notion.

Theorem 3.9. *If $\mathcal{N}$ is a mad family whose partitioner algebra is isomorphic to $\mathfrak{B}$, then there exist a function F that induces an isomorphism from $\mathfrak{B}$ to $P_\mathcal{N}$ and a proper forcing notion $\mathbb{D}$ such that, in $V^\mathbb{D}$, there exist a good coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \omega_1\}$ and a sequence $\Sigma = \{\sigma_\alpha : \alpha \in \omega_1\}$ in $\lim \omega_1$ such that the following are true.*

(i) *For all $\alpha \in \omega_1$, $\mathcal{A}_\alpha$ is an r-sequence and $\mathcal{A}$ can be extended by some subset X of ω.*

(ii) *F, $\mathcal{A}$, $C(\mathcal{A}, F)$ and Σ satisfy the conclusions of Lemma 3.8.*

Where $C(\mathcal{A}, F) = \{(c_{\alpha+k}, e_{\alpha+k}) \colon \alpha \in \lim \omega_1 \text{ and } 0 < k \leq \omega\}$ is the derived *sequence of $\mathcal{A}$ and F*

4. Separating Mad Families

In this section we will prove Theorem 4.1, which will be the foundation for the whole proof of Theorem 3.9.

We are motivated by the picture that either the $\{N_{i,j}\}$'s or J_n's listed therein are like the columns in $\omega \times \omega$ and the $F(a_\alpha)$'s are like the points below a graph of a function. By the construction of $\mathfrak{B}$, if F induces an isomorphism from $\mathfrak{B}$ to some partitioner algebra, then $\{(F(a_\alpha), F(a'_\alpha)) : \alpha < \omega_2\}$ will be very similar to an unfilled gap. However, by PFA, each (ω_2, ω_2^*)-gap can be filled. It is essentially this situation which led us to believe that $\mathfrak{B}$ should not be representable.

Theorem 4.1. *If $\mathcal{N}$ is a mad family on ω such that $\mathfrak{B}$ is isomorphic to the partitioner algebra $P_{\mathcal{N}}$ of $\mathcal{N}$, then there exists a map F from $[\omega \cup (\omega \times \{0\})]^\omega$ to $[\omega]^\omega$ such that F induces an isomorphism from $\mathfrak{B}$ to $P_{\mathcal{N}}$ and there exist $\tilde{\mathcal{N}} = \{N_{i,j} : i, j \in \omega\} \subseteq \mathcal{N}$ and a partition of ω , $\tilde{J} = \{J_n : n \in \omega\}$ such that either (i)-(ii) hold or (iii)-(v) hold where (i)-(v) are as follows:*

For $\alpha \in \omega_2$, let $X_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a_\alpha)\}$.

- (i) *$X_\alpha \cap \{N_{i,j} : j \in \omega\}$ is finite for all $\alpha \in \omega_2$, $i \in \omega$;*
- (ii) *If $I \in [\mathcal{N}]^\omega$ and there are infinitely many $i \in \omega$ such that $I \cap \{N_{i,j} : j \in \omega\} \neq \emptyset\}$, then there exists an α such that $I \cap X_\alpha \neq^* \emptyset$;*
- (iii) *for all $\alpha \in \omega_2$, for all $m \in \omega$, $F(a_\alpha) \cap J_m =^* \emptyset$;*
- (iv) *if $a \subseteq \omega$ and there are infinitely many $m \in \omega$ such that $a \cap J_m \neq \emptyset$, then there exists an α such that $a \cap F(a_\alpha) \neq^* \emptyset$;*
- (v) *For any $\mathcal{N}' \in [\bigcup_{\alpha\in\omega_2} X_\alpha]^\omega$, there is a β such that $\mathcal{N}' \cap X_\beta$ is infinite.*

4.1. **Aligning the mad family $\mathcal{N}$.** Suppose $\mathcal{N}$ is a mad family on ω which represents $\mathfrak{B}$ and that $\varphi : \mathfrak{B} \to P_{\mathcal{N}}$ is an isomorphism induced by $F : [\omega \cup (\omega \times \{0\})]^\omega \to [\omega]^\omega$. Obviously, if F induces φ and if $F' : [\omega \cup (\omega \times \{0\})]^\omega \to [\omega]^\omega$ is such that for any non trivial partitioner $A \in \widetilde{\mathfrak{B}}$, $F(A)$ is a partitioner of $\mathcal{N}$ and $F(A) =_{\mathcal{N}} F'(A)$, then F' also induces φ.

We will prove that either there exists F_1 and $\tilde{\mathcal{N}}$ satisfying the properties (i) and (ii) in Theorem 4.1 or there exist F_2 and $\tilde{J}$ satisfying the properties (iii)-(v).

For the rest of this chapter we assume that $\mathcal{N}$ is a mad family representing $\mathfrak{B}$ and we fix an isomorphism φ from $\mathfrak{B}$ to $P_{\mathcal{N}}$. If $F : [\omega \cup (\omega \times \{0\})]^\omega \to [\omega]^\omega$ induces φ, we set $X^F_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a_\alpha)\}$ and $Z^F_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a'_\alpha)\}$. It follows that $\{X^F_\alpha : \alpha < \omega_2\}$ and $\{Z^F_\alpha : \alpha < \omega_2\}$ are increasing, mod finite, subsets of $\mathcal{N}$.

For any subset A of ω, we will also use $\mathcal{N}(A)$ to denote the set $\{N \in \mathcal{N} : N \subseteq^* A\}$. The next lemma produces a set P which looks very much like a partitioner of $\mathcal{N}$ which should not exist (i.e. this P corresponds to the illegal partitioner $[\omega]$ of $\mathfrak{B}$ in $P_{\widetilde{\mathcal{M}}}$). Recall we have PFA$^+$ and $\diamondsuit(S^2_0)$ available.

Lemma 4.2. *There is a function $F : [\omega \cup (\omega \times \{0\})]^\omega \to [\omega]^\omega$ that induces φ and for which the following hold.*

1. *There exists $\{Y_n : n \in \omega\} \subseteq \mathcal{P}(\mathcal{N})$ and $\{Y'_n : n \in \omega\}$ such that Y'_n is a countable subset of Y_n and,*
   (a) *for any countable subset K of $\mathcal{N}$, if K is almost disjoint with X^F_α for all α in ω_2, then $K \subseteq^* Y_n$ for some n;*
   (b) *For all $\alpha \in \omega_2$, $X^F_\alpha \cap Y_n = X^F_\alpha \cap Y'_n$ is finite.*
2. *for all $\lambda \in \omega_2$, there exists μ_λ such that*
$$\bigcup_{\xi<\lambda} X^F_\xi \setminus X^F_{\mu_\lambda} \text{ is countable.}$$
3. *there exists $P \subseteq \omega$, such that*
$$(\forall \alpha \in \omega_2) F(a_\alpha) \subseteq P \text{ and } F(a'_\alpha) \cap P = \emptyset.$$

Proof. Fix any function f which induces the isomorphism φ and such that, for each $\alpha \in \omega_2$, $f(a_\alpha) \cap f(a'_\alpha)$ is empty. Let $\mathcal{I} = \{I \in [\mathcal{N}]^\omega : (\exists\alpha) I \subseteq^* X^f_\alpha\}$ and let $\mathcal{J} = \{I \in [\mathcal{N}]^\omega : (\exists\alpha) I \subseteq^* Z^f_\alpha\}$. It follows that $\mathcal{I}$ and $\mathcal{J}$ are P-ideals and we now verify that $\mathcal{I}$ and $\mathcal{J}$ are ω_1-separated, i.e. given sequences $\{I_\alpha : \alpha < \omega_1\}$ and $\{J_\alpha : \alpha < \omega_1\}$ from $\mathcal{I}$ and $\mathcal{J}$ respectively, there is an H such that for all $\alpha \in \omega_1$, $I_\alpha \subseteq^* H$ and $H \cap J_\alpha$ is finite. First pick $\xi < \omega_2$ such that , for all $\alpha < \omega_1$, $I_\alpha \subseteq^* X^f_\xi$ and $J_\alpha \subseteq^* Z^f_\xi$. Since $X^f_\xi \cap Z^f_\xi$ is finite, simply set $H = X^f_\xi$. By Theorem 2.8, $\mathcal{I}$ and $\mathcal{J}$ are strongly countably separated. Therefore there are $\{X(n) : n \in \omega\}$ such that for any $I \in \mathcal{I}$ and $J \in \mathcal{J}$, there exists n such that $I \subseteq^* X(n)$ and $X(n) \cap J$ is finite. Actually, since $\mathcal{I}$ and $\mathcal{J}$ are both P-ideals, there is a single n_0 such that $X(n_0)$ separates $\mathcal{I}$ and $\mathcal{J}$. To see this, suppose that it is not true. Then, for any n in ω, there exists $I_n \in \mathcal{I}$ and $J_n \in \mathcal{J}$ such that either $I_n \not\subseteq^* X(n)$ or $X(n) \cap J_n$ is infinite. Since they are P-ideals, fix $I \in \mathcal{I}$ and $J \in \mathcal{J}$ so that, for all $n \in \omega$, $I_n \subseteq^* I$ and $J_n \subseteq^* J$. But now there is m such that $X(m)$ separates I and J which implies that $X(m)$ separates I_m and J_m which contradicts the choice of I_m and J_m.

Let X be $X(n_0)$. Obviously, for all $\alpha \in \omega_2$, $X^f_\alpha \subseteq^* X$ and $X \cap Z^f_\alpha$ is finite. We may therefore assume $X^f_\alpha \subseteq X$ and $X \cap Z^f_\alpha$ is empty for all $\alpha < \omega_2$ because we can replace f by f' where $f'(a_\alpha) = f(a_\alpha) \setminus \bigcup(X^f_\alpha \setminus X)$ and $f'(a'_\alpha) = f(a'_\alpha) \setminus \bigcup(Z^f_\alpha \cap X)$.

Let $\mathcal{K} = \{I \in [X]^\omega : (\forall\alpha \in \omega_2)\ I \cap X^f_\alpha =^* \emptyset\}$. Again $\mathcal{I}$ and $\mathcal{K}$ are ω_1-separated. By Theorem 2.8, $\mathcal{I}$ and $\mathcal{K}$ can be strongly countably separated, say, by $\{Y_n : n \in \omega\}$. Hence, for each $\alpha \in \omega_2$, $X^f_\alpha \cap Y_n$ is finite and each $K \in \mathcal{K}$, K is almost contained in some Y_n.

Claim 1 There are $Y'_n \in [Y_n]^\omega$ such that, for all $\alpha \in \omega_2$, $X^f_\alpha \cap \bigcup_{n\in\omega} Y_n \setminus Y'_n$ is finite.

Proof of the Claim 1: If this were not the case, we can construct an ω_1-sequence $\{\alpha_\xi : \xi < \omega_1\}$ in ω_2 such that, for all $\eta < \omega_1$,

$$(X^f_{\alpha_\eta} \setminus \bigcup_{\xi<\eta} X^f_{\alpha_\xi}) \cap \bigcup_{n\in\omega} Y_n \text{ is infinite.}$$

The construction is straightforward. Take any $\alpha \in \omega_2$ as α_0. At stage η, let $\tilde{Y}_n = Y_n \cap \bigcup_{\xi<\eta} X_{\alpha_\xi}$. If the claim fails, there must be an α such that $X^f_\alpha \cap \bigcup_{n\in\omega} Y_n \setminus \tilde{Y}_n$ is infinite, i.e., $(X^f_\alpha \setminus \bigcup_{\xi<\eta} X^f_{\alpha_\xi}) \cap \bigcup_{n\in\omega} Y_n$ is infinite. Take such α as α_η .

Let λ be $\sup\{\alpha_\xi : \xi \in \omega_1\}$. Then , for all $\eta < \omega_1$,

$$\Gamma_\eta = (X^f_\lambda \cap X^f_{\alpha_\eta} \setminus \bigcup_{\xi<\eta} X^f_{\alpha_\xi}) \cap \bigcup_{n\in\omega} Y_n \text{ is infinite.}$$

But the Γ_η's are disjoint. Hence $X^f_\lambda \cap \bigcup_{n\in\omega} Y_n$ is uncountable. This contradicts the choice of Y_n's, which finishes the proof of Claim 1.

By again modifying f, we can assume that, for all α, $X^f_\alpha \cap \bigcup_{n\in\omega}(Y_n \setminus Y'_n) = \emptyset$, i.e., for all α , $X^f_\alpha \cap Y_n \subseteq Y'_n$.

Claim 2. For all $\lambda \in \omega_2$, there exists $\mu_\lambda \geq \lambda$ such that $\bigcup_{\alpha<\lambda} X^f_\alpha \setminus X^f_{\mu_\lambda}$ and $\bigcup_{\alpha<\lambda} Z^f_\alpha \setminus Z^f_{\mu_\lambda}$ are both countable.

Proof of Claim 2: We only give the proof for the X^f_α's.

Fix a $\lambda \in \omega_2$ and let S_λ be $\bigcup_{\alpha<\lambda} X^f_\alpha \setminus X^f_\lambda$. Then $|S_\lambda| \leq \aleph_1$ and we just have to consider the case when $|S_\lambda| = \aleph_1$. Set $S_\lambda = \{N_\beta : \beta < \omega_1\}$. Since Chang's Conjecture holds and by Lemma 2, there are no strong chains of length ω_2 on S_λ. Therefore there is some μ_0 such that, for all $\mu > \mu_0$, $(X^f_\mu \setminus X^f_{\mu_0}) \cap S_\lambda$ is countable.

Subclaim There is a countable subset S of S_λ and $\mu_\lambda > \mu_0$ such that

$$\text{for all } \mu > \mu_\lambda,\ (S_\lambda \setminus S) \cap (X^f_\mu \setminus X^f_{\mu_\lambda}) \text{ is finite.}$$

Proof of Subclaim: Suppose otherwise. We can then easily define a sequence $\{\alpha^0_\xi, \alpha^1_\xi, \mu^0_\xi, \mu^1_\xi : \xi \in \omega_1\}$ such that

1. for $\xi < \eta < \omega_1$, $\alpha^0_\xi < \alpha^1_\xi < \alpha^0_\eta; \mu_0 < \mu^0_\xi < \mu^1_\xi < \mu^0_\eta$;
2. $(X^f_{\mu^1_\xi} \setminus X^f_{\mu^0_\xi}) \cap \{N_\beta : \beta \in (\alpha^0_\xi, \alpha^1_\xi)\}$ is infinite.

Let μ_{ω_1} be $\sup\{\mu^1_\xi : \xi < \omega_1\}$. Then, for all $\xi < \omega_1$, $(X^f_{\mu_{\omega_1}} \setminus X^f_{\mu_0}) \cap \{N_\beta : \beta \in (\alpha^0_\xi, \alpha^1_\xi)\}$ is infinite. Hence, $(X^f_{\mu_{\omega_1}} \setminus X^f_{\mu_0}) \cap S_\lambda$ is uncountable contradicting the choice of μ_0. This finishes the proof of Subclaim.

Let A be $S_\lambda \setminus (S \cup X^f_{\mu_\lambda})$. We prove that A is countable. We have for any $\mu < \omega_2$, $A \cap X^f_\mu$ is finite. For any $I \in [A]^\omega$, there is an n such that $I \subseteq^* Y_n$. But $A \cap Y_n \subseteq Y'_n$, therefore, $I \subseteq^* Y'_n$. Thus $A \subseteq^* \bigcup_{n\in\omega} Y'_n$ and A is countable. Therefore, $\bigcup_{\alpha<\lambda} X^f_\alpha \setminus X^f_{\mu_\lambda} =^* S_\lambda \setminus X^f_{\mu_\lambda} \subseteq (A \cup S)$ is countable. This finishes the proof of the Claim 2.

Observe that (1) and (2) of the lemma holds for (our revised) f and $\{Y_n, Y'_n : n \in \omega\}$. Next we modify f to some F such that (1)-(3) of the lemma holds for F and $\{Y_n, Y'_n : n \in \omega\}$

Let $T = \{(a, b) \in \mathcal{P}(\omega) \times \mathcal{P}(\omega) : a \cap b = \emptyset \text{ and } \exists \alpha\, a \subseteq^* f(a_\alpha), b \subseteq f(a'_\alpha)\}$ Let $K_0 = \{\{(a, b), (c, d)\} \in [T]^2 : a \cap d \neq \emptyset \text{ or } c \cap b \neq \emptyset\}$. Let $K_1 = [X]^2 \setminus K_0$. K_0 is open with respect to the topology on T inherited from $\mathcal{P}(\omega)$.

Claim 3 There is no 0-homogeneous set with cardinality $\aleph_1$

Proof of Claim 3 Suppose there is an $\aleph_1$ sized 0-homogeneous set $\{(c_\alpha, d_\alpha) : \alpha < \omega_1\}$. For each $\alpha \in \omega_1$, let $\xi_\alpha < \omega_2$ be such that $c_\alpha \subseteq^* f(a_{\xi_\alpha})$ and $d_\alpha \subseteq^* f(a'_{\xi_\alpha})$. Let $\lambda = \sup\{\xi_\alpha : \alpha < \omega_1\}$. By Claim 2, there is a μ_λ such that $\bigcup_{\alpha\in\omega_1} X^f_{\xi_\alpha} \setminus X^f_{\mu_\lambda}$ and $\bigcup_{\alpha\in\omega_1} Z^f_{\xi_\alpha} \setminus Z^f_{\mu_\lambda}$ are countable. Therefore there exist an uncountable subset Γ of ω_1 and finite subsets $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{N}$, such that for each $\alpha \in \Gamma$, $X^f_{\xi_\alpha} \setminus X^f_{\mu_\lambda} = \mathcal{N}_1$ and $Z^f_{\xi_\alpha} \setminus Z^f_{\mu_\lambda} = \mathcal{N}_2$. Thus for each $\alpha \in \Gamma$, $f(a_{\xi_\alpha}) \subseteq^* f(a_{\mu_\lambda}) \cup \bigcup \mathcal{N}_1$ and $f(a'_{\xi_\alpha}) \subseteq^* f(a'_{\mu_\lambda}) \cup \bigcup \mathcal{N}_2$. Of course, for all $\alpha \in \Gamma$, $\bigcup \mathcal{N}_1 \subseteq^* f(a_{\xi_\alpha})$ and $\bigcup \mathcal{N}_2 \subseteq^* f(a'_{\xi_\alpha})$. Let $e_1 = (f(a_{\mu_\lambda}) \cup \bigcup \mathcal{N}_1) \setminus \bigcup \mathcal{N}_2$ and $e_2 = (f(a'_{\mu_\lambda}) \cup \bigcup \mathcal{N}_2) \setminus \bigcup \mathcal{N}_1$. Obviously $e_1 \cap e_2$ is finite and, for all $\alpha \in \Gamma$, $c_\alpha \subseteq^* e_1$ and $d_\alpha \subseteq^* e_2$.

By shrinking Γ, if necessary, we can assume that there exist n and disjoint subsets L_1, L_2 of n such that, for each $\alpha \in \Gamma$, $c_\alpha \setminus n \subseteq e_1$, $d_\alpha \setminus n \subseteq e_2$, $c_\alpha \cap n = L_1$ and $d_\alpha \cap n = L_2$.

It is easy to see that for any α, β in Γ, $c_\alpha \cap d_\beta = c_\beta \cap d_\alpha = \emptyset$. This contradicts the assumption that $\{(c_\alpha, d_\alpha), (c_\beta, d_\beta)\} \in K_0$ and finishes the proof of Claim 3.

By OCA, T can be written as $\bigcup_{n\in\omega} T(n)$, where each $T(n)$ is a 1-homogeneous set. Since, for any $\alpha \in \omega_2$, $f(a_\alpha) \cap f(a'_\alpha)$ is empty, $(f(a_\alpha), f(a'_\alpha)) \in T$. There exists an n_0 such that there are cofinally many α in ω_2, such that $(f(a_\alpha), f(a'_\alpha)) \in T(n_0)$. Let $P = \bigcup\{f(a_\alpha) : (f(a_\alpha), f(a'_\alpha)) \in T(n_0)\}$. Then it is easy to see that, for all $\alpha \in \omega_2$, $f(a_\alpha) \subseteq_{\mathcal{N}} P$ and $f(a'_\alpha) \cap P =_{\mathcal{N}} \emptyset$. So, for all $\alpha \in \omega_2$, there are finite subsets $\mathcal{N}^1_\alpha, \mathcal{N}^2_\alpha$ such that $f(a_\alpha) \setminus \bigcup \mathcal{N}^1_\alpha \subseteq^* P$ and $(f(a'_\alpha) \setminus \bigcup \mathcal{N}^2_\alpha) \cap P$ is finite. Now define F so that $F(a_\alpha) = P \cap f(a_\alpha) \setminus \bigcup \mathcal{N}^1_\alpha$ and $F(a'_\alpha) = f(a'_\alpha) \setminus (\bigcup \mathcal{N}^2_\alpha \cap P)$ and for any other $A \in \operatorname{dom} f$, $F(A) = f(A)$. Obviously F and P satisfy the conclusions in (3) of the lemma. Also, since, for all $\alpha \in \omega_2$, $X^F_\alpha \subseteq X^f_\alpha =^* X^F_\alpha$, it follows that (1) and (2) of the lemma still hold for F and $\{Y_n, Y'_n : n \in \omega\}$. $\square$

Remark: For any F' such that for all A, $F'(A) \subseteq F(A)$ and $F(A) \setminus F'(A) \in \mathcal{I}_{\mathcal{N}}$, F' still satisfies Lemma 4.2.

We finish this section with a proof of Theorem 4.1.

Proof of Theorem 4.1:

Let F, P, $\{Y_n, Y'_n : n \in \omega\}$ satisfies the Lemma 4.2 and for each $\alpha < \omega_2$, let $X_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a_\alpha)\}$ We proceed by cases.

Case 1 : There are infinitely many n such that $Y_n \setminus \bigcup_{k<n} Y_k$ is infinite.

Let $\{n_i : i \in \omega\} = \{n : Y_n \setminus \bigcup_{k<n} Y_k$ is infinite $\}$. For each i, fix an infinite countable set $\mathcal{N}_i$ such that

$$Y'_{n_i} \setminus \bigcup_{k<n_i} Y_k \subseteq \mathcal{N}_i \subseteq Y_{n_i} \setminus \bigcup_{k<n_i} Y_k$$

Let $\mathcal{N}_i = \{N_{i,j} : j \in \omega\}$. Obviously, (i) and (ii) hold.

Case 2: There are only finitely many n such that $Y_n \setminus \bigcup_{k<n} Y_k$ is infinite.

In this case, there is an $m \in \omega$, such that, for all $n > m$, $Y_n \subseteq^* \bigcup_{k\leq m} Y_k$. Therefore, for all countable subsets I of $\mathcal{N}$, if $I \cap X_\alpha$ is finite for all $\alpha \in \omega_2$, then $I \subseteq^* \bigcup_{k\leq m} Y_k$. Since each $X_\alpha \cap \bigcup_{k\leq m} Y_k$ is finite, we can assume that, for all $\alpha \in \omega_2$, $X_\alpha \cap \bigcup_{k\leq m} Y_k$ is empty (i.e. by replacing $F(a_\alpha)$ with $F(a_\alpha) \setminus \bigcup (X_\alpha \cap \bigcup_{k\leq m} Y_k)$).

Let $X = \bigcup_{\alpha\in\omega_2} X_\alpha$ and $Y = \bigcup_{k\leq m} Y_k$.

Observe that for any infinite countable subset I of X, I is not mod finite contained in Y and so there is an α such that $I \cap X_\alpha$ is infinite. We record this as claim 1.

Claim 1: For all $I \in [X]^\omega$, there exists α such that $I \cap X_\alpha$ is infinite.

Now we are ready to find $\{J_n : n \in \omega\}$. Let $T = \{(a,b) \in [\omega]^\omega \times [\omega]^\omega : (\exists\alpha) a \subseteq^* F(a_\alpha)$ and $(\forall\alpha) b \cap F(a_\alpha) =^* \emptyset$ and $a \cap b = \emptyset\}$. Let $K_0 = \{\{(a,b),(c,d)\} \in [T]^2 : a \cap d \neq \emptyset$ or $c \cap b \neq \emptyset\}$. K_0 is an open coloring for the natural separable metric topology on T. With a similar argument as in the proof of Claim 3 in Lemma 4.2, we can easily get the following

Claim 2: There is no uncountable 0-homogeneous set in T

By OCA we let $T = \bigcup_{n\in\omega} T(n)$ where each $T(n)$ is 1-homogeneous. For all n, let $S(n) = \bigcup\{b : (\exists a)(a,b) \in T(n)\}$. Let E be the set of all n such that there are cofinally many α in ω_2 such that there exists b with $(F(a_\alpha), b) \in T(n)$. For any b such that b is almost disjoint with each $F(a_\alpha)$, there exist k and n such that there are cofinally many α in ω_2 such that $(F(a_\alpha), b \setminus k) \in T(n)$. Hence $n \in E$ and $b \subseteq^* S(n)$. It is easy to see that for any $n \in E$, and any $\alpha \in \omega_2$, there exists $\beta > \alpha$ such that $(F(a_\beta), b) \in T(n)$ for some $b \subseteq \omega$, which implies $F(a_\beta) \cap S(n)$ is empty. Since $F(a_\alpha) \subseteq_{\mathcal{N}} F(a_\beta)$, $F(a_\alpha) \cap S(n)$ is in $\mathcal{I}_{\mathcal{N}}$.

Claim 3: For all $n \in E$, there is $a \in \mathcal{I}_{\mathcal{N}}$ such that, for all $\alpha < \omega_2$, $F(a_\alpha) \cap S(n) \subseteq^* a$.

Proof of Claim 3 Suppose the claim fails for some $n \in E$. Then we can find an infinite set $\{N_i : i \in \omega\} \subseteq \mathcal{N}$ such that, for all i in ω, $N_i \cap S(n)$ is infinite and N_i is contained in some $F(a_{\alpha_i})$. Clearly $\{N_i : i \in \omega\} \subseteq X$. Therefore, by Claim 1, there is an α such that $\{N_i : i \in \omega\} \cap X_\alpha$ is infinite. Without loss of generality, we assume that $N_i \in X_\alpha$ for all i, i.e. for all $i \in \omega$, $N_i \subseteq^* F(a_\alpha)$. Since each N_i intersects $S(n)$ on an infinite set, $F(a_\alpha) \cap S(n) \notin \mathcal{I}_{\mathcal{N}}$. This contradicts the fact that $n \in E$.

According to Claim 3, for each $n \in E$, there exists a finite subset $\mathcal{N}'_n$ of X, such that, for all $\alpha \in \omega_2$, $F(a_\alpha) \cap S(n) \subseteq^* \bigcup \mathcal{N}'_n$. Now for $n \in E$, let $S^0(n) = S(n) \setminus \bigcup \mathcal{N}'_n$. Suppose $b \subseteq \omega$ is such that, for all $\alpha \in \omega_2$, $b \cap F(a_\alpha)$ is finite. Let $n \in E$ be such that $b \subseteq^* S(n)$. Since $\mathcal{N}'_n$ $(n \in E)$ are subsets of X, $b \cap \bigcup \mathcal{N}'_n$ is finite and so $b \subseteq^* S^0(n)$.

We enumerate E as $\{n_k : k < \rho\}$, where $\rho \leq \omega$. For $k < \rho$, let $J_k = S^0(n_k) \setminus \bigcup_{i<k} S^0(n_i)$. Obviously for all $\alpha \in \omega_2$ and $k < \rho$, $F(a_\alpha) \cap S^0(n_k)$ is finite, and for any b such that, for all $\alpha \in \omega_2$, $b \cap F(a_\alpha)$ is finite, there is $k < \rho$, such that $b \subseteq^* \bigcup_{i \leq k} J_i$.

The next Claim will finish the proof of the Theorem.

Claim 4: There are infinitely many $k \in \rho$ such that J_k is infinite.

Proof of Claim 4. Assume there is $k_0 \in \omega$ such that, for all $k \in \rho$,

$$S^0(n_k) \subseteq^* \bigcup_{i \leq k_0} S^0(n_i).$$

Let $Q = \bigcup_{i \leq k_0} S^0(n_i)$. For any $\alpha \in \omega_2$, $F(a_\alpha) \cap Q$ is finite. By (3) of Lemma 4.2, $F(a'_\alpha) \cap F(a_\beta)$ is empty for all α and β in ω_2 . Therefore $F(a'_\alpha) \subseteq^* Q$ for all $\alpha \in \omega_2$. If Q is a partitioner of $\mathcal{N}$, then there is a partitioner A of $\widetilde{M}$ such that $A \in \mathfrak{B}'$ and $F(A) =_{\mathcal{N}} Q$. Note that since $F(A) =_{\mathcal{N}} Q$, it follows that for all $\alpha \in \omega_2$, $a_\alpha \cap A \in \mathcal{I}_{\widetilde{M}}$ and $a'_\alpha \setminus A \in \mathcal{I}_{\widetilde{M}}$. Therefore, by Fact 3.4, we have that Q is not a partitioner of $\mathcal{N}$. On the other hand, for any fix $N \in \mathcal{N}$, if $N \in X$, then there is some α such that $N \subseteq^* F(a_\alpha)$, so $N \cap Q \subseteq^* F(a_\alpha) \cap Q$ is finite. If $N \notin X$, then N is disjoint with $F(a_\alpha)$ for all $\alpha \in \omega_2$. In this case, there is k such that $N \subseteq^* S(n_k)$. Since $N \notin X$ and $\mathcal{N}'_{n_k} \subseteq X$, $N \notin \mathcal{N}'_{n_k}$. Therefore $N \subseteq^* S^0(n_k) \subseteq^* Q$. Thus Q is a partitioner of $\mathcal{N}$; this contradiction completes the proof. $\square$

Henceforth we assume F, $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{J}}$ satisfy the conclusion of Theorem 4.1.

Since $\widetilde{\mathcal{N}}$ and $\widetilde{J}$ are countable, we will assume, for convenience, that they are members of $\mathfrak{M}_\lambda$ for all λ.

5. Extending Coherent Sequences

5.1. **An Extension Lemma.** Recall that (See page 2) for a coherent sequence $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha \in \beta\}$ on ω, $\mathbb{P}_{\mathcal{A}}$ is the poset consisting of the partial functions p from ω to $\{0,1\}$ such that for any $p \in \mathbb{P}_{\mathcal{A}}$, there is an $\alpha < \beta$ such that $p =^* \mathrm{Ch}^{c_\alpha}_{d_\alpha}$. For p and q in $\mathbb{P}_{\mathcal{A}}$, $p < q$ if $p \supseteq q$.

In this section, we prove a preparatory lemma which will allow us to construct a good coherence sequence in the next section.

Recall that our strategy is to find a good coherent sequence $\mathcal{A}_{\omega_1}$ such that there is a partitioner X of $\mathcal{M}$ such that $[X]$ is in $\mathfrak{B}$ and, for all $\alpha \in \omega_1$, $X \cap a_{\mu(\alpha)} =^* b_\alpha$ while there is no $Y \subseteq \omega$ such that, for all $\alpha \in \omega_1$, $Y \cap F(a_{\mu(\alpha)}) =_{\mathcal{N}} F(b_\alpha)$. This, of course, will contradict the fact that F induces an isomorphism from $\mathfrak{B}$ to the partitioner algebra $\mathcal{P}_{\mathcal{N}}$. The preparatory lemma will ensure that certain good coherent sequences can be extended while keeping certain commitments.

In order to ensure that, in $V^{\mathbb{P}_{\mathcal{A}}}$, no partitioner of $\mathcal{N}$ will extend, mod $\mathcal{I}_{\mathcal{N}}$, the image of $\mathcal{A}_{\omega_1}$ under F, we have to consider all possible partitioners of $\mathcal{N}$. That is, we will need to prove that for all possible names τ of partitioners of $\mathcal{N}$,

$$\Vdash_{\mathbb{P}_{\mathcal{A}_{\omega_1}}} \text{“}\tau \cap F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) \neq_{\mathcal{N}} F(b_{\alpha+1}) \setminus F(b_\alpha)\text{”} .$$

In constructing the coherent sequences $\mathcal{A}_{\omega_1}$, we will handle one name, τ, at a time. Lemma 5.1 will guarantee that, under certain situations, we can extend $\mathcal{A}_\alpha$ to $\mathcal{A}_{\alpha+1}$ and take care of $\tau = \tau_\alpha$ from a suitable list.

Lemma 5.1. *Suppose μ is a limit in ω_2. Let $\mathcal{A} = \{(b_\xi, a_{\mu(\xi)}) : \xi \leq \beta\}$ be a good coherent sequence such that $\mu = \mu(\beta)$ and $\mathcal{A} \in \mathfrak{M}_\mu$. Let $\tau \in \mathfrak{M}_\mu$ be a nice $\mathbb{P}_\mathcal{A}$-name for a subset of ω. Then there exist $p \in \mathbb{P}_\mathcal{A}$ and a $\mathbb{P}_\mathcal{A}$-generic, g, over $\mathfrak{M}_\mu$ and such that $g \in \mathfrak{M}_{\mu+1}$, $p \subseteq g$ and for any $\mu' \geq \mu + 2$, there are $\lambda' < \lambda < \omega_2$ with $\mu' \leq \lambda'$ and $\{R_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ such that the following hold.*

1. *$\mathcal{A}' = \mathcal{A}^\frown(g^{-1}(1) \cap a_\lambda, a_\lambda)$ is a good coherent sequence,*
2. *$\forall A \subseteq \omega$, if $\exists^\infty n\, A \cap R_n \neq^* \emptyset$, then $A \notin \mathcal{I}_\mathcal{N}$,*
3. *$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}}$ "$\forall^\infty n\, R_n \cap (\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda'}) \setminus F(a_{\mu'})) \notin \mathcal{I}_\mathcal{N}$",*
4. *$\{R_n : n \in \omega\} \cup \{F(g^{-1}(1)), F(a_{\lambda'}), F(a_{\mu'})\} \subseteq \mathfrak{M}_\lambda$.*

Note if $g \in \mathfrak{M}_{\mu+1}$ is $\mathbb{P}_\mathcal{A}$-generic over $\mathfrak{M}_\mu$, then $g^{-1}(1)$ is a partitioner of $\mathcal{M}$. Therefore if a is a partitioner of $\mathcal{M}$ and $a_{\mu(\beta)} \subseteq^* a$, then $\mathcal{A}^\frown(g^{-1}(1) \cap a, a)$ is a good coherent sequence . We will denote $\mathcal{A}^\frown(g^{-1}(1) \cap a, a)$ by $\mathcal{A}(g, a)$. Since (1) is guaranteed by the fact that $g \in \mathfrak{M}_{\mu+1}$ and that g is $\mathbb{P}_\mathcal{A}$-generic over $\mathfrak{M}_\mu$, we will not verify condition (1) in the proofs. Also, by condition (2), condition (3) is actually saying that

$$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}_{\alpha+1}}} \text{“}\tau \cap F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) \neq_\mathcal{N} F(b_{\alpha+1}) \setminus F(b_\alpha)\text{”} .$$

It will be clear when applying this lemma in the next section that other conditions are simply there to help preserve the above forcing statement when $p \cup g \restriction a_\lambda$ is considered as an element in $\mathbb{P}_{\mathcal{A}_{\omega_1}}$ and the forcing notion is $\mathbb{P}_{\mathcal{A}_{\omega_1}}$ in the above forcing statement.

We break the proof into three Lemmas. We will skip the proof of Lemma 5.2 from Lemma 5.3 since it is an straightforward pressing-down argument. We will first prove Lemma 5.1 from Lemma 5.2. We give, in the next section, the longest proof, that of Lemma 5.3, which, using the structures from Theorem 4.1, basically tells us why F cannot induce an isomorphism from $\mathcal{B}$ to the partitioner algebra of $\mathcal{N}$.

Lemma 5.2. *Let $\mathcal{A}$, μ, and τ be as in the Lemma 5.1. Then there exist $g \in \mathfrak{M}_{\mu+1}$ and $p \in \mathbb{P}_\mathcal{A}$ such that g is $\mathbb{P}_\mathcal{A}$-generic over $\mathfrak{M}_\mu$, $p \subseteq g$ and there are cofinally many λ's in ω_2 satisfying the following property:*

(†) *$\forall \rho, \gamma$ such that $\mu + 2 \leq \rho < \lambda \leq \gamma$*

$$p \cup g \restriction a_\gamma \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \notin \mathcal{I}_\mathcal{N}\text{”} ,$$

where $\mathcal{A}' = \mathcal{A}(g, a_\gamma)$

Lemma 5.3. *Let $\mathcal{A}$, μ, τ be as in Lemma 5.1. There exist a $\mathbb{P}_\mathcal{A}$-generic g over $\mathfrak{M}_\mu$ and $p \in \mathbb{P}_\mathcal{A}$ such that $g \in \mathfrak{M}_{\mu+1}$ and $p \subseteq g$. Moreover, for all $\rho \geq \mu$, $q \in \mathbb{P}_\mathcal{A}$, and $m \in \omega$, there is $\lambda_0 = \lambda_0(\rho, q) \in \omega_2$ such that for all $\lambda > \lambda_0$, $\gamma \geq \lambda$, $m \in \omega$, and $\mathcal{N}' \in [\mathcal{N}]^{<\omega}$, either $p \cup q \cup g \restriction a_\gamma$ is not a condition of $\mathbb{P}_{\mathcal{A}'}$ or*

$$p \cup q \cup g \restriction a_\gamma \not\Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq m \cup \bigcup \mathcal{N}'\text{”} ,$$

where $\mathcal{A}' = \mathcal{A}(g, a_\gamma)$

Now we prove Lemma 5.1

Proof of Lemma 5.1 from Lemma 5.2: Let p, g be as the Lemma 5.2. Let $\mu' = \mu + 2$. Take any increasing sequence $\{\lambda_n : n \in \omega\}$ such that $\lambda_0 > \mu'$ and each λ_n satisfies (†) in Lemma 5.2. Let $\lambda' = \sup\{\lambda_n : n \in \omega\}$ and fix $\lambda > \lambda'$ large enough so that $\mathfrak{M}_\lambda$ contains $\{F(a_{\lambda_i}) : i \in \omega\} \cup \{F(a_{\lambda'}), F(a_\mu), F(a_{\mu'}), F(g^{-1}(1))\}$.

For each $n \in \omega$, let $R_n = F(a_{\lambda_{n+1}}) \setminus \bigcup_{k \leq n} F(a_{\lambda_k})$. Since each R_n is a non-trivial partitioner of $\mathcal{N}$ and $\{R_n : n \in \omega\}$ is pairwise disjoint, $\{R_n : n \in \omega\}$ satisfies (2). Obviously $\{R_n : n \in \omega\}$ is a member of $\mathfrak{M}_\lambda$. Therefore (4) is also true. Next we prove that (3) holds.

For each n in ω, it is easy to see that $R_n \cap F(a_{\lambda'}) \setminus F(a_{\mu'}) =_{\mathcal{N}} F(a_{\lambda_{n+1}}) \setminus F(a_{\lambda_n})$. Since λ_{n+1} satisfies (†), we can set $\lambda = \gamma$ and $\rho = \lambda_n$ and $\mathcal{A}' = \mathcal{A}(g, a_\lambda)$ and conclude that

$$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“} (\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda_{n+1}}) \setminus F(a_{\lambda_n})) \notin \mathcal{I}_{\mathcal{N}} \text{”} \ .$$

Therefore

$$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“} R_n \cap (\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda'}) \setminus F(a_{\mu'})) \notin \mathcal{I}_{\mathcal{N}} \text{”} \ .$$

□

The following is an easy corollary of the above proof.

Corollary 5.4. *If $\{R_n : n \in \omega\}$ satisfies (1) and (2) of Lemma 5.1, then for any $\lambda \geq \lambda'' \geq \lambda'$,*

$$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“} \forall^\infty n \, R_n \cap (\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda''}) \setminus F(a_{\mu'})) \notin \mathcal{I}_{\mathcal{N}} \text{”} \ ,$$

where $\mathcal{A} = \mathcal{A}(g, a_\lambda)$. □

5.2. **Proof of the main lemma.** We need some preliminaries to prove Lemma 5.3.

Lemma 5.5. *Suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha \leq \beta\}$ is a coherent sequence with length $\beta + 1$. Then for any $g : \omega \to \{0,1\}$, g is a $\mathbb{P}_{\mathcal{A}}$-generic if and only if $g \restriction d_\beta =^* \mathrm{Ch}_{d_\beta}^{c_\beta}$ and $g \restriction \omega \setminus d_\beta$ is a Cohen real on $\omega \setminus d_\beta$.*

Proof. Trivial. □

Suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha \leq \beta\}$ is a coherent sequence and $g : \omega \to \omega$ is such that

$$g \restriction d_\beta =^* \mathrm{Ch}_{d_\beta}^{c_\beta} \ .$$

Let $\mathcal{D}$ be a family of predense subset of $\mathbb{P}_{\mathcal{A}}$. We say g is *$\mathcal{D}$-generic* if for each $E \in \mathcal{D}$, there is $e \in E$ such that $e \subseteq g$. Hence if g is $\mathbb{P}_{\mathcal{A}}$-generic over some model $\mathfrak{M}$ which contains $\mathcal{A}$ as an element, then g is a $\mathcal{D}_{\mathfrak{M}}$-generic, where $\mathcal{D}_{\mathfrak{M}}$ is the set of all predense subset of $\mathbb{P}_{\mathcal{A}}$ in $\mathfrak{M}$.

Let E be a subset of $\mathbb{P}_{\mathcal{A}}$. We use $E(n)$ to denote the set $\{r \in \mathbb{P}_{\mathcal{A}} : n \subseteq \mathrm{dom}\, r \text{ and } \exists\, q \in E, r \leq q\}$. Obviously, we have

Fact 5.6. *E is a predense subset of $\mathbb{P}_{\mathcal{A}}$ if and only if $E(n)$ is a dense and open subset of $\mathbb{P}_{\mathcal{A}}$.* □

For any bijection $\pi : \{0,1\}^n \to \{0,1\}^n$, we use $E(\pi)$ to denote the following set

$$\{r \in \mathbb{P}_{\mathcal{A}} : n \subseteq \mathrm{dom}\, r \text{ and } \pi(r \restriction n) \,\overline{\cup}\, r \in E(n)\} \ .$$

So $E(\pi)$ is obtained from $E(n)$ by changing the roots of the elements of $E(n)$ according to π. The following fact is obvious.

Fact 5.7. *$E(n)$ is dense in $\mathbb{P}_{\mathcal{A}}$ if and only if $E(\pi)$ is dense in $\mathbb{P}_{\mathcal{A}}$.* □

We call a family $\mathcal{D}$ of predense subsets of $\mathbb{P}_{\mathcal{A}}$ *closed* if for any $E \in \mathcal{D}$, $n \in \omega$, $E(n) \in \mathcal{D}$ and for any permutation π on $\{0,1\}^n$, $E(\pi) \in \mathcal{D}$. For any family of predense subsets $\mathcal{D}$ of $\mathbb{P}_{\mathcal{A}}$, we use $\operatorname{cl}\mathcal{D}$ to denote the smallest closed set containing $\mathcal{D}$. It is easy to see that $\operatorname{cl}\mathcal{D}$ is definable from $\mathcal{D}$. In particular, if $\mathcal{D}$ is in some elementary submodel $\mathfrak{M}$ of some H_λ, then $\operatorname{cl}\mathcal{D}$ is also in $\mathfrak{M}$. Obviously, for all α such that $\mathcal{A} \in \mathfrak{M}_\alpha$, if $\mathcal{D}$ is the set of all predense subset of $\mathbb{P}_{\mathcal{A}}$ in $\mathfrak{M}_\alpha$, then $\mathcal{D}$ must be closed. Following lemma is the main reason to introduce the closeness here.

Lemma 5.8. *Suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha < \beta\}$ is a coherent sequence and $\mathcal{D}$ is a family of predense subsets of $\mathbb{P}_{\mathcal{A}}$. If g is a $\mathcal{D}$-generic and $\mathcal{D}$ is closed, then*

1. *any finite modification of g is still a $\mathcal{D}$-generic,*
2. *for any $E \in \mathcal{D}$, E is still predense in $\mathbb{P}_{\mathcal{A}(g,d)}$ for any $d \subseteq \omega$ such that for all $\alpha < \beta$, $d_\alpha \subseteq^* d$.*

Proof. Obvious. □

The following corollary collects these results together.

Corollary 5.9. *Let $\mathfrak{M}$ be an elementary submodel of H_θ and suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha < \beta\} \in \mathfrak{M}$ is a coherent sequence. If g is $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}$, then any finite modification of g is also $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}$ and for any predense subset E of $\mathbb{P}_{\mathcal{A}}$ in $\mathfrak{M}$, E remains predense in $\mathbb{P}_{\mathcal{A}'}$ whenever $\mathcal{A}' = \mathcal{A}(g,d)$ and d almost contains each d_α.* □

We will also need the following easy lemma. For any subset A of a poset P, we will use $A^\perp$ to denote the set of all elements of P that are incompatible with each element in A.

Lemma 5.10. *Suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha < \beta\}$ is a coherent sequence. Let τ be a nice $\mathbb{P}_{\mathcal{A}}$-name for a subset of some set S. Let g be $\mathcal{D}$ generic for some closed family $\mathcal{D}$ of predense subsets of $\mathbb{P}_{\mathcal{A}}$. For $s \in S$, let $A_s = \{p \in \mathbb{P}_{\mathcal{A}} : (\check{s}, p) \in \tau\}$ and $B_s = A_s^\perp$. If, for all $s \in S$, $A_s \cup B_s$ is in $\mathcal{D}$, then for any $p \in \mathbb{P}_{\mathcal{A}}$, and any $d \subseteq \omega$ such that d almost contains d_α for all $\alpha < \beta$,*

$$p \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{“} s \in \tau \text{”} \text{ iff } p \Vdash_{\mathbb{P}_{\mathcal{A}(g,d)}} \text{“} s \in \tau \text{”}$$

and

$$p \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{“} s \notin \tau \text{”} \text{ iff } p \Vdash_{\mathbb{P}_{\mathcal{A}(g,d)}} \text{“} s \notin \tau \text{”}$$

Proof. Trivial. □

Now we are ready to prove Lemma 5.3.

Proof of Lemma 5.3: Let μ, $\mathcal{A}$ and τ be as given in the Lemma 5.1. Recall that we have fixed $\tilde{\mathcal{N}}$, $\tilde{\mathcal{J}}$ and F satisfying the conclusions of Theorem 4.1 and that $X_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a_\alpha)\}$ $(\alpha \in \omega_2)$.

We remind the reader that we must show that there exist a $\mathbb{P}_{\mathcal{A}}$-generic g over $\mathfrak{M}_\mu$ and $p \in \mathbb{P}_{\mathcal{A}}$ such that $g \in \mathfrak{M}_{\mu+1}$ and $p \subseteq g$. Moreover, for all $\rho \geq \mu$ and $q \in \mathbb{P}_{\mathcal{A}}$, there is $\lambda_0 = \lambda_0(\rho, q) \in \omega_2$ such that for all $\lambda > \lambda_0$, $\gamma \geq \lambda$, $m \in \omega$ and $\mathcal{N}' \in [\mathcal{N}]^{<\omega}$, either $p \cup q \cup g \restriction a_\gamma$ is not a condition of $\mathbb{P}_{\mathcal{A}'}$ or

$$p \cup q \cup g \restriction a_\gamma \not\Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq m \cup \bigcup \mathcal{N}' \text{”} .$$

where $\mathcal{A}' = \mathcal{A}(g, a_\gamma)$.

Our plan is to find infinitely many $N_i \in \mathcal{N}$ $(i \in \omega)$ and λ_0 such that each N_i meet $\tau \triangle F(g^{-1}(1))$ on an infinite set and all but finitely N_i are contained in $F(a_\lambda) \setminus F(a_\rho)$ for all $\lambda \geq \lambda_0$. The existence of such λ_0 will be guaranteed by the choice of N_i together with (i) or (v) of Theorem 4.1. The choice of g will depend on whether τ is "small" or "big". If τ is small, we choose g such that $g^{-1}(0)$ is a subset of $a_{\mu+1}$, then $F(g^{-1}(1))$ will be big enough to contain mod finite all N_i which is disjoint with τ. If τ is "big", we choose g such that $g^{-1}(1)$ is a subset of $a_{\mu+1}$, then $F(g^{-1}(1))$ will be small enough to avoid each N_i which meet τ on an infinite set. We will proceed by different cases in Theorem 4.1 combined with the "small-big" possibilities.

Case 1: $\{N_{i,j} : i,j \in \omega\}$ satisfies (i) and (ii) of the Theorem 4.1.
Subcase 1. There is a $p \in \mathbb{P}_{\mathcal{A}}$ such that

$$p \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{"} \exists^\infty i \exists^\infty j\ \tau \cap N_{i,j} =^* \emptyset \text{"} .$$

This is the case when τ is small. Let $g_0 = g_0^\mu \in \mathfrak{M}_{\mu+1}$ be the Cohen real over $\mathfrak{M}_\mu$ defined in the construction of section 3 and then $g_0^{-1}(0) \subseteq^* a_{\mu+1}$. Let $g = p \overline{\cup} \mathrm{Ch}^{b_\beta}_{a_{\mu(\beta)}} \overline{\cup} g_0$. Then according to Lemma 5.5, g is $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_\mu$. Clearly $g^{-1}(0) \subseteq^* a_{\mu+1}$. Consider any $\rho \geq \mu$ and $q \in \mathbb{P}_{\mathcal{A}}$. If p and q are incompatible, there is nothing to prove. Therefore we assume that p and q are compatible. Let $\tilde{g} = q \overline{\cup} g$. Then $\tilde{g}$ is also $\mathbb{P}_{\mathcal{A}}$ generic over $\mathfrak{M}_\mu$ and $\tilde{g} \in \mathfrak{M}_{\mu+1}$. By elementarity, $(p \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{"}\exists^\infty i \exists^\infty j\ \tau \cap N_{i,j} =^* \emptyset\text{"})^{\mathfrak{M}_\mu}$. Since $p \subset \tilde{g} \in \mathfrak{M}_{\mu+1}$ and $(p \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{"}\exists^\infty i \exists^\infty j\ \tau \cap N_{i,j} =^* \emptyset\text{"})^{\mathfrak{M}_\mu}$, there are $\{i_k : k \in \omega\} \subseteq \omega$ and infinite subsets d_k of ω, for each $k \in \omega$, such that, for all $j \in d_k$, $\tau_{\tilde{g}} \cap N_{i_k,j}$ is finite, where $\tau_{\tilde{g}} = \mathrm{val}(\tau, \tilde{g})$. Since $g^{-1}(1)$ is in $\mathfrak{M}_{\mu+1}$, by Fact 3.3, $g^{-1}(1)$ is a partitioner of $\mathcal{M}$. Consequently $F(g^{-1}(1))$ and $F(g^{-1}(0))$ are partitioners of $\mathcal{N}$. Recall that for any subset A of ω we use $\mathcal{N}(A)$ to denote the set $\{N \in \mathcal{N} : N \subseteq^* A\}$.

Since $X_\rho \cap \{N_{i,j} : j \in \omega\}$ and $X_{\mu+2} \cap \{N_{i,j} : j \in \omega\}$ are finite, there is an h such that, for all $i \in \omega$,

$$\mathcal{N}(F(a_\rho) \cup F(a_{\mu+2}) \cup F(g^{-1}(0)) \cup \omega \setminus F(g^{-1}(1))) \cap \{N_{i,j} : j \in \omega\} \subseteq \{N_{i,j} : j < h(i)\}$$

For all $k \in \omega$, let $j_k \in d_k \setminus h(i_k)$, then there exists λ_0 such that $\{N_{i_k,j_k} : k \in \omega\} \cap X_{\lambda_0}$ is infinite. For all $\lambda > \lambda_0$, $X_{\lambda_0} \subseteq^* X_\lambda$, hence $\{N_{i_k,j_k} : k \in \omega\} \cap X_\lambda$ is infinite.

For all λ, $\gamma \in \omega_2$, $m \in \omega$ and $\mathcal{N}' \in [\mathcal{N}]^{<\omega}$ such that $\gamma \geq \lambda > \lambda_0$, there is $k \in \omega$ such that $N = N_{i_k,j_k} \in X_\lambda \setminus \mathcal{N}'$. Then $N \subseteq^* F(a_\lambda) \setminus F(a_\rho)$ and $N \cap \tau_{\tilde{g}}$ is finite. Since $N \subseteq^* F(g^{-1}(1))$, there exists l in the set

$$N \cap (F(g^{-1}(1)) \setminus \tau_{\tilde{g}}) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus (m \cup \bigcup \mathcal{N}') .$$

Since $l \notin \tau_{\tilde{g}}$, there exists $r \subseteq \tilde{g} = q \overline{\cup} g$ such that $r \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{"}l \notin \tau\text{"}$. Let $\mathcal{A}' = \mathbb{P}_{\mathcal{A}(g,a_\gamma)}$. Since g is $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_\mu$ and τ, $\mathbb{P}_{\mathcal{A}}$ are in $\mathfrak{M}_\mu$, by Lemma 5.10,

$$r \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{"}l \notin \tau\text{"} .$$

Therefore,

$$r \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{"}l \in (\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus (m \cup \bigcup \mathcal{N}')\text{"} .$$

Since, $r \subseteq q \overline{\cup} g$ and q and p are compatible, either $p \cup q \cup g \restriction a_\gamma$ is not a condition or r is compatible with it. Thus either $p \cup q \cup g \restriction a_\gamma$ is not a condition or

$$p \cup q \cup g \restriction a_\gamma \nVdash_{\mathbb{P}_{\mathcal{A}'}} \text{"}l \notin (\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus (m \cup \bigcup \mathcal{N}')\text{"} .$$

Hence,

$$p \cup q \cup g \restriction a_\gamma \nVdash_{\mathbb{P}_{\mathcal{A}'}} \text{"}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq (m \cup \bigcup \mathcal{N}')\text{"} .$$

Subcase 2: $\exists p \in \mathbb{P}_{\mathcal{A}}\, \exists k\, (i > k \rightarrow p \Vdash \text{“}\forall^\infty j\ \tau \cap N_{i,j} \text{ infinite ”})$.

Let $g \in \mathfrak{M}_{\mu+1}$ be a $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_\mu$ such that $p \subseteq g$ and $g^{-1}(1) \subseteq^* a_{\mu+1}$. As in subcase 1, g is made using g_1^μ. Let $\rho \geq \mu$, $q \in \mathbb{P}_{\mathcal{A}}$. Again we assume that p and q are compatible. Let $\tilde{g}$ be $q \overline{\cup} g$. Then $\tilde{g}$ is also $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_\mu$ containing p. As in the previous subcase, we can find a function $f_0 \in \mathfrak{M}_{\mu+1} \cap {}^\omega\omega$ such that

$$\forall i > k\, \forall j \geq f_0(i)\ \tau_{\tilde{g}} \cap N_{i,j} \text{ is infinite.}$$

Take an h, greater than f_0, such that for all $i \in \omega$,

$$\{N_{i,j} : j \in \omega\} \cap \mathcal{N}(A) \subseteq \{N_{i,j} : j < h^*(i)\},$$

where $A = F(g^{-1}(1)) \cup (\omega \setminus F(g^{-1}(0))) \cup F(a_\rho) \cup F(a_{\mu+1}) \cup F(a_{\mu+2})$. Again since $\tilde{\mathcal{N}}$ satisfies (ii) of Theorem 4.1, there exists λ_0 such that $X_{\lambda_0} \cap \{N_{i,h^*(i)} : i \in \omega\}$ is infinite and, for all $\lambda > \lambda_0$, $X_\lambda \cap \{N_{i,h^*(i)} : i \in \omega\}$ is also infinite. The rest of the proof is similar to the subcase 1.

Case 2 $\{J_n : n \in \omega\}$ satisfies (iii) to (v) of Theorem 4.1.

Observe that in this case, for any $N \in \mathcal{N}$, either there are only finitely many i such that $N \cap J_i$ is not empty or N is almost disjoint with all J_n $(n \in \omega)$. To see this, let $N \in \mathcal{N}$. If there exists λ such that $N \subseteq^* F(a_\lambda)$, then by property (iii), N is almost disjoint with each J_n $(n \in \omega)$. Otherwise, since each $F(a_\lambda)$ is a partitioner, $N \cap F(a_\lambda)$ is finite for all $\lambda < \omega_2$. By property (iv), there are at most finitely many $i \in \omega$ such that $N \cap J_i$ is not empty.

We will only give the proof for the case when there exists $p \in \mathbb{P}_{\mathcal{A}}$, $p \Vdash$ “$\forall^\infty n J_n \subseteq^* \tau$” because the proof for the other case is almost the same.

As in Case 1, we can take $g \in \mathfrak{M}_{\mu+1}$, a $\mathbb{P}_{\mathcal{A}}$-generic real over $\mathfrak{M}_\mu$ such that $g^{-1}(1) \subseteq^* a_{\mu+1}$. Again g is made using g_1^μ. As in the previous case, $F(g^{-1}(1))$ is a partioner of $\mathcal{N}$. Let $\rho \geq \mu, q \in \mathbb{P}_{\mathcal{A}}$. Let $h \in \mathfrak{M}_{\mu+2}$ such that ${}^\omega\omega \cap \mathfrak{M}_{\mu+1} <^* h$. Let $A = F(g^{-1}(1)) \cup (\omega \setminus F(g^{-1}(0)))$, then $A \subseteq_{\mathcal{N}} F(a_{\mu+1})$, i.e. $A \setminus F(a_{\mu+1}) \in \mathcal{I}_{\mathcal{N}}$. Hence there are at most finitely many i such that $A \setminus F(a_{\mu+2}) \cap J_i$ is not finite. Take $f_0 \in {}^\omega\omega$ and $k \in \omega$, such that, for all $n > k$,

$$J_n \cap (F(g^{-1}(1)) \cup (\omega \setminus F(g^{-1}(0))) \cup F(a_\rho)) \subseteq f_0(n)\cdot$$

Let $h^* \in {}^\omega\omega$ such that $h^* > \{f_0, h\}$.

Define $\{A_i, \beta_i, N_i : i \in \omega\}$ as follows.

Let $A_0 = \{\min(J_n \setminus h^*(n)) : n \in \omega\}$. Since $\mathcal{N}$ is a mad family, there exists N_0 such that $N_0 \cap A_0$ is infinite. Obviously, there are infinitely many n such that $N_0 \cap A_0 \cap J_n$ is not empty. According to (iv) of Theorem 4.1, there is a β_0 such that $N_0 \cap A_0 \cap F(a_{\beta_0})$ is infinite. Hence, $N_0 \subseteq^* F(a_{\beta_0})$, i.e. $N_0 \in X_{\beta_0}$. At stage i, suppose we have defined $\{A_j, \beta_j, N_j : j < i\}$ such that $N_j \subseteq^* F(a_{\beta_j})$ and $A_j \setminus F(g^{-1}(1)) \cup F(a_\rho)$ is infinite. Now let $A_i = \{\min(J_n \setminus (h^*(n) \cup \bigcup_{j<i} F(a_{\beta_j}))) \colon n \in \omega\}$. Let $N_i \in \mathcal{N}$ such that $N_i \cap A_i$ is infinite. Similarly, we can find a β_i such that $N_i \subseteq^* F(a_{\beta_i})$. Obviously for each $i \in \omega$ and $l \in \omega$, $N_i \cap (\bigcup_{n>l} J_n \setminus h^*(n))$ is infinite and $N_i \setminus (F(g^{-1}(1)) \cup F(a_\rho))$ is infinite, thus $N_i \cap (F(g^{-1}(1)) \cup F(a_\rho))$ is finite.

By (v) of the Theorem 4.1, there is a λ_0 such that $\{N_i : i \in \omega\} \cap X_{\lambda_0}$ is infinite. Consequently, for all $\lambda > \lambda_0$, $\{N_i : i \in \omega\} \cap X_\lambda$ is infinite. For any $\lambda > \lambda_0$, $\gamma \geq \lambda$ and $\mathcal{N}' \in [\mathcal{N}]^{<\omega}$ and $m \in \omega$, take i such that $N = N_i \in X_\lambda \setminus \mathcal{N}'$. Let $\tilde{g} = q \overline{\cup} g$. If q is not compatible with p, we are done. So assume q is compatible with p. In this case, $p \subseteq \tilde{g}$. Since $p \Vdash$ “$\forall^\infty n\ J_n \subseteq^* \tau$” there exist k and $f \in \mathfrak{M}_{\mu+1}$ such that, for all $n > k$, $J_n \setminus \tau_{\tilde{g}} \subseteq f(n)$. Therefore, since $h^* > f$, there is a k, such that

$$\bigcup_{n>k} J_n \setminus h^*(n) \subseteq \tau_{\tilde{g}} \ .$$

It follows that $N \cap \tau_{\tilde{g}}$ is infinite. By the choice of N_i, $N = N_i \subseteq^* (F(a_\lambda) \setminus F(a_\rho)) \setminus (F(g^{-1}(1)) \cup \bigcup \mathcal{N}')$. Hence $a = (\tau_{\tilde{g}} \setminus F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus (m \cup \bigcup \mathcal{N}')$ is infinite. Take $l \in a$ and choose $r \subseteq \tilde{g}$ such that $r \Vdash_{\mathbb{P}_{\mathcal{A}}}$ "$l \in \tau$". Hence, as did in the first case, we have

$$r \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“} l \in (\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus m \cup \bigcup \mathcal{N}' \text{”} \ ,$$

where $\mathcal{A}' = \mathcal{A}(g, a_\gamma)$. If $q \cup p \cup g \restriction a_\gamma$ is a function, then r is compatible with it, thus

$$q \cup p \cup g \restriction a_\gamma \not\Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq m \cup \bigcup \mathcal{N}' \text{”} \ .$$

□

6. Collapsing $\aleph_2$ and Good Coherent Sequence

In this section, we will find a proper forcing notion $\mathbb{D}$ which forces that there is a coherent sequence with the properties stated in the Theorem 3.9. At the end of this section we will summarize the results and complete the proof of Theorem 3.9.

6.1. **Collapsing $\aleph_2$.** Let $\mathbb{P} = \mathrm{Fn}(\omega_1, \omega_2, \aleph_1)$ with the reverse inclusion as its partial order. This poset $\mathbb{P}$ is often called the countable condition collapse of ω_2 to ω_1.

We will prove several results to get ready to prove Theorem 3.9. The following theorem is the culmination of this Chapter.

Theorem 6.1. *If G is a $\mathbb{P}$-generic filter, then in $V[G]$, there is a good coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha < \omega_1\}$ such that*

(a) $\mathbb{P}_{\mathcal{A}}$ *is* ccc*,*

(b) $\mathbb{P}_{\mathcal{A}}$ *forces that for all $Y \subseteq \omega$,*

$$\{\alpha : Y \cap F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) \neq_{\mathcal{N}} F(b_{\alpha+1}) \setminus F(b_\alpha)\} \text{ is uncountable}$$

(c) $\{(b_\xi, a_{\mu(\xi)}) : \xi < \alpha\} \in \mathfrak{M}_{\mu(\alpha)}$.

In this subsection, we prove some lemmas and then prove Theorem 6.1.

Since $\mathfrak{c} = \omega_2$, we can enumerate ${}^\omega 2$ as $\{d_\alpha : \alpha < \omega_2\}$ such that each function $d \in {}^\omega 2$ appears in the enumeration only once. Let $\mathbb{Q}$ be the set of countable partial functions f from $\{(\alpha, \beta) : \beta < \alpha < \omega_1\}$ to $\{0, 1\}$ such that $\mathrm{dom}\, f = \bigcup_{\xi \in A} \{\xi\} \times \xi$ for some countable subset A of ω_1. $\mathbb{Q}$ is a poset with the reverse inclusion as its order. It is well-known that $\mathfrak{c} = \omega_2$ implies that $\mathbb{Q}$ is forcing equivalent to $\mathbb{P}$. However, for the purpose of the following lemma, we define an isomorphism e from $\mathbb{P}$ to $\mathbb{Q}$ as follows.

For each $\alpha < \omega_1$, let $\pi_\alpha : \alpha \to \omega$ be a bijection. Define for $f \in \mathbb{P}$, $e(f)$ so that $\mathrm{dom}\, e(f)$ is $\bigcup_{\alpha \in \mathrm{dom}(f)} \{\alpha\} \times \alpha$ and for $\alpha \in \mathrm{dom}\, f$, $\beta < \alpha$,

$$e(f)(\alpha, \beta) = d_{f(\alpha)}(\pi_\alpha(\beta)) \ .$$

It is easy to see that e is an isomorphism. Obviously e is order preserving. To see it is a bijection, we define a function $c : \mathbb{Q} \to \mathbb{P}$ such that $e \circ c = \mathrm{id}_{\mathbb{Q}}$ and $c \circ e = \mathrm{id}_{\mathbb{P}}$. For any $g \in \mathbb{Q}$, let A be the countable subset of ω_1 such that $\mathrm{dom}\, g = \bigcup_{\alpha \in A} \{\alpha\} \times \alpha$. For all $\alpha \in A$, let $h^\alpha : \omega \to 2$ be such that

$$h^\alpha(n) = g(\alpha, \pi_\alpha^{-1}(n))$$

and define $c(g) \in \mathbb{P}$ such that $\operatorname{dom} c(g) = A$ and $c(g)(\alpha)$ is the unique ordinal η such that $h^\alpha = d_\eta$. Obviously, for all $\alpha \in A$, and $\beta < \alpha$, $e(c(g))(\alpha, \beta) = d_{c(g)(\alpha)}(\pi_\alpha(\beta)) = h^\alpha(\pi_\alpha(\beta)) = g(\alpha, \pi_\alpha^{-1}\pi_\alpha(\beta)) = g(\alpha, \beta)$. Therefore $e(c(g)) = g$. It is also easy to see that for any $f \in \mathbb{P}$, $c(e(f)) = f$.

For any filter G on $\mathbb{P}$, $g = \bigcup G : \omega_1 \to \omega_2^V$ is a partial function. We can define $e(g) = \bigcup\{e(f) : f \in G\}$. Obviously G is $\mathbb{P}$-generic iff $e[G]$ is $\mathbb{Q}$-generic.

When G is a $\mathbb{P}$-generic we also call g a $\mathbb{P}$-generic and $e(g)$ a $\mathbb{Q}$-generic.

We fix a $\mathbb{P}$-generic $g : \omega_1 \to \omega_2^V$. Let $s_\alpha = \{\beta < \alpha : e(g)(\alpha, \beta) = 1\}$, i.e. $s_\alpha = g_\alpha^{-1}(1)$, where $g_\alpha : \alpha \to \{0, 1\}$ is defined by $g_\alpha(\beta) = e(g)(\alpha, \beta)$. Let $\delta_\alpha = \sup\{g(\xi) : \xi < \alpha\}$.

Let $\dot{g}$ be the $\mathbb{P}$-name for the $\mathbb{P}$-generic g. Let $\dot{s_\alpha}$ and $\dot{\delta_\alpha}$ be the names for s_α and δ_α respectively. The proof of the following Lemma is again standard; we will omit it.

Lemma 6.2. *If G is $\mathbb{P}$-generic, then in $V[G]$, $\{s_\alpha : \alpha \in \omega_1$ and $\operatorname{cf} \alpha = \omega\}$ is a $\diamondsuit$-sequence on $\{\alpha \in \omega_1 : \operatorname{cf} \alpha = \omega\}$ such that for any $X \subseteq \omega_1$ and $f : X \to \omega_2^V$ there are stationarily many α such that $X \cap \alpha = s_\alpha$ and $f \restriction s_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$.* $\square$

The following Corollary is what will actually be used the the proof of Theorem 6.1.

Corollary 6.3. *If G is $\mathbb{P}$-generic, then in $V[G]$, there is a $\diamondsuit$-sequence $\{(S_\alpha, T_\alpha) \in \mathcal{P}(\omega_1) \times \mathcal{P}(\omega_1) : \alpha \in \omega_1$ and $\operatorname{cf} \alpha = \omega\}$ such that for any X, $Y \subset \omega_1$ and $f_0 : X \to Z$, $f_1 : Y \to Z$, where Z is in $\mathfrak{M}_0$ with cardinality ω_2 in V, then there are stationarily many α such that $X \cap \alpha = S_\alpha$, $Y \cap \alpha = T_\alpha$, $f_0 \restriction S_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$ and $f_1 \restriction T_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$.* $\square$

6.2. **Forcing a Good Coherent Sequence.** We list some notations and properties in the next lemma to make the proof of Theorem 6.1 shorter.

Lemma 6.4. *Suppose $\mathcal{A}_\alpha \prec \mathcal{A}_\beta$ $(\alpha < \beta)$ are coherent sequences, $\mathcal{R} = \{R_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ and τ is a nice $\mathbb{P}_{\mathcal{A}_\alpha}$-name. Then the following are true:*

1. *Each of the following are predense subsets of $\mathbb{P}_{\mathcal{A}_\alpha}$,*
   (i) *$D_n(\mathcal{A}_\alpha, \mathcal{R}, \tau) = D_n^0(\mathcal{A}_\alpha, \mathcal{R}, \tau) \cup (D_n^0(\mathcal{A}_\alpha, \mathcal{R}, \tau))^\perp$, where $D_n^0(\mathcal{A}_\alpha, \mathcal{R}, \tau) = \{q \in \mathbb{P}_{\mathcal{A}_\alpha} : \exists i > n,\, q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ "$\tau \cap R_i \neq^* \emptyset$"$\}$.*
   (ii) *$E_{i,n}(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ for each i, n, where,*

$$E_{i,n}(\mathcal{A}_\alpha, \mathcal{R}, \tau) = \{q \in \mathbb{P}_{\mathcal{A}_\alpha} : q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}} \text{“}\tau \cap R_i \subseteq n\text{”} \\ \text{or } \exists k \in R_i \setminus n,\, q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}} \text{“}k \in \tau\text{”}\} .$$

   (iii) *$H_n^\tau = A_n^\tau \cup (A_n^\tau)^\perp$, where $A_n^\tau = \{p \in \mathbb{P}_{\mathcal{A}_\alpha} \colon (n, p) \in \tau\}$.*
2. *If $p \in \mathbb{P}_{\mathcal{A}_\alpha}$ and*
$$p \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}} \text{“}\exists^\infty i\, R_i \cap \tau \neq^* \emptyset\text{”} ,$$
   then, for all n in ω, $D_n^0(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ is dense below p.
3. *If $D_n(\mathcal{A}_\alpha, \mathcal{R}, \tau)$, $E_{i,n}(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ and H_n^τ are predense in $\mathbb{P}_{\mathcal{A}_\beta}$, and*
$$p \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}} \text{“}\exists^\infty i\, R_i \cap \tau \neq^* \emptyset\text{”} ,$$
   then
$$p \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}} \text{“}\exists^\infty i\, R_i \cap \tau \neq^* \emptyset\text{”} .$$

4. *If H_n^τ are predense in $\mathbb{P}_{\mathcal{A}_\beta}$, then for any subsets A, B of ω,*

$$\mathbf{1} \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}} \text{“}\tau' = (\tau \cap A) \triangle B\text{”} ,$$

where τ' is the $\mathbb{P}_{\mathcal{A}_\alpha}$-name defined as follows,

$$\tau' = \{\langle n, p\rangle \in \tau, n \in A \setminus B\} \cup \{\langle n, \mathbf{1}\rangle : n \in B \setminus A\} \\ \cup \{\{n\} \times (A_n^\tau)^\perp : n \in A \cap B\}.$$

Proof. (1) and (2) are trivial. (4) is also straightforward. We prove (3) by contradiction. Take $p' \in \mathbb{P}_{\mathcal{A}_\beta}$ and $n \in \omega$ such that

$$\forall i > n\, p' \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}} \text{“}R_i \cap \tau =^* \emptyset\text{”} .$$

Since $D_n(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ is predense in $\mathbb{P}_{\mathcal{A}_\beta}$, there exists $q \in D_n(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ such that p' and q are compatible. Since q is compatible with p and $p \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$\exists^\infty i\, R_i \cap \tau \neq^* \emptyset$”, q must be in $D_n^0(\mathcal{A}_\alpha, \mathcal{R}, \tau)$. Therefore, there is $i > n$ such that $q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$\tau \cap R_i \neq^* \emptyset$”. By the choice of p', we can find $r \in \mathbb{P}_{\mathcal{A}_\beta}$ and $m \in \omega$ such that $r \leq q \cup p'$ and $r \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}}$ “$\tau \cap R_i \subseteq m$”. Since $E_{i,m}$ is predense in $\mathbb{P}_{\mathcal{A}_\beta}$, there is some $s \in E_{i,m}$ which is compatible with r. Since s is compatible with q and $q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$\tau \cap R_i \neq^* \emptyset$”, $s \not\Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$\tau \cap R_i \subseteq m$”. Therefore there is a $k \in R_i \setminus m$ such that $s \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$k \in \tau$”. It follows that A_k^τ is predense below s in $\mathbb{P}_{\mathcal{A}_\alpha}$. We claim that A_k^τ is also predense below s in $\mathbb{P}_{\mathcal{A}_\beta}$. Indeed, since H_k^τ is predense in $\mathbb{P}_{\mathcal{A}_\beta}$, it is sufficient to see that any element in $(A_k^\tau)^\perp$ is incompatible with s. But this is obvious from the definition of $(A_k^\tau)^\perp$. Therefore, we have $s \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}}$ “$k \in \tau$”. However, from the choice of r , $s \cup r \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}}$ “$\tau \cap R_i \subseteq m$”. Contradiction. □

In the sequel, we will use $\mathcal{A}_\alpha$ to denote $\{(b_\xi, a_{\mu(\xi)}) : \xi < \alpha\} \in \mathfrak{M}_{\mu(\alpha)}$. Let $\mathcal{D}$ denote the set of all the countable partial functions from ω to $\{0,1\}$. Recall that $|\mathcal{D}|$ is ω_2. Since $\mathbb{P}$ is ω_1-closed, the definition of $\mathcal{D}$ is absolute between V and $V^{\mathbb{P}}$. Since $\mathbb{P}$ collapses ω_2^V to ω_1, we can enumerate, in $V^{\mathbb{P}}$, $\mathcal{D}$ as $\{f_\alpha : \alpha < \omega_1\}$.

Proof of Theorem 6.1: Let G be a $\mathbb{P}$-generic over V and let g be $\bigcup G$. Recall we use δ_α to denote $\sup\{g(\beta) : \beta < \alpha\}$. Fix a $\diamondsuit$-sequence satisfying Corollary 6.3. Since $\mathbb{P}$ collapses ω_2^V to ω_1, we can enumerate all the τ of the form

$$\bigcup_{n\in\omega} \{n\} \times A_n$$

with $A_n \in [\mathcal{D}]^{\leq\omega}$ as $\{\tau_\alpha : \alpha < \omega_1\}$ such that for each τ, $\{\alpha \in \lim\omega_1 : \tau = \tau_\alpha\}$ is uncountable.

Note that since $\mathbb{P}$ is ω_1-closed, there is no new countable coherent sequence in $V^{\mathbb{P}}$ and for any countable coherent sequence $\mathcal{A}'$, there is no new countable nice $\mathbb{P}_{\mathcal{A}'}$-name.

In $V[G]$, we will define, by induction on α, $\mathcal{A}_\alpha$, $\mathcal{D}_\alpha$, $\mathcal{R}^\alpha = \{R_n^\alpha : n \in \omega\}$, ρ'_α, ρ''_α, p_α, τ'_α and g_α such that $\mathcal{A}_\alpha = \{(b_\beta, a_{\mu(\beta)}) : \beta < \alpha\}$ is a good coherent sequence. For each infinite countable ordinal α, $\mathcal{D}_\alpha$ is a family of predense subsets of $\mathbb{P}_{\mathcal{A}_\alpha}$ which is closed. Moreover, for each infinite ordinal α and β the following are true.

1. if $\alpha < \beta$, then $\mathcal{A}_\alpha$ is an initial segment of $\mathcal{A}_\beta$.
2. $\mathcal{A}_{\alpha+1} \in \mathfrak{M}_{\mu(\alpha)}$, and $b_\alpha = g_\alpha^{-1}(1) \cap a_{\mu(\alpha)}$; if $\alpha = \beta + 1$, then $g_\alpha \in \mathfrak{M}_{\mu(\beta)+1}$.
3. if $\alpha \in \lim\omega_1$ and $\tau_\alpha \in \mathfrak{M}_{\mu(\alpha)}$ is a $\mathbb{P}_{\mathcal{A}_{\alpha+1}}$-name, then $\mu(\alpha) + 2 < \rho''_\alpha < \rho'_\alpha < \mu(\alpha + 1)$ and the following are true,

(i) $F(a_{\mu(\alpha)})$, $F(a_{\rho'_\alpha})$, $F(a_{\rho''_\alpha})$, $F(g^{-1}_{\alpha+1}(1)\cap a_{\rho'_\alpha})$, $F(g^{-1}_{\alpha+1}(1)\cap a_{\rho''_\alpha})$, $F(b_\alpha)$, p_α, τ'_α and $\{R^\alpha_n : n\in\omega\}$ are elements of $\mathfrak{M}_{\mu(\alpha+1)}$ and τ'_α is a $\mathbb{P}_{\mathcal{A}_{\alpha+2}}$ name for a subset of ω defined as in Lemma 6.4 with $A = F(a_{\rho'_\alpha})\setminus F(a_{\rho''_\alpha})$ and $B = F(g^{-1}_{\alpha+1}(1)\cap a_{\rho'_\alpha})\setminus F(g^{-1}_{\alpha+1}(1)\cap a_{\rho''_\alpha})$. Thus **1** forces that

$$\tau'_\alpha = (\tau_\alpha\cap(F(a_{\rho'_\alpha})\setminus F(a_{\rho''_\alpha})))\triangle(F(g^{-1}_{\alpha+1}(1)\cap a_{\rho'_\alpha})\setminus F(g^{-1}_{\alpha+1}(1)\cap a_{\rho''_\alpha}))\ .$$

Moreover, $p_\alpha\in\mathbb{P}_{\mathcal{A}_{\alpha+1}}$ is such that $p_\alpha\cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}}\in\mathbb{P}_{\mathcal{A}_{\alpha+2}}$ and

$$p_\alpha\cup\mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}}\Vdash_{\mathbb{P}_{\mathcal{A}_{\alpha+2}}} \text{“}\exists^\infty i,\ R^\alpha_i\cap\tau'_\alpha\neq^*\emptyset\text{”}\ .$$

(ii) For all $A\subseteq\omega$, if $\exists^\infty i,\ A\cap R^\alpha_i\neq^*\emptyset$, then $A\notin\mathcal{I}_{\mathcal{N}}$.

4. (i) $\omega<\alpha\le\beta\to\mathcal{D}_\alpha\subseteq\mathcal{D}_\beta$; if β is a limit, then $\mathcal{D}_\beta=\bigcup_{\alpha<\beta}\mathcal{D}_\alpha$.
(ii) $\mathcal{D}_{\alpha+1}\subseteq\mathfrak{M}_{\mu(\alpha)}$.
(iii) If α,τ'_α, τ_α, ρ'_α, ρ''_α and $\mathcal{R}^\alpha$ and $g_{\alpha+1}$ are as in (3), then for all n and i in ω, $D_n(\mathcal{A}_{\alpha+2},\mathcal{R}^{\alpha+1},\tau'_\alpha)$, $E_{i,n}(\mathcal{A}_{\alpha+2},\mathcal{R}^{\alpha+1},\tau'_\alpha)$ and $H^{\tau'_\alpha}_n$ are in $\mathcal{D}_{\alpha+2}$.
(iv) If α is a limit and $S_\alpha, T_\alpha\in\mathfrak{M}_{\mu+1}$, where $\mu=\sup\{\mu(\xi):\xi<\alpha\}$ and $\mathcal{A}_\alpha,\{f_\xi:\xi\in T_\alpha\}\in\mathfrak{M}_{\mu+1}$ and $\{f_\xi:\xi\in T_\alpha\}$ is predense in $\mathbb{P}_{\mathcal{A}_\alpha}$, then

$$\{f_\xi:\xi\in T_\alpha\}\in\mathcal{D}_{\alpha+1}\ .$$

(v) For $\alpha>\omega$, g_α is a $\mathcal{D}_\alpha$-generic.

5. $\mu(\alpha)>\delta_\alpha$.

The basic idea involved in (3) and (4) is to keep various predense subsets of $\mathbb{P}_{\mathcal{A}_\alpha}$ still predense in $\mathbb{P}_{\mathcal{A}_\beta}$ when $\alpha<\beta$. It is not difficult to see that (4) will guarantee that the final poset $\mathbb{P}_{\mathcal{A}}$ is ccc. To achieve (b), we handle each Y at stage $\alpha+2$ when τ_α is a name for Y. In (3) we pose a condition which ensures that $\mathbb{P}_{\mathcal{A}_{\alpha+2}}$ will force that $Y\cap F(a_{\mu(\alpha+1)})\setminus F(a_{\mu(\alpha)})$ is not equal to $F(b_{\alpha+1})\setminus F(b_\alpha)\}$ mod $\mathcal{I}_{\mathcal{N}}$. We then preserve this condition by preserving certain predense subsets of $\mathbb{P}_{\mathcal{A}_{\alpha+2}}$.

First of all, we show that if $\mathcal{A}=\{(b_\alpha,a_{\mu(\alpha)}):\alpha<\omega_1\}$ satisfies (1) to (5), then $\mathcal{A}$ satisfies (a) to (c). To check that $\mathbb{P}_{\mathcal{A}}$ is ccc, suppose K is a maximal antichain in $\mathbb{P}_{\mathcal{A}}$, and $K=\{f_\alpha:\alpha\in\Gamma\}$. Let $\Sigma=\{\alpha: f_\alpha\in\mathbb{P}_{\mathcal{A}}\}$. Recall $\mathcal{D}=\{f_\alpha\colon\alpha<\omega_1\}$ is an enumeration of all countable partial functions from ω to $\{0,1\}$ (see the discussion after Lemma 6.4). Let $f:\omega_1\to\mathcal{D}$ be defined by $f(\alpha)=f_\alpha$. Then by Corollary 6.3, $E=\{\alpha:\Sigma\cap\alpha=S_\alpha,\ \Gamma\cap\alpha=T_\alpha,\ f\restriction S_\alpha\in\mathfrak{M}_{\delta_\alpha+1}$ and $f\restriction T_\alpha\in\mathfrak{M}_{\delta_\alpha+1}\}$ is stationary in ω_1. Let C_1 be the set of α for which $\mathbb{P}_{\mathcal{A}_\alpha}=\{f_\xi:\xi\in\Sigma\cap\alpha\}$ and for which $K\cap\mathbb{P}_{\mathcal{A}_\alpha}$ is a maximal antichain in $\mathbb{P}_{\mathcal{A}}$. $C_2=\{\alpha\in\omega_1:\delta_\alpha=\sup\{\mu(\beta):\beta<\alpha\}\}$. Obviously C_1 and C_2 are cubs on ω_1. Take $\alpha\in E\cap C_1\cap C_2$, then $\Sigma\cap\alpha=S_\alpha$ and $\Gamma\cap\alpha=T_\alpha$. Moreover $\mathbb{P}_{\mathcal{A}_\alpha}=\{f_\xi:\xi\in\Sigma\cap\alpha\}=\operatorname{ran} f\restriction S_\alpha\in\mathfrak{M}_{\delta_\alpha+1}$. Similarly, $K\cap\mathbb{P}_{\mathcal{A}_\alpha}=\operatorname{ran} f\restriction T_\alpha\in\mathfrak{M}_{\delta_\alpha+1}$. By (i) and (iv) of (4), $K\cap\mathbb{P}_{\mathcal{A}_\alpha}$ is predense in $\mathbb{P}_{\mathcal{A}_\beta}$ for all $\beta>\alpha$. Therefore $K\cap\mathbb{P}_{\mathcal{A}_\alpha}$ is predense in $\mathbb{P}_{\mathcal{A}}$. This means $K=K\cap\mathbb{P}_{\mathcal{A}_\alpha}$ and so is countable. Condition (c) is immediate by (2).

Now we show, by contradiction, that (b) is true. Since $\mathbb{P}_{\mathcal{A}}$ is homogeneous, we have **1** forces that

$$\exists x\ \subseteq\omega\{\alpha: x\cap F(a_{\mu(\alpha+1)})\setminus F(a_{\mu(\alpha)})\neq_{\mathcal{N}} F(b_{\alpha+1})\setminus F(b_\alpha)\}\text{ is countable }\ .$$

Since $\mathbb{P}_{\mathcal{A}}$ is ccc, there exist a nice $\mathbb{P}_{\mathcal{A}}$-name τ for a subset of ω and $\eta\in\omega_1$ such that, for all $\alpha>\eta$,

$$\mathbf{1}\Vdash\text{“}\tau\cap F(a_{\mu(\alpha+1)})\setminus F(a_{\mu(\alpha)})=_{\mathcal{N}} F(b_{\alpha+1})\setminus F(b_\alpha)\text{”}\ .$$

Since $\mathbb{P}$ is ccc, τ is a countable name. Therefore τ is in V. Therefore there are uncountably many limit α such that $\tau = \tau_\alpha$. By (5), $\{\mu(\alpha) : \alpha < \omega_1\}$ is cofinal in ω_2^V. Therefore, there is a limit α such that $\tau = \tau_\alpha \in \mathfrak{M}_{\mu(\alpha)}$ and $\alpha > \eta$.

We claim that $p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \in \mathbb{P}_{\mathcal{A}_{\alpha+2}}$ and

$$p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}\tau \cap (F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) \neq_\mathcal{N} F(b_{\alpha+1}) \setminus F(b_\alpha))\text{”}$$

This will contradict that $\alpha > \eta$.

We have to prove that

$$p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}(\tau \cap (F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)})))\triangle(F(b_{\alpha+1}) \setminus F(b_\alpha)) \notin \mathcal{I}_\mathcal{N}\text{”} .$$

By (4) of Lemma 6.4 and (i) and (iii) of (4) in the above construction, we have

$$\mathbf{1} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}\tau'_\alpha = (\tau_\alpha \cap (F(a_{\rho'_\alpha}) \setminus F(a_{\rho''_\alpha})))\triangle(F(g^{-1}_{\alpha+1}(1) \cap a_{\rho'_\alpha}) \setminus F(g^{-1}_{\alpha+1}(1) \cap a_{\rho''_\alpha}))\text{”} .$$

Since $\mathcal{A}$ is a good coherent sequence, it is easy to see that $\mathbf{1}$ forces that τ'_α is contained, mod $\mathcal{I}_\mathcal{N}$, in $(\tau_\alpha \cap (F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)})))\triangle(F(b_{\alpha+1}) \setminus F(b_\alpha))$. Therefore it is sufficient to prove

$$p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}\tau'_\alpha \notin \mathcal{I}_\mathcal{N}\text{”} .$$

It is obvious that property (ii) of (3) is absolute. Therefore, it suffices to prove that

$$p \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}\exists^\infty i\, \tau'_\alpha \cap R_i \neq^* \emptyset\text{”} .$$

Since

$$p \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_{\mathcal{A}_{\alpha+2}}} \text{“}\exists^\infty i\, \tau'_\alpha \cap R_i \neq^* \emptyset\text{”} ,$$

by Lemma 6.4, it is sufficient to see that $D_n(\mathcal{A}_{\alpha+2}, \mathcal{R}^\alpha, \tau'_\alpha)$, $E_{i,n}(\mathcal{A}_{\alpha+2}, \mathcal{R}^\alpha, \tau'_\alpha)$ and $H^{\tau'}_n$ are predense in $\mathbb{P}_\mathcal{A} = \mathbb{P}_{\mathcal{A}_{\omega_1}}$. However by (i) and (iii) of (4), for all $\beta > \alpha + 2$, $D_n(\mathcal{A}_{\alpha+2}, \mathcal{R}^\alpha, \tau'_\alpha)$, $E_{i,n}(\mathcal{A}_{\alpha+2}, \mathcal{R}^\alpha, \tau'_\alpha)$ and $H^{\tau'_\alpha}_n$ are predense in $\mathbb{P}_{\mathcal{A}_\beta}$, therefore they are predense in $\mathbb{P}_{\mathcal{A}_{\omega_1}}$.

Next we construct the sequences satisfying (1) to (5). We will proceed by induction. Notice that for $\mathcal{R}^\alpha$, ρ'_α, ρ''_α and τ'_α we only have to define them when α is of the form $\beta + 1$ for β a limit ordinal. For simplicity, we will not mention them in any other case in the following construction. For g_α, we will define it at stage $\alpha + 1$. We will construct p_α at stage $\alpha + 2$.

We let $\mathcal{A}_0$ be $\emptyset$. At stage α we proceed according to the following cases.

Case 1 α is a limit.

We let $\mathcal{A}_\alpha = \bigcup_{\xi<\alpha} \mathcal{A}_\xi = \{(b_\xi, a_{\mu(\xi)}) : \xi < \alpha\}$. Let μ be the least ordinal bigger than $\sup\{\mu(\xi) : \xi < \alpha\}$ such that $\mathcal{A}_\alpha \in \mathfrak{M}_\mu$. Let $\mathcal{D}_\alpha = \bigcup_{\beta<\alpha} \mathcal{D}_\beta$. Obviously, $\mathcal{D}_\alpha$ is closed. For all $\xi < \alpha$ and $D \in \mathcal{D}_\xi$, we prove that D is predense in $\mathbb{P}_{\mathcal{A}_\alpha}$. Take any $p \in \mathbb{P}_{\mathcal{A}_\alpha}$, we can find $\eta < \alpha$ such that $\xi < \eta$ and $p \in \mathbb{P}_{\mathcal{A}_\eta}$. But by the induction hypothesis, D is predense in $\mathbb{P}_{\mathcal{A}_\eta}$, therefore there is $q \in D$ such that p and q are compatible.

Case 2 $\alpha = \beta + 1$ and $\mathcal{A}_\beta \in \mathfrak{M}_{\mu+1}$, where β is a limit and $\mu = \sup\{\mu(\xi) : \xi \in \beta\}$.

By the construction of $\{\mathfrak{M}_\alpha : \alpha \in \omega_2\}$ in Subsection 3.1, there is a g in $\mathfrak{M}_{\mu+2}$ which is $\mathrm{Fn}(\omega, 2)$-generic over $\mathfrak{M}_{\mu+1}$. Since $\mathbb{P}_{\mathcal{A}_\beta}$ and $\mathrm{Fn}(\omega, 2)$ are both countable and atomless, they have isomorphic dense subsets. Therefore, there is a $\mathbb{P}_{\mathcal{A}_\beta}$-generic g_β over $\mathfrak{M}_{\mu+1}$ such that $g_\beta \in \mathfrak{M}_{\mu+2}$. Let $\mu(\beta) = \mu + 2$ and $b_\beta = g^{-1}_\beta(1) \cap a_{\mu(\beta)}$. By Fact 3.3, $g^{-1}_\beta(1)$ is a partitioner of $\mathcal{M}$. Thus b_β is a partitioner of $\mathcal{M}$. Since g_β is generic over $\mathfrak{M}_{\mu+1}$, by Lemma 5.8, every set D in $\mathfrak{M}_{\mu+1}$ which is predense in

$\mathbb{P}_{\mathcal{A}_\beta}$ is still predense in $\mathbb{P}_{\mathcal{A}_{\beta+1}}$. However, by the induction hypothesis, $\mathcal{D}_\beta \subseteq \mathfrak{M}_{\mu+1}$. Therefore, every member D of $\mathcal{D}_\beta$ is still predense in $\mathbb{P}_{\mathcal{A}_{\beta+1}}$. Now let $\mathcal{D}_{\beta+1}$ be all the predense subsets of $\mathbb{P}_{\mathcal{A}_{\beta+1}}$ which are in $\mathfrak{M}_{\mu(\beta)} = \mathfrak{M}_{\mu+2}$. Obviously, $\mathcal{D}_{\beta+1}$ is closed and $\mathcal{D}_\beta \subseteq \mathcal{D}_{\beta+1}$. It is also clear that $\{f_\xi : \xi \in T_\beta\} \in \mathcal{D}_{\beta+1}$ provided $\{f_\xi : \xi \in T_\beta\}$ is predense in $\mathbb{P}_{\mathcal{A}_\beta}$ and it is in $\mathfrak{M}_{\mu+1}$.

Case 3 $\alpha = \beta + 1$, β is a limit and $\mathcal{A}_\beta \notin \mathfrak{M}_{\mu+1}$, where $\mu = \sup\{\mu(\xi) : \xi < \beta\}$.

Note $\xi < \beta$ implies that $\mathcal{A}_\xi$ is in $\mathfrak{M}_\mu$. Let g in $\mathfrak{M}_{\mu+1}$ be a Cohen generic real over $\mathfrak{M}_\mu$. Let λ be big enough such that $\{\mathcal{A}_\beta, \mathcal{D}_\beta\} \subseteq \mathfrak{M}_\lambda$ and $\lambda > \delta_\beta$. Choose a function $f \in {}^\omega\omega \cap \mathfrak{M}_{\lambda+1}$ which dominates $\mathfrak{M}_\lambda$. Also, in $\mathfrak{M}_\lambda$, choose $\{\beta_n : n \in \omega\}$ cofinal in β. We define $h \in {}^\omega\omega$ such that $h(n+1) = f(h(n)+1)$. Since $f \in \mathfrak{M}_{\lambda+1}$, $h \in \mathfrak{M}_{\lambda+1}$. Observe that for each increasing f', if $f' <^* f$ then, for almost all n, $h(n) < f'(h(n)) < f'(h(n)+1) < h(n+1)$. Let $x_n = a_{\mu(\beta_n)} \setminus \bigcup_{j<n} a_{\mu(\beta_j)}$ and $x = \bigcup_{n\in\omega} x_n \setminus h(n)$. Let $g_\beta \in {}^\omega\omega$ be such that g_β agrees with g on $\omega \setminus x$ and, for each n, agrees with g_{β_n} on $x_n \setminus h(n)$. Clearly g_β is in $\mathfrak{M}_{\lambda+1}$. Let $\mu(\beta)$ be $\lambda+1$ and b_β be $a_{\lambda+1} \cap g_\beta^{-1}(1)$. Let $\mathcal{D}_{\beta+1} = \mathcal{D}_\beta$. We check that $\mathcal{A}_{\beta+1}$ is a good coherent sequence and g_β is a $\mathcal{D}_\beta$-generic. It will then follow that $E \in \mathcal{D}_\beta$, E will still be predense in $P_{\mathcal{A}_{\beta+1}}$. Let us use g_n to denote the restriction of the function g_{β_n} on x_n.

First we prove that $\mathcal{A}_{\beta+1}$ is a good coherent sequence. It suffices to prove that $g_\beta^{-1}(1)$ is a partitioner of $\mathcal{M}$. Let N be in $\mathcal{M}$. We need to show that $g_\beta^{-1}(1)$ does not split N. Let ξ be such that N first appears in $M_{\xi+1}$. Since $g_\beta \in M_{\lambda+1}$, by Fact 3.3, $g_\beta^{-1}(1)$ does not split any member of $\mathcal{M}$ which is not in $\mathfrak{M}_{\lambda+1}$. Hence we can assume that $\xi < \lambda$. We may assume that N is not contained, mod finite, in a_β for any $\beta < \mu$ since then N is contained in $a_{\mu(\beta_n)}$ for some n and g_β almost agrees with g_n on $a_{\mu(\beta_n)}$. Therefore, N is almost disjoint with each x_n, in which case $\xi > \mu$. Consequently, $\xi \geq \mu+2$. Now f and h are dominating over $\mathfrak{M}_\lambda$. Thus N is almost disjoint from x. On the other hand, since $g^{-1}(1)$ is in $\mathfrak{M}_{\mu+1}$ and N is not in $\mathfrak{M}_{\mu+1}$, $g^{-1}(1)$ does not split N. Finally, noting that the restrictions of g and g_β on $\omega \setminus x$ are the same, we deduce that $g_\beta^{-1}(1)$ does not split N.

Next we prove that g_β is a $\mathcal{D}_\beta$-generic.

Given E in $\mathcal{D}_\beta$, we must prove that there exists an e in E, such that $e \subseteq g_\beta$. Let $\xi \in \beta$ such that $E \in \mathcal{D}_\xi$. Take an $m \in \omega$ such that $\beta_m > \xi$. Then, for all $e \in E$, dom(e) is almost contained in $a_{\mu(\beta_m)}$. Define, in $\mathfrak{M}_\lambda$, a strictly increasing function F_E from ω to ω such that

1. $F_E(0) = m$,
2. If $F_E(n) = l$, then for all $s \in Fn(l,2)$ and k such that $m \leq k \leq l$, there is an $e \in E$ such that

$$e \subseteq s \mathbin{\overline{\cup}} \bigcup_{j\leq k} g_j \mathbin{\overline{\cup}} g \text{ and } \mathrm{dom}(e) \setminus \bigcup_{j\leq m} x_j \subseteq F_E(n+1) .$$

We first show that if we have F_E in $\mathfrak{M}_\lambda$ with above property, then we can find an e such that $e \subseteq g_\beta$. Indeed, since $F_E <^* f$, for almost all n, there is a k in ω such that $h(n) \leq F_E(k) < F_E(k+1) < h(n+1)$. Choose such $n > m$ and k. Let $s = g \restriction \bigcup_{j\leq n} h(j) \cap x_j$, then $s \in \mathrm{Fn}(F_E(k), 2)$ and

$$g_\beta \restriction \bigcup_{j\leq n} x_j \mathbin{\overline{\cup}} g = s \mathbin{\overline{\cup}} \bigcup_{j\leq n} g_j \mathbin{\overline{\cup}} g .$$

Therefore there is an $e \in E$ such that

$$e \subseteq g_\beta \restriction \bigcup_{j \leq n} x_j \,\overline{\cup}\, g$$

and

$$\mathrm{dom}(e) \setminus \bigcup_{j \leq m} x_j \subseteq F_E(k+1) < h(n+1) \ .$$

Since $\mathrm{dom}(e) \setminus \bigcup_{j \leq m} x_j \subseteq h(n+1)$ and $g_\beta \restriction (h(n+1) \setminus \bigcup_{j \leq n} x_j) = g \restriction (h(n+1) \setminus \bigcup_{j \leq n} x_j)$, it is clear that $e \subseteq g_\beta$.

The construction of F_E is straightforward. Suppose we have defined $F_E(n) = l$. For each $s \in \mathrm{Fn}(l, 2)$ and k such that $m \leq k \leq l$, since $s \,\overline{\cup} \bigcup_{j \leq k} g_j \,\overline{\cup}\, g$ is a $P_{\mathcal{A}_{\beta_k+1}}$ generic over $\mathfrak{M}_{\mu(\beta_j)}$, we can find an $e(s,k) \in E$ such that $e(s,k) \subseteq s \,\overline{\cup} \bigcup_{j \leq k} g_j \,\overline{\cup}\, g$. Recall that $\mathrm{dom}(e(s,k)) \subseteq^* a_{\mu_{\beta_m}} =^* \bigcup_{j \leq m} x_j$. Let $F_E(n+1)$ be bigger than l and large enough such that for all $s \in \mathrm{Fn}(l, 2)$ and k with $m \leq k \leq l$, $F_E(n+1)$ contains $\mathrm{dom}(e(s,k)) \setminus \bigcup_{j \leq m} x_j$.

Case 4 $\quad \alpha = \beta + 2$ and β is a limit ordinal

We have to define $\mu(\beta+1)$, $g_{\beta+1}$, p_β, ρ'_β, ρ''_β and $\mathcal{R}^\beta$ satisfying the required properties. Let τ be a nice $\mathbb{P}_{\mathcal{A}_{\beta+1}}$-name for a subset of ω such that if τ_β is a nice $\mathbb{P}_{\mathcal{A}_{\beta+1}}$-name for a subset of ω and $\tau_\beta \in \mathfrak{M}_{\mu(\beta)}$, then τ is τ_β.

Let μ be $\mu(\beta)$. By Lemma 5.1, there are $p \in \mathbb{P}_{\mathcal{A}_{\beta+1}}$ and g in $\mathfrak{M}_{\mu+1}$ such that g is a $\mathbb{P}_{\mathcal{A}_{\beta+1}}$-generic over $\mathfrak{M}_\mu$ and there are μ', λ' and λ such that $\mu + 2 < \mu' < \lambda' < \lambda$ and $\delta_{\beta+1} < \lambda$ and $\mathcal{A}_{\beta+1} \in \mathfrak{M}_\lambda$. There are also $\{R_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ such that

(i) For all $A \subseteq \omega$, if $\exists^\infty n \ A \cap R_n \neq^* \emptyset$, then $A \notin \mathcal{I}_\mathcal{N}$;
(ii) $p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}}$ "$\forall^\infty n \ R_n \cap ((\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda'}) \setminus F(a_{\mu'}))) \notin \mathcal{I}_\mathcal{N}$", where $\mathcal{A}' = \mathcal{A}_{\beta+1}(g, a_\lambda)$.
(iii) $\{R_n : n \in \omega\} \cup \{F(g^{-1}(1)), F(a_{\mu'}), F(a_{\lambda'})\} \subseteq \mathfrak{M}_\lambda$.

As argued in the previous cases, $g^{-1}(1)$ is a partitioner of $\mathcal{M}$. Now let $\mu(\beta+1) = \lambda$, $g_{\beta+1} = g$, $b_{\beta+1} = g^{-1}(1) \cap a_\lambda$, $\rho'_\beta = \lambda'$ and $\rho''_\beta = \mu'$. Let $p_\alpha = p$ and for each n, $R^\beta_n = R_n$. If $\tau_\beta \in \mathfrak{M}_{\mu(\beta)}$ is a nice $\mathbb{P}_{\mathcal{A}_\beta}$-name, then τ'_α is defined as in Lemma 6.4. It is easy to see that

$$\mathbf{1} \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{"} \tau'_\beta =_\mathcal{N} (\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda'}) \setminus F(a_{\mu'}))\text{"} \ .$$

Otherwise, let $\tau'_\beta = \emptyset$. Since τ'_β is definable from the sets in $\mathfrak{M}_{\mu(\beta+1)}$, τ'_β is also in $\mathfrak{M}_{\mu(\beta+1)}$. For the same reason, all the following dense subsets of $\mathbb{P}_{\mathcal{A}_{\beta+2}}$ are in $\mathfrak{M}_{\mu(\beta+1)}$: $D_n(\mathcal{A}_{\beta+2}, \mathcal{R}^\beta, \tau'_\beta)$, $E_{i,n}(\mathcal{A}_{\beta+2}, \mathcal{R}^\beta, \tau'_\beta)$ and $H^{\tau'_\beta}_n$.

Since $\mathcal{D}_{\beta+1} \subseteq \mathfrak{M}_{\mu(\beta)}$, $g_{\beta+1}$ is a $\mathcal{D}_{\beta+1}$-generic. Thus each member of $\mathcal{D}_{\beta+1}$ is still predense in $\mathbb{P}_{\mathcal{A}_{\beta+2}}$. Let $\mathcal{D}_{\beta+2}$ be the set of all of predense subsets of $\mathbb{P}_{\mathcal{A}_{\beta+2}}$ in $\mathfrak{M}_{\mu(\beta+1)}$.

Case 5 $\quad \alpha = \beta + n + 1$, where β is a limit ordinal and $n \geq 2$.

Since $\mathcal{A}_{\beta+n} \in \mathfrak{M}_{\mu(\beta+n-1)}$, there is a g in $\mathfrak{M}_{\mu(\beta+n-1)+1}$ which is $\mathbb{P}_{\mathcal{A}_{\beta+n}}$-generic over $\mathfrak{M}_{\mu(\beta+n-1)}$. Take any limit λ such that λ is bigger than both $\delta_{\beta+n}$ and $\mu(\beta+n-1)$ and $\mathfrak{M}_\lambda$ contains $\mathcal{A}_{\beta+n}$ and $\mathcal{D}_{\beta+n}$. Let $\mu(\beta+n)$ be λ, $g_{\beta+n}$ be g, and let $\mathcal{D}_{\beta+n+1}$ be the set of all predense subsets of $\mathbb{P}_{\mathcal{A}_{\beta+n+2}}$ which are in $\mathfrak{M}_\lambda$. It is easily checked that all the conditions from (1) to (5) are satisfied. $\square$

By Theorem 6.1, if we take the poset $\mathbb{P} * \mathbb{P}_\mathcal{A}$ and the good coherent sequence $\mathcal{A}$, we are almost done provided $\mathcal{A}$ is in V. We try to use PFA, via a dense-subset argument, to show that such $\mathcal{A}$ can exist in V. However, the property (b) can not be decided by $\aleph_1$ many dense subsets of $\mathbb{P} * \mathbb{P}_{\mathcal{A}}$. There are actually $\aleph_2$ many reals in the forcing extension. We fix this in the rest of the section.

In the next section, we work on an improved good coherent sequence which will be given by the next theorem. We will basically force, by a ccc forcing notion $\mathbb{Q}$, countably many Hausdorff-Luzin type gaps simultaneously which will ensure (b) is true. However, any Hausdorff-Luzin type gap can be decided by $\aleph_1$ many dense subsets. We start with the next theorem.

Theorem 6.5. *Let G be a $\mathbb{P}$ generic filter over V. Suppose $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \omega_1\}$ satisfies the conclusion of Theorem 6.1. Then in $V[G]$, there is a subsequence $\{\delta(\alpha) : \alpha < \omega_1\}$ of ω_1 such that*

(i) $\forall \alpha, \widetilde{\mathcal{A}}_\alpha = \{(b_{\delta(\xi)}, a_{\mu(\delta(\xi))}) : \xi < \alpha\}$ *is in* $\mathfrak{M}_{\mu(\delta(\alpha))}$,

(ii) $\mathbb{P}_{\widetilde{\mathcal{A}}_{\omega_1}}$ *forces that for all $Y \subseteq \omega$ there exists β in ω_1 such that either $\forall \xi > \eta > \beta$,*

$$Y \cap [F(a_{\mu(\delta(\xi))}) \setminus (F(a_{\mu(\delta(\eta))}) \cup F(b_{\mu(\delta(\xi))}))] \notin \mathcal{I}_{\mathcal{N}} \ ;$$

or $\forall \xi > \eta > \beta$,

$$(F(b_{\mu(\delta(\xi))}) \setminus F(b_{\mu(\delta(\eta))})) \setminus Y \notin \mathcal{I}_{\mathcal{N}} \ .$$

We denote the least such β by β_Y.

(iii) $\mathbb{P}_{\widetilde{\mathcal{A}}_{\omega_1}}$ *is ccc.*

Proof. Let H be a $\mathbb{P}_{\mathcal{A}}$-generic filter over $V[G]$. Since CH holds in $V[G]$ and $\mathbb{P}_{\mathcal{A}}$ is a ccc poset with size ω_1, CH still holds in $V[G][H]$. In $V[G][H]$, let $\{Y_\alpha : \alpha \in \omega_1\}$ enumerate $\mathcal{P}(\omega)$. For $\alpha \in \omega_1$, let

$$\begin{aligned}
\Gamma_\alpha =& \{\xi \in \omega_1 : Y_\alpha \cap (F(a_{\mu(\xi+1)}) \setminus F(a_{\mu(\xi)})) \neq_{\mathcal{N}} F(b_{\xi+1}) \setminus F(b_\xi)\},\\
\Gamma^0_\alpha =& \{\xi \in \Gamma_\alpha : Y_\alpha \cap [F(a_{\mu(\xi+1)}) \setminus (F(a_{\mu(\xi)}) \cup F(b_{\xi+1}))] \notin \mathcal{I}_{\mathcal{N}}\},\\
\Gamma^1_\alpha =& \{\xi \in \Gamma_\alpha : (F(b_{\xi+1}) \setminus F(b_\xi)) \setminus Y_\alpha \notin \mathcal{I}_{\mathcal{N}}\}.
\end{aligned}$$

Clearly $\Gamma_\alpha = \Gamma^0_\alpha \cup \Gamma^1_\alpha$. Therefore, for each α, there is $i(\alpha) \in \{0,1\}$ such that $\Gamma^{i(\alpha)}_\alpha$ is uncountable. Let C_α be the set of all the limit points of $\Gamma^{i(\alpha)}_\alpha$. Then C_α is a cub in ω_1 and there is an element of $\Gamma^{i(\alpha)}_\alpha$ between each two elements of C_α. It is easy to see that if $i(\alpha) = 0$, then for any $\eta < \xi \in C_\alpha$,

$$Y_\alpha \cap [F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi))] \notin \mathcal{I}_{\mathcal{N}} \ .$$

Similarly, if $i(\alpha) = 1$, then for all $\eta < \xi \in C_\alpha$,

$$(F(b_\xi) \setminus F(b_\eta)) \setminus Y_\alpha \notin \mathcal{I}_{\mathcal{N}} \ .$$

Set $C = \mathop{\triangle}\limits_{\delta<\omega_1} C_\delta = \{\xi : \forall \delta < \xi, \xi \in C_\delta\}$. For all $\alpha < \omega_1$, if $\eta < \xi \in C$ and $\alpha < \eta$ then

$$i(\alpha) = 0 \rightarrow Y_\alpha \cap F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \notin \mathcal{I}_{\mathcal{N}} \ ,$$

and

$$i(\alpha) = 1 \rightarrow (F(b_\xi) \setminus F(b_\eta)) \setminus Y_\alpha \notin \mathcal{I}_{\mathcal{N}} \ .$$

Since C is a cub in $V[G][H]$ and $\mathbb{P}_{\mathcal{A}}$ is ccc, C contains a cub C^0 in $V[G]$. Furthermore, since $\mathbb{P}$ is a countably closed forcing, any countable subset of $\mathcal{A}$ is actually in V. By the construction of $\mathcal{A}$, $\{\mu(\alpha) : \alpha < \omega_1\}$ is cofinal in ω_2^V. Therefore we can find a subsequence $C^1 = \{\delta(\alpha) : \alpha < \omega_1\}$ of C^0 such that for all $\alpha \in \omega_1$,

$$\{(b_{\delta(\xi)}, a_{\mu(\delta(\xi))}) : \xi \in \alpha\} \in \mathfrak{M}_{\mu(\delta(\alpha))} \ .$$

Obviously $\mathbb{P}_{\mathcal{A}} = \mathbb{P}_{\widetilde{\mathcal{A}}}$ and $\mathbb{P}_{\widetilde{\mathcal{A}}}$ is ccc and condition (ii) holds. □

From this point on we will only use the sequence $\widetilde{\mathcal{A}}$ in Theorem 6.5, but for convenience we simply assume $\mathcal{A} = \widetilde{\mathcal{A}}$.

Therefore we have the following

Theorem 6.6. *In $V^{\mathbb{P}}$, there is a good coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \omega_1\}$ such that*

(i) *$\forall\alpha$, $\mathcal{A}_\alpha$ is an r-sequence.*
(ii) *$\mathbb{P}_{\mathcal{A}}$ is ccc,*
(iii) *$\mathbb{P}_{\mathcal{A}}$ forces that, for all $Y \subseteq \omega$, there exists β such that either*

$$\forall \eta, \xi > \beta, (\eta < \xi \rightarrow Y \cap F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \notin \mathcal{I}_{\mathcal{N}}),$$

or

$$\forall \eta, \xi > \beta, (\eta < \xi \rightarrow (F(b_\xi) \setminus F(b_\eta)) \setminus Y \notin \mathcal{I}_{\mathcal{N}}).$$

□

For the good coherent sequence $\mathcal{A}$, $\mathbb{P}_{\mathcal{A}}$ adds an $X \subseteq \omega$ such that $X \cap a_{\mu(\alpha)} =^* b_\alpha$ for all $\alpha \in \omega_1$. We will show that there is a ccc forcing $\mathbb{Q}$ in $V^{\mathbb{P} * \mathbb{P}_{\mathcal{A}}}$ such that $\mathbb{Q}$ adds some Hausdorff-Luzin type gap which will imply that there is no $Y \subseteq \omega$ such that $F(X) = Y$. We intend to apply PFA as follows. Since $\mathbb{P} * \mathbb{P}_{\mathcal{A}} * \mathbb{Q}$ is proper and we only need $\aleph_1$ many dense subsets to determine the above $\mathcal{A}$, X and these Hausdorff-Luzin gaps as well as the related properties, we can get them in V.

In the next subsection we will modify the usual techniques in order to add what might be called countably many simultaneous Hausdorff-Luzin type gaps.

6.3. **Hausdorff-Luzin type gaps.**

Definition 6.7. Suppose $I = \{c_\alpha : \alpha \in \omega_1\}$ and $J = \{e_\alpha : \alpha \in \omega_1\}$ are two sequences of infinite subsets of ω. We say that I is *near to* J if for any subset Y of ω, either $\{\alpha : c_\alpha \subseteq^* Y\}$ is countable or $\{\alpha : Y \cap e_\alpha$ is finite $\}$ is countable. We say I and J are tight or $\{(c_\alpha, e_\alpha) : \alpha \in \omega_1\}$ is tight if, for any pair Σ and Γ of uncountable subsets of ω_1 and $n \in \omega$, there exist uncountable subsets Σ' and Γ' of Σ and Γ respectively such that

$$\left| \bigcap_{\alpha \in \Sigma'} c_\alpha \cap \bigcap_{\alpha \in \Gamma'} e_\alpha \right| \geq n \ .$$

Fact 6.8. *If $\{c_\alpha : \alpha \in \omega_1\}$ is near to $\{e_\alpha : \alpha \in \omega_1\}$, then $\{c_\alpha : \alpha \in \omega_1\}$ and $\{e_\alpha : \alpha \in \omega_1\}$ are tight .*

Proof. Suppose Σ and Γ are uncountable subsets of ω_1 and $n \in \omega$. We will define uncountable sets $\{\Sigma_i, \Gamma_i : i \leq n\}$ and integers $\{m_i : i < n\}$ such that

1. $\Sigma_n \subseteq \cdots \subseteq \Sigma_0 \subseteq \Sigma$, $\Gamma_n \subseteq \cdots \subseteq \Gamma_0 \subseteq \Gamma$ and $m_0 < \cdots < m_{n-1}$.
2. $m_i \in \bigcap_{\alpha \in \Sigma_{i+1}} c_\alpha \cap \bigcap_{\alpha \in \Gamma_{i+1}} e_\alpha$

Obviously, if we have the above sequence and we let, $\Sigma' = \Sigma_n$ and $\Gamma' = \Gamma_n$ then

$$\{m_0, \ldots, m_{n-1}\} \subseteq \bigcap_{\alpha\in\Sigma'} c_\alpha \cap \bigcap_{\alpha\in\Gamma'} e_\alpha \ .$$

Let $\Sigma_0 = \Sigma$ and $\Gamma_0 = \Gamma$. At stage $i > 0$, let Y be the set of $n \in \omega$ which is in c_ξ for uncountably many ξ in Σ_i. Then the set of $\xi \in \Sigma_i$ for which $a_\xi \not\subseteq^* Y$ is countable. Since $\{c_\alpha : \alpha \in \omega_1\}$ is near to $\{e_\alpha : \alpha \in \omega_1\}$, for all but countably many ξ in Γ_i, $Y \cap e_\xi$ is infinite. Let $\Gamma_i' = \{\xi \in \Gamma_i : Y \cap e_\xi \neq^* \emptyset\}$. Γ_i' is uncountable. For each $\xi \in \Gamma_i'$, take $n_\xi \in Y \cap e_\xi \setminus \max\{m_j : j < i\}$. There is a n^* such that $n^* = n_\xi$ for uncountably many ξ. Let $m_i = n^*$ and $\Sigma_{i+1} = \{\xi \in \Sigma_i : n \in c_\xi\}$ and $\Gamma_{i+1} = \{\xi \in \Gamma_i' : n^* = n_\xi\}$. ☐

Fact 6.9. *If $\{(c_\alpha, e_\alpha) : \alpha \in \omega_1\}$ is tight, then there is a ccc forcing $\mathbb{Q}$ with size ω_1 such that* **1** *forces*

$$\exists I \in [\omega_1]^{\omega_1}, \forall k \in \omega, \{(c_{\alpha+k}\, e_{\alpha+k}) : \alpha \in I\} \textit{ is a Hausdorff-Luzin type family.}$$

Proof. Let $\mathbb{R} = \{(H, n) : H \in [\omega_1]^{<\omega}, n \in \omega\}$. Define $(H, n) < (K, m)$ if the following holds

1. $K \subseteq H$ and $m \leq n$;
2. if $\beta < \alpha$, $\alpha \in K$, and $\beta \in H \setminus K$ then for all $k < m$, $|c_{\alpha+k} \cap e_{\beta+k}| > m$.

It is easy to see that $(\mathbb{R}, <)$ is a poset.

Claim 1 $(\mathbb{R}, <)$ is ccc.

Suppose $\mathbb{R}$ has an uncountable antichain A. By the $\triangle$-system Lemma, we can assume A has the form $\{(H_\alpha, n) : \alpha \in \omega_1\}$ for some n and that $\{H_\alpha : \alpha \in \omega_1\}$ forms a $\triangle$-system with root H. We can also assume $H_\alpha = H \cup \{\xi_1^\alpha, \ldots, \xi_m^\alpha\}$ for all α and that, for $\alpha < \beta$, $H < \{\xi_1^\alpha, \ldots, \xi_m^\alpha\} < \{\xi_1^\beta, \ldots, \xi_m^\beta\}$. By applying the fact that $\{(c_\alpha, e_\alpha) : \alpha \in \omega\}$ is tight finitely many times, we can find uncountable subsets Γ_1 and Γ_2 of ω_1 such that for each $k < n$ and for all $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$ and $i, j \in \{1, \ldots, m\}$,

$$|c_{\xi_i^\alpha+k} \cap e_{\xi_j^\beta+k}| > n \text{ and } |e_{\xi_i^\alpha+k} \cap c_{\xi_j^\beta+k}| > n \ .$$

Now it is easy to see that for any $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$, $(H_\alpha \cup H_\beta, n)$ is stronger than both (H_α, n) and (H_β, n).

Next we prove that forcing with a subset of $\mathbb{R}$ will add many Hausdorff-Luzin gaps as described. Let $\mathbb{R}_\alpha = \{(H, n) \in \mathbb{R}, H \cap \alpha = \emptyset\}$.

Since $\mathbb{R}$ is ccc, the proof of the following claim is straightforward.

Claim 2 There exits α such that for all $\beta > \alpha$, $D_\beta^\alpha = \{(H, n) \in \mathbb{R}_\alpha : H \not\subseteq \beta\}$ is dense in $\mathbb{R}_\alpha$.

Fix an α^* which satisfies Claim 2. Let $\mathbb{Q}$ be $\mathbb{R}_{\alpha^*}$. Obviously $\mathbb{Q}$ is ccc and $\mathbb{Q}$ adds a subset I as required. We are done. ☐

Working in $V^{\mathbb{P}}$, let $\{(c_{\alpha+k}, e_{\alpha+k}) : \alpha \in \lim \omega_1 \text{ and } 1 < k \leq \omega\}$ be the derived sequence of $\mathcal{A}$ and F (See the paragraph following Lemma 3.8 for the definition). We first prove the following lemma.

Lemma 6.10. *In $V^{\mathbb{P}*\dot{\mathbb{P}}_{\mathcal{A}}}$, $\{c_\alpha : \alpha \text{ is a successor}\}$ is near to $\{e_\alpha : \alpha \text{ is a successor}\}$.*

Proof. Let $Y \subseteq \omega$. Suppose there are uncountably many successor ordinals α such that $c_\alpha \subseteq Y$. Recall that β_Y is the minimal ordinal β such that either

$$\forall \eta, \xi > \beta\ (\eta < \xi \rightarrow Y \cap F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \notin \mathcal{I}_{\mathcal{N}}),$$

or

$$\forall \eta, \xi > \beta \ (\eta < \xi \rightarrow (F(b_\xi) \setminus F(b_\eta)) \setminus Y \notin \mathcal{I}_\mathcal{N}).$$

We will show that for any successor $\alpha > \beta_Y$, $e_\alpha \cap Y$ is infinite. Take $\alpha > \beta_Y$ such that $c_\alpha \subseteq Y$. Since $c_\alpha =_\mathcal{N} F(b_\alpha) \setminus F(b_{\alpha-1})$, it is not true that

$$\forall \eta, \xi > \beta_Y \ (\eta < \xi \rightarrow (F(b_\xi) \setminus F(b_\eta)) \setminus Y \notin \mathcal{I}_\mathcal{N}).$$

Therefore

$$\forall \eta, \xi > \beta_Y, (\eta < \xi \rightarrow Y \cap F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \notin \mathcal{I}_\mathcal{N}),$$

and so for all successor $\alpha > \beta_Y$,

$$Y \cap F(a_{\mu(\alpha)}) \setminus (F(a_{\mu(\alpha-1)}) \cup F(b_\alpha)) \notin \mathcal{I}_\mathcal{N}.$$

But

$$e_\alpha =_\mathcal{N} F(a_{\mu(\alpha)}) \setminus (F(a_{\mu(\alpha-1)}) \cup F(b_\alpha)) \ ,$$

hence $Y \cap e_\alpha$ is infinite since it is not in $\mathcal{I}_\mathcal{N}$. □

By Fact 6.9, we have

Theorem 6.11. *In $V^{\mathbb{P} * \dot{\mathbb{P}}_\mathcal{A}}$, there is a ccc forcing $\mathbb{Q}$ with size ω_1 such that*

$$\mathbf{1} \Vdash_\mathbb{Q} \text{“}\exists I \in [\omega_1]^{\omega_1} \text{ such that } \forall k \in \mathbb{N},$$
$$\{(c_{\alpha+k}, e_{\alpha+k}) : \alpha \in I\} \text{ is a Hausdorff-Luzin type family”}$$

□

By Theorem 6.6 and Theorem 6.11, we now are ready to prove Theorem 3.9.

Proof of Theorem 3.9: Let $\dot{Q}$ be the $\mathbb{P} * \mathbb{P}_\mathcal{A}$-name for the poset in Theorem 6.11 and let $\mathbb{D}$ be the the poset $\mathbb{P} * \mathbb{P}_\mathcal{A} * \dot{Q}$. Obviously in $V^\mathbb{P}$, $\mathbb{P}_\mathcal{A}$ adds an X which extends $\mathcal{A}$. Let $\{(c_{\alpha+k}, e_{\alpha+k}) \colon \alpha \in \lim \omega_1 \text{ and } k \in \mathbb{N} \text{ or } k = \omega\}$ be the derived sequence obtained before Lemma 6.10 and Σ be the sequence $I \in V^\mathbb{D}$ obtained by Theorem 6.11. We enumerate I by $\{\sigma_\alpha : \alpha \in \omega_1\}$. It is easily checked that these sequences and sets satisfy the statement of the theorem. □

This completes the proof that PFA$^+$ $+\diamondsuit(S^2_0)$ implies there is a subalgebra of a representable algebra which is not representable. We should remark that we have no information as to whether the hypotheses can be weakened. The most interesting questions are to ask if PFA$^+$ can be weakened to PFA and, still more important, if the consistency can be established relative to ZFC without any large cardinal assumption.

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