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Abstract

We prove that some problems in the theory of compactifications of the integers are not decided by the usual axioms of set theory.

We prove in Chapter 2 through Chapter 5 that it is consistent that there be a partitioner algebra on ω which has a subalgebra which is not representable by any partitioner algebra. The present construction makes use of a supercompact cardinal.

In Chapter 6, we prove that the Continuum Hypothesis implies that, for every ultrafilter p on ω, the ultrafilter space $\omega \cup \{p\}$ can be embedded into some pseudoradial zero-dimensional Hausdorff space.

In Chapter 7, it is established that in any forcing extension by adding enough Cohen reals the space $\omega \cup \{p\}$ can not be embedded into any regular pseudoradial space, for any P-point p.

In Chapter 8, we prove that in some forcing extension as in Chapter 7 there is a special ultrafilter p, such that $\omega \cup \{p\}$ can be embedded into some pseudoradial and zero-dimensional Hausdorff space.

In Chapter 9, we construct, in ZFC, an ultrafilter p on ω with a special almost disjoint refinement and a strong Noetherian base.

In Chapter 10, we dicsuss some possible implications of our results and our new techniques and give a list of open problems.

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Contents

CHAPTER 1

Introduction and Preliminaries

This thesis is an investigation into some problems arising from the general study of topological compactifications of the integers. This study occupies a central place in the field because it impacts so broadly on the class of compact spaces with countable convergence properties. Topology itself has two separate roots: geometry and analysis. One may regard topology's role in analysis as the language of convergence. In this context the need to fully understand countable convergence in compact spaces is of paramount importance.

Let us briefly explain the notions of limit points and convergence of sequences which are what ever Calculus is based on. When a point is infinitesimally close to a set , the point is described as a limit point of the set. Topology has made precise the notion of infinitesimal closeness (but has had to abandon as insufficient the naive notion of distance in the conventional sense). Putting a topological structure on a set is actually to define which points are limit points of which sets. A simple converging sequence is modeled on the intuition that the elements in the sequence are continually getting closer to the limit. However with no adequate notion of "closer" available, let us recognize that this simply means that every infinite subset of the set still has the point as a limit. This simplicity is very helpful and should perhaps be contrasted with the extreme case we study in Chapter 6 in which one has that if the set is partitioned into two, precisely one of them still has the point as a limit.

Simple countable converging sequences were the motivation to study topological space in order to offer calculus a firm base which does not depend on the geometric intuition when various pathological phenomena in convergence of trigonometric series were discovered early last century. However, the notion of countable convergence sequence is not general enough to encapsulate the conceptual ideas of convergence both in the countable and the uncountable settings. This was soon discovered. It is a constant challenge to understand to what degree the notion of converging sequence is sufficient. The question is even more relevant today as the new development in the study of the compact spaces with countable convergence properties tightness has influential applications in other branches of mathematics, particularly, Functional Analysis.

Several important classes of spaces arising in Functional Analysis do not have simple a convergence structure as one might expect. For example, when one starts with even the unit interval and considers the topology of pointwise convergence on the space of continuous real-valued functions, the space, while quite nice in many respects, has no globally adequate notion of distance, is not *first countable* (i.e. countable local basis at each point). It is not *Fréchet* (i.e. if x is a limit of A then there is a sequence from A converging to x) and it is not even *sequential* (i.e. if A is not closed, there should be a sequence from A converging to a point outside of A). This space does have the property called *countable tightness* (i.e. if x is a limit of A, then it is a limit of a countable subset of A).

However compactness dramatically improves the situation (Grothendieck): a countably compact subset is compact and Fréchet. This is true of all Eberlein compact spaces and many of its generalizations. Each of them is known to have countable tightness but not much more beyond countable tightness can be said universally.

While we know in certain situations, compact spaces with countable tightness do have a large supply of converging sequence, we do not really understand this class of spaces quite well. Working from below we have metric spaces, Fréchet spaces and sequential spaces whose topology is determined by the countable converging sequences. Working from above, we try to extrapolate those properties which guarantee a large supply of the convergence sequence, such as sequentially compact, in order to analyze how large a class of such spaces is to gain more insight into the properties of function spaces which distinguish them from the wider class of spaces with countable tightness.

These are some of the reasons that the class of compact spaces with countable tightness or countable convergence properties are both fundamental and interesting in the study of topology. The even more basic concept is lurking in the background and this is what is called *an almost disjoint family* of countable sets.

A family of infinite subset of a countable set is an almost disjoint family if each two of its members have finite intersection. If an almost disjoint family is not a sub family of any other almost disjoint family, then it is called a maximal almost disjoint family. To give a simple but illustrative example, let us note that any family of converging sequences of rational numbers which are convergent to different points is actually an almost disjoint family. Thus we see the countable set of rational numbers supports rather large almost disjoint families. One can see the similarities in the following brief explanation about the important roles of the almost disjoint families in the study of convergence. The first step in computing the set of limit points for a set in an abstract space in our classes is best viewed just in this way: consider the maximal almost disjoint family of converging subsequences. The structure of this almost disjoint family determines which points are the limit points of the set.

We now recall some of the more explicit appearances of almost disjoint families in the topological constructions and their significant role in set theory. Mrówka constructed what is now called the Ψ-space associated with a maximal almost disjoint family to give the first example of a pseudocompact space which is not countably compact. Later on Terasawa and Mrówka [Ter81, Mro77] constructed various maximal almost disjoint families and, via the Ψ-spaces, solved a number of problems in dimension theory and the theory of compactifications of ω. P. Simon's elegant result on the product of Fréchet spaces is another such example. P. Simon constructed an example of two compact Fréchet spaces whose product is not Fréchet. However, the vital part of the constructions need a subtle analysis of general maximal almost disjoint families. Both examples actually demonstrated intimate intrinsic connections between the properties of an almost disjoint family and those of a topological space. Another place where the almost disjoint families play an equally important role is the study of $\beta\omega$ (the Stone-Céch compactification of the countable discrete set of integers) or equivalently the combinatorial structure on the Boolean algebra of infinite subsets of ω. Since all countable discrete spaces are viewed as the same spaces, we will use the set of natural numbers ω as our standard countable discrete space. We know that any fundamental result about $\beta\omega$ will enjoy a substantial long-term degree of significance because of the diverse and numerous ways in which $\beta\omega$ arises in mathematics.

The notion of almost disjoint family dates back to the work of Cantor at the turn of last century ([Lus47]) and was investigated in more details by great mathematicians like Erdös, Sierpiński and Tarski early this century. The theory of the almost disjoint families on a countable set has deep consequences in Set-Theory and Model Theory (§12 of [CN74]). However, the properties of almost disjoint families are very problematic to determine and sensitive to the axioms of set theory. For example, the minimal size of a maximal almost disjoint family varies with models but Martin's Axiom decides that each maximal almost disjoint family must have cardinality $\mathfrak{c}$.

Our first result focuses primarily on the relatively recent attempts to classify almost disjoint families. Baumgartner and Weese [BW82] had the interesting idea to investigate the properties of maximal almost disjoint families (henceforth *mad* families) by virtue of their algebra of partitioners (see page 21). In actual fact this study began with Mrowka's ground breaking result that the partitioner algebra could be trivial. This was also cleverly modified by Terasawa (see [Mro77, Ter81]). However Baumgartner and Weese initiated the general study of these partitioner algebras. In terms of partitioner algebras, Terasawa and Mrówka basically constructed various mad families whose partitioner algebras have required properties. In [BW82], Boolean algebras isomorphic to some partitioner algebras are called representable. There have been many interesting results since that time aimed at determining which Boolean algebras are representable. (see [DN92], [DF91], [Fra85], [FZ90] and [FZ88].) In Baumgartner and Weese's paper [BW82], there were a number of open problems regarding representability. Several have been answered but not till now, the first question which is to determine if each subalgebra of a representable algebra is still representable. By the results in [BW82], the answer is trivially yes under CH. However, the question of whether the answer could be negative in some model of set theory remained open for more than a decade. In this thesis our contribution towards the study of representable algebras is to answer this question. We prove that from the existence of a supercompact cardinal a model of Martin's Axiom can be constructed in which the answer is negative. We actually prove that in any model of PFA$^+$ and $\diamondsuit(S^2_0)$ there exists a partitioner algebra containing a subalgebra which is not representable (see 2.4.1).

Later chapters mainly discuss the possibility of embedding subspaces of the

Stone-Céch compactification $\beta\omega$ of the form $\omega\cup\{p\}$ into some pseudoradial spaces. Before we explain why this is an interesting topic, let us first remark that the above mentioned spaces $\omega \cup \{p\}$ have no converging sequence while the pseudoradial spaces are expected to have sufficiently many converging sequences to decide their structure.

A radial space is such a space where one can get the all limit point of a subset by adding all the limit points of well-ordered sequences. In a pseudoradial spaces, one has to iteratively taking such limit points. Radial and pseudoradial spaces were first introduced by Herrlich [Her67] in 1967 as generalizations of Fréchet spaces and sequential spaces. The same classes of topological spaces were then studied by a number of topologists such as Mrówka, Rajagopalan and Arhangel′skiĭ (see, for example, [DMR67], [SMS73], [Arh78b], [Arh78a] and [AIG86]). However, pseudoradial spaces has recently received new attention, particularly after Shapirovskiĭ's remarkable theorem on the equivalence of sequential compactness and pseudoradialness in compact spaces (see [Sha90]). Shapirovskiĭ's result stimulated several interesting results regarding sequential compactness and pseudoradialness (see [DJSS96], [JS93]). By studying Fréchet and sequential spaces, topology offers a powerful tool in study of the convergent sequences. The development of the study of radial spaces and pseudoradial spaces show that the transfinite sequence not only serves as a natural more general concept than the countable convergent sequences but also plays an important role which is especially useful in inductive constructions.

Many properties known to be true for Fréchet or sequential spaces are not clear for radial or pseudoradial spaces. In this thesis our contributions are towards the fundamental problem of deciding how pseudoradial spaces differ from other topological spaces. In doing so, we solve several open problems.

In [AIG86], the authors posed a problem of finding an internal characterization of those spaces which are subspaces of pseudoradial spaces. It is know that each topological space, X, is a quotient space of the topological sum of its so-called prime spaces $\{X_p\colon p \in X\}$, where X_p is obtained from X by making each point except p isolated while p retains its neighbourhoods. Since quotient maps preserve sub-pseudoradialness, it naturally becomes the first step to examine the spaces with only one non isolated point. Further more, by a theorem in [Zho94], it is often sufficient to examine all countable subspace of X. Therefore, the countable spaces with only one non isolated point becomes very important. Among them, the subspaces $\omega \cup \{p\}$ of $\beta\omega$, again become most interesting in this area, where p are ultrafilters on ω. Our work again involves the questions and techniques in study of the Stone-Céch compactification $\beta\omega$ including manipulation of almost disjoint families on the countable set ω.

In [AIG86], the authors asked whether, for each ultrafilter p on ω, the space $\omega \cup \{p\}$ is a subspace of some pseudoradial space (sub pseudoradial space), where $\omega \cup \{p\}$ takes the subspace topology in $\beta\omega$, i.e. p is the only non-isolated point and a neighbourhood of p is $A \cup \{p\}$ for $A \in p$. It is proved in [Zho94] that, under the assumption $\mathfrak{p} = \mathfrak{c}$, $\omega \cup \{p\}$ is a subspace of a pseudoradial Hausdorff space for $p \in \beta\omega \setminus \omega$. As a consequence, under $\mathfrak{p} = \mathfrak{c}$, each space with countable tightness is a subspace of some pseudoradial space. However the pseudoradial space constructed in [Zho94] is not regular. J. Vaughan [Vau90] asked if it is possible to make the pseudoradial space in [Zho94] regular. In [Nyi92] P. Nyikos asked whether each topological space can be embedded into a pseudoradial space (Problem 6.22). These are clearly very natural questions for which one needs to know the answer in order to understand the subspaces of pseudoradial spaces. In this thesis we will settle these problems in the class of regular spaces and, for brevity, we will not specifically mention that the spaces under discussion are assumed to be regular.

It is proved that there is a forcing extension in which for some ultrafilter p, the space $\omega \cup \{p\}$ can be embedded into some regular pseudoradial space while the same is not true for all ultrafilters, specifically not for P-point ultrafilters. It is also proved that, under CH, the space $\omega \cup \{p\}$ can be embedded into a zero-dimensional Hausdorff pseudoradial space for every ultrafilter p on ω.

Therefore, in the class of regular spaces, for ultrafilters p on ω, the sub pseudoradialness of $\omega \cup \{p\}$ is totally independent of ZFC and there are plenty of spaces which are not sub pseudoradial.

As a by-product, we produce, in ZFC, an ultrafilter on ω with a strong Noetherian base (see page 109 for definition). By our results, it appears very likely that $\mathfrak{c} \leq \omega_2$ implies that $\omega \cup \{p\}$ is sub pseudoradial for this particular ultrafilter p.

All our results and proofs occurs in Chapter 2 through Chapter 9. In Chapter 10, we will give a more specific summary of our work as well as a list of open problems. We will also try to explore possible implications of our results and our new techniques.

This chapter also contains the notations and preliminaries. The section 1.2 can be skipped. The reader can come back when some theorems or notations are used in reading the later chapters.

In Chapter 2 through chapter 5, we deal with representable algebras.

In Chapter 2, we will construct a partitioner algebra and find a subalgebra $\mathfrak{B}$ which is not representable. We will prove that $\mathfrak{B}$ is not representable based on a theorem whose proof will take the following three chapters.

In Chapter 3, we prove a structural theorem which reflects our motivations in constructing the non representable algebra $\mathfrak{B}$ in the way we did.

In Chapter 4, we will prove a key lemma used in the construction of a coherent sequence (see page 11) in Chapter 5. The lemma will show that we can extend a coherent sequence with certain requirements.

In Chapter 5, we give the proof of the theorem cited in Chapter 2 to prove that $\mathfrak{B}$ is not representable. We construct a special coherent sequence in some forcing extension by a proper forcing.

In Chapter 6 through Chapter 8, we prove some results about embedding the spaces $\omega \cup \{p\}$ into some pseudoradial spaces.

In Chapter 6, we give the relevant definitions and prove that under CH, for every ultrafilter p on ω, the space $\omega \cup \{p\}$ with the usual topology can be embedded into some pseudoradial zero-dimensional Hausdorff space.

In Chapter 7, we prove that in any forcing extension by adding $\aleph_2$ many Cohen reals to a model of CH, for all P-point p, the space $\omega \cup \{p\}$ can not be embedded into any regular pseudoradial spaces.

In Chapter 8, we first introduce the notion of a *reflecting self-indexed stationary set* and then show that the existence of such a stationary set together with the assumption $\mathfrak{c} = \omega_2$ will imply that for the ultrafilter p which will be discussed in Chapter 9, the space $\omega \cup \{p\}$ is a subspace of some pseudoradial and zero-dimensional Hausdorff space. We then deduce that there is a forcing extension in which, for all P-points p, the space $\omega \cup \{p\}$ can not be embedded into any regular pseudoradial spaces while for our special ultrafilter p, $\omega \cup \{p\}$ can be embedded into some pseudoradial and zero-dimensional Hausdorff space.

In Chapter 9, we construct, in ZFC, an ultrafilter p with a special almost disjoint refinement (see page 101) and a strong Noetherian base.

We give some basic definitions and notations and the preliminaries in the next two sections.

1.1. Definitions and Notations

Our notations and definitions are standard. Most of the notations and definitions for set theory can be found in [Kun80]. Notations and definitions for general topology can usually be found in R. Engelking's book General Topology [Eng77]. Other good references are [KV84] and [She82]. The terminologies in model theory can be found in [CK73]

For any set S, we use $|S|$ to denote the cardinality of S. For any cardinal number κ, $[S]^\kappa$ is the set of subsets of S with cardinality κ. When we write $f\colon A \to B$, we mean f is a function or a partial function from the set A to the set B.

For a set S and a family $\mathcal{I}$ of subsets of S, $\mathcal{I}$ is an *ideal* on S if, (1) for any I in $\mathcal{I}$, any subset of I is in $\mathcal{I}$, (2) for any I and J in $\mathcal{I}$, $I \cup J$ is also in $\mathcal{I}$. For a family $\mathcal{A}$ of subsets of S, we use $\mathcal{I}_\mathcal{A}$ to denote the minimal ideal on S containing $\mathcal{A}$. For an ideal $\mathcal{I}$ on S and two subsets A and B of S, we write $A \subseteq_\mathcal{I} B$ if $A \setminus B \in \mathcal{I}$. Naturally, $A =_\mathcal{I} B$ means that $A \triangle B \in \mathcal{I}$.

We will use Fin to denote the ideal of finite subsets of ω. When S is ω and $\mathcal{I}$ is Fin, we have the following standard notations.

If A and B are subsets of ω, then $A \subseteq^* B$ means that $A \setminus B$ is finite. We write $A =^* B$ when $A \triangle B$ is finite. Hence $A \neq^* \emptyset$ means that A is an infinite set. If $A \cap B =^* \emptyset$, i.e. $A \cap B$ is finite, then we say that A and B are *almost disjoint*. An *almost disjoint family* $\mathcal{M}$ on ω is a subset of $\mathcal{P}(\omega)$ such that the members of $\mathcal{M}$ are pairwise almost disjoint. We also call an almost disjoint family an *ad* family and a maximal almost disjoint family a *mad* family. We will call a sequence $\{a_\alpha\colon \alpha < \lambda\}$ in $\mathcal{P}(\omega)$ *increasing (decreasing) mod finite* if for $\alpha < \beta < \lambda$, $a_\alpha \subseteq^* a_\beta$ $(a_\alpha{}^* \supseteq a_\beta)$. We will also use the natural generalizations to other countable sets.

For a subset A of ω, A is *co-infinite* if $\omega \setminus A$ is infinite and A is *co-finite* if $\omega \setminus A$

is finite. If $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathcal{P}(\omega)$ we write $\mathcal{A} < \mathcal{B}$ or $\mathcal{B} > \mathcal{A}$ if for all $a \in \mathcal{A}$ and for all $b \in \mathcal{B}$, $a \subseteq^* b$. If $\mathcal{A}$ is a singleton containing only the element a, then we write $a < \mathcal{B}$ instead of $\{a\} < \mathcal{B}$. Similarly we will write $\mathcal{A} < b$ or $b > \mathcal{A}$, etc. In case $a < \mathcal{B}$, we call a a pseudo-intersection of $\mathcal{B}$.

If a and b are subsets of ω, we say *a splits b* if both $b \cap a$ and $b \setminus a$ are infinite. If a does not split any member of $\mathcal{M}$, we call a a *partitioner* of $\mathcal{M}$.

We will sometimes call a function from ω to ω a real. We use $\mathbb{N}$ to denote all positive integers, i.e. $\mathbb{N} = \omega \setminus \{0\}$.

We call a partially ordered set a poset and sometimes a forcing notion. If P is a poset and E_1, $E_2 \subseteq P$, we write $E_1 \perp E_2$ when, for all $p \in E_1$ and $q \in E_2$, q and p are incompatible (p and q have no common extension). In case E_1 is singleton say, $\{p\}$, we write $p \perp E_2$ instead of $\{p\} \perp E_2$. We use $E^{\perp}$ to denote the set $\{q \in P\colon q \perp E\}$. Thus $E_1 \perp E_2$ iff $E_1 \subseteq (E_2)^{\perp}$.

Our standard references for forcing are [Kun80] and [She82]. We always assume a poset has a largest element $\mathbf{1}$. We use $\check{x}$ to denote a canonical name for x in the ground model. To make notations less complicated, we will sometimes use x in place of $\check{x}$ in the discussions. Consequently, for a poset $\mathbb{P}$ and a set X in the ground model , if τ is a *nice* $\mathbb{P}$-name for a subset of X, we will write τ in the following form, $\bigcup\{\{x\}\times\pi_x\colon x \in X\}$ instead of the formally correct one $\bigcup\{\{\check{x}\}\times\pi_x\colon x \in X\}$, where each π_x is an antichain of $\mathbb{P}$. This abuse of terminology is quite standard.

Definition 1.1.1. A sequence $\{(a_\alpha, b_\alpha) : \alpha \in \lambda\}$ is called a *coherent sequence* on ω provided that both $\{a_\alpha : \alpha \in \lambda\}$ and $\{b_\alpha : \alpha \in \lambda\}$ are increasing mod finite sequences on ω and for each $\alpha < \beta < \lambda$,

- $a_\alpha \subseteq b_\alpha$,
- $a_\alpha =^* a_\beta \cap b_\alpha$.

A set X is said to *uniformize* $\mathcal{A}$, if for all $\alpha < \lambda$, $X \cap b_\alpha =^* a_\alpha$.

If $a \subseteq b$ are two sets, we adopt Shelah's notation in using Ch^a_b to denote the function r from b to $\{0,1\}$ such that $r^{-1}(1) = a$.

DEFINITION 1.1.2. Suppose $\mathcal{A} = \{(a_\alpha, b_\alpha) : \alpha \in \lambda\}$ is a coherent sequence on ω. $\mathbb{P}_\mathcal{A}$ is the poset consisting of the partial functions p from ω to $\{0,1\}$ such that there is an $\alpha < \lambda$ such that $p =^* \mathrm{Ch}^{a_\alpha}_{b_\alpha}$, i.e the following holds,

(1) $\operatorname{dom} p =^* b_\alpha$ and
(2) $p^{-1}(1) =^* a_\alpha$

For p and q in $\mathbb{P}_\mathcal{A}$, $p \leq q$ if $p \supseteq q$.

Obviously, $\mathbb{P}_\mathcal{A}$ adds a subset X of ω which uniformizes $\mathcal{A}$.

If $\mathcal{A} = \{(a_\alpha, b_\alpha) : \alpha \in \lambda\}$ and $\mathcal{B} = \{(c_\alpha, d_\alpha) : \alpha \in \rho\}$ are coherent sequences, then $\mathcal{A}^\frown\mathcal{B}$ is the sequence $\{(u_\alpha, v_\alpha) : \alpha < \lambda + \rho\}$ where $(u_\alpha, v_\alpha) = (a_\alpha, b_\alpha)$ when $\alpha < \lambda$ and $(u_\alpha, v_\alpha) = (c_\xi, d_\xi)$ when $\alpha = \lambda + \xi$ for some $\xi < \rho$. It is easy to see that $\mathcal{A}^\frown\mathcal{B}$ will also be a coherent sequence if for all $\alpha < \lambda$, $a_\alpha =^* c_0 \cap b_\alpha$ and $b_\alpha \subseteq^* d_0$. In the case $\mathcal{B}$ is the singleton $\{(c_0, d_0)\}$, we write $\mathcal{A}^\frown(c_0, d_0)$ instead of $\mathcal{A}^\frown\{(c_0, d_0)\}$.

DEFINITION 1.1.3. For sets I and J and an infinite cardinal λ,
$\mathrm{Fn}(I, J, \lambda) = \{p : |p| < \lambda \wedge p \text{ is a function } \wedge \operatorname{dom} p \subseteq I\}$.
Order $\mathrm{Fn}(I, J, \lambda)$ by: $p \leq q \leftrightarrow q \subseteq p$. $\mathrm{Fn}(I, J)$ is $\mathrm{Fn}(I, J, \omega)$.

To simplify the notations, we will sometimes use logical symbols $\forall$, $\exists$, $\rightarrow$, $\leftrightarrow$, $\vee$, $\wedge$ as well as $\neg$ as logical connectives in the standard way. Also, when we write $\exists^\infty$ we mean "there are infinitely many" and by $\forall^\infty$ we mean "for all but only finitely many". More generally, suppose λ is a cardinal number, then $\exists^\lambda$ means "there exists λ many"; $\forall^\lambda$ means "for all but less than λ many". We give one more notation to finish this section. If f and g are any two functions, we use $f \overline{\cup} g$

to denote the function

$$f \cup g \restriction (\operatorname{dom} g \setminus \operatorname{dom} f).$$

1.2. Preliminaries

In this section, we state some axioms and theorems in set theory which will be needed. We will also introduce more notions and definitions related to these theorems.

DEFINITION 1.2.1. [Kun80] A poset $\mathbb{P}$ has the *countable chain condition* (ccc) iff every antichain in $\mathbb{P}$ is countable. More generally $\mathbb{P}$ satisfies the λ-cc if every antichain in $\mathbb{P}$ has cardinality less than λ. Therefore, ω_1-cc is simply ccc.

Suppose λ is a cardinal. A poset $\mathbb{P}$ is λ-*directed closed* if whenever $D \subseteq P$ is directed and $|D| < \lambda$, then there is a $p \in \mathbb{P}$ with $p \leq q$ for all $q \in D$. Obviously $\mathbb{P}$ is λ-*directed closed* iff whenever $\gamma < \lambda$ and $\{p_\xi \colon \xi < \gamma\}$ is a decreasing sequence of elements of $\mathbb{P}$, then there is a $p \in \mathbb{P}$ such that $p \leq p_\xi$ for all $\xi < \gamma$. In case λ is ω_1, $\mathbb{P}$ is said to be countably closed.

DEFINITION 1.2.2. Suppose S is a set. $C \subseteq [S]^{\leq\omega}$ is called a *cub* in $[S]^{\leq\omega}$, if

(1) for any countable subset B of S, there is an $A \in C$ such that $B \subseteq A$, and
(2) for any countable increasing sequence $\{A_n \colon n \in \omega\}$ in C,

$$\bigcup_{n\in\omega} A_n \text{ is in } C\ .$$

$E \subseteq [S]^{\leq\omega}$ is called stationary, if E meets every cub on $[S]^{\leq\omega}$.

For a function f from $[A]^{<\omega}$ to A, we use $C(f)$ to denote the set $\{B \in [A]^{\leq\omega} \colon \forall F \in [B]^{<\omega}, f(F) \in B\}$. Obviously, $C(f)$ is a cub on $[A]^{\omega}$. It is also not difficult to prove that C contains a cub on A iff there is a function f from $[A]^{<\omega}$ to A such that $C(f) \subseteq C$.

PROPOSITION 1.2.3. *Suppose $\mathbb{P}$ is a ccc poset and A is any set. If $\dot{C}$ is a $\mathbb{P}$-name for a cub on $[A]^{<\omega}$ and G is a $\mathbb{P}$-generic over V, then there is a cub C_0 on $[A]^{<\omega}$ such that C_0 is from V and $C_0 \subseteq \mathrm{val}(\dot{C}, G)$.*

Proof: Let $C = \mathrm{val}(\dot{C}, G)$. Let f be a function from $[A]^{<\omega}$ to A such that $C(f) \subseteq C$. Since $\mathbb{P}$ is ccc, by Lemma 6.5.5 of [Kun80], there is a function F from $[A]^{<\omega}$ to $[A]^{\omega}$ such that for any $B \in [A]^{<\omega}$, $f(B) \subseteq F(B)$ and F is in the ground model. Let C_0 be the set of countable subsets B of A such that for each $E \in [B]^{<\omega}$, $F(E) \subseteq B$. Clearly C_0 is a cub and is a subset of C. □

DEFINITION 1.2.4. [She82] A poset $\mathbb{P}$ is called *proper* if, for any cardinal number λ and any stationary subset S of $[\lambda]^{\leq\omega}$, S remains stationary in the forcing extension $V[G]$ for any $\mathbb{P}$-generic G over V.

By the above proposition, every ccc poset is proper. It is also easy to prove that every countably closed poset is proper.

DEFINITION 1.2.5. PFA is the following statement:

> Suppose P is a proper poset, $\{D_\alpha \colon \alpha \in \omega_1\}$ are dense subsets of P. Then there exists a filter $G \subseteq P$, such that
> $$\forall \alpha < \omega_1 \, G \cap D_\alpha \neq \emptyset \; .$$

PFA$^+$ is the following statement:

> Suppose P is a proper poset, $\{D_\alpha \colon \alpha \in \omega_1\}$ are dense subsets of P and $\{\tau_\alpha \colon \alpha \in \omega_1\}$ are P-names for a subset of ω_1 such that $\forall \alpha < \omega_1 \Vdash_P$ "τ_α is stationary subset of ω_1". Then there exists a filter $G \subseteq P$, such that
> (1) $\forall \alpha < \omega_1 \, G \cap D_\alpha \neq \emptyset$,
> (2) $\forall \alpha < \omega_1 \, S_\alpha(G) = \{\xi < \omega_1 \colon \exists p \in G \, p \Vdash_P \text{"}\xi \in \tau_\alpha\text{"}\}$ is stationary in ω_1.

For a cardinal κ, and a stationary set E on κ, $\diamondsuit(E)$ is the following statement:

> There exists a sequence $\{S_\alpha\colon \alpha \in E\}$ such that for any subset X of κ, there are stationary many $\alpha \in E$ such that $X \cap \alpha = S_\alpha$.

We call such sequence a $\diamondsuit(E) -$ *sequence*. Therefore, $\diamondsuit$ is $\diamondsuit(\omega_1)$. We use S^2_0 to denote the stationary set of all ordinals in ω_2 with countable cofinality. Thus a $\diamondsuit(S^2_0)$-sequence is of the form

$$\{S_\alpha\colon \alpha \in \omega_2 \text{ and } \operatorname{cf}\alpha = \omega\}$$

and such that for any subset X of ω_2, for stationary many $\alpha \in S^2_0$, $X \cap \alpha = S_\alpha$.

The following theorem is well-known, so we only give an outline of its proof.

THEOREM 1.2.6. *If there is a supercompact cardinal, then there is a model V of ZFC such that*

$$V \models \text{“PFA}^+ + \diamondsuit(S^2_0)\text{”}.$$

Proof: Suppose there exists a supercompact cardinal κ. By the Theorem in [Lav78], we can assume κ is indestructible, i.e. for any forcing notion Q which is κ -directed closed, κ remains supercompact in V^Q.

Let $P = \operatorname{Fn}(\kappa, 2, \kappa)$. Then it is easy to show that in V^P there is a $\diamondsuit$-sequence on $\{\xi\colon \operatorname{cf}\kappa = \omega\}$. Since P is κ-directed closed, κ is still supercompact. In V^P, let Q be the usual proper forcing notion to get a model of PFA in which $\mathfrak{c} = \kappa = \omega_2$. (for example, see [Dev83], Section 5). It is easy to verify that forcing with Q actually results in a model of PFA^+. Hence, in $V^{P*\dot{Q}}$, PFA^+ is true. Since Q is also κ-cc with cardinality κ, there is again a $\diamondsuit$-sequence on $\{\xi \in \kappa\colon \operatorname{cf}\xi = \omega\}$ in $V^{P*\dot{Q}}$. □

We will need a result of Koszmider which is a consequence of a principle called Chang's Conjecture.

Chang's Conjecture:

> For any structure of the form $\mathcal{A} = \langle \omega_2, \omega_1, (f_i)_{i<\omega} \rangle$, there is an elementary submodel $\mathcal{B}$ of $\mathcal{A}$ such that the universe B of $\mathcal{B}$ has cardinality ω_1 and $A \cap \omega_1$ is countable.

The reader can refer to [CK73] for more details. We recall results in [Bek91] about the relationship between PFA$^+$ and Chang's Conjecture.

Definition 1.2.7. An ω_1 ϵ-chain of submodels of H_θ is a sequence $\langle N_\zeta \colon \zeta < \omega_1 \rangle$ such that

(1) $\langle N_\theta \colon \zeta < \eta \rangle \in N_{\eta+1}$
(2) $N_\zeta = \bigcup_{\xi<\zeta} N_\xi$ for ζ limit,
(3) N_ζ is a countable submodel of H_θ

Definition 1.2.8. We say $S \subseteq [H_\theta]^\omega$ reflects if there is an ω_1 ϵ-chain $\langle N_\zeta \colon \zeta < \omega_1 \rangle$ such that $\{\zeta \colon N_\zeta \in S\}$ is stationary.

The following theorem is a combination of Theorem 3.25 and Theorem 3.27 in [Bek91].

Theorem 1.2.9 (S. Todorcěvić). *If for all cardinals $\theta > \omega_1$, each stationary set in $[H_\theta]^\omega$ reflects, then Chang's Conjecture holds.* □

It is essentially folklore that PFA$^+$ implies that every stationary set in $[H_\theta]^\omega$ reflects for any cardinal $\theta > \omega_1$. Therefore we have,

Proposition 1.2.10 (S. Todorcěvić). PFA$^+$ *implies Chang's Conjecture.* □

For our application of Chang's Conjecture we need the following notion.

Definition 1.2.11. A sequence $\{X_\alpha : \alpha \in \lambda\}$ in $[\omega_1]^{\omega_1}$ is called a *strong chain* if for any α and β in λ such that, for $\alpha < \beta$, $X_\alpha \setminus X_\beta$ is finite and $X_\beta \setminus X_\alpha$ is uncountable.

Here is the theorem by P. Koszmider.

LEMMA 1.2.12. [Kos] *Under the Chang's Conjecture, there is no strong chain of length ω_2 in $[\omega_1]^{\omega_1}$.* □

We still need more consequences of PFA.

THEOREM 1.2.13 (THEOREM 3.16, [Bek91]). PFA *implies* $2^{\aleph_0} = \aleph_2$. □

A *weak Kurepa tree* is a tree of height ω_1 and cardinality $\aleph_1$ with more than $\aleph_1$ many $\aleph_1$ branches. The weak Kurepa Hypothesis (wKH) asserts that there is a weak Kurepa tree.

THEOREM 1.2.14 (THEOREM 7.10, [Bau84]). PFA *implies that every tree of height ω_1 and cardinality $\aleph_1$ is special and therefore wKH is false.* □

If G is a $\mathrm{Fn}(\omega, 2)$-generic over V, then we call $g = \bigcup G$ or $g^{-1}(0)$ a Cohen real over V. If A is a ground model subset of ω, then $G \cap \mathrm{Fn}(A, 2)$ is a $\mathrm{Fn}(A, 2)$-generic. Therefore we also call $g \restriction A$ or $g^{-1}(1) \cap A$ a Cohen real on A.

If $\mathfrak{M}$ is a model of a suitable fragment of ZFC (eg. $\mathrm{ZFC}^- = \mathrm{ZFC}$ - Power Set Axiom), we call a real $r \in {}^{\omega}2$ a Cohen generic real over $\mathfrak{M}$, if $\{p \in \mathrm{Fn}(\omega, 2) \colon p \subseteq r\}$ a generic filter on $\mathrm{Fn}(\omega, 2)$ over $\mathfrak{M}$.

We say that a family $\mathcal{A}$ of subsets of ω is *independent on a subset* B of ω if for any two disjoint finite subsets F and H of $\mathcal{A}$,

$$B \cap \left(\bigcap F \setminus \bigcup H\right) \text{ is infinte.}$$

We will need the following fact. The proof is a standard application of MA so we will omit it.

PROPOSITION 1.2.15 (MA). *Suppose $\mathfrak{M}$ is a model of ZFC^- with size $\aleph_1$. Then there are countably many Cohen generics over $\mathfrak{M}$, $\{r_n \colon n \in \omega\}$, such that for any $a \subseteq \omega$ in $\mathfrak{M}$, $\{r_n^{-1}(1) \colon n \in \omega\}$ is independent on a.* □

We will denote this by saying that $\{r_n : n \in \omega\}$ are mutually Cohen over $\mathfrak{M}$.

We will need to use the Open Coloring Axiom (OCA). The OCA is the following statement

If X is a separable metric space and

$$[X]^2 = K_0 \cup K_1$$

is a partition of $[X]^2$ with K_0 open, i.e. for any $\{x, y\} \in K_0$, there are open neighbourhoods U_0 and U_1 of x and y respectively, such that, for any $x' \in U_0$ and $y' \in U_1$, $\{x', y'\} \in K_0$.

Then *either*

(1) there exists an uncountable 0-homogeneous subset A of X, i.e. $[A]^2 \subseteq K_0$.

or

(2) X can be covered by countably many 1-homogeneous subsets, $\{A_n : n \in \omega\}$, i.e. for each n, $[A_n]^2 \subseteq K_1$.

THEOREM 1.2.16 (S. TODORCĚVIĆ, [Tod89]). PFA *implies* OCA.

We will need the following lemma to be able to apply OCA.

LEMMA 1.2.17. *Suppose* $T \subseteq \mathcal{P}(\omega)\times\mathcal{P}(\omega)$ *and* $K_0 = \{\{(a, b), (c, d)\} \in [T]^2 : a \cap d \neq \emptyset$ *or* $c \cap b \neq \emptyset\}$. *If* T *inherits the usual topology from* $\mathcal{P}(\omega)$, *then* K_0 *is an open set in* $[T]^2$ *with respect to this separable metric topology on* T.

Proof: Let σ be the topology on $\mathcal{P}(\omega)$ generated by

$$\{[p] : p \in \mathrm{Fn}(\omega, 2)\}$$

where $[p] = \{A \subseteq \omega : A \cap \operatorname{dom} p = p^{-1}(0)\}$. It is obvious that $(\mathcal{P}(\omega), \sigma)$ is homomorphic to the Cantor set. Hence $(\mathcal{P}(\omega) \times \mathcal{P}(\omega), \sigma \times \sigma)$ is separable metric space. Let τ be the subspace topology of T in $(\mathcal{P}(\omega)\times\mathcal{P}(\omega), \sigma\times\sigma)$. Let $x = (a, b)$ and $y = (c, d)$ and assume $\{x, y\} \in K_0$. We have to prove that there are open sets U and V containing x and y respectively such that, for any $x' \in U$ and $y' \in V$, $\{x', y'\} \in K_0$. Without loss of generality, we assume that $a \cap d$ is not empty. Choose $m \in a \cap d$. Let $p = \{\langle m, 1\rangle\}$. Since $x \neq y$, there are disjoint open sets U' and V' such that $x \in U'$ and $y \in V'$. Let $U = U' \cap ([p] \times \mathcal{P}(\omega))$ and $V = V' \cap (\mathcal{P}(\omega) \times [p])$. Clearly $\{\{x', y'\} \colon x' \in U \text{ and } y' \in V\} \cap [T]^2 \subseteq K_0$. □

An ideal $\mathcal{I}$ is a *P-ideal* if for any countable sequence $\{I_n : n \in \omega\}$ in $\mathcal{I}$, there exists I in $\mathcal{I}$, such that, for all $n \in \omega$, I_n is almost contained in I.

DEFINITION 1.2.18. A family $\{(a_\alpha, b_\alpha) \colon \alpha \in \omega_1\}$ is a *gap/* if for each $\alpha < \omega_1, \beta < \omega_1$, $a_\alpha \cap b_\beta$ is finite and $a_\alpha \cap b_\alpha$ is empty. For a set a, we say *a fills* the gap if for all α, $a_\alpha \subseteq^* a$ and $b_\alpha \cap a$ is finite. In this case we say that the gap can be filled.

DEFINITION 1.2.19 (DEFINITION 2.6, [Dow97]). A family $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$ is a Hausdorff-Luzin type family if for each $\alpha < \omega_1$ and $m \in \omega$,

$$\{\beta < \alpha : |a_\alpha \cap b_\beta| < m\} \text{ is finite.}$$

If, in addition, $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$ is a gap, then we call it a Hausdorff-Luzin type gap.

It is easy to see that the properties of being a Hausdorff-Luzin type family is absolute for models preserving ω_1. It is also easy to see that any Hausdorf-Luzin type gap can not be filled.

DEFINITION 1.2.20 (DEF 2.1, [Dow97]). Two ideals $\mathcal{I}$ and $\mathcal{J}$ can be *countably separated* if there are countably many sets X_n such that for each pair $(I, J) \in \mathcal{I} \times \mathcal{J}$, there is an n such that I is almost disjoint from X_n and J is almost contained in X_n. We will say that the ideals are *strongly countably separated* if the X_n's have the additional property that $X_n \cap I$ is finite for each $I \in \mathcal{I}$ and each $n \in \omega$ (i.e. $[X_n]^\omega \cap \mathcal{I} = \emptyset$ for each n).

If we say that a set of pairs $\{(I_\alpha, J_\alpha) : \alpha \in \omega_1\}$ are (strongly) countably separated then we mean that the ideal generated by family $\{I_\alpha : \alpha \in \omega_1\}$ is (strongly) countably separated from the one generated by $\{J_\alpha : \alpha \in \omega_1\}$.

We quote two results about separating ideals to finish this chapter.

LEMMA 1.2.21 (LEMMA 2.2, [Dow97]). *If $\mathcal{I}$ and $\mathcal{J}$ are countably separated and $\mathcal{I}$ is a P-ideal, then they are strongly countably separated.*

THEOREM 1.2.22 (THEOREM 2.3, [Dow97]). *Suppose PFA holds. If $\mathcal{I}$ and $\mathcal{J}$ are ideals of countable sets with $\mathcal{I}$ a P-ideal then either*

(1) *$\mathcal{I}$ and $\mathcal{J}$ can be countably separated,*

or

(2) *there are $\aleph_1$-generated subideals $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{J}' \subset \mathcal{J}$ which cannot be countably separated.*

CHAPTER 2

A Non Representable Boolean Algebra

In [BW82], the first open problem is to decide if each subalgebra of a representable algebra is also representable. Baumgartner and Weese proved in [BW82] that, under CH, any Boolean Algebra with cardinality $\leq \mathfrak{c}$ is representable. Therefore, under CH, the answer to their first question is positive. By assuming the existence of a supercompact cardinal, we will show that it is consistent that the answer to the above question is negative. In this chapter, we will construct a partitioner algebra with a non representable subalgebra. Then by applying a technical theorem we will prove that subalgebra is not representable. However, the proof of the technical theorem will take the next three chapters.

2.1. Definitions for Representable Algebras

Let $\mathcal{M}$ be a mad family on ω. For $a \subseteq \omega$, if for each $M \in \mathcal{M}$, either $M \subseteq^* a$ or $M \cap a =^* \emptyset$, then a is called a *partitioner* of $\mathcal{M}$. A partitioner a is a *non trivial partitioner* of $\mathcal{M}$, if a is not in $\mathcal{I}_{\mathcal{M}}$, the ideal generated by the finite subsets of ω and the mad family $\mathcal{M}$.

The set $\mathfrak{A}$ of all partitioners of $\mathcal{M}$ forms a subalgebra of $\mathcal{P}(\omega)$ which is sometimes denoted by $\mathfrak{A}_{\mathcal{M}}$. $\mathfrak{A}/\mathcal{I}_{\mathcal{M}}$ is called the *partitioner algebra* of $\mathcal{M}$. We also use $P_{\mathcal{M}}$ to denote the partitioner algebra of $\mathcal{M}$. For any non trivial partitioner A of

$\mathcal{M}$, we use $[A]$ to denote the corresponding element in $P_{\mathcal{M}}$, i.e. the equivalence class containing A.

A Boolean algebra $\mathcal{A}$ is called *representable* if it is isomorphic to a partitioner algebra of some mad family on ω. In this case, we say that the mad family represents $\mathcal{A}$.

Obviously, we can replace ω in the above definitions with any countably infinite set. Therefore when we mention a partitioner algebra we will not be concerned if the underlying set for the mad family is ω as long as it is a countable set.

2.2. Constructions

By Theorem 1.2.6, if there is a supercompact cardinal, then there is a model in which PFA$^+$ and $\diamondsuit(S^2_0)$ hold. Hence, from this chapterto chapter 5, we will assume that PFA$^+$ and $\diamondsuit(S^2_0)$ hold in V. Therefore by Theorem 1.2.13, 1.2.10 and 1.2.14 the following are true in our universe V:

(1) Chang's Conjecture;
(2) $\mathfrak{c} = \omega_2$;
(3) there is no weak Kurepa tree.

We now start the actual construction of the algebra $\mathfrak{B}$. We will spend considerable efforts to prove that it works. We will first construct a mad family $\mathcal{M}$ and an increasing mod finite sequence $\{a_\alpha \colon \alpha \in \omega_2\}$ of non trivial partitioners of $\mathcal{M}$. The sequence $\{a_\alpha \colon \alpha \in \omega_2\}$ will generate a maximal ideal on ω, i.e $\{\omega \setminus a_\alpha \colon \in \omega_2\}$ is a base for an ultrafilter on ω. It is easy to see that $\mathcal{P}_{\mathcal{M}} \times \mathcal{P}_{\mathcal{M}}$ is isomorphic to the partitioner algebra of mad family $\widetilde{\mathcal{M}}$ defined as follows. $\widetilde{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}'$, where $\mathcal{M}' = \{M \times \{0\} \colon M \in \mathcal{M}\}$. We will use $P_{\widetilde{\mathcal{M}}}$ instead of $\mathcal{P}_{\mathcal{M}} \times \mathcal{P}_{\mathcal{M}}$. $\mathfrak{B}$ will be, in some sense, generated by $\{a_\alpha, a_\alpha \times \{0\} \colon \alpha \in \omega_2\}$. Formally we let $\mathfrak{B}$ be the set of equivalence classes in $P_{\widetilde{\mathcal{M}}}$ of thoses partitioners x for which there in an $\alpha < \omega_2$ such that either x or its complement contained in $a_\alpha \cup \alpha \times \{0\}$, i.e.

$\mathfrak{B} = \{[x] \in P_{\widetilde{\mathcal{M}}} \colon \exists \alpha < \omega_2\ x \text{ or } (\omega \cup \omega \times \{0\}) \setminus x \subseteq_{I_{\widetilde{\mathcal{M}}}} a_\alpha \cup a_\alpha \times \{0\}\}$. Obviously ω is a nontrivial partitioner of $\widetilde{\mathcal{M}}$ which is not in $\mathfrak{B}$. Our motivation is to ensure that if $\mathfrak{B}$ is a partitioner algebra of some mad family on $\omega \cup \omega \times \{0\}$, then ω will actually be a non trivial partitioner of that mad family. This ensures that $\mathfrak{B}$ is not equal to the partitioner algebra. To ensure that $\mathfrak{B}$ is not representable, we will actually ensure that if a mad family $\mathcal{N}$ represents $\mathfrak{B}$, then some element analogous to $[\omega]$ will be a partitioner of $\mathcal{N}$.

Let us first fix an enumeration, $\{C_\alpha : \alpha \in \lim \omega_2\}$, of $[\omega]^\omega$ and choose a sufficiently large regular cardinal θ.

We will define for each $\alpha < \omega_2$, $\mathfrak{M}_\alpha$, $\mathcal{M}_\alpha$ and a_α such that $\{\mathfrak{M}_\alpha : \alpha < \omega_2\}$ is a $\in$-chain of elementary submodels of H_θ each with cardinality $\aleph_1$, $\{\mathcal{M}_\alpha : \alpha \in \omega_2\}$ is an increasing sequence of almost disjoint families on ω and $\{a_\alpha : \alpha \in \omega_2\}$ is an increasing, mod finite, sequence of infinite subsets of ω. Moreover, the following properties will hold.

(1) $\mathfrak{M}_0$ contains a $\diamondsuit(S^2_0)$ sequence and $\{C_\alpha : \alpha \in \lim \omega_2\}$ as elements, thus if $\alpha \in \mathfrak{M}_\alpha$, then $C_\alpha \in \mathfrak{M}_\alpha$.

(2) $[T_\alpha] \subseteq \mathfrak{M}_{\alpha+1}$, where T_α is the tree consisting of all countable coherent sequences in $\mathfrak{M}_\alpha$. The ordering on T_α is end extension, i.e. if $\mathcal{B}$ and $\mathcal{A}$ are in T_α and $\mathcal{A} = \{(c_\xi, d_\xi) : \xi < \beta\}$, then $\mathcal{B} \prec \mathcal{A}$ iff for some $\gamma \leq \beta$, $\mathcal{B} = \{(c_\xi, d_\xi) : \xi < \gamma\}$. $[T_\alpha]$ is the set of all $\aleph_1$ branches of T_α.

(3) $\forall \alpha \in \omega_2$, a_α is co-infinite and $\mathcal{M}_\alpha < a_\alpha$, i.e. $\forall M \in \mathcal{M}_\alpha.\ M \subseteq^* a_\alpha$

(4) For all α, $\alpha \in \mathfrak{M}_\alpha$, $\mathcal{M}_\alpha \in \mathfrak{M}_\alpha$, $a_\alpha \in \mathfrak{M}_\alpha$, and if $\alpha < \beta$, then $\mathcal{M}_\beta \cap \mathfrak{M}_\alpha = \mathcal{M}_\alpha$.

(5) For limit α, $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ and

$$\mathcal{M}_\alpha = \mathcal{M}_{\alpha+1} = \mathcal{M}_{\alpha+2} \text{ and } \mathcal{M}_{\alpha+n} = \mathcal{M}_{\alpha+3} (n \geq 3)$$

and if $a \subseteq \omega$ and $a \in \mathfrak{M}_{\alpha+2}$, then a is a partitioner of $\mathcal{M}_{\alpha+3} \setminus \mathcal{M}_\alpha$.

(6) For limit α, there exists $M \in \mathcal{M}_{\alpha+3}$ such that $C_\alpha \cap M \neq^* \emptyset$ and either $\omega \setminus C_\alpha \subseteq^* a_{\alpha+1}$ or $C_\alpha \subseteq^* a_{\alpha+1}$.

(7) For all $\alpha \in \omega_2$, $\omega \setminus a_\alpha$ is infinite and there exists, in $\mathfrak{M}_{\alpha+1}$, Cohen generic reals g_0^α and g_1^α over $\mathfrak{M}_\alpha$ such that

$$(g_0^\alpha)^{-1}(0) \cup (g_1^\alpha)^{-1}(1) \subseteq a_{\alpha+1}$$

It will be clear during the construction that most oft the conditions will be trivially satisfied. For example, to get condition (1) and (2), we just put the necessary sets into $\mathfrak{M}_\alpha$. Condition (6) ensures that $\{a_\alpha\colon \alpha < \omega_2\}$ will generate a maximal ideal and that $\bigcup_{\alpha<\omega_2} \mathcal{M}_\alpha$ is a mad family. The Cohen reals in Condition (7) are, in some sense to guarantee, that the mad family $\bigcup_{\alpha<\omega_2} \mathcal{M}_\alpha$ and its partitioner algebra are complicated enough to make sure that $\mathfrak{B}$ is not representable. Conditions (1) - (5) are mainly technical ones which will be used in the next three chapters.

We will actually do the induction along the non successor ordinals, i.e. for each limit α, we describe how to get $\{\mathfrak{M}_{\alpha+i} : i \in \omega\}$, $\{\mathcal{M}_{\alpha+i} : i \in \omega\}$ and $\{a_{\alpha+i} : i \in \omega\}$. Suppose we are at stage α and α is not a successor.

Case 1: $\alpha = 0$ Take any elementary submodel $\mathfrak{M}$ of H_θ as $\mathfrak{M}_0$ such that $\mathfrak{M}$ contains a $\diamondsuit(S_0^2)$ sequence and $\{C_\alpha\colon \alpha \in \lim \omega_2\}$ as elements. Let $\mathcal{M}_0 = \emptyset$. Let $a_0 \in \mathfrak{M}_0$ be any infinite and co-infinite subset of ω.

Case 2: $\alpha \in \lim \omega_2$ Let $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$. Since $|\mathcal{M}_\alpha| \leq \aleph_1$ and PFA holds, there exists $a_\alpha \subseteq \omega$ such that $\omega \setminus a_\alpha$ is infinite and

$$\forall \beta < \alpha, a_\beta \subseteq^* a_\alpha \ .$$

Let $\mathfrak{M}_\alpha$ be an elementary submodel of H_θ such that $|\mathfrak{M}_\alpha| \leq \aleph_1$ and $\{\mathfrak{M}_\beta, \beta < \alpha\}, \{\mathcal{M}_\beta, \beta < \alpha\}$ and $\{a_\beta : \beta < \alpha\}$ and $\{C_\alpha, a_\alpha\}$ are elements of $\mathfrak{M}_\alpha$. It is easy to see that condition (1)-(7) are satisfied. Next we define, by induction on i, $\{\mathfrak{M}_{\alpha+i}, \mathcal{M}_{\alpha+i}, a_{\alpha+i} : 0 < i \in \omega\}$.

Subcase 1 : $i = 1$ We have to find $\mathfrak{M}_{\alpha+1}$, $\mathcal{M}_{\alpha+1}$, $a_{\alpha+1}$ and g_0^α and g_1^α. Let

$$T_\alpha = \{\mathcal{A} \in \mathfrak{M}_\alpha : \mathcal{A} \text{ is a coherent sequence of countable length}\} .$$

Then T_α is a tree under the partial order $\prec$ defined above. Obviously T_α has cardinality not greater than $\aleph_1$ and height $\aleph_1$. By Theorem 1.2.14, weak Kurepa Hypothesis fails, thus T_α has at most $\aleph_1$ many $\aleph_1$ branches, i.e. $|[T_\alpha]| \leq \aleph_1$. By the Proposition 1.2.15, there are countably many Cohen reals over $\mathfrak{M}_\alpha$, say, $\{r_n : n \in \omega\}$ such that for any real $a \subseteq \omega$ in $\mathfrak{M}_\alpha$, $\{r_n^{-1}(0) \colon n \in \omega\}$ is independent on a.

Now let $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha$ and $\mathfrak{M}_{\alpha+1}$ be an elementary submodel of H_θ with cardinality $\aleph_1$ and $\{[T_\alpha], \{r_n : n \in \omega\}, \mathfrak{M}_\alpha\} \subseteq \mathfrak{M}_{\alpha+1}$. Note that since ω_1 is both a subset of $\mathfrak{M}_{\alpha+1}$ and an element of $\mathfrak{M}_{\alpha+1}$, the sets $[T_\alpha], \{r_n : n \in \omega\}$ and $\mathfrak{M}_\alpha$ are also subsets of $\mathfrak{M}_{\alpha+1}$. Since $a_\alpha \in \mathfrak{M}_\alpha$, $r_0^{-1}(0) \cup r_1^{-1}(1) \cup a_\alpha$ is co-infinite. If $\omega \setminus C_\alpha$ is almost contained in a_α, then we let $a_{\alpha+1}$ be $r_0^{-1}(0) \cup r_1^{-1}(1) \cup a_\alpha$. Otherwise, $a_\alpha \cup C_\alpha$ is co-infinite, therefore, $r_0^{-1}(0) \cup r_1^{-1}(1) \cup a_\alpha \cup C_\alpha$ is also co-infinite. Let $a_{\alpha+1} = r_0^{-1}(0) \cup r_1^{-1}(1) \cup a_\alpha \cup C_\alpha$. Let $g_0^\alpha = r_0$ and $g_1^\alpha = r_1$. Obviously, condition (1)-(7) are satisfied.

Subcase 2 : $i = 2$

We have to find $\mathfrak{M}_{\alpha+2}, \mathcal{M}_{\alpha+2}$, $a_{\alpha+2}$, $g_0^{\alpha+1}$ and $g_1^{\alpha+1}$. Let $\mathcal{M}_{\alpha+2} = \mathcal{M}_\alpha$. Take any elementary submodel $\mathfrak{M}$ of H_θ containing a sequence of reals $\{r_{1,n} : n \in \omega\}$ which are mutually Cohen generic over $\mathfrak{M}_{\alpha+1}$. Let $\mathfrak{M}$ also contain all the $\aleph_1$ branches $[T_{\alpha+1}]$ of $T_{\alpha+1}$ and $\mathfrak{M}_{\alpha+1}$. Let $\mathfrak{M}_{\alpha+2}$ be $\mathfrak{M}$. Let $a_{\alpha+2} = r_{1,0}^{-1}(0) \cup r_{1,1}^{-1}(1) \cup a_{\alpha+1}$. Let $g_0^{\alpha+1} = r_{1,0}$ and $g_1^{\alpha+1} = r_{1,1}$

Subcase 3: $i = 3$

We have to find $\mathfrak{M}_{\alpha+3}$, $\mathcal{M}_{\alpha+3}$, $a_{\alpha+3}$, $g_0^{\alpha+2}$, and $g_1^{\alpha+2}$.

Take an ultrafilter $\mathcal{V}$ on ω such that $\mathcal{V} \in \mathfrak{M}_{\alpha+2}$ and $\{\omega \setminus M : M \in \mathcal{M}_\alpha\} \subseteq \mathcal{V}$ and $C_\alpha \in \mathcal{V}$ in the case that, for all $M \in \mathcal{M}_\alpha$, $C_\alpha \cap M =^* \emptyset$. Again PFA implies there exists a pseudo-intersection M^* of $\mathcal{V} \cap \mathfrak{M}_{\alpha+2}$. Let

$$\mathcal{M}_{\alpha+3} = \mathcal{M}_{\alpha+2} \cup \{M^*\} .$$

Since M^* is a pseudo-intersection of $\mathcal{V} \cap \mathcal{M}_{\alpha+2}$, no subset of ω in $\mathfrak{M}_{\alpha+2}$ splits M^*. Clearly, M^* is not in $\mathfrak{M}_{\alpha+2}$.

Let $\{r_{2,n} : n \in \omega\}$ be mutually Cohen reals over $\mathfrak{M}_{\alpha+2}$. Let $\mathfrak{M}_{\alpha+3}$ be an elementary submodel of H_θ with cardinality $\aleph_1$ such that $\{\mathfrak{M}_{\alpha+2},\ M^*,\ \{r_{2,n} : n \in \omega\},\ [T_{\alpha+2}]\} \subseteq \mathfrak{M}_{\alpha+3}$

By the choice of M^*, it is easy to see that $a_{\alpha+2} \cup M^*$ is contained in some co-infinite subset of a in $\mathfrak{M}_{\alpha+2}$. Therefore, $a_{\alpha+2} \cup M^* \cup r_{2,0}^{-1}(0) \cup r_{2,1}^{-1}(1)$ is co-infinite. Let $a_{\alpha+3} = a_{\alpha+2} \cup M^* \cup r_{2,0}^{-1}(0) \cup r_{2,1}^{-1}(1)$ and $g_0^{\alpha+2} = r_{2,0}$ and $g_1^{\alpha+2} = r_{2,1}$.

Subcase 4: $i > 3$

In this case we let $\mathcal{M}_{\alpha+i} = \mathcal{M}_{\alpha+3}$ and simply choose $\mathfrak{M}_{\alpha+i} \prec H_\theta$ such that $\{\mathfrak{M}_{\alpha+i-1},\ [T_{\alpha+i-1}],\ \{r_{i-1,j} : j \in \omega\}\} \subseteq \mathfrak{M}_{\alpha+i}$, where $\{r_{i-1,j} : j \in \omega\}$ are reals which are mutually Cohen over $\mathfrak{M}_{\alpha+i-1}$. Let $a_{\alpha+i} = a_{\alpha+i-1} \cup r_{i-1,0}^{-1}(0) \cup r_{i-1,1}^{-1}(1)$ and $g_0^{\alpha+i-1} = r_{i-1,0}$ and $g_1^{\alpha+i-1} = r_{i-1,1}$.

Finally let $\mathcal{M} = \bigcup_{\alpha<\omega_2} \mathcal{M}_\alpha$.

2.3. Basic Facts and $\mathfrak{B}$

We first state some basic facts about what we have constructed.

FACT 2.3.1. *$\mathcal{M}$ is a mad family.*

Proof: This is obvious by (6). □

FACT 2.3.2. *$\{a_\alpha : \alpha \in \omega_2\}$ generates a maximal ideal on in $\mathcal{P}(\omega)$.*

Proof: By (6), for each subset a of ω, either a or $\omega \setminus a$ is contained in some a_β.

FACT 2.3.3. (1) *If $a \in \mathfrak{M}_\alpha$, then a is a partitioner of $\mathcal{M} \setminus \mathcal{M}_\alpha$.*

(2) *each a_α is a non trivial partitioner of $\mathcal{M}$.*

Proof: Since $\mathcal{M}_\alpha < a_\alpha$, (1) implies (2), so we prove (1). Let $M \in \mathcal{M} \setminus \mathcal{M}_\alpha$ and set β to be the minimal ordinal such that $M \in \mathcal{M}_\beta$. Then by (4), β must be a successor ordinal and β must be of the form $\xi + 3$ with ξ a limit. Since $\alpha < \beta = \xi + 3$, $a \in \mathfrak{M}_{\xi+2}$. By (4), a is a partitioner of $\mathcal{M}_{\xi+3} \setminus \mathcal{M}_\xi$ which contains M according to the choice of ξ. Therefore a does not split M. $\square$

Recall that if $\mathcal{M}$ is an ad family on a countable subset S , we use $\mathcal{I}_\mathcal{M}$ to denote the ideal on $\mathcal{P}(S)$ generated by $\mathcal{M}$.

We need some additional notation to define $\mathfrak{B}$ which will actually be defined using a partitioner algebra on $\omega \cup \omega \times \{0\}$.

We use a'_α to denote $a_\alpha \times \{0\}$. Let $\mathcal{M}' = \{M \times \{0\} : M \in \mathcal{M}\}$ and $\widetilde{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}'$. Obviously, $\widetilde{\mathcal{M}}$ is a mad family on $\omega \cup \omega \times \{0\}$ and $P_\mathcal{M} \times P_{\widetilde{\mathcal{M}}}$ is isomorphic to the partitioner algebra of $\widetilde{\mathcal{M}}$. Let $\mathfrak{A}$ denote the partitioner algebra of $\widetilde{\mathcal{M}}$.

Let $\widetilde{\mathfrak{B}} = \{a \subseteq \omega \cup \omega \times \{0\} : a$ is a partitioner of $\widetilde{\mathcal{M}}$ and $\exists\, \beta\ a \subseteq^* a_\beta \cup a'_\beta$ or $(\omega \cup \omega \times \{0\}) \setminus a \subseteq^* a_\beta \cup a'_\beta\}$.

CLAIM 2.3.1. $\widetilde{\mathfrak{B}}$ forms a subalgebra of $\mathcal{P}(\omega \cup \omega \times \{0\})$.

Proof: We verify that $\widetilde{\mathfrak{B}}$ is closed under the Boolean operations. Obviously, $a \in \widetilde{\mathfrak{B}}$ iff $\omega \setminus a \in \mathfrak{B}'$. We check that it is closed under the intersection operation. Let a and b be in $\widetilde{\mathfrak{B}}$. Let α and β witnesses that a and b are in $\widetilde{\mathfrak{B}}$ and without loss of generality, assume that $\alpha < \beta$. If $a \subset_{\mathcal{I}_{\widetilde{\mathcal{M}}}} a_\alpha \cup a'_\alpha$ or $b \subset_{\mathcal{I}_{\widetilde{\mathcal{M}}}} a_\beta \cup a'_\beta$, then $a \cap b \subset_{\mathcal{I}_{\widetilde{\mathcal{M}}}} a_\beta \cup a'_\beta$. Otherwise, $\omega \setminus a \subset_{\mathcal{I}_{\widetilde{\mathcal{M}}}} a_\alpha \cup a'_\alpha$ and $\omega \setminus b \subset_{\mathcal{I}_{\widetilde{\mathcal{M}}}} a_\beta \cup a'_\beta$, thus $\omega \setminus (a \cap b) \subset_{\mathcal{I}_{\widetilde{\mathcal{M}}}} a_\beta \cup a'_\beta$. Therefore β witnesses that $a \cap b$ is in $\widetilde{\mathfrak{B}}$. $\square$

Let $\mathfrak{B} = \widetilde{\mathfrak{B}}/\mathcal{I}_{\widetilde{\mathcal{M}}}$. Obviously, $\mathfrak{B}$ is the subalgebra of the partitioner algebra $\mathfrak{A}$ and we have the following.

FACT 2.3.4. *There does not exist* $a \in \widetilde{\mathfrak{B}}$ *such that, for all* $\alpha \in \omega_2$, $a_\alpha \setminus a \in \mathcal{I}_{\widetilde{\mathcal{M}}}$ *and* $a'_\alpha \cap a \in \mathcal{I}_{\widetilde{\mathcal{M}}}$ □

Since $\{a_\alpha : \alpha \in \omega_2\}$ generates a maximal ideal on ω, $\{\omega \setminus a_\alpha : \alpha \in \omega_2\}$ is a base for a P_{ω_2}-ultrafilter on ω. Let us call this ultrafilter z.

Let X, X_0 and X_1 denote the Stone spaces of $P_{\mathcal{M}} \times P_{\mathcal{M}}$, $P_{\mathcal{M}}$ and $P_{\mathcal{M}'}$ respectively. Then X is the topological union of X_0 and X_1. Let z_0 and z_1 be the copies of the P_{ω_2} point z in X_0 and X_1 respectively. In X we identify z_0 and z_1 to get a quotient space Y. Hence another way to describe $\mathfrak{B}$ is that it is the clopen algebra of the topological space Y.

We need some notations in the following discussion.

DEFINITION 2.3.5. Let $\mathcal{A}$ be a coherent sequence. We call $\mathcal{A}$ a *good coherent sequence* if $\mathcal{A}$ is of the following form:

$$\{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \rho\} \ ,$$

where $a_{\mu(\alpha)}$ are the non trivial partitioners of $\mathcal{M}$ in the above construction and b_α is also a nontrivial partitioner of $\mathcal{M}$ for all $\alpha \in \rho$. A good coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha < \rho\}$ is called a *reasonable sequence* if for all $\alpha < \rho$,

$$\mathcal{A}_\alpha = \{(b_\xi, a_{\mu(\xi)}) : \xi < \alpha\} \in \mathfrak{M}_{\mu(\alpha)} \ .$$

We use CSQ (GCSQ, RCSQ) to denote the set of all the coherent sequences (good coherent sequences, reasonable sequences) with countable length.

2.4. $\mathfrak{B}$ is not representable

As the title indicates, we prove the following theorem in this section.

THEOREM 2.4.1. *The Boolean Algebra* $\mathfrak{B}$ *is not representable.*

DEFINITION 2.4.2. Let F be a function from $[\omega \cup \omega \times \{0\}]^{\omega}$ to $[\omega]^{\omega}$ and let $\mathcal{N}$ be a mad family on ω. We say that F induces an isomorphism from $\mathfrak{B}$ to $P_{\mathcal{N}}$ if there is a isomorphism φ from $\mathfrak{B}$ to $P_{\mathcal{N}}$ such that for any partitioner of A of $\mathcal{M}'$ with $[A] \in \mathfrak{B}$, $F(A)$ is a non trivial partitioner of $\mathcal{N}$ and $\varphi[A] = [F(A)]$, i.e, the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{F} & F(A) \\ {\scriptstyle\varphi_0}\Big\downarrow & & {\scriptstyle\varphi_1}\Big\downarrow \\ [A] & \xrightarrow{\varphi} & \varphi([A]) \end{array}$$

Where $\varphi_0 \colon \widetilde{\mathfrak{B}} \to \mathfrak{B}$ and $\varphi_1 \colon \mathfrak{A}_{\mathcal{N}} \to P_{\mathcal{N}}$ are the natural maps.

It will be clear from the proof of Theorem 2.4.1 that the following Theorem essentially says that there is a coherent sequence $\mathcal{A}$ uniformized by some set X while the image of $\mathcal{A}$ under F can not be uniformized modulo $\mathcal{I}_{\mathcal{N}}$ by any subset of ω. The c_α's and e_α's are used to guarantee that no partitioner of $\mathcal{N}$ can exist as the image of X.

THEOREM 2.4.3. *If $\mathcal{N}$ is a mad family whose partitioner algebra is isomorphic to $\mathfrak{B}$, then there is a function $F \colon [\omega \cup (\omega \times \{0\})]^{\omega} \to [\omega]^{\omega}$ and a reasonable coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \omega_1\}$ such that F induces an isomorphism from $\mathfrak{B}$ to $P_{\mathcal{N}}$ and $\mathcal{A}$ is uniformized by some set $X \subseteq \omega$. Moreover, there exists a sequence $C = \{(c_{\alpha+k}, e_{\alpha+k}) \colon \alpha \in \lim \omega_1, 0 < k \le \omega\}$ and an increasing sequence $\Sigma = \{\sigma_\alpha : \alpha \in \omega_1\}$ in $\lim \omega_1$ such that c_ξ and e_ξ are disjoint non trivial partitioners of $\mathcal{N}$ and the following holds*

(1) $\forall \alpha \in \lim \omega_1, k,\ l \in \mathbb{N}$, *$c_{\alpha+k} \subseteq c_{\alpha+\omega}$ and $e_{\alpha+k} \subseteq e_{\alpha+\omega}$ and $k < l$ implies*

$c_{\alpha+k}$, $e_{\alpha+k}$, $c_{\alpha+l}$ and $e_{\alpha+l}$ are mutually disjoint and following holds.

$$\begin{aligned}
c_{\alpha+\omega} &=_{\mathcal{I}_{\mathcal{N}}} F(b_{\alpha+\omega}) \\
e_{\alpha+\omega} &=_{\mathcal{I}_{\mathcal{N}}} F(a_{\mu(\alpha+\omega)}) \setminus F(b_{\alpha+\omega}) \\
c_{\alpha+k} &=_{\mathcal{I}_{\mathcal{N}}} F(b_{\alpha+k}) \setminus F(b_{\alpha+k-1}) \\
e_{\alpha+k} &=_{\mathcal{I}_{\mathcal{N}}} F(a_{\mu(\alpha+k)}) \setminus (F(a_{\mu(\alpha+k-1)}) \cup F(b_{\alpha+k}))
\end{aligned}$$

(2) $\forall k \in \mathbb{N}$,

$$\{(c_{\sigma_\alpha+k}, e_{\sigma_\alpha+k}) \colon \alpha \in \omega_1\}$$

is a Hausdorff-Luzin type family.

Note that the role of the sequence C is simply to ensure that there is no suitable real which uniformizes the image of $\mathcal{A}$ modulo $\mathcal{I}_{\mathcal{N}}$. In the theorem 2.4.4 below, we will obtain such a coherent sequence in some proper forcing extension. It turns out that it is very difficult to get such proper forcing notion. Then, since we are using PFA to produce such coherent sequence in the ground model, we are only allowed to use ω_1 many dense subsets to determine the fact no suitable subsets of ω uniformizes the image of $\mathcal{A}$ modulo $\mathcal{I}_{\mathcal{N}}$. However, a Hausdorff-Luzin type family can be determined via ω_1 many dense subsets. We need a notation to simplify the following discussion.

Let $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \lambda\}$ be a good coherent sequence with λ a limit ordinal such that $\lambda \neq \beta+\omega$ for any β. Suppose F induces an isomorphism from $\mathfrak{B}$ to some partitioner algebra $P_{\mathcal{N}}$. Then the derived sequence $C(\mathcal{A}, F) = \{(c_{\alpha+k}, e_{\alpha+k}) \colon \alpha \in$

$\lim \lambda$ and $0 < k \leq \omega\}$ of $\mathcal{A}$ and F is defined by the following formulas,

$$\begin{aligned}
d_{\alpha+\omega} &= F(a_{\mu(\alpha+\omega)}) \\
c_{\alpha+\omega} &= F(b_{\alpha+\omega}) \cap d_{\alpha+\omega}, \\
e_{\alpha+\omega} &= d_{\alpha+\omega} \setminus c_{\alpha+\omega} \\
d_{\alpha+k} &= (F(a_{\mu(\alpha+k)}) \setminus \bigcup_{i<k} F(a_{\mu(\alpha+i)})) \cap d_{\alpha+\omega}\ , \\
c_{\alpha+k} &= c_{\alpha+\omega} \cap d_{\alpha+k} \\
e_{\alpha+k} &= d_{\alpha+k} \setminus c_{\alpha+k}
\end{aligned}$$

It is obvious that the derived sequence satisfies (1) in the Theorem 2.4.3.

THEOREM 2.4.4. *If $\mathcal{N}$ is a mad family whose partitioner algebra is isomorphic to $\mathfrak{B}$, then there exist a function F which induces an isomorphism from $\mathfrak{B}$ to $P_{\mathcal{N}}$ and a proper forcing notion $\mathbb{D}$ such that, in $V^{\mathbb{D}}$, there exist a good coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \omega_1\}$ and a sequence $\Sigma = \{\sigma_\alpha : \alpha \in \omega_1\}$ in* $\lim \omega_1$ *such that the following are true.*

- **(i):** *For all $\alpha \in \omega_1$, $\mathcal{A}_\alpha \in$ RCSQ and $\mathcal{A}$ can be uniformized by some subset X of ω.*
- **(ii):** *F, $\mathcal{A}$, $C(\mathcal{A}, F)$ and Σ satisfy the conclusions of Theorem* 2.4.3.

*Where $C(\mathcal{A}, F) = \{(c_{\alpha+k}, e_{\alpha+k})\colon \alpha \in$ $\lim \omega_1$ *and* $0 < k \leq \omega\}$ *is the* derived *sequence of $\mathcal{A}$ and F*

We first complete the proof of Theorem 2.4.1 from Theorem 2.4.3. The proof of Theorem 2.4.3 from Theorem 2.4.4 is a standard dense sets argument. We will give that proof at the end of this chapter. The reader can choose to skip it, which will not result in any difficulty in reading other parts of this thesis. However, the proof of Theorem 2.4.4 will occupy from Chapter 3 to Chapter 5.

Proof of Theorem 2.4.1: Suppose that $\mathfrak{B}$ is representable. We will derive a contradiction. Let $\mathcal{N}$ be a mad family which represents $\mathfrak{B}$. Let F, $\mathcal{A}$, C, Σ be as in Theorem 2.4.3. Let $\mu = \sup\{\mu(\beta)\colon \beta < \omega_1\}$. Observe that $\{\mathcal{A}_\alpha\colon \alpha < \omega_1\} \in [T_\mu]$. so $\mathcal{A} \in \mathfrak{M}_{\mu+1}$. By elementarity we can assume that $X \in \mathfrak{M}_{\mu+1}$. We can also assume that that $X \subseteq a_\mu$. Otherwise we use $X \cap a_\mu$ to replace X. Let $Y = F(X)$.

Since $X \in \mathfrak{M}_{\mu+1}$, it follows from Fact 2.3.3 that X does not split any $M \in \mathcal{M}\backslash\mathcal{M}_{\mu+1}$. However for any $M \in \mathcal{M}_{\mu+1}$, there exists $\alpha < \omega_1$ such that $M \subseteq^* a_{\mu(\alpha)}$. Since X uniformizes $\mathcal{A}$, $X \cap a_{\mu(\alpha)} =^* b_\alpha$ and recall that b_α is a partitioner of $\mathcal{M}$. Therefore X does not split M. We have proven that X is a partitioner of $\mathcal{M}$. Furthermore, since $X \subseteq a_{\mu+1}$, $X \in \widetilde{\mathfrak{B}}$, i.e $[X] \in \mathfrak{B}$.

Therefore Y must be a non-trivial partitioner of $\mathcal{N}$.

For $\alpha \in \omega_1$, since X uniformizes $\mathcal{A}$ and F induces an isomorphism from $\mathfrak{B}$ to $P_{\mathcal{N}}$,

$$F(X) \cap F(a_{\mu(\alpha)}) =_{\mathcal{I}_{\mathcal{N}}} F(b_\alpha)\ .$$

Therefore, $c_{\alpha+\omega} \subseteq_{\mathcal{I}_{\mathcal{N}}} Y$ and $e_{\alpha+\omega} \cap Y =_{\mathcal{I}_{\mathcal{N}}} \emptyset$.

For each $k \in \mathbb{N}$, if $c_{\alpha+k} \setminus Y$ is infinite, then there exists $N_k \in \mathcal{N}$ such that $N_k \subseteq^* c_{\alpha+k} \setminus Y \subseteq c_{\alpha+\omega} \setminus Y$. Since $c_{\alpha+k}\,(k \in \omega)$ are mutually disjoint, the N_k's are distinct. Since $c_{\alpha+\omega} \subseteq_{\mathcal{I}_{\mathcal{N}}} Y$, there are at most finitely many such N_k. Therefore, there are at most finitely many k such that $c_{\alpha+k} \not\subseteq^* Y$. Similarly there are only finitely many k such that $e_{\alpha+k} \cap Y \neq^* \emptyset$. Therefore we can find an uncountable subset Γ of ω_1 and $k \in \mathbb{N}$ and finite sets H, K such that for all $\alpha \in \Gamma$,

$$c_{\sigma_\alpha+k} \setminus Y = H \text{ and } e_{\sigma_\alpha+k} \cap Y = K\ .$$

Let $m > \max(H \cup K)$, and $\alpha \in \Gamma$ such that $\alpha \cap \Gamma$ is infinite. Then, for all $\beta \in \alpha \cap \Gamma$, $c_{\sigma_\alpha+k} \cap e_{\sigma_\beta+k} \subseteq m$. Therefore $\{\beta < \alpha\colon c_{\sigma_\alpha+k} \cap e_{\sigma_\beta+k} \subseteq m\}$ is infinite. But according to (2) of Theorem 2.4.3, $\{(c_{\sigma_\alpha+k}, e_{\sigma_\alpha+k})\colon \alpha \in \omega_1\}$ is a Hausdorff-Luzin type family which is the contradiction we seek. $\square$

Finally we prove Theorem 2.4.3 from Theorem 2.4.4. We use PFA to show that there exist $\mathcal{A}$, X, $\{\sigma_\alpha : \alpha \in \omega_1\}$ and a derived sequence $\{(c_{\alpha+k}, e_{\alpha+k}) : \alpha \in \lim\omega_1$ and $0 < i \leq \omega\}$ of F and $\mathcal{A}$, all in the ground model V, such that the conclusions of Theorem 2.4.4 are satisfied. The proof is tedious but standard.

Proof of Theorem 2.4.3: Let $\dot{\mathcal{A}}$, $\dot{X}$, $\dot{\Sigma}$, $\dot{C}$ be the $\mathbb{D}$-names for $\mathcal{A}$, X, Σ and C such that $\mathbf{1}$ forces that $\dot{\mathcal{A}}$, $\dot{X}$, $\dot{\Sigma}$, and $\dot{C}$ have the properties stated in the Theorem 2.4.4. Let $\dot{\mathcal{S}}$ be a $\mathbb{D}$-name for the sequence $\{\dot{\mathcal{A}}_\alpha : \alpha < \omega_1\}$, i.e. $\dot{\mathcal{S}}$ is a $\mathbb{D}$-name for a function from ω_1 to RCSQ. To simplify the notation we will write x in the place of $\check{x}$.

We assume

$$\dot{\mathcal{S}} = \bigcup\{\{(\alpha, S)\} \times \pi^{\dot{\mathcal{S}}}_{\alpha,S} : \alpha \in \omega_1,\ S \in \text{RCSQ and } l(S) = \alpha\}\ ,$$

where $\pi^{\dot{\mathcal{S}}}_{\alpha,S}$ is a subset of $\mathbb{D}$. We can assume that $\pi^{\dot{\mathcal{S}}}_{\alpha,S}$ is regular open, i.e. for $q \in \mathbb{D}$, if $\pi^{\dot{\mathcal{S}}}_{\alpha,S}$ is predense below q, then q is in $\pi^{\dot{\mathcal{S}}}_{\alpha,S}$. We can do so because if we replace $\pi^{\dot{\mathcal{S}}}_{\alpha,S}$ by $\{q : \pi^{\dot{\mathcal{S}}}_{\alpha,S}$ is predense below $q\}$ we get a $\mathbb{D}$-name $\dot{\mathcal{S}}_1$ such that $\mathbf{1} \Vdash \dot{\mathcal{S}}_1 = \dot{\mathcal{S}}$.

Since $\mathbf{1} \Vdash$ "$\dot{\mathcal{S}} : \omega_1 \to \text{RCSQ}$ and $\alpha < \beta \to \dot{\mathcal{S}(\alpha)} < \dot{\mathcal{S}(\beta)}$". Obviously we have

(i): $\forall \alpha \in \omega_1$ and $S_1, S_2 \in \text{RCSQ}$

$$S_1 \neq S_2 \to \pi^{\dot{\mathcal{S}}}_{\alpha,S_1} \perp \pi^{\dot{\mathcal{S}}}_{\alpha,S_2}\ ,$$

(ii): $\forall \alpha$,

$$D^{\dot{\mathcal{S}}}_\alpha = \bigcup\{\pi^{\dot{\mathcal{S}}}_{\alpha,S} : S \in \text{RCSQ}\}$$

is dense in $\mathbb{D}$,

(iii): $\forall \alpha < \beta \in \omega_1$,

$$E^{\dot{\mathcal{S}}}_{\alpha,\beta} = \bigcup\{\pi^{\dot{\mathcal{S}}}_{\alpha,S} \cap \pi^{\dot{\mathcal{S}}}_{\beta,S'} : S \prec S' \in \text{RCSQ}\}$$

is dense in $\mathbb{D}$.

It is also obvious that conditions (i)-(iii) imply above forcing statement.

Similarly we can assume that

$$\dot{\Sigma} = \bigcup\{\{(\alpha,\beta)\} \times \pi^{\dot{\Sigma}}_{\alpha,\beta} \colon \alpha \in \omega_1 \text{ and } \beta \in \lim \omega_1\} \ ,$$

where each $\pi^{\dot{\Sigma}}_{\alpha,\beta}$ is regular open and

(1) $\forall \alpha < \omega_1,\ \beta_1,\ \beta_2 \in \omega_1$,

$$\beta_1 \neq \beta_2 \to \pi^{\dot{\Sigma}}_{\alpha,\beta_1} \perp \pi^{\dot{\Sigma}}_{\alpha,\beta_2} \ ,$$

(2) $\forall \alpha$,

$$D^{\dot{\Sigma}}_{\alpha} = \bigcup\{\pi^{\dot{\Sigma}}_{\alpha,\beta} \colon \beta \in \lim \omega_1\}$$

is dense,

(3) $\forall \alpha < \beta \in \omega_1$,

$$E^{\dot{\Sigma}}_{\alpha,\beta} = \bigcup\{\pi^{\dot{\Sigma}}_{\alpha,\xi} \cap \pi^{\dot{\Sigma}}_{\beta,\eta} \colon \xi < \eta \in \lim \omega_1\}$$

is dense in $\mathbb{D}$.

Let $\dot{H}$ be a name for a function from $\omega_1 \times \mathbb{N} \times \mathbb{N}$ to $[\omega_1]^{<\omega}$ such that, for all $(\alpha, k, n) \in \omega_1 \times \mathbb{N} \times \mathbb{N}$,

$$\mathbf{1} \Vdash \text{“} \dot{H}(\alpha,k,n) = \{\beta \in \alpha \colon |\dot{c}_{\sigma_\alpha + k} \cap \dot{e}_{\sigma_\beta + k}| \leq n\} \text{ is finite”} \ .$$

We assume that

$$\dot{H} = \bigcup\{\{(t,J)\} \times \pi^{\dot{H}}_{t,J} \colon t \in \omega_1 \times \mathbb{N} \times \mathbb{N},\ J \in [\omega_1]^{<\omega_1}\} \ ,$$

where each $\pi^{\dot{H}}_{t,j}$ is a regular open subset of $\mathbb{D}$. Again, we have that

(i): $\forall t \in \omega_1 \times \mathbb{N} \times \mathbb{N},\ \forall J_1, J_2 \in [\omega_1]^{<\omega}$,

$$J_1 \neq J_2 \to \pi^{\dot{H}}_{t,J_1} \perp \pi^{\dot{H}}_{t,J_2} \ .$$

(ii): $\forall t \in \omega_1 \times \mathbb{N} \times \mathbb{N}$,

$$D_t^{\dot{H}} = \bigcup \{\pi_{t,J}^{\dot{H}} \colon J \in [\omega_1]^{<\omega}\}$$

is dense in $\mathbb{D}$,

If $S = \{(b_\beta, a_{\mu(\beta)}) \colon \beta \leq \xi + \omega\} \in \mathrm{RCSQ}$ with length $\xi + \omega + 1$ and $\eta + \omega \leq \xi$, we say that (η, ξ, S) is (k,n)-big if

$$|c_{\xi+k} \cap e_{\eta+k}| > n \ .$$

where

$$\begin{aligned} c_{\xi+k} &= F(b_{\xi+\omega}) \cap F(a_{\mu(\xi+\omega)}) \cap F(a_{\mu(\xi+k)}) \setminus \bigcup_{i<k} F(a_{\mu(\xi+i)}) \\ e_{\eta+k} &= F(a_{\mu(\eta+\omega)}) \cap (F(a_{\mu(\eta+k)}) \setminus \bigcup_{i<k} F(a_{\mu(\eta+i)})) \setminus F(b_{\eta+\omega}) \end{aligned}$$

For $\alpha, \xi, \beta, \eta \in \omega$ and $S \in \mathrm{RCSQ}$, let $E(\alpha, \xi, \beta, \eta, S)$ be

$$\pi_{\alpha,\xi}^{\dot{\Sigma}} \cap \pi_{\beta,\eta}^{\dot{\Sigma}} \cap \pi_{\xi+\omega+1,S}^{\dot{\mathcal{S}}} \ .$$

Let $D_\beta(\alpha, k, n, J)$ be the following set

$$\begin{aligned} (\pi_{(\alpha,k,n),J}^{\dot{H}})^\perp \cup \bigcup \{E(\alpha, \xi, \beta, \eta, S) \colon \eta < \xi, \\ S \in \mathrm{RCSQ}, l(S) = \xi + \omega + 1 \text{ and } (\xi, \eta, S) \text{ is } (k,n)\text{-big } \} \end{aligned}$$

Claim: For all $\alpha \in \omega_1$, $k, n \in \mathbb{N}$, $J \in [\omega_1]^{<\omega}$, if $\beta \in \alpha \setminus J$, then

$$D_\beta(\alpha, k, n, J) \text{ is dense in } \mathbb{D} \ .$$

Proof of the claim: Let p be an arbitrary element of $\mathbb{D}$. Let $t = (\alpha, k, n)$. Since $D_t^{\dot{H}}$ is dense, there exists $q \in D_t^{\dot{H}}$ such that $q \leq p$. Now choose $J' \in [\omega_1]^{<\omega}$ such that $q \in \pi_{t,J'}^{\dot{H}}$. If $J' \neq J$, then $q \perp \pi_{t,J}^{\dot{H}}$, (hence $q \in D_\beta(\alpha, k, n, J)$) otherwise, by the choice of q,

$$q \Vdash \text{“} J = \dot{H}(\alpha, k, n)\text{”} \ .$$

Therefore,

$$q \Vdash \text{“} |\dot{c}_{\sigma_\alpha + k} \cap \dot{e}_{\sigma_\beta + k}| > n \text{”} .$$

Since $E^{\dot{\Sigma}}_{\beta,\alpha}$ is dense, choose $r_0 \le q$ such that $r_0 \in E^{\dot{\Sigma}}_{\beta,\alpha}$. Therefore there are $\eta < \xi \in \lim \omega_1$, such that

$$r_0 \in \pi^{\dot{\Sigma}}_{\alpha,\xi} \cap \pi^{\dot{\Sigma}}_{\beta,\eta} .$$

Since $D^{\dot{\mathcal{S}}}_{\xi+\omega+1}$ is dense , there is an r below r_0 such that $r \in D^{\dot{\mathcal{S}}}_{\xi+\omega+1}$. Fix $S \in$ RCSQ such that $r \in \pi^{\dot{\mathcal{S}}}_{\xi+\omega+1,S}$. Since $\pi^{\dot{\Sigma}}_{\alpha,\xi}$, $\pi^{\dot{\Sigma}}_{\beta,\eta}$ are both open,

$$r \in \pi^{\dot{\Sigma}}_{\alpha,\xi} \cap \pi^{\dot{\Sigma}}_{\beta,\eta} \cap \pi^{\dot{\mathcal{S}}}_{\xi+\omega+1,S} = E(\alpha, \xi, \beta, \eta, S) .$$

By the choice of r, we have

$$r \Vdash \text{“} \dot{\sigma}_\alpha = \xi \text{ and } \dot{\sigma}_\beta = \eta \text{ and } \dot{\mathcal{A}}_{\xi+\omega+1} = S \text{”} .$$

Let $S = \{(b_\rho, a_{\mu(\rho)}) \colon \rho \le \xi + \omega\}$. Since

$$q \Vdash \text{“} |\dot{c}_{\sigma_\alpha + k} \cap \dot{e}_{\sigma_\beta + k}| > n \text{”} ,$$

we have

$$r \Vdash \text{“} |\dot{c}_{\sigma_\alpha + k} \cap \dot{e}_{\sigma_\beta + k}| > n \text{”} .$$

However, $r \Vdash$ “ $\dot{c}_{\sigma_\alpha+k} = c_{\xi+k}$ and $\dot{e}_{\sigma_\beta+k} = e_{\eta+k}$”. Where $c_{\xi+k}$ and $e_{\eta+k}$ are from the derived sequence of F and S. Therefore

$$|c_{\xi+k} \cap e_{\eta+k}| > n .$$

This completes the proof that (ξ, η, S) is (k, n)-big. Therefore, $r \in D_\beta(\alpha, k, n, J)$ and finishes the proof of the claim.

We assume $\dot{X} = \{\{n\} \times \pi^{\dot{X}}_n : n \in \omega\}$ and again we assume that each $\pi^{\dot{X}}_n$ is regular open. Therefore, $p \Vdash$ “$n \in \dot{X}$” if and only if $p \in \pi^{\dot{X}}_n$. For each n let $D_n = \pi^{\dot{X}}_n \cup (\pi^{\dot{X}}_n)^\perp$. Let $E_n = \bigcup_{i \ge n} \pi^{\dot{X}}_i$. Obviously each D_n is dense in $\mathbb{D}$. Since $\mathbb{D}$ forces that $\dot{X}$ is infinite, each E_n is dense in $\mathbb{D}$.

Let $\dot{h}$ be a $\mathbb{D}$-name for a function from ω_1 to ω such that for each $\alpha \in \omega_1$,

$$\mathbf{1} \Vdash \text{“}(\dot{X} \cap \dot{a}_{\mu(\alpha)}) \triangle \dot{b}_\alpha \subseteq \dot{h}(\alpha)\text{”} .$$

As usual, we assume that

$$\dot{h} = \bigcup \{ \{(\alpha, n)\} \times \pi^{\dot{h}}_{\alpha,n} \colon \alpha \in \omega_1,\, n \in \omega \} ,$$

where each $\pi^{\dot{h}}_{\alpha,n}$ is regular open and

(i): $\forall \alpha \in \omega_1,\, n_1, n_2 \in \omega,$

$$n_1 \neq n_2 \to \pi^{\dot{h}}_{\alpha,n_1} \perp \pi^{\dot{h}}_{\alpha,n_2} ,$$

and

(ii): $\forall \alpha \in \omega_1,$

$$D^{\dot{h}}_\alpha = \bigcup \{ \pi^{\dot{h}}_{\alpha,n} \colon n \in \omega_1 \} \text{ is dense} \quad .$$

Obviously for any $p \in \mathbb{D}$, $\alpha \in \omega_1$ and $n \in \omega$,

$$p \Vdash \text{“}\dot{h}(\alpha) = n\text{”} \leftrightarrow p \in \pi^{\dot{h}}_{\alpha,n} .$$

Let

$$\begin{aligned}
\mathcal{D}_1 =& \{ D^{\dot{\mathcal{S}}}_\alpha \colon \alpha \in \omega_1 \} \cup \{ E^{\dot{\mathcal{S}}}_{\alpha,\beta} \colon \alpha < \beta \in \omega_1 \} \\
\mathcal{D}_2 =& \{ D^{\dot{\Sigma}}_\alpha \colon \alpha \in \omega_1 \} \cup \{ E^{\dot{\Sigma}}_{\alpha,\beta} \colon \alpha < \beta \in \omega_1 \} \\
\mathcal{D}_3 =& \{ D_\beta(\alpha, k, n, J) \colon \alpha \in \omega_1, k, n \in \omega, J \in [\omega_1]^{<\omega}, \beta \in \alpha \setminus J \} \\
& \cup \{ D^{\dot{H}}_t \colon t \in \omega_1 \times \mathbb{N} \times \mathbb{N} \} \\
\mathcal{D}_4 =& \{ D^{\dot{h}}_\alpha \colon \alpha \in \omega_1 \} \\
\mathcal{D}_5 =& \{ E_n : n \in \omega \} \\
\mathcal{D}_6 =& \{ D_n : n \in \omega \}
\end{aligned}$$

Let $\mathcal{D} = \{ \bigcap \mathcal{D}' \colon \mathcal{D}' \in [\bigcup_{i=1}^{6} \mathcal{D}_i]^{<\omega} \}$.

By PFA, there is a filter $G \subseteq \mathbb{D}$ such that G meets every dense set in $\mathcal{D}$.

It is easy to see that $\mathrm{val}(\dot{\mathcal{S}}, G)$ is an increasing sequence in $(\mathrm{RCSQ}, \prec)$. Let $\mathrm{val}(\dot{\mathcal{S}}, G)$ be $\{\mathcal{A}_\alpha \colon \alpha \in \omega_1\}$ and $\mathcal{A} = \bigcup_{\alpha\in\omega_1} \mathcal{A}_\alpha = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha < \omega_1\}$. Obviously $\mathcal{A}$ is a good coherent sequence. Since G is $\mathcal{D}_2$ generic, $\mathrm{val}(\dot{\Sigma}, G)$ is an increasing sequence in $\lim\omega_1$ with length ω_1, so let $\mathrm{val}(\dot{\Sigma}, G)$ be $\{\sigma_\alpha \colon \alpha \in \omega_1\}$. Let $X = \mathrm{val}(\dot{X}, G)$ and $h = \mathrm{val}(\dot{h}, G)$. Finally let $C = \{(c_{\alpha+k}, e_{\alpha+k}) \colon \alpha \in \lim\omega_1, 0 < k \leq \omega\}$ be the derived sequence of F and $\mathcal{A}_{\omega_1}$. It is easy to see that X is an infinite subset of ω and $h \colon \omega_1 \to \omega$.

We have to show that X uniformizes $\mathcal{A}$ and that $\mathcal{A}$, C, Σ satisfies (1) and (2) in the Theorem 2.4.3.

First we prove that X uniformizes $\mathcal{A}$. Let $\alpha \in \omega_1$, we have to prove that $X \cap a_{\mu(\alpha)} =^* b_\alpha$.

Since G is a $\mathcal{D}$-generic, there exists $p \in G \cap D^{\dot{h}}_\alpha \cap D^{\dot{\mathcal{S}}}_{\alpha+\omega+1}$. It is then obvious that $p \in \pi^{\dot{\mathcal{S}}}_{\alpha+\omega+1,S}$, where $S = \mathcal{A}_{\alpha+\omega+1}$and so $p \Vdash$ "$\dot{a}_{\mu(\alpha)} = a_{\mu(\alpha)}$ and $\dot{b}_\alpha = b_\alpha$". Since $p \in D^{\dot{h}}_\alpha$, there exists a unique $n \in \omega$ such that $p \in \pi^{\dot{h}}_{\alpha,n}$. Obviously $h(\alpha) = n$. Since $p \Vdash$ "$\dot{h}(\alpha) = n$", we have that $p \Vdash$ "$(\dot{X} \cap a_{\mu(\alpha)}) \triangle b_\alpha \subseteq n$". Therefore, for all $m \in a_{\mu(\alpha)} \setminus n$, $m \in b_\alpha$ if and only if $p \Vdash$ "$m \in \dot{X}$", i.e. $p \in \pi^{\dot{X}}_m$. Since any $q \leq p$ forces what p forces, it follows that, for the above m, $m \in b_\alpha$ if and only if $q \Vdash$ "$m \in \dot{X}$". Thus $m \notin b_\alpha$ will imply $p \perp \pi^{\dot{X}}_m$. Therefore $p \in \pi^{\dot{X}}_m$ if and only if $G \cap \pi^{\dot{X}}_m$ is not empty and this is equivalent to m being in X. Thus we proved that $(X \cap a_{\mu(\alpha)}) \triangle b_\alpha \subseteq n$.

Next we check that the properties (1) and (2) in Theorem 2.4.3 hold. (1) follows immediately from the definition of C. We are only left to verify (2). Let $\alpha \in \omega_1$ and $k \in \mathbb{N}$; we have to show that, for all $n \in \omega$,

$$\{\beta < \alpha \colon |c_{\sigma_\alpha+k} \cap e_{\sigma_\beta+k}| \leq n\} \text{ is finite .}$$

Let $t = (\alpha, k, n)$. Since $D^{\dot{H}}_t \cap G$ is not empty, pick $J \in [\omega_1]^{<\omega}$ and $q \in G \cap \pi^{\dot{H}}_{t,J}$. Now for any $\beta \in \alpha \setminus J$, pick $r \in G \cap D_\beta(\alpha, k, n, J)$. Since $G \cap \pi^{\dot{H}}_{t,J}$ is not empty,

r is compatible with some element in $\pi^{\dot{H}}_{t,J}$. Therefore, there exists $\eta < \xi \in \lim \omega_1$ and $S \in$ RCSQ such that (ξ, η, S) is (k,n)-big, and $r \in E(\alpha, \xi, \beta, \eta, S)$. Since $r \in G$ and by the definition of $E(\alpha, \xi, \beta, \eta, S)$, it must be the case that $\xi = \sigma_\alpha$, $\eta = \sigma_\beta$ and $S = \mathcal{A}_{\xi+\omega+1}$. Consequently, $\{(c_{\gamma+i}, e_{\gamma+i}) \in C : \gamma + i \leq \xi + \omega + 1\}$ is the derived sequence of F and S. Now it is easy to see that

$$|c_{\sigma_\alpha+k} \cap e_{\sigma_\beta+k}| = |c_{\xi+k} \cap e_{\eta+k}| > n \ .$$

Thus we proved that $\{\beta < \alpha \colon |c_{\sigma_\alpha+k} \cap e_{\sigma_\beta+k}| \leq n\}$ is contained in the finite set J. We are done. $\square$

For a quick reading of the proof for Theorem 2.4.3, the reader can skip next two chapters and read Chapter 5 directly. However, the theorems in Chapter 3 and Chapter 4 are not just purely technical, they actually reflect the motivations behind the proof for Theorem 2.4.3.

CHAPTER 3

Separating Mad Families

In this chapter we will prove Theorem 3.0.1, which will be the foundation for the whole proof of Theorem 2.4.3.

We are motivated by the picture that either the $\{N_{i,j}\}$'s or J_n's listed therein are like the columns in $\omega \times \omega$ and the $F(a_\alpha)$'s are like the points below a graph of a function. By the construction of $\mathfrak{B}$, if F induces an isomorphism from $\mathfrak{B}$ to some partitioner algebra, then $\{(F(a_\alpha), F(a'_\alpha))\colon \alpha < \omega_2\}$ will be very similar to an unfilled gap. However, by PFA, each (ω_2, ω_2^*)-gap can be filled. It is essentially this situation which led us to believe that $\mathfrak{B}$ should not be representable.

THEOREM 3.0.1. *If $\mathcal{N}$ is a mad family on ω such that $\mathfrak{B}$ is isomorphic to the partitioner algebra $P_{\mathcal{N}}$ of $\mathcal{N}$, then there exists a map F from $[\omega \cup (\omega \times \{0\})]^\omega$ to $[\omega]^\omega$ such that F induces an isomorphism from $\mathfrak{B}$ to $P_{\mathcal{N}}$ and there exist $\tilde{\mathcal{N}} = \{N_{i,j} : i, j \in \omega\} \subseteq \mathcal{N}$ and a partition of ω , $\tilde{J} = \{J_n : n \in \omega\}$ such that either (i)-(ii) hold or (iii)-(v) hold where (i)-(v) are as follows:*

For $\alpha \in \omega_2$, let $X_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a_\alpha)\}$.

(i): *$X_\alpha \cap \{N_{i,j} : j \in \omega\}$ is finite for all $\alpha \in \omega_2$, $i \in \omega$;*

(ii): *If $I \in [\mathcal{N}]^\omega$ and there are infinitely many $i \in \omega$ such that $I \cap \{N_{i,j} : j \in \omega\} \neq \emptyset\}$, then there exists an α such that $I \cap X_\alpha \neq^* \emptyset$.*

(iii): *for all $\alpha \in \omega_2$, for all $m \in \omega$, $F(a_\alpha) \cap J_m =^* \emptyset$; and*

(iv): *if $a \subseteq \omega$ and there are infinitely many $m \in \omega$ such that $a \cap J_m \neq \emptyset$, then there exists an α such that $a \cap F(a_\alpha) \neq^* \emptyset$.*

(v): *For any $\mathcal{N}' \in [\bigcup_{\alpha\in\omega_2} X_\alpha]^\omega$, there is a β such that $\mathcal{N}' \cap X_\beta$ is infinite.*

3.1. Rearranging the mad family $\mathcal{N}$

Suppose $\mathcal{N}$ is a mad family on ω which represents $\mathfrak{B}$ and that $\varphi : \mathfrak{B} \to P_{\mathcal{N}}$ is an isomorphism induced by $F : [\omega \cup (\omega \times \{0\})]^\omega \to [\omega]^\omega$. Obviously, if F induces φ and if $F' : [\omega \cup (\omega \times \{0\})]^\omega \to [\omega]^\omega$ is such that for any non trivial partitioner $A \in \widetilde{\mathfrak{B}}$, $F(A)$ is a partitioner of $\mathcal{N}$ and $F(A) =_{\mathcal{I}_{\mathcal{N}}} F'(A)$, then F' also induces φ.

We will prove that either there exists F_1 and $\tilde{\mathcal{N}}$ satisfying the properties (i) and (ii) in the Theorem 3.0.1 or there exist F_2 and $\tilde{J}$ satisfying the properties (iii)-(v).

For the rest of this chapter we assume that $\mathcal{N}$ is a mad family representing $\mathfrak{B}$ and we fix an isomorphism φ from $\mathfrak{B}$ to $P_{\mathcal{N}}$. If $F : [\omega\cup(\omega\times\{0\})]^\omega \to [\omega]^\omega$ induces φ, we set $X^F_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a_\alpha)\}$ and $Z^F_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a'_\alpha)\}$. It follows that $\{X^F_\alpha : \alpha < \omega_2\}$ and $\{Z^F_\alpha : \alpha < \omega_2\}$ are increasing, mod finite, subsets of $\mathcal{N}$.

For any subset A of ω, we will also use $\mathcal{N}(A)$ to denote the set $\{N \in \mathcal{N} : N \subseteq^* A\}$. The next lemma produces a set P which looks very much like a partitioner of $\mathcal{N}$ which should not exist (i.e. this P corresponds to the illegal partitioner $[\omega]$ of $\mathfrak{B}$ in $P_{\widetilde{\mathcal{M}}}$). Recall we have PFA$^+$ and $\diamondsuit(S^2_0)$ available.

LEMMA 3.1.1. *There is a function $F : [\omega \cup (\omega \times \{0\})]^\omega \to [\omega]^\omega$ which induces φ for which the following hold.*

(1) *There exists* $\{Y_n : n \in \omega\} \subseteq \mathcal{P}(\mathcal{N})$ *and* $\{Y'_n : n \in \omega\}$ *such that* Y'_n *is a countable subset of* Y_n *and,*

(a): *for any countable subset* K *of* $\mathcal{N}$*, if* K *is almost disjoint with* X^F_α *for all* α *in* ω_2*, then* $K \subseteq^* Y_n$ *for some* n*;*

(b): *For all* $\alpha \in \omega_2$*,* $X^F_\alpha \cap Y_n = X^F_\alpha \cap Y'_n$ *is finite.*

(2) *for all* $\lambda \in \omega_2$*, there exists* μ_λ *such that*

$$\bigcup_{\xi<\lambda} X^F_\xi \setminus X^F_{\mu_\lambda} \text{ is countable.}$$

(3) *there exists* $P \subseteq \omega$*, such that*

$$(\forall \alpha \in \omega_2) F(a_\alpha) \subseteq P \text{ and } F(a'_\alpha) \cap P = \emptyset.$$

Proof: Fix any function f which induces the isomorphism φ and such that, for each $\alpha \in \omega_2$, $f(a_\alpha) \cap f(a'_\alpha)$ is empty. Let $\mathcal{I} = \{I \in [\mathcal{N}]^\omega : (\exists\alpha) I \subseteq^* X^f_\alpha\}$ and let $\mathcal{J} = \{I \in [\mathcal{N}]^\omega : (\exists\alpha) I \subseteq^* Z^f_\alpha\}$. It follows that $\mathcal{I}$ and $\mathcal{J}$ are P-ideals and we now verify that $\mathcal{I}$ and $\mathcal{J}$ are ω_1-separated, i.e. given sequences $\{I_\alpha : \alpha < \omega_1\}$ and $\{J_\alpha : \alpha < \omega_1\}$ from $\mathcal{I}$ and $\mathcal{J}$ respectively, there is an H such that for all $\alpha \in \omega_1$, $I_\alpha \subseteq^* H$ and $H \cap J_\alpha$ is finite. First pick $\xi < \omega_2$ such that , for all $\alpha < \omega_1$, $I_\alpha \subseteq^* X^f_\xi$ and $J_\alpha \subseteq^* Z^f_\xi$. Since $X^f_\xi \cap Z^f_\xi$ is finite, simply set $H = X^f_\xi$. By Theorem 1.2.22 and Lemma 1.2.21, $\mathcal{I}$ and $\mathcal{J}$ are strongly countably separated. Therefore there are $\{X(n) : n \in \omega\}$ such that for any $I \in \mathcal{I}$ and $J \in \mathcal{J}$, there exists n such that $I \subseteq^* X(n)$ and $X(n) \cap J$ is finite. Actually, since $\mathcal{I}$ and $\mathcal{J}$ are both P-ideals, there is a single n_0 such that $X(n_0)$ separates $\mathcal{I}$ and $\mathcal{J}$. To see this, suppose that it is not true. Then, for any n in ω, there exists $I_n \in \mathcal{I}$ and $J_n \in \mathcal{J}$ such that either $I_n \not\subseteq^* X(n)$ or $X(n) \cap J_n$ is infinite. Since they are P-ideals, fix $I \in \mathcal{I}$ and $J \in \mathcal{J}$ so that, for all $n \in \omega$, $I_n \subseteq^* I$ and $J_n \subseteq^* J$. But now there is m such that $X(m)$ separates I and J which implies that $X(m)$ separates I_m and J_m which contradicts the choice of I_m and J_m.

Let X be $X(n_0)$. Obviously, for all $\alpha \in \omega_2$, $X^f_\alpha \subseteq^* X$ and $X \cap Z^f_\alpha$ is finite. We may therefore assume $X^f_\alpha \subseteq X$ and $X \cap Z^f_\alpha$ is empty for all $\alpha < \omega_2$ because we can replace f by f' where $f'(a_\alpha) = f(a_\alpha) \setminus \bigcup(X^f_\alpha \setminus X)$ and $f'(a'_\alpha) = f(a'_\alpha) \setminus \bigcup(Z^f_\alpha \cap X)$.

Let $\mathcal{K} = \{I \in [X]^\omega : (\forall \alpha \in \omega_2)\ I \cap X^f_\alpha =^* \emptyset\}$. Again $\mathcal{I}$ and $\mathcal{K}$ are ω_1-separated. By Theorem 1.2.22 and Lemma 1.2.21, $\mathcal{I}$ and $\mathcal{K}$ can be strongly countably separated, say, by $\{Y_n : n \in \omega\}$. Hence, for each $\alpha \in \omega_2$, $X^f_\alpha \cap Y_n$ is finite and each $K \in \mathcal{K}$, K is almost contained in some Y_n.

Claim 1 There are $Y'_n \in [Y_n]^\omega$ such that, for all $\alpha \in \omega_2$, $X^f_\alpha \cap \bigcup_{n\in\omega} Y_n \setminus Y'_n$ is finite.

Proof of the Claim 1: If this were not the case, we can construct an ω_1-sequence $\{\alpha_\xi : \xi < \omega_1\}$ in ω_2 such that, for all $\eta < \omega_1$,

$$(X^f_{\alpha_\eta} \setminus \bigcup_{\xi<\eta} X^f_{\alpha_\xi}) \cap \bigcup_{n\in\omega} Y_n \text{ is infinite.}$$

The construction is straightforward. Take any $\alpha \in \omega_2$ as α_0. At stage η, let $\tilde{Y}_n = Y_n \cap \bigcup_{\xi<\eta} X_{\alpha_\xi}$. If the claim fails, there must be an α such that $X^f_\alpha \cap \bigcup_{n\in\omega} Y_n \setminus \tilde{Y}_n$ is infinite, i.e., $(X^f_\alpha \setminus \bigcup_{\xi<\eta} X^f_{\alpha_\xi}) \cap \bigcup_{n\in\omega} Y_n$ is infinite. Take such α as α_η .

Let λ be $\sup\{\alpha_\xi : \xi \in \omega_1\}$. Then , for all $\eta < \omega_1$,

$$\Gamma_\eta = (X^f_\lambda \cap X^f_{\alpha_\eta} \setminus \bigcup_{\xi<\eta} X^f_{\alpha_\xi}) \cap \bigcup_{n\in\omega} Y_n \text{ is infinite.}$$

But the Γ_η's are disjoint. Hence $X^f_\lambda \cap \bigcup_{n\in\omega} Y_n$ is uncountable. This contradicts the choice of Y_n's, which finishes the proof of Claim 1.

By again modifying f, we can assume that, for all α, $X^f_\alpha \cap \bigcup_{n\in\omega}(Y_n \setminus Y'_n) = \emptyset$, i.e., for all α , $X^f_\alpha \cap Y_n \subseteq Y'_n$.

Claim 2. For all $\lambda \in \omega_2$, there exists $\mu_\lambda \geq \lambda$ such that $\bigcup_{\alpha<\lambda} X^f_\alpha \setminus X^f_{\mu_\lambda}$ and $\bigcup_{\alpha<\lambda} Z^f_\alpha \setminus Z^f_{\mu_\lambda}$ are both countable.

Proof of Claim 2: We only give the proof for the X^f_α's.

Fix a $\lambda \in \omega_2$ and let S_λ be $\bigcup_{\alpha<\lambda} X^f_\alpha \setminus X^f_\lambda$. Then $|S_\lambda| \leq \aleph_1$ and we just have to consider the case when $|S_\lambda| = \aleph_1$. Set $S_\lambda = \{N_\beta : \beta < \omega_1\}$. By Theorem 1.2.10, Chang's Conjecture holds and by Lemma 1.2.12, there are no strong chains of length ω_2 on S_λ. Therefore there is some μ_0 such that , for all $\mu > \mu_0$, $(X^f_\mu \setminus X^f_{\mu_0}) \cap S_\lambda$ is countable.

Subclaim There is a countable subset S of S_λ and $\mu_\lambda > \mu_0$ such that

$$\text{for all } \mu > \mu_\lambda,\ (S_\lambda \setminus S) \cap (X^f_\mu \setminus X^f_{\mu_\lambda}) \text{ is finite.}$$

Proof of Subclaim: Suppose otherwise. We can then easily define a sequence $\{\alpha^0_\xi, \alpha^1_\xi, \mu^0_\xi, \mu^1_\xi : \xi \in \omega_1\}$ such that

(1) for $\xi < \eta < \omega_1$, $\alpha^0_\xi < \alpha^1_\xi < \alpha^0_\eta; \mu_0 < \mu^0_\xi < \mu^1_\xi < \mu^0_\eta$;

(2) $(X^f_{\mu^1_\xi} \setminus X^f_{\mu^0_\xi}) \cap \{N_\beta : \beta \in (\alpha^0_\xi, \alpha^1_\xi)\}$ is infinite.

Let μ_{ω_1} be $\sup\{\mu^1_\xi : \xi < \omega_1\}$. Then, for all $\xi < \omega_1$, $(X^f_{\mu_{\omega_1}} \setminus X^f_{\mu_0}) \cap \{N_\beta : \beta \in (\alpha^0_\xi, \alpha^1_\xi)\}$ is infinite. Hence, $(X^f_{\mu_{\omega_1}} \setminus X^f_{\mu_0}) \cap S_\lambda$ is uncountable contradicting the choice of μ_0. This finishes the proof of Subclaim.

Let A be $S_\lambda \setminus (S \cup X^f_{\mu_\lambda})$. We prove that A is countable. We have for any $\mu < \omega_2$, $A \cap X^f_\mu$ is finite. For any $I \in [A]^\omega$, there is an n such that $I \subseteq^* Y_n$. But $A \cap Y_n \subseteq Y'_n$, therefore, $I \subseteq^* Y'_n$. Thus $A \subseteq^* \bigcup_{n\in\omega} Y'_n$ and A is countable. Therefore, $\bigcup_{\alpha<\lambda} X^f_\alpha \setminus X^f_{\mu_\lambda} =^* S_\lambda \setminus X^f_{\mu_\lambda} \subseteq (A \cup S)$ is countable. This finishes the proof of the Claim 2.

Observe that (1) and (2) of the lemma holds for (our revised) f and $\{Y_n, Y'_n : n \in \omega\}$. Next we modify f to some F such that (1)-(3) of the lemma holds for F and $\{Y_n, Y'_n : n \in \omega\}$

Let $T = \{(a, b) \in \mathcal{P}(\omega) \times \mathcal{P}(\omega) : a \cap b = \emptyset \text{ and } \exists\alpha\, a \subseteq^* f(a_\alpha), b \subseteq f(a'_\alpha)\}$ Let $K_0 = \{\{(a, b), (c, d)\} \in [T]^2 : a \cap d \neq \emptyset \text{ or } c \cap b \neq \emptyset\}$. Let $K_1 = [X]^2 \setminus K_0$. By Lemma 1.2.17, K_0 is open with respect to the topology on T inherited from $\mathcal{P}(\omega)$.

Claim 3 There is no 0-homogeneous set with cardinality $\aleph_1$

Proof of Claim 3 Suppose there is an $\aleph_1$ sized 0-homogeneous set $\{(c_\alpha, d_\alpha) : \alpha < \omega_1\}$. For each $\alpha \in \omega_1$, let $\xi_\alpha < \omega_2$ be such that $c_\alpha \subseteq^* f(a_{\xi_\alpha})$ and $d_\alpha \subseteq^* f(a'_{\xi_\alpha})$. Let $\lambda = \sup\{\xi_\alpha : \alpha < \omega_1\}$. By Claim 2, there is a μ_λ such that $\bigcup_{\alpha\in\omega_1} X^f_{\xi_\alpha} \setminus X^f_{\mu_\lambda}$ and $\bigcup_{\alpha\in\omega_1} Z^f_{\xi_\alpha} \setminus Z^f_{\mu_\lambda}$ are countable. Therefore there exists an uncountable subset Γ of ω_1 and finite subsets $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{N}$, such that for each $\alpha \in \Gamma$, $X^f_{\xi_\alpha} \setminus X^f_{\mu_\lambda} = \mathcal{N}_1$. $Z^f_{\xi_\alpha} \setminus Z^f_{\mu_\lambda} = \mathcal{N}_2$. Thus for each $\alpha \in \Gamma$, $f(a_{\xi_\alpha}) \subseteq^* f(a_{\mu_\lambda}) \cup \bigcup \mathcal{N}_1$ and $f(a'_{\xi_\alpha}) \subseteq^* f(a'_{\mu_\lambda}) \cup \bigcup \mathcal{N}_2$. Of course, for all $\alpha \in \Gamma$, $\bigcup \mathcal{N}_1 \subseteq^* f(a_{\xi_\alpha})$ and $\bigcup \mathcal{N}_2 \subseteq^* f(a'_{\xi_\alpha})$. Let $e_1 = (f(a_{\mu_\lambda}) \cup \bigcup \mathcal{N}_1) \setminus \bigcup \mathcal{N}_2$ and $e_2 = (f(a'_{\mu_\lambda}) \cup \bigcup \mathcal{N}_2) \setminus \bigcup \mathcal{N}_1$. Obviously $e_1 \cap e_2$ is finite and, for all $\alpha \in \Gamma$, $c_\alpha \subseteq^* e_1$ and $d_\alpha \subseteq^* e_2$.

By shrinking Γ, if necessary, we can assume that there exist n and disjoint subsets L_1, L_2 of n such that, for each $\alpha \in \Gamma$, $c_\alpha \setminus n \subseteq e_1$, $d_\alpha \setminus n \subseteq e_2$, $c_\alpha \cap n = L_1$ and $d_\alpha \cap n = L_2$. It is easy to see that for any α, β in Γ, $c_\alpha \cap d_\beta = c_\beta \cap d_\alpha = \emptyset$. This contradicts the assumption that $\{(c_\alpha, d_\alpha), (c_\beta, d_\beta)\} \in K_0$ and finishes the proof of Claim 3.

By OCA, T can be written as $\bigcup_{n\in\omega} T(n)$, where each $T(n)$ is a 1-homogeneous set. Since, for any $\alpha \in \omega_2$, $f(a_\alpha) \cap f(a'_\alpha)$ is empty, $(f(a_\alpha), f(a'_\alpha)) \in T$. There exists an n_0 such that there are cofinally many α in ω_2, such that $(f(a_\alpha), f(a'_\alpha)) \in T(n_0)$. Let $P = \bigcup\{f(a_\alpha) : (f(a_\alpha), f(a'_\alpha)) \in T(n_0)\}$. Then it is easy to see that, for all $\alpha \in \omega_2$, $f(a_\alpha) \subseteq_{\mathcal{I}_\mathcal{N}} P$ and $f(a'_\alpha) \cap P =_{\mathcal{I}_\mathcal{N}} \emptyset$. So, for all $\alpha \in \omega_2$, there are finite subsets $\mathcal{N}^1_\alpha, \mathcal{N}^2_\alpha$ such that $f(a_\alpha) \setminus \bigcup \mathcal{N}^1_\alpha \subseteq^* P$ and $(f(a'_\alpha) \setminus \bigcup \mathcal{N}^2_\alpha) \cap P$ is finite. Now define F so that $F(a_\alpha) = P \cap f(a_\alpha) \setminus \bigcup \mathcal{N}^1_\alpha$ and $F(a'_\alpha) = f(a'_\alpha) \setminus (\bigcup \mathcal{N}^2_\alpha \cap P)$ and for any other $A \in \operatorname{dom} f$, $F(A) = f(A)$. Obviously F and P satisfy the conclusions in (3) of the lemma. Also, since, for all $\alpha \in \omega_2$, $X^F_\alpha \subseteq X^f_\alpha =^* X^F_\alpha$, it follows that (1) and (2) of the lemma still hold for F and $\{Y_n, Y'_n : n \in \omega\}$. $\square$

Remark: For any F' such that for all A, $F'(A) \subseteq F(A)$ and $F(A) \setminus F'(A) \in \mathcal{I}_\mathcal{N}$, F' still satisfies Lemma 3.1.1.

Next section is a proof for Theorem 3.0.1.

3.2. Proof of Theorem 3.0.1

Let F, P, $\{Y_n, Y'_n : n \in \omega\}$ satisfies the Lemma 3.1.1 and for each $\alpha < \omega_2$, let $X_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a_\alpha)\}$ We proceed by cases.

Case 1 : There are infinitely many n such that $Y_n \setminus \bigcup_{k<n} Y_k$ is infinite.

Let $\{n_i : i \in \omega\} = \{n : Y_n \setminus \bigcup_{k<n} Y_k$ is infinite $\}$. For each i, fix an infinite countable set $\mathcal{N}_i$ such that

$$Y'_{n_i} \setminus \bigcup_{k<n_i} Y_k \subseteq \mathcal{N}_i \subseteq Y_{n_i} \setminus \bigcup_{k<n_i} Y_k$$

Let $\mathcal{N}_i = \{N_{i,j} : j \in \omega\}$. Obviously, (i) and (ii) hold.

Case 2: There are only finitely many n such that $Y_n \setminus \bigcup_{k<n} Y_k$ is infinite.

In this case, there is an $m \in \omega$, such that, for all $n > m$, $Y_n \subseteq^* \bigcup_{k\leq m} Y_k$. Therefore, for all countable subsets I of $\mathcal{N}$, if $I \cap X_\alpha$ is finite for all $\alpha \in \omega_2$, then $I \subseteq^* \bigcup_{k\leq m} Y_k$. Since each $X_\alpha \cap \bigcup_{k\leq m} Y_k$ is finite, we can assume that, for all $\alpha \in \omega_2$, $X_\alpha \cap \bigcup_{k\leq m} Y_k$ is empty (i.e. by replacing $F(a_\alpha)$ with $F(a_\alpha) \setminus \bigcup(X_\alpha \cap \bigcup_{k\leq m} Y_k)$).

Let $X = \bigcup_{\alpha\in\omega_2} X_\alpha$ and $Y = \bigcup_{k\leq m} Y_k$.

Observe that for any infinite countable subset I of X, I is not mod finite contained in Y and so there is an α such that $I \cap X_\alpha$ is infinite. We record this as claim 1.

Claim 1: For all $I \in [X]^\omega$, there exists α such that $I \cap X_\alpha$ is infinite.

Now we are ready to find $\{J_n : n \in \omega\}$. Let $T = \{(a, b) \in [\omega]^\omega \times [\omega]^\omega : (\exists\alpha) a \subseteq^* F(a_\alpha)$ and $(\forall\alpha) b \cap F(a_\alpha) =^* \emptyset$ and $a \cap b = \emptyset\}$. Let $K_0 = \{\{(a, b), (c, d)\} \in [T]^2 : a \cap d \neq \emptyset$ or $c \cap b \neq \emptyset\}$. By Lemma 1.2.17, K_0 is an open coloring for some separable metric topology on T .

Claim 2: There is no uncountable 0-homogeneous set in T

Proof of Claim 2. The proof is almost identical to the proof of Claim 3 in the proof of Lemma 3.1.1. We give a sketch of the proof. Suppose $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\}$ is an uncountable 0-homogeneous subset of T. For all $\alpha \in \omega_1$, let $\xi_\alpha \in \omega_2$ be such that $c_\alpha \subseteq^* F(a_{\xi_\alpha})$. By Lemma 3.1.1, there is a $\lambda > \sup\{\xi_\alpha : \alpha \in \omega_1\}$ such that

$$\bigcup_{\alpha<\omega_1} X_{\xi_\alpha} \setminus X_\lambda$$

is countable. There is a finite subset $\mathcal{N}'$ of $\mathcal{N}$ and an uncountable subset Γ of ω_1, such that, for all $\alpha \in \Gamma$, $X_{\xi_\alpha} \setminus X_\lambda = \mathcal{N}'$. Therefore, $F(a_\lambda) \cup \bigcup \mathcal{N}'$ will separate $\{c_\alpha : \alpha \in \Gamma\}$ and $\{d_\alpha : \alpha \in \Gamma\}$. Again this will contradict the fact that $\{(c_\alpha, d_\alpha) : \alpha \in \Gamma\}$ is a 0-homogeneous set.

By OCA we let $T = \bigcup_{n\in\omega} T(n)$ where each $T(n)$ is 1-homogeneous. For all n, let $S(n) = \bigcup\{b : (\exists a)(a, b) \in T(n)\}$. Let E be the set of all n such that there are cofinally many α in ω_2 such that there exists b with $(F(a_\alpha), b) \in T(n)$. For any b such that b is almost disjoint with each $F(a_\alpha)$, there exists k and n such that there are cofinally many α in ω_2 such that $(F(a_\alpha), b \setminus k) \in T(n)$. Hence $n \in E$ and $b \subseteq^* S(n)$. It is easy to see that for any $n \in E$, and any $\alpha \in \omega_2$, there exists $\beta > \alpha$ such that $(F(a_\beta), b) \in T(n)$ for some $b \subseteq \omega$, which implies $F(a_\beta) \cap S(n)$ is empty. Since $F(a_\alpha) \subseteq_{\mathcal{I}_\mathcal{N}} F(a_\beta)$, $F(a_\alpha) \cap S(n)$ is in $\mathcal{I}_\mathcal{N}$.

Claim 3: For all $n \in E$, there is $a \in \mathcal{I}_\mathcal{N}$ such that, for all $\alpha < \omega_2$, $F(a_\alpha) \cap S(n) \subseteq^* a$.

Proof of Claim 3 Suppose the claim fails for some $n \in E$. Then we can find an infinite set $\{N_i : i \in \omega\} \subseteq \mathcal{N}$ such that, for all i in ω , $N_i \cap S(n)$ is infinite and N_i is contained in some $F(a_{\alpha_i})$. Clearly $\{N_i : i \in \omega\} \subseteq X$. Therefore by Claim 1 there is an α such that $\{N_i : i \in \omega\} \cap X_\alpha$ is infinite. Without loss of generality, we assume that $N_i \in X_\alpha$ for all i, i.e. for all $i \in \omega$, $N_i \subseteq^* F(a_\alpha)$. Since each N_i intersects $S(n)$ on an infinite set, $F(a_\alpha) \cap S(n) \notin \mathcal{I}_\mathcal{N}$. This contradicts the fact that $n \in E$.

According to Claim 3, for each $n \in E$, there exists a finite subset $\mathcal{N}'_n$ of X, such that, for all $\alpha \in \omega_2$, $F(a_\alpha) \cap S(n) \subseteq^* \bigcup \mathcal{N}'_n$. Now for $n \in E$, let

$S^0(n) = S(n) \setminus \bigcup \mathcal{N}'_n$. Suppose $b \subseteq \omega$ is such that, for all $\alpha \in \omega_2$, $b \cap F(a_\alpha)$ is finite. Let $n \in E$ be such that $b \subseteq^* S(n)$. Since $\mathcal{N}'_n\,(n \in E)$ are subsets of X, $b \cap \bigcup \mathcal{N}'_n$ is finite and so $b \subseteq^* S^0(n)$.

We enumerate E as $\{n_k : k < \rho\}$, where $\rho \leq \omega$. For $k < \rho$, let $J_k = S^0(n_k) \setminus \bigcup_{i<k} S^0(n_i)$. Obviously for all $\alpha \in \omega_2$ and $k < \rho$, $F(a_\alpha) \cap S^0(n_k)$ is finite, and for any b such that, for all $\alpha \in \omega_2$, $b \cap F(a_\alpha)$ is finite, there is $k < \rho$, such that $b \subseteq^* \bigcup_{i \leq k} J_i$.

The next Claim will finish the proof of the Theorem.

Claim 4: There are infinitely many $k \in \rho$ such that J_k is infinite.

Proof of Claim 4. Assume there is $k_0 \in \omega$ such that, for all $k \in \rho$,

$$S^0(n_k) \subseteq^* \bigcup_{i \leq k_0} S^0(n_i).$$

Let $Q = \bigcup_{i \leq k_0} S^0(n_i)$. For any $\alpha \in \omega_2$, $F(a_\alpha) \cap Q$ is finite. By (3) of Lemma 3.1.1, $F(a'_\alpha) \cap F(a_\beta)$ is empty for all α and β in ω_2 . Therefore $F(a'_\alpha) \subseteq^* Q$ for all $\alpha \in \omega_2$. If Q is a partitioner of $\mathcal{N}$, then there is a partitioner A of $\widetilde{M}$ such that $A \in \mathfrak{B}'$ and $F(A) =_{\mathcal{I}_\mathcal{N}} Q$. Note that since $F(A) =_{\mathcal{I}_\mathcal{N}} Q$, it follows that for all $\alpha \in \omega_2$, $a_\alpha \cap A \in \mathcal{I}_{\widetilde{M}}$ and $a'_\alpha \setminus A \in \mathcal{I}_{\widetilde{M}}$. Therefore, by Fact 2.3.4, we have that Q is not a partitioner of $\mathcal{N}$. On the other hand, fix any $N \in \mathcal{N}$. If $N \in X$, then there is some α such that $N \subseteq^* F(a_\alpha)$, so $N \cap Q \subseteq^* F(a_\alpha) \cap Q$ is finite. If $N \notin X$, then N is disjoint with $F(a_\alpha)$ for all $\alpha \in \omega_2$. In this case, there is k such that $N \subseteq^* S(n_k)$. Since $N \notin X$ and $\mathcal{N}'_{n_k} \subseteq X$, $N \notin \mathcal{N}'_{n_k}$. Therefore $N \subseteq^* S^0(n_k) \subseteq^* Q$. Thus Q is a partitioner of $\mathcal{N}$; this contradiction completes the proof. $\square$

Henceforth we assume F, $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{J}}$ satisfy the conclusion of Theorem 3.0.1.

Since $\widetilde{\mathcal{N}}$ and $\tilde{J}$ are countable, we will assume, for convenience, that they are members of $\mathfrak{M}_\lambda$ for all λ.

CHAPTER 4

Extending Coherent Sequences

4.1. An Extension Lemma

Recall that (See page 12) for a coherent sequence $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha \in \beta\}$ on ω, $\mathbb{P}_\mathcal{A}$ is the poset consisting of the partial functions p from ω to $\{0,1\}$ such that for any $p \in \mathbb{P}_\mathcal{A}$, there is an $\alpha < \beta$ such that $p =^* \mathrm{Ch}_{d_\alpha}^{c_\alpha}$. For p and q in $\mathbb{P}_\mathcal{A}$, $p < q$ if $p \supseteq q$.

In this chapter, we prove a preparatory lemma which will allow us to construct a good coherence sequence in the next chapter.

Recall that our strategy is to find a good coherent sequence $\mathcal{A}_{\omega_1}$ such that there is a partitioner X of $\mathcal{M}$ such that $[X]$ is in $\mathfrak{B}$ and, for all $\alpha \in \omega_1$, $X \cap a_{\mu(\alpha)} =^* b_\alpha$ while there is no $Y \subseteq \omega$ such that, for all $\alpha \in \omega_1$, $Y \cap F(a_{\mu(\alpha)}) =_{\mathcal{I}_\mathcal{N}} F(b_\alpha)$. This, of course, will contradict the fact that F induces an isomorphism from $\mathfrak{B}$ to the partitioner algebra $\mathcal{P}_\mathcal{N}$. The preparatory lemma will ensure that certain good coherent sequence can be extended while keeping certain commitments.

In order to ensure that, in $V^{\mathbb{P}_\mathcal{A}}$, no partitioner of $\mathcal{N}$ will uniformize, mod $\mathcal{I}_\mathcal{N}$, the image of $\mathcal{A}_{\omega_1}$ under F, we have to consider all possible partitioners of $\mathcal{N}$. That is, we will need to prove that for all possible names τ of partitioners of $\mathcal{N}$,

$$\Vdash_{\mathbb{P}_{\mathcal{A}_{\omega_1}}} \text{“}\,\tau \cap F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) \neq_{\mathcal{I}_\mathcal{N}} F(b_{\alpha+1}) \setminus F(b_\alpha)\text{”}\ .$$

In constructing the coherent sequences $\mathcal{A}_{\omega_1}$, we will handle one name τ at a time. Lemma 4.1.1 will guarantee that, under certain situations, we can extend $\mathcal{A}_\alpha$ to $\mathcal{A}_{\alpha+1}$ and take care of name τ_α from a suitable list.

LEMMA 4.1.1. *Suppose μ is a limit in ω_2. Let $\mathcal{A} = \{(b_\xi, a_{\mu(\xi)}) : \xi \leq \beta\}$ be a good coherent sequence such that $\mu = \mu(\beta)$ and $\mathcal{A} \in \mathfrak{M}_\mu$. Let $\tau \in \mathfrak{M}_\mu$ be a nice $\mathbb{P}_\mathcal{A}$-name for a subset of ω. Then there exist $p \in \mathbb{P}_\mathcal{A}$ and a $\mathbb{P}_\mathcal{A}$-generic, g, over $\mathfrak{M}_\mu$ and such that $g \in \mathfrak{M}_{\mu+1}$, $p \subseteq g$ and for any $\mu' \geq \mu + 2$, there are $\lambda' < \lambda < \omega_2$ with $\mu' \leq \lambda'$ and $\{R_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ such that the following hold.*

(1) *$\mathcal{A}' = \mathcal{A}^\frown(g^{-1}(1) \cap a_\lambda, a_\lambda)$ is a good coherent sequence,*
(2) *$\forall A \subseteq \omega$, if $\exists^\infty n\, A \cap R_n \neq^* \emptyset$, then $A \notin \mathcal{I}_\mathcal{N}$,*
(3) *$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}}$ "$\forall^\infty n\, R_n \cap (\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda'}) \setminus F(a_{\mu'})) \notin \mathcal{I}_\mathcal{N}$",*
(4) *$\{R_n : n \in \omega\} \cup \{F(g^{-1}(1)), F(a_{\lambda'}), F(a_{\mu'})\} \subseteq \mathfrak{M}_\lambda$.*

Note if $g \in \mathfrak{M}_{\mu+1}$ is $\mathbb{P}_\mathcal{A}$-generic over $\mathfrak{M}_\mu$, then $g^{-1}(1)$ is a partitioner of $\mathcal{M}$. Therefore if a is a partitioner of $\mathcal{M}$ and $a_{\mu(\beta)} \subseteq^* a$, then $\mathcal{A}^\frown(g^{-1}(1) \cap a, a)$ is a good coherent sequence . We will denote $\mathcal{A}^\frown(g^{-1}(1) \cap a, a)$ by $\mathcal{A}(g, a)$. Since (1) is guaranteed by the fact that $g \in \mathfrak{M}_{\mu+1}$ and g is $\mathbb{P}_\mathcal{A}$-generic over $\mathfrak{M}_\mu$ we will not verify the condition (1) in the proofs. Also, by condition (2), condition (3) is actually saying that

$$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}_{\alpha+1}}} \text{“} \tau \cap F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) \neq_{\mathcal{I}_\mathcal{N}} F(b_{\alpha+1}) \setminus F(b_\alpha)\text{”} .$$

It will be clear when applying this lemma in the next chapter that other conditions are simply there to help preserve the above forcing statement when $p \cup g \restriction a_\lambda$ is considered as an element in $\mathbb{P}_{\mathcal{A}_{\omega_1}}$ and the forcing notion is $\mathbb{P}_{\mathcal{A}_{\omega_1}}$ in the above forcing statement.

We break the proof into three Lemmas. We will first prove Lemma 4.1.2 from Lemma 4.1.3 and then prove Lemma 4.1.1 from Lemma 4.1.2. Finally we give, in the next section, the longest proof, that of Lemma 4.1.3, which, using the structures from Theorem 3.0.1, basically tells us why F can not induce an isomorphism from $\mathcal{B}$ to the partitioner algebra of $\mathcal{N}$.

Lemma 4.1.2. *Let $\mathcal{A}$, μ, τ be as in the Lemma 4.1.1. Then there exist $g \in \mathfrak{M}_{\mu+1}$ and $p \in \mathbb{P}_{\mathcal{A}}$ such that g is $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_{\mu}$, $p \subseteq g$ and there are cofinally many λ's in ω_2 satisfying the following property:*

(†) $\forall \rho, \gamma$ *such that* $\mu + 2 \leq \rho < \lambda \leq \gamma$

$$p \cup g \restriction a_\gamma \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \notin \mathcal{I}_{\mathcal{N}}\text{”},$$

where $\mathcal{A}' = \mathcal{A}(g, a_\gamma)$

Lemma 4.1.3. *Let $\mathcal{A}$, μ, τ be as in Lemma 4.1.1. There exist a $\mathbb{P}_{\mathcal{A}}$-generic g over $\mathfrak{M}_\mu$ and $p \in \mathbb{P}_{\mathcal{A}}$ such that $g \in \mathfrak{M}_{\mu+1}$ and $p \subseteq g$. Moreover, for all $\rho \geq \mu$, $q \in \mathbb{P}_{\mathcal{A}}$, and $m \in \omega$, there is $\lambda_0 = \lambda_0(\rho, q) \in \omega_2$ such that for all $\lambda > \lambda_0$, $\gamma \geq \lambda$, $m \in \omega$, and $\mathcal{N}' \in [\mathcal{N}]^{<\omega}$, either $p \cup q \cup g \restriction a_\gamma$ is not a condition of $\mathbb{P}_{\mathcal{A}'}$ or*

$$p \cup q \cup g \restriction a_\gamma \nVdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq m \cup \bigcup \mathcal{N}'\text{”},$$

where $\mathcal{A}' = \mathcal{A}(g, a_\gamma)$

Proof of Lemma 4.1.2 from Lemma 4.1.3: Let $\mu' \geq \mu + 2$. Let $g \in \mathfrak{M}_{\mu+1}$, $p \in \mathbb{P}_{\mathcal{A}}$ be chosen by Lemma 4.1.3. We will actually prove that there are cub many λ in ω_2 which satisfy the property (†). Assume that this is not the case, i.e. there are stationary many λ such that (†) is false for λ. Then for stationary many λ, there are $\rho_\lambda, \gamma_\lambda$, $p_\lambda \in \mathbb{P}_{\mathcal{A}}$, $\mathcal{N}_\lambda \in [\mathcal{N}]^{<\omega}$ and $m_\lambda \in \omega$ such that $\rho_\lambda < \lambda \leq \gamma_\lambda$, p_λ is compatible with $g \restriction a_\gamma \cup p$ and

$$g \restriction a_{\gamma_\lambda} \cup p \cup p_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_{\rho_\lambda})) \subseteq m_\lambda \cup \bigcup \mathcal{N}_\lambda\text{”},$$

where $\mathcal{A}' = \mathcal{A}(g, a_{\gamma_\lambda})$. By the Pressing Down Lemma, there exist ρ and a stationary set S such that, for all $\lambda \in S$, ρ_λ is equal to ρ. Since the cardinality of $\mathbb{P}_{\mathcal{A}}$

is less then ω_2, we can also assume that there are q and m such that $p_\lambda = q$ and $m_\lambda = m$ for all $\lambda \in S$. Let $\lambda_0 = \lambda_0(\rho, q)$ be obtained by applying Lemma 4.1.3 to ρ and q. Then, for $\lambda \in S \setminus \lambda_0$, according to Lemma 4.1.3 we have

$$p \cup q \cup g \restriction a_{\gamma_\lambda} \nVdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq m \cup \bigcup \mathcal{N}_\lambda\text{”} ,$$

where $\mathcal{A}' = \mathcal{A}(g, a_{\gamma_\lambda})$. This contradicts the choice of λ, γ_λ, q, m and $\mathcal{N}_\lambda$. □

Now we prove Lemma 4.1.1

Proof of Lemma 4.1.1 from Lemma 4.1.2: Let p, g be as the Lemma 4.1.2. Let μ'

$mu + 2$. Take any increasing sequence $\{\lambda_n : n \in \omega\}$ such that $\lambda_0 > \mu'$ and each λ_n satisfies (†) in Lemma 4.1.2. Let $\lambda' = \sup\{\lambda_n : n \in \omega\}$ and fix $\lambda > \lambda'$ large enough so that $\mathfrak{M}_\lambda$ contains $\{F(a_{\lambda_i}) : i \in \omega\} \cup \{F(a_{\lambda'}), F(a_\mu), F(a_{\mu'}), F(g^{-1}(1))\}$.

For each $n \in \omega$, let $R_n = F(a_{\lambda_{n+1}}) \setminus \bigcup_{k \leq n} F(a_{\lambda_k})$. Since each R_n is a non-trivial partitioner of $\mathcal{N}$ and $\{R_n : n \in \omega\}$ is pairwise disjoint, $\{R_n : n \in \omega\}$ satisfies (2). Obviously $\{R_n : n \in \omega\}$ is a member of $\mathfrak{M}_\lambda$. Therefore (4) is also true. Next we prove that (3) holds.

For each n in ω, it is easy to see that $R_n \cap F(a_{\lambda'}) \setminus F(a_{\mu'}) =_{\mathcal{I}_\mathcal{N}} F(a_{\lambda_{n+1}}) \setminus F(a_{\lambda_n})$. Since λ_{n+1} satisfies (†), we can set $\lambda = \gamma$ and $\rho = \lambda_n$ and $\mathcal{A}' = \mathcal{A}(g, a_\lambda)$ and conclude that

$$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda_{n+1}}) \setminus F(a_{\lambda_n})) \notin \mathcal{I}_\mathcal{N}\text{”} .$$

Therefore

$$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}R_n \cap (\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda'}) \setminus F(a_{\mu'})) \notin \mathcal{I}_\mathcal{N}\text{”} .$$

□

The following is an easy corollary of the above proof.

COROLLARY 4.1.4. *If $\{R_n : n \in \omega\}$ satisfies (1) and (2) of Lemma 4.1.1, then for any $\lambda \geq \lambda'' \geq \lambda'$,*

$$p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}\forall^\infty n\, R_n \cap (\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda''}) \setminus F(a_{\mu'})) \notin \mathcal{I}_{\mathcal{N}}\text{”}\,,$$

where $\mathcal{A} = \mathcal{A}(g, a_\lambda)$. □

4.2. Proof of the main lemma

We need some preliminaries to prove Lemma 4.1.3.

LEMMA 4.2.1. *Suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha \leq \beta\}$ is a coherent sequence with length $\beta + 1$. Then for any $g : \omega \to \{0,1\}$, g is a $\mathbb{P}_{\mathcal{A}}$-generic if and only if $g \restriction d_\beta =^* \mathrm{Ch}_{d_\beta}^{c_\beta}$ and $g \restriction \omega \setminus d_\beta$ is a Cohen real on $\omega \setminus d_\beta$.*

Proof: Trivial. □

Suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha \leq \beta\}$ is a coherent sequence and $g : \omega \to \omega$ is such that

$$g \restriction d_\beta =^* \mathrm{Ch}_{d_\beta}^{c_\beta}\ .$$

Let $\mathcal{D}$ be a family of predense subset of $\mathbb{P}_{\mathcal{A}}$. We say g is *$\mathcal{D}$-generic* if for each $E \in \mathcal{D}$, there is $e \in E$ such that $e \subseteq g$. Hence if g is $\mathbb{P}_{\mathcal{A}}$-generic over some model $\mathfrak{M}$ which contains $\mathcal{A}$ as an element, then g is a $\mathcal{D}_{\mathfrak{M}}$-generic, where $\mathcal{D}_{\mathfrak{M}}$ is the set of all predense subset of $\mathbb{P}_{\mathcal{A}}$ in $\mathfrak{M}$.

Let E be a subset of $\mathbb{P}_{\mathcal{A}}$. We use $E(n)$ to denote the set $\{r \in \mathbb{P}_{\mathcal{A}} : n \subseteq \operatorname{dom} r \text{ and } \exists\, q \in E, r \leq q\}$. Obviously, we have

FACT 4.2.2. *E is a predense subset of $\mathbb{P}_{\mathcal{A}}$ if and only if $E(n)$ is a dense and open subset of $\mathbb{P}_{\mathcal{A}}$.* □

For any bijection $\pi : \{0,1\}^n \to \{0,1\}^n$, we use $E(\pi)$ to denote the following set

$$\{r \in \mathbb{P}_{\mathcal{A}} \colon n \subseteq \operatorname{dom} r \text{ and } \pi(r \restriction n) \,\overline{\cup}\, r \in E(n)\}\ .$$

So $E(\pi)$ is obtained from $E(n)$ by changing the roots of the elements of $E(n)$ according to π.

FACT 4.2.3. *$E(n)$ is dense in $\mathbb{P}_{\mathcal{A}}$ if and only if $E(\pi)$ is dense in $\mathbb{P}_{\mathcal{A}}$.*

Proof: It is easy to see that $E(n)(n) = E(n)$ and $E(\pi)(n) = E(\pi)$. It is also easy to see that $E(\pi)(\pi^{-1}) = E(n)$. Therefore, it suffices to prove the "if" part. So assume that $E(n)$ is dense and let $r \in \mathbb{P}_{\mathcal{A}}$. Without loss of generality, we assume that $n \subseteq \operatorname{dom} r$. Let $q = \pi(r \restriction n) \mathbin{\overline{\cup}} r$. Since $E(n)$ is dense, choose $q' \in E(n)$ such that $q' \leq q$. Let $r' = r \restriction n \mathbin{\overline{\cup}} q'$, then $r' \in E(\pi)$ and $r' \leq r$. $\square$

We call a family $\mathcal{D}$ of predense subsets of $\mathbb{P}_{\mathcal{A}}$ *closed* if for any $E \in \mathcal{D}$, $n \in \omega$, $E(n) \in \mathcal{D}$ and for any permutation π on $\{0,1\}^n$, $E(\pi) \in \mathcal{D}$. For any family of predense subsets $\mathcal{D}$ of $\mathbb{P}_{\mathcal{A}}$, we use $\operatorname{cl}\mathcal{D}$ to denote the smallest closed set containing $\mathcal{D}$. It is easy to see that $\operatorname{cl}\mathcal{D}$ is definable from $\mathcal{D}$. In particular, if $\mathcal{D}$ is in some elementary submodel $\mathfrak{M}$ of some H_λ, then $\operatorname{cl}\mathcal{D}$ is also in $\mathfrak{M}$. Obviously, for all α such that $\mathcal{A} \in \mathfrak{M}_\alpha$, if $\mathcal{D}$ is the set of all predense subset of $\mathbb{P}_{\mathcal{A}}$ in $\mathfrak{M}_\alpha$, then $\mathcal{D}$ must be closed.

LEMMA 4.2.4. *Suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha < \beta\}$ is a coherent sequence and $\mathcal{D}$ is a family of predense subsets of $\mathbb{P}_{\mathcal{A}}$. If g is a $\mathcal{D}$-generic and $\mathcal{D}$ is closed, then*

(1) *any finite modification of g is still a $\mathcal{D}$-generic,*

(2) *for any $E \in \mathcal{D}$, E is still predense in $\mathbb{P}_{\mathcal{A}(g,d)}$ for any $d \subseteq \omega$ such that for all $\alpha < \beta$, $d_\alpha \subseteq^* d$.*

Proof: We first show that (1) implies (2). Suppose (1) is true. Let $E \in \mathcal{D}$ and $p \in \mathbb{P}_{\mathcal{A}(g,d)}$. We have to find an $e \in E$ such that e and p are compatible. By the definition of $\mathbb{P}_{\mathcal{A}(g,d)}$, we can assume that $p = q \mathbin{\overline{\cup}} g \restriction d$, where $q \in \mathbb{P}_{\mathcal{A}}$. Let $g' = q \mathbin{\overline{\cup}} g$. By (1), g' is also $\mathcal{D}$-generic. Therefore, there is an $e \in E$ such that $e \subseteq g'$. Obviously, e is compatible with $q \mathbin{\overline{\cup}} g \restriction d = p$. Next we prove (1).

Let g' be any finite modification of g, i.e. there is an $n \in \omega$ such that g and g' agree on $\omega \setminus n$. Let $E \in \mathcal{D}$. We have to prove that there is an $e \in E$ such that $e \subseteq g'$. Let π be the bijection $\{0,1\}^n$ such that $\pi(g \restriction n) = g' \restriction n$. Since $\mathcal{D}$ is closed, $E(\pi)$ is in $\mathcal{D}$. Therefore, since g is a $\mathcal{D}$-generic, there is an $r \in E(\pi)$ such that $r \subseteq g$. By the definition of $E(\pi)$, $\pi(r \restriction n) \mathbin{\overline{\cup}} r \in E(n)$, i.e. $g' \restriction n \mathbin{\overline{\cup}} r \in E(n)$. Since $g \restriction \omega \setminus n = g' \restriction \omega \setminus n$ and $r \subseteq g$, $g' \restriction n \mathbin{\overline{\cup}} r \subset g'$. But there is some $e \in E$ such that $e \subseteq g' \restriction n \mathbin{\overline{\cup}} r$. Hence $e \subseteq g'$. □

The following corollary collects these results together.

COROLLARY 4.2.5. *Let $\mathfrak{M}$ be an elementary submodel of H_θ and suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha < \beta\} \in \mathfrak{M}$ is a coherent sequence. If g is $\mathbb{P}_\mathcal{A}$-generic over $\mathfrak{M}$, then any finite modification of g is also $\mathbb{P}_\mathcal{A}$-generic over $\mathfrak{M}$ and for any predense subset E of $\mathbb{P}_\mathcal{A}$ in $\mathfrak{M}$, E remains predense in $\mathbb{P}_{\mathcal{A}'}$ whenever $\mathcal{A}' = \mathcal{A}(g,d)$ and d almost contain each d_α.* □

We will also need the following easy lemma.

LEMMA 4.2.6. *Suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha < \beta\}$ is a coherent sequence. Let τ be a nice $\mathbb{P}_\mathcal{A}$-name for a subset of some set S. Let g be $\mathcal{D}$ generic for some closed family $\mathcal{D}$ of predense subsets of $\mathbb{P}_\mathcal{A}$. For $s \in S$, let $A_s = \{p \in \mathbb{P}_\mathcal{A} : (\check{s}, p) \in \tau\}$ and $B_s = A_s^\perp$. If, for all $s \in S$, $A_s \cup B_s$ is in $\mathcal{D}$, then for any $p \in \mathbb{P}_\mathcal{A}$, and any $d \subseteq \omega$ such that d almost contains d_α for all $\alpha < \beta$,*

$$p \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}s \in \tau\text{”} \textit{ iff } p \Vdash_{\mathbb{P}_{\mathcal{A}(g,d)}} \text{“}s \in \tau\text{”}$$

and

$$p \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}s \notin \tau\text{”} \textit{ iff } p \Vdash_{\mathbb{P}_{\mathcal{A}(g,d)}} \text{“}s \notin \tau\text{”}$$

Proof: Let $\mathcal{A}'$ be $\mathcal{A}(g,d)$. Since $A_s \cup B_s$ is predense in $\mathbb{P}_{\mathcal{A}'}$, it is easy to see that for any $p \in \mathbb{P}_\mathcal{A}$, A_s (respectively B_s) is predense below p in $\mathbb{P}_\mathcal{A}$ iff A_s (respectively B_s) is predense below p in $\mathbb{P}_{\mathcal{A}'}$. However, it is also easy to see that $p \Vdash_{\mathbb{P}_{\mathcal{A}}}$ "$s \in S$" if and only if A_s is predense below p in $\mathbb{P}_{\mathcal{A}}$ and $p \Vdash_{\mathbb{P}_{\mathcal{A}'}}$ "$s \in S$" if and only if A_s is predense below p in $\mathbb{P}_{\mathcal{A}'}$. Similarly, we have that $p \Vdash_{\mathbb{P}_{\mathcal{A}}}$ "$s \notin S$" if and only if B_s is predense below p in $\mathbb{P}_{\mathcal{A}}$ and that $p \Vdash_{\mathbb{P}_{\mathcal{A}'}}$ "$s \notin S$" if and only if B_s is predense below p in $\mathbb{P}_{\mathcal{A}'}$. We are done. $\square$

Now we are ready to prove Lemma 4.1.3

Proof of Lemma 4.1.3: Let μ, $\mathcal{A}$ and τ be as given in the Lemma 4.1.1. Recall that we have fixed $\tilde{\mathcal{N}}$, $\tilde{\mathcal{J}}$ and F satisfying the conclusion of Theorem 3.0.1 and that $X_\alpha = \{N \in \mathcal{N} : N \subseteq^* F(a_\alpha)\}\,(\alpha \in \omega_2)$.

We remind the reader that we must show that there exist a $\mathbb{P}_{\mathcal{A}}$-generic g over $\mathfrak{M}_\mu$ and $p \in \mathbb{P}_{\mathcal{A}}$ such that $g \in \mathfrak{M}_{\mu+1}$ and $p \subseteq g$. Moreover, for all $\rho \geq \mu$ and $q \in \mathbb{P}_{\mathcal{A}}$, there is $\lambda_0 = \lambda_0(\rho, q) \in \omega_2$ such that for all $\lambda > \lambda_0$, $\gamma \geq \lambda$, $m \in \omega$, and $\mathcal{N}' \in [\mathcal{N}]^{<\omega}$, either $p \cup q \cup g \restriction a_\gamma$ is not a condition of $\mathbb{P}_{\mathcal{A}'}$ or

$$p \cup q \cup g \restriction a_\gamma \not\Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq m \cup \bigcup \mathcal{N}'\text{”} .$$

where $\mathcal{A}' = \mathcal{A}(g, a_\gamma)$

Case 1: $\{N_{i,j} : i, j \in \omega\}$ satisfies (i) and (ii) of the Theorem 3.0.1.

Subcase 1. There is a $p \in \mathbb{P}_{\mathcal{A}}$ such that

$$p \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{“}\exists^\infty i \exists^\infty j\ \tau \cap N_{i,j} =^* \emptyset\text{”} .$$

Let $g_0 = g_0^\mu \in \mathfrak{M}_{\mu+1}$ be the Cohen real over $\mathfrak{M}_\mu$ defined in the construction of Chapter 2 and then $g_0^{-1}(0) \subseteq^* a_{\mu+1}$. Let $g = p \,\overline{\cup}\, \mathrm{Ch}^{b_\beta}_{a_{\mu(\beta)}} \overline{\cup} g_0$. Then according to Lemma 4.2.1, g is $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_\mu$. Clearly $g^{-1}(0) \subseteq^* a_{\mu+1}$. Consider any $\rho \geq \mu$ and $q \in \mathbb{P}_{\mathcal{A}}$. If p and q are incompatible, there is nothing to prove. Therefore we assume that p and q are compatible. Let $\tilde{g} = q \,\overline{\cup}\, g$. Then $\tilde{g}$ is also $\mathbb{P}_{\mathcal{A}}$ generic over $\mathfrak{M}_\mu$ and $\tilde{g} \in \mathfrak{M}_{\mu+1}$. Let $h \in \mathfrak{M}_{\mu+2} \cap {}^\omega\omega$ be such that h dominates all reals in $\mathfrak{M}_{\mu+1}$, i.e.

$$\forall f \in {}^\omega\omega \cap \mathfrak{M}_{\mu+1}, f <^* h .$$

By elementarity, $(p \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{“}\exists^\infty i \exists^\infty j\ \tau \cap N_{i,j} =^* \emptyset\text{”})^{\mathfrak{M}_\mu}$. Since $p \subset \tilde{g} \in \mathfrak{M}_{\mu+1}$ and $(p \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{“}\exists^\infty i \exists^\infty j\ \tau \cap N_{i,j} =^* \emptyset\text{”})^{\mathfrak{M}_\mu}$, there are $\{i_k : k \in \omega\} \subseteq \omega$ and infinite subsets d_k of ω, for each $k \in \omega$, such that, for all $j \in d_k$, $\tau_{\tilde{g}} \cap N_{i_k,j}$ is finite, where $\tau_{\tilde{g}} = \mathrm{val}(\tau, \tilde{g})$. Since $g^{-1}(1)$ is in $\mathfrak{M}_{\mu+1}$, by Fact 2.3.3, $g^{-1}(1)$ is a partitioner of $\mathcal{M} \setminus \mathcal{M}_{\mu+2}$. But since $g^{-1}(1) \cap a_\mu = g^{-1}(1) \cap a_{\mu(\beta)} =^* b_\beta$, $g^{-1}(1)$ is also a partitioner of $\mathcal{M}_\mu = \mathcal{M}_{\mu+2}$. Therefore, $g^{-1}(1)$ is a partitioner of $\mathcal{M}$. Consequently $F(g^{-1}(1))$ and $F(g^{-1}(0))$ are partitioners of $\mathcal{N}$. Recall that for any subset A of ω we use $\mathcal{N}(A)$ to denote the set $\{N \in \mathcal{N} \colon N \subseteq^* A\}$.

Since $X_\rho \cap \{N_{i,j} \colon j \in \omega\}$ and $X_{\mu+2} \cap \{N_{i,j} \colon j \in \omega\}$ are finite, there is an $h^* > h$ such that, for all $i \in \omega$,

$$\begin{aligned}\mathcal{N}(F(a_\rho) \cup F(a_{\mu+2}) \cup F(g^{-1}(0)) \cup \omega \setminus F(g^{-1}(1))) \cap \{N_{i,j} : j \in \omega\} \\ \subseteq \{N_{i,j} : j < h^*(i)\}\end{aligned}$$

For all $k \in \omega$, let $j_k \in d_k \setminus h^*(i_k)$, then there exists λ_0 such that $\{N_{i_k,j_k} : k \in \omega\} \cap X_{\lambda_0}$ is infinite. For all $\lambda > \lambda_0$, $X_{\lambda_0} \subseteq^* X_\lambda$, hence $\{N_{i_k,j_k} : k \in \omega\} \cap X_\lambda$ is infinite.

For all λ, $\gamma \in \omega_2$, $m \in \omega$ and $\mathcal{N}' \in [\mathcal{N}]^{<\omega}$ such that $\gamma \geq \lambda > \lambda_0$, there is $k \in \omega$ such that $N = N_{i_k,j_k} \in X_\lambda \setminus \mathcal{N}'$. Then $N \subseteq^* F(a_\lambda) \setminus F(a_\rho)$ and $N \cap \tau_{\tilde{g}}$ is finite. Since $N \subseteq^* F(g^{-1}(1))$, there exists l in the set

$$N \cap (F(g^{-1}(1)) \setminus \tau_{\tilde{g}}) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus (m \cup \bigcup \mathcal{N}')\ .$$

Since $l \notin \tau_{\tilde{g}}$, there exists $r \subseteq \tilde{g} = q \overline{\cup} g$ such that $r \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{“}\, l \notin \tau\text{”}$. Let $\mathcal{A}' = \mathbb{P}_{\mathcal{A}(g,a_\gamma)}$. Since g is $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_\mu$ and τ, $\mathbb{P}_{\mathcal{A}}$ are in $\mathfrak{M}_\mu$, by Lemma 4.2.6,

$$r \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}\, l \notin \tau\text{”}\ .$$

Therefore,

$$r \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}\, l \in (\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus (m \cup \bigcup \mathcal{N}')\text{”}\ .$$

Since, $r \subseteq q \overline{\cup} g$ and q and p are compatible, either $p \cup q \cup g \restriction a_\gamma$ is not a condition or r is compatible with it. Thus either $p \cup q \cup g \restriction a_\gamma$ is not a condition or

$$p \cup q \cup g \restriction a_\gamma \not\Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“} l \notin (\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus (m \cup \bigcup \mathcal{N}') \text{”} .$$

Hence,

$$p \cup q \cup g \restriction a_\gamma \not\Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“} (\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq (m \cup \bigcup \mathcal{N}') \text{”} .$$

Subcase 2: $\exists p \in \mathbb{P}_{\mathcal{A}} \, \exists k \, (i > k \rightarrow p \Vdash \text{“} \forall^\infty j \; \tau \cap N_{i,j} \text{ infinite ”})$.

Let $g \in \mathfrak{M}_{\mu+1}$ be a $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_\mu$ such that $p \subseteq g$ and $g^{-1}(1) \subseteq^* a_{\mu+1}$. As in subcase 1, g is made using g_1^μ. Let $\rho \geq \mu$, $q \in \mathbb{P}_{\mathcal{A}}$. Again we assume p and q are compatible. Let $\tilde{g}$ be $q \overline{\cup} g$. Then $\tilde{g}$ is also $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_\mu$ containing p. As in the previous subcase, we can find a function $f_0 \in \mathfrak{M}_{\mu+1} \cap {}^\omega\omega$ such that

$$\forall i > k \, \forall j \geq f_0(i) \; \tau_{\tilde{g}} \cap N_{i,j} \text{ is infinite } .$$

Let $h \in {}^\omega\omega \cap \mathfrak{M}_{\mu+2}$ dominate ${}^\omega\omega \cap \mathfrak{M}_{\mu+1}$. Take an h^*, greater than both h and f_0, such that for all $i \in \omega$,

$$\{N_{i,j} : j \in \omega\} \cap \mathcal{N}(A) \subseteq \{N_{i,j} : j < h^*(i)\}, \tag{2}$$

where $A = F(g^{-1}(1)) \cup (\omega \setminus F(g^{-1}(0))) \cup F(a_\rho) \cup F(a_{\mu+1}) \cup F(a_{\mu+2})$. Again since $\tilde{\mathcal{N}}$ satisfies (ii) of Theorem 3.0.1, there exists λ_0 such that $X_{\lambda_0} \cap \{N_{i,h^*(i)} : i \in \omega\}$ is infinite and, for all $\lambda > \lambda_0$, $X_\lambda \cap \{N_{i,h^*(i)} : i \in \omega\}$ is also infinite.

For all $\lambda \geq \lambda_0$, $\gamma \geq \lambda$, $m \in \omega$ and $\mathcal{N}'$ a finite subset of $\mathcal{N}$, let $i > k$ be such that $N = N_{i,h^*(i)} \notin \mathcal{N}'$ and $N \in X_\lambda$. Since $i > k$, $\tau_{\tilde{g}} \cap N$ is infinite. By (2), $N \cap (F(g^{-1}(1)) \cup F(a_\rho))$ is finite. Hence,

$$N \cap [(\tau_{\tilde{g}} \setminus F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus \bigcup \mathcal{N}'] \text{ is infinite } .$$

Let $l > m$ be in the above set. Then since $l \in \tau_{\tilde{g}}$, there exists $r \subseteq \tilde{g}$ such that

$$r \Vdash_{\mathbb{P}_{\mathcal{A}}} \text{“} l \in \tau \text{”}$$

Since $\mathbb{P}_{\mathcal{A}}$ and τ are in $\mathfrak{M}_\mu$ and g is a $\mathbb{P}_{\mathcal{A}}$-generic over $\mathfrak{M}_\mu$, again by Lemma 4.2.6,

$$r \Vdash_{\mathbb{P}_{\mathcal{A}(g,a_\gamma)}} \text{“} l \in \tau \text{”}.$$

Hence

$$r \Vdash \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \not\subseteq m \cup \bigcup \mathcal{N}' \text{”} .$$

But $r \subseteq \tilde{g} = q \,\overline{\cup}\, g$ and q is compatible with p. Therefore if $q \cup p \cup g \restriction a_\gamma$ is a function, then r is compatible with it. Therefore either $q \cup p \cup g \restriction a_\gamma$ is not a function or

$$q \cup p \cup g \restriction a_\gamma \nVdash_{\mathbb{P}_{\mathcal{A}(g,a_\gamma)}} \text{“}(\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq m \cup \bigcup \mathcal{N}' \text{”}$$

Case 2 $\{J_n : n \in \omega\}$ satisfies the (iii) to (v) of Theorem 3.0.1.

Observe that in this case, for any $N \in \mathcal{N}$, either there are only finitely many i such that $N \cap J_i$ is not empty or N is almost disjoint with all $J_n\,(n \in \omega)$. To see this, let $N \in \mathcal{N}$. If there exists λ such that $N \subseteq^* F(a_\lambda)$, then by property (iii), N is almost disjoint with each $J_n\,(n \in \omega)$. Otherwise, since each $F(a_\lambda)$ is a partitioner, $N \cap F(a_\lambda)$ is finite for all $\lambda < \omega_2$. By the property (iv), there are at most finitely many $i \in \omega$ such that $N \cap J_i$ is not empty.

Subcase 1 $\exists p \in \mathbb{P}_{\mathcal{A}}$, $p \Vdash$ “$\forall^\infty n\; J_n \subseteq^* \tau$”

As in Case 1, we can take $g \in \mathfrak{M}_{\mu+1}$, a $\mathbb{P}_{\mathcal{A}}$-generic real over $\mathfrak{M}_\mu$ such that $g^{-1}(1) \subseteq^* a_{\mu+1}$. Again g is made using g_1^μ. As in the previous case, $F(g^{-1}(1))$ is a partioner of $\mathcal{N}$. Let $\rho \geq \mu, q \in \mathbb{P}_{\mathcal{A}}$. Let $h \in \mathfrak{M}_{\mu+2}$ such that ${}^\omega\omega \cap \mathfrak{M}_{\mu+1} <^* h$. Let $A = F(g^{-1}(1)) \cup (\omega \setminus F(g^{-1}(0)))$, then $A \subseteq_{\mathcal{I}_{\mathcal{N}}} F(a_{\mu+1})$, i.e. $A \setminus F(a_{\mu+1}) \in \mathcal{I}_{\mathcal{N}}$. Hence there are at most finitely many i such that $A \setminus F(a_{\mu+2}) \cap J_i$ is not finite. Take $f_0 \in {}^\omega\omega$ and $k \in \omega$, such that, for all $n > k$,

$$J_n \cap (F(g^{-1}(1)) \cup (\omega \setminus F(g^{-1}(0))) \cup F(a_\rho)) \subseteq f_0(n)\cdot$$

Let $h^* \in {}^\omega\omega$ such that $h^* > \{f_0, h\}$.

Define $\{A_i, \beta_i, N_i : i \in \omega\}$ as follows.

Let $A_0 = \{\min(J_n \setminus h^*(n)) : n \in \omega\}$. Since $\mathcal{N}$ is a mad family, there exists N_0 such that $N_0 \cap A_0$ is infinite. Obviously, there are infinitely many n such that $N_0 \cap A_0 \cap J_n$ is not empty. According to (iv) of Theorem 3.0.1, there is a β_0 such that $N_0 \cap A_0 \cap F(a_{\beta_0})$ is infinite. Hence, $N_0 \subseteq^* F(a_{\beta_0})$, i.e. $N_0 \in X_{\beta_0}$. At stage i, suppose we have defined $\{A_j, \beta_j, N_j \colon j < i\}$ such that $N_j \subseteq^* F(a_{\beta_j})$ and $A_j \setminus F(g^{-1}(1)) \cup F(a_\rho)$ is infinite. Now let $A_i = \{\min(J_n \setminus (h^*(n) \cup \bigcup_{j<i} F(a_{\beta_j}))) \colon n \in \omega\}$. Let $N_i \in \mathcal{N}$ such that $N_i \cap A_i$ is infinite. Similarly, we can find a β_i such that $N_i \subseteq^* F(a_{\beta_i})$. Obviously for each $i \in \omega$ and $l \in \omega$, $N_i \cap (\bigcup_{n>l} J_n \setminus h^*(n))$ is infinite and $N_i \setminus (F(g^{-1}(1)) \cup F(a_\rho))$ is infinite, thus $N_i \cap (F(g^{-1}(1)) \cup F(a_\rho))$ is finite.

By (v) of the Theorem 3.0.1, there is a λ_0 such that $\{N_i : i \in \omega\} \cap X_{\lambda_0}$ is infinite. Consequently, for all $\lambda > \lambda_0$, $\{N_i : i \in \omega\} \cap X_\lambda$ is infinite. For any $\lambda > \lambda_0$, $\gamma \geq \lambda$ and $\mathcal{N}' \in [\mathcal{N}]^{<\omega}$ and $m \in \omega$, take i such that $N = N_i \in X_\lambda \setminus \mathcal{N}'$. Let $\tilde{g} = q \,\overline{\cup}\, g$. If q is not compatible with p, we are done. So assume q is compatible with p. In this case, $p \subseteq \tilde{g}$. Since $p \Vdash$ "$\forall^\infty n\ J_n \subseteq^* \tau$" there is k and $f \in \mathfrak{M}_{\mu+1}$ such that, for all $n > k$, $J_n \setminus \tau_{\tilde{g}} \subseteq f(n)$. Therefore, since $h^* > f$, there is a k, such that

$$\bigcup_{n>k} J_n \setminus h^*(n) \subseteq \tau_{\tilde{g}} \ .$$

It follows that $N \cap \tau_{\tilde{g}}$ is infinite. By the choice of N_i, $N = N_i \subseteq^* (F(a_\lambda) \setminus F(a_\rho)) \setminus (F(g^{-1}(1)) \cup \bigcup \mathcal{N}')$. Hence $a = (\tau_{\tilde{g}} \setminus F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus (m \cup \bigcup \mathcal{N}')$ is infinite. Take $l \in a$ and choose $r \subseteq \tilde{g}$ such that $r \Vdash_{\mathbb{P}_{\mathcal{A}}}$ "$l \in \tau$". Hence, as did in the first case, we have

$$r \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}\, l \in (\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \setminus m \cup \bigcup \mathcal{N}' \,\text{”} \ ,$$

where $\mathcal{A}' = \mathcal{A}(g, a_\gamma)$. If $q \cup p \cup g \restriction a_\gamma$ is a function, then r is compatible with it, thus

$$q \cup p \cup g \restriction a_\gamma \nVdash_{\mathbb{P}_{\mathcal{A}'}} \text{“}\, (\tau \triangle F(g^{-1}(1))) \cap (F(a_\lambda) \setminus F(a_\rho)) \subseteq m \cup \bigcup \mathcal{N}' \,\text{”} \ .$$

Subcase 2 $\exists p \in \mathbb{P}_{\mathcal{A}}\ p \Vdash$ "$\exists^\infty n\ J_n \not\subseteq^* \tau$".

Take a $\mathbb{P}_{\mathcal{A}}$-generic g over $\mathfrak{M}_\mu$ such that $p \subseteq g$, $g^{-1}(0) \subseteq a_{\mu+1}$ and $g \in \mathfrak{M}_{\mu+1}$. Let $\rho \geq \mu$, $q \in \mathbb{P}_{\mathcal{A}}$. We assume, for the non-trivial case, q and p are compatible. Let h in ${}^\omega\omega \cap \mathfrak{M}_{\mu+2}$ dominate ${}^\omega\omega \cap \mathfrak{M}_{\mu+1}$. For similar reasons as in subcase 1, we can find a $f_0 \in {}^\omega\omega$ and $k \in \omega$, such that for all $n > k$,

$$((\omega \setminus F(g^{-1}(1))) \cup F(g^{-1}(0)) \cup F(a_\rho)) \cap J_n \subseteq f_0(n) \ .$$

and $f_0 > h$. Therefore,

$$F(g^{-1}(1)) \supseteq \bigcup_{n>k} J_n \setminus f_0(n) \ .$$

Let $\tilde{g} = q \,\overline{\cup}\, g$; since q and p are compatible, $p \subseteq \tilde{g}$, and so there are $\{n_j : j \in \omega\} \in \mathfrak{M}_{\mu+1}$ such that $J_{n_j} \setminus \tau_{\tilde{g}}$ is infinite. Using the method used in subcase 1, we can find $\{A_i, N_i, \beta_i : i \in \omega\}$ such that

(1) $A_i \cap J_{n_j}$ is finite for all $i, j \in \omega$,

(2) $A_i \subseteq \bigcup_{j\in\omega} J_{n_j} \setminus (\tau_{\tilde{g}} \cup f_0(n_j) \cup \bigcup_{j<i} F(a_{\beta_j}))$,

(3) $N_i \cap A_i$ is infinite,

(4) $N_i \subseteq F(a_{\beta_i})$.

Consequently, there is a λ_0 such that, for $\lambda > \lambda_0$, $\{N_i : i \in \omega\} \cap X_\lambda$ is infinite. Now it is obvious that for each N_i, $N_i \setminus \tau_{\tilde{g}}$ is infinite and, if $N_i \in X_\lambda$, then $N_i \cap (F(g^{-1}(1)) \cap (F(a_\lambda) \setminus F(a_\rho)))$ is infinite. Being a partitioner of $\mathcal{M}$, $F(g^{-1}(1)) \cap (F(a_\lambda) \setminus F(a_\rho))$ almost contains N_i. The rest of the proof is identical to the previous subcase. $\square$

CHAPTER 5

Collapsing $\aleph_2$ and a Good Coherent Sequence

In this chapter, we will find a proper forcing notion $\mathbb{D}$ whose extension will contain a coherent sequence with the properties stated in the Theorem 2.4.4. At the end of this chapter we will summarize the results and complete the proof of Theorem 2.4.4.

5.1. Collapsing $\aleph_2$

Recall that $\mathrm{Fn}(\omega_1, \omega_2, \aleph_1)$ denotes the set of all partial functions from ω_1 to ω_2 with countable domain. Let $\mathbb{P} = \mathrm{Fn}(\omega_1, \omega_2, \aleph_1)$ with the reverse inclusion as its partial order, i.e. for p and q in $\mathbb{P}$, $p \leq q$ if p extends q. This poset $\mathbb{P}$ is also often denoted by $\mathrm{Coll}(\omega_2, < \omega_1)$ because it will collapse ω_2 to ω_1.

We will prove several results to get ready to prove Theorem 2.4.4. The following theorem is the culmination of this Chapter.

Theorem 5.1.1. *If G is a $\mathbb{P}$ generic filter, then in $V[G]$, there is a good coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha < \omega_1\}$ such that*

(a): $\mathbb{P}_{\mathcal{A}}$ *is* ccc*,*

(b): $\mathbb{P}_{\mathcal{A}}$ *forces that for all* $Y \subseteq \omega$,

$$\{\alpha : Y \cap F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) \neq_{\mathcal{I}_{\mathcal{N}}} F(b_{\alpha+1}) \setminus F(b_\alpha)\} \text{ is uncountable}$$

(c): $\{(b_\xi, a_{\mu(\xi)}) : \xi < \alpha\} \in \mathfrak{M}_{\mu(\alpha)}$.

In this section, we prove some lemmas before we move to next section to prove Theorem 5.1.1.

Since $\mathfrak{c} = \omega_2$, we can enumerate ${}^{\omega}2$ as $\{d_\alpha : \alpha < \omega_2\}$ such that each function $d \in {}^{\omega}2$ appears in the enumeration only once. Let $\mathbb{Q}$ be the set of countable partial functions f from $\{(\alpha, \beta) : \beta < \alpha < \omega_1\}$ to $\{0, 1\}$ such that $\operatorname{dom} f = \bigcup_{\xi \in A} \{\xi\} \times \xi$ for some countable subset A of ω_1. $\mathbb{Q}$ is a poset with the reverse inclusion as its order. It is well known tha $\mathfrak{c} = \omega_2$ implies that $\mathbb{Q}$ is forcing equivalent to $\mathbb{P}$. However, for the purpose of the following lemma, we define an isomorphism e from $\mathbb{P}$ to $\mathbb{Q}$ as follows.

For each $\alpha < \omega_1$, let $\pi_\alpha : \alpha \to \omega$ be a bijection. Define for $f \in \mathbb{P}$, $e(f)$ so that $\operatorname{dom} e(f)$ is $\bigcup_{\alpha \in \operatorname{dom}(f)} \{\alpha\} \times \alpha$ and for $\alpha \in \operatorname{dom} f$, $\beta < \alpha$,

$$e(f)(\alpha, \beta) = d_{f(\alpha)}(\pi_\alpha(\beta)) \ .$$

It is easy to see that e is an isomorphism. Obviously e is order preserving. To see it is bijection, we define a function $c : \mathbb{Q} \to \mathbb{P}$ such that $e \circ c = \mathrm{id}_{\mathbb{Q}}$ and $c \circ e = \mathrm{id}_{\mathbb{P}}$. For any $g \in \mathbb{Q}$, let A be the countable subset of ω_1 such that $\operatorname{dom} g = \bigcup_{\alpha \in A} \{\alpha\} \times \alpha$. For all $\alpha \in A$, let $h^\alpha : \omega \to 2$ be such that

$$h^\alpha(n) = g(\alpha, \pi_\alpha^{-1}(n))$$

and define $c(g) \in \mathbb{P}$ such that $\operatorname{dom} c(g) = A$ and $c(g)(\alpha)$ is the unique ordinal η such that $h^\alpha = d_\eta$. Obviously, for all $\alpha \in A$, and $\beta < \alpha$, $e(c(g))(\alpha, \beta) = d_{c(g)(\alpha)}(\pi_\alpha(\beta)) = h^\alpha(\pi_\alpha(\beta)) = g(\alpha, \pi_\alpha^{-1}\pi_\alpha(\beta)) = g(\alpha, \beta)$. Therefore $e(c(g)) = g$. It is also easy to see that for any $f \in \mathbb{P}$, $c(e(f)) = f$.

For any filter G on $\mathbb{P}$, $g = \bigcup G : \omega_1 \to \omega_2^V$ is a partial function. We can define $e(g) = \bigcup\{e(f) : f \in G\}$. Obviously G is $\mathbb{P}$-generic iff $e[G]$ is $\mathbb{Q}$-generic.

When G is a $\mathbb{P}$-generic we also call g a $\mathbb{P}$-generic and $e(g)$ a $\mathbb{Q}$-generic.

We fix a $\mathbb{P}$-generic $g : \omega_1 \to \omega_2^V$. Let $s_\alpha = \{\beta < \alpha : e(g)(\alpha, \beta) = 1\}$, i.e. $s_\alpha = g_\alpha^{-1}(1)$, where $g_\alpha : \alpha \to \{0, 1\}$ is defined by $g_\alpha(\beta) = e(g)(\alpha, \beta)$. Let $\delta_\alpha = \sup\{g(\xi) : \xi < \alpha\}$.

Let $\dot{g}$ be the $\mathbb{P}$-name for the $\mathbb{P}$-generic g. Let $\dot{s_\alpha}$ and $\dot{\delta_\alpha}$ be the names for s_α and δ_α respectively.

Lemma 5.1.2. *If G is $\mathbb{P}$-generic, then in $V[G]$, $\{s_\alpha : \alpha \in \omega_1$ and $\operatorname{cf} \alpha = \omega\}$ is a $\diamondsuit$-sequence on $\{\alpha \in \omega_1 : \operatorname{cf} \alpha = \omega\}$ such that for any $X \subseteq \omega_1$ and $f : X \to \omega_2^V$ there are stationarily many α such that $X \cap \alpha = s_\alpha$ and $f \restriction s_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$.*

Proof: Since our construction in chapter 2 guaranteed that $\mathfrak{M}_0$ contains a $\diamondsuit(S_0^2)$ sequence, by a standard argument, we can fix, in $\mathfrak{M}_0$, a sequence $J = \{J_\alpha : \alpha \in S_0^2\}$ such that, for each α, $J_\alpha = (X_\alpha, Y_\alpha, Z_\alpha, W_\alpha) \in (\mathcal{P}(\alpha))^4$ and for all X, Y, Z and $W \subseteq \omega_2$, there are stationarily many α such that $X \cap \alpha = X_\alpha$, $Y \cap \alpha = Y_\alpha$, $Z \cap \alpha = Z_\alpha$ and $W \cap \alpha = W_\alpha$. Let $p_0 \in \mathbb{P}$ and suppose that $\dot{X}$, $\dot{C}$ and $\dot{f}$ are $\mathbb{P}$-names such that

$$p_0 \Vdash \text{“}\dot{X} \subset \omega_1 \text{ and } \dot{C} \text{ is a cub on } \omega_1 \text{ and } \dot{f} : \dot{X} \to \check{\omega_2}\text{”} .$$

We will find $q \leq p_0$ and $\alpha \in \omega_1$ such that

$$(\dagger) \qquad q \Vdash \text{“}\dot{X} \cap \alpha = \dot{s_\alpha} \text{ and } \dot{f} \restriction \dot{s_\alpha} \in \mathfrak{M}_{\dot{\delta}_\alpha+1} \text{ and } \alpha \in \dot{C}\text{”} .$$

Let $\chi(\dot{X})$ be the $\mathbb{P}$-name for the characteristic function of $\dot{X}$. For convenience, we assume that the name $\chi(\dot{X})$ is of the form $\{(\xi_j, i_j, p_j) : j \in I\}$ where for any $p \in \mathbb{P}$ and $\xi \in \omega_1$,

$$p \Vdash \text{“}\xi \in \dot{X}\text{”} \to (\xi, 1, p) \in \chi(\dot{X})$$

and

$$p \Vdash \text{“}\xi \notin \dot{X}\text{”} \rightarrow (\xi, 0, p) \in \chi(\dot{X}) \ .$$

To simplify the notation, we identify $\dot{X}$ with its characteristic function $\chi(\dot{X})$. We adopt the same convention on $\dot{C}$ and $\dot{f}$. Thus we will have, for example,

$$p \Vdash \text{“}\xi \in \dot{X}\text{”} \leftrightarrow (\xi, 1, p) \in \dot{X}$$

and

$$p \Vdash \text{“}\dot{f}(\alpha) \neq \beta\text{”} \leftrightarrow (\alpha, \beta, 0, p) \in \dot{f} \ .$$

Using the fact that $\mathfrak{c} = \aleph_2$, we fix, in $\mathfrak{M}_0$, bijections h_0, h_1, h_2 each with the same range, ω_2, and domains $\omega_1 \times \{0,1\} \times \mathbb{P}, \omega_1 \times \omega_2 \times \{0,1\} \times \mathbb{P}$ and $\mathbb{P}$ respectively.

We intend to identify $\mathbb{P}$-names $\dot{X}, \dot{C}, \dot{f}$ and $\mathbb{P}$ as subsets of ω_2 so that we can use the diamond sequence J. Let $X = h_0''\dot{X}$, $Y = h_0''\dot{C}$, $Z = h_1''\dot{f}$ and $W = h_2''\mathbb{P}$ respectively.

Take a sufficiently large cardinal κ and a continuous $\in$-chain $\{N_\alpha : \alpha \in \omega_2\}$ of elementary submodels of H_κ such that N_0 contains $\{\mathbb{P}, \mathbb{Q}, p_0, J, \dot{X}, \dot{C}, \dot{f}, \dot{g}, e, c, \mathfrak{M}_0\}$ and each N_α has cardinality $\aleph_1$. Note that N_0 also contains $\mathfrak{M}_0$ as a subset and so h_0, h_1 and h_2 are all in N_0. Now take δ such that $\operatorname{cf}\delta = \omega$, $\delta = N_\delta \cap \omega_2$ and

$$J_\delta = (X \cap \delta, Y \cap \delta, Z \cap \delta, W \cap \delta) \ .$$

Since J and δ are in $\mathfrak{M}_{\delta+1}$, $J_\delta \in \mathfrak{M}_{\delta+1}$. Therefore $X \cap \delta$, $Y \cap \delta$, $Z \cap \delta$ and $W \cap \delta$ are elements of $\mathfrak{M}_{\delta+1}$. Let $\dot{X}_\delta = \dot{X} \cap N_\delta$, $\dot{C}_\delta = \dot{C} \cap N_\delta$, $\dot{f}_\delta = \dot{f} \cap N_\delta$ and $\mathbb{P}_\delta = \mathbb{P} \cap N_\delta$.

Claim 1 $\dot{X}_\delta = \dot{X} \cap N_\delta$, $\dot{C}_\delta = \dot{C} \cap N_\delta$, $\dot{f}_\delta = \dot{f} \cap N_\delta$ and $\mathbb{P}_\delta = \mathbb{P} \cap N_\delta$ are in $\mathfrak{M}_{\delta+1}$.

Proof of Claim 1: We only prove that $\dot{X}_\delta = \dot{X} \cap N_\delta \in \mathfrak{M}_{\delta+1}$ since the proofs for the others are similar. Since $X \cap \delta$ and h_0 are in $\mathfrak{M}_{\delta+1}$, it is sufficient to show that $\dot{X}_\delta = h_0^{-1}(X \cap \delta)$, i.e. $h_0''(\dot{X} \cap N_\delta) = (h_0''\dot{X}) \cap \delta$. However, for any $(\xi, i, p) \in \dot{X}$, since $h_0 \in N_\delta$, $(\xi, i, p) \in N_\delta$ iff $h_0((\xi, i, p)) \in N_\delta$. But $N_\delta \cap \omega_2 = \delta$. Hence $(\xi, i, p) \in N_\delta$ iff $h_0((\xi, i, p)) \in (h_0''\dot{X}) \cap \delta$. We are done.

Take, in $\mathfrak{M}_{\delta+1}$, an increasing sequence $\{\eta_n : n \in \omega\}$ converging to δ.

Claim 2: There exist $\{p_n : n \in \omega\} \subseteq \mathbb{P}_\delta$, $\{\gamma_n : n \in \omega\}$ increasing in ω_1, $\{x_n : n \in \omega\}$ and $\{r_n : n \in \omega\}$ such that the following hold:

(0) r_n is a function from x_n to ω_2.

(1) $(\gamma_{n+1}, 1, p_{n+1}) \in \dot{C}_\delta$;

(2) $x_{n+1} \subseteq \gamma_n$ and for all $\xi < \gamma_n$, $\xi \in x_{n+1} \leftrightarrow (\xi, 1, p_{n+1}) \in \dot{X}_\delta$ and $\xi \notin x_{n+1} \leftrightarrow (\xi, 0, p_{n+1}) \in \dot{X}_\delta$;

(3) for all $\alpha \in x_{n+1}$, $(\alpha, r_n(\alpha), 1, p_{n+1}) \in \dot{f}_\delta$

(4) $\operatorname{dom} p_n \in \omega_1$, $\gamma_n < \operatorname{dom} p_n < \gamma_{n+1}$, $\eta_n < p_{n+1}(\operatorname{dom} p_n) < \delta$ and $\sup \operatorname{ran} p_n < p_{n+1}(\operatorname{dom} p_n)$.

We first show that Claim 2 will give us q and α which satisfies (†). By elementarity, we can assume the sequences $\{p_n : n \in \omega\}$, $\{\gamma_n : n \in \omega\}$, $\{x_n : n \in \omega\}$ and $\{r_n : n \in \omega\}$ are in $\mathfrak{M}_{\delta+1}$. Let $p = p_\omega = \bigcup_{n\in\omega} p_n$, $\alpha = \sup_n \gamma_n$. Let $r = \bigcup_{n\in\omega} r_n$ and $x = \bigcup_{n\in\omega} x_n$. Of course, x and r are in $\mathfrak{M}_{\delta+1}$.

(1) implies that, for all n, $p \Vdash$ "$\gamma_{n+1} \in \dot{C}$". Therefore $p \Vdash$ "$\alpha \in \dot{C}$". (4) implies that $p \Vdash$ "$\delta = \dot{\delta}_\alpha$". (2) implies $p \Vdash$ "$\dot{X} \cap \alpha = x$". (3) implies that $p \Vdash$ "$\dot{f} \restriction x = r$". The only thing left to do is to find a q below p such that $q \Vdash$ "$\dot{s}_\alpha = x$".

Recall π_α is a fixed bijection from α to ω. Define $d : \omega \to \{0, 1\}$ such that $d(n) = 1$ iff $\pi_\alpha^{-1}(n) \in x$. Let $\eta \in \omega_2$ be the unique ordinal such that $d_\eta = d$. Let $q = p \cup \{(\alpha, \eta)\}$. By the definition of s_α it is easy to see that

$$\begin{aligned} q \Vdash \text{“}\dot{s}_\alpha &= \{\xi < \alpha : e(\dot{g})(\alpha, \xi) = 1\} \\ &= \{\xi < \alpha : d_{\dot{g}(\alpha)}(\pi_\alpha(\xi)) = 1\} \\ &= \{\xi < \alpha : d_\eta(\pi_\alpha(\xi)) = 1\} \\ &= \{\xi < \alpha : \xi \in x\} = x\text{”} \end{aligned}$$

Since $q \leq p$, it is clear that q and α satisfies (†). Next we prove Claim 2.

We will construct the sequences by induction on ω. To keep the induction going, we add one more condition in the induction hypothesis.

(5) x_n, $r_n \in N_\delta$.

Let $\gamma_0 = 0$, $x_0 = \emptyset$, $r_0 = \emptyset$. At stage n, we need to find p_{n+1}, γ_{n+1}, x_{n+1} and r_{n+1}. Since $p_n \in N_\delta$, $\alpha = \operatorname{dom} p_n < \omega_1$ and $\beta = \sup \operatorname{ran} p_n < \delta$. Let $p_n' = p_n^\frown(\alpha, \max\{\eta_n + 1, \beta + 1\}) \in N_\delta$. Since

$$p_n' \Vdash \text{“} \dot{C} \text{ is a cub ”} ,$$

there exists $p_n'' \leq p_n'$ and $\gamma_n' > \alpha$ such that

$$p_n'' \Vdash \text{“} \gamma_n' \in \dot{C} \setminus \alpha \text{”} .$$

By elementarity, we can assume that p_n'' and γ_n' are in N_δ. Since $\mathbb{P}$ is countably closed, there exists $\tilde{p}_n \leq p_n''$ such that $\operatorname{dom} \tilde{p}_n$ is an ordinal and $\tilde{p}_n$ decides $\dot{X} \cap \gamma_n$ and $\dot{f} \restriction (\dot{X} \cap \gamma_n)$. So there are x_{n+1} and r_{n+1} such that

$$\tilde{p}_n \Vdash \text{“} \dot{X} \cap \gamma_n = x_{n+1} \text{”}$$

and

$$\tilde{p}_n \Vdash \text{“} \dot{f} \restriction x_{n+1} = r_{n+1} \text{”} .$$

By elementarity, we can assume that x_{n+1}, r_{n+1} and $\tilde{p}_n$ are all in N_δ.

Now let $p_{n+1} = \tilde{p}_n$ and $\gamma_{n+1} = \gamma_n'$. We prove (1)-(5) holds at this stage. Since $p_{n+1} \Vdash$ “$\gamma_{n+1} \in \dot{C}$”, our assumption on the name $\dot{C}$ implies $(\gamma_{n+1}, 1, p_{n+1}) \in \dot{C}$. But γ_{n+1} and p_{n+1} are in N_δ. Therefore $(\gamma_{n+1}, 1, p_{n+1}) \in \dot{C}_\delta$. Thus (1) is true. Since $p_{n+1} \Vdash$ “$\dot{X} \cap \gamma_n = x_{n+1}$”, for $\xi < \gamma_n$, $\xi \in x_{n+1}$ implies that $p_{n+1} \Vdash$ “$\xi \in \dot{X}$” and $\xi \in \gamma_n \setminus x_{n+1}$ implies that $p_{n+1} \Vdash$ “$\xi \notin \dot{X}$”. Similar to the proof of (1) we get $\xi \in x_{n+1}$ implies $(\xi, 1, p_{n+1}) \in \dot{X}_\delta$ and $\xi \notin x_{n+1}$ implies $(\xi, 0, p_{n+1}) \in \dot{X}_\delta$. Therefore we can get (2). (3) is proved in a same way we proved (2). (4) and (5) are obvious. We are done $\square$

The following Corollary is what will actually be used the the proof of Theorem 5.1.1.

COROLLARY 5.1.3. *If G is $\mathbb{P}$-generic, then in $V[G]$, there is a $\diamondsuit$-sequence $\{(S_\alpha, T_\alpha) \in \mathcal{P}(\omega_1) \times \mathcal{P}(\omega_1) : \alpha \in \omega_1$ and $\mathrm{cf}\,\alpha = \omega\}$ such that for any X, $Y \subset \omega_1$ and $f_0 : X \to Z$, $f_1 : Y \to Z$, where Z is in $\mathfrak{M}_0$ with cardinality ω_2 in V, then there are stationarily many α such that $X \cap \alpha = S_\alpha$, $Y \cap \alpha = T_\alpha$, $f_0 \restriction S_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$ and $f_1 \restriction T_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$*

Proof: Since there is a bijection in $\mathfrak{M}_0$ between ω_2^V and Z, we can assume that $Z = \check{\omega}_2$. Let $\{s_\alpha : \alpha \in \omega_1, \mathrm{cf}\,\alpha = \omega\}$ be the $\diamondsuit$-sequence in the Lemma 5.1.2. Let p_0 denote the projection from $\omega_1 \times \omega_1$ to ω_1 defined by the formula $p_0(\alpha, \beta) = \alpha$. In $\mathfrak{M}_0$, take a bijection $\pi : \omega_1 \times \omega_1 \to \omega_1$. Let C be the cub $\{\alpha < \omega_1 : \pi''\alpha \times \alpha = \alpha\}$. For all $\alpha \in C$ with $\mathrm{cf}\,\alpha = \omega$, let $S_\alpha = p_0''(\pi^{-1}(s_\alpha) \cap \omega_1 \times \{0\})$ and $T_\alpha = p_0''(\pi^{-1}(s_\alpha) \cap \omega_1 \times \{1\})$. We check that $\{(S_\alpha, T_\alpha) : \alpha \in C$ and $\mathrm{cf}\,\alpha = \omega\}$ works. Let $X, Y \subseteq \omega_1$ and $f_0 : X \to \check{\omega}_2$ and $f_1 : Y \to \check{\omega}_2$. Let $B = X \times \{0\} \cup Y \times \{1\}$ and let $A = \pi''B$. Now define $F : A \to \check{\omega}_2$ as follows,

$$F(\xi) = \begin{cases} f_0(\alpha) & \xi = \pi(\alpha, 0) \text{ for some } \alpha \text{ in } X \\ f_1(\alpha) & \xi = \pi(\alpha, 1) \text{ for some } \alpha \text{ in } Y \end{cases}$$

Let E be all $\alpha \in C$ such that $A \cap \alpha = s_\alpha$ and $F \restriction s_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$. Of course, E is stationary. For any $\alpha \in E$, it is easy to see that $S_\alpha = X \cap \alpha$ and $T_\alpha = Y \cap \alpha$. To see that $f_0 \restriction S_\alpha$ is in $\mathfrak{M}_{\delta_\alpha+1}$, let $\sigma : X \cap \alpha \to (X \cap \alpha) \times \{0\}$ be such that $\sigma(\alpha) = (\alpha, 0)$. It is easy to see that $S_\alpha = X \cap \alpha \in \mathfrak{M}_{\delta_\alpha+1}$. Indeed

$$X \cap \alpha = p_0''(\pi^{-1}(s_\alpha) \cap (\omega_1 \times \{0\})) \in \mathfrak{M}_{\delta_\alpha+1} \ .$$

Therefore, $\sigma \in \mathfrak{M}_{\delta_\alpha+1}$. Observe that

$$f_0 \restriction S_\alpha = (F \restriction s_\alpha) \circ \pi \circ \sigma \ .$$

Therefore, $f_0 \restriction S_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$. Similarly, $f_1 \restriction T_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$. $\square$

We will also need the following notion.

DEFINITION 5.1.4 (PAGE 245 E1, [Kun80]). A poset $(P, <)$ is called *almost homogeneous* if, for all p, q in P, there exists an automorphism i of P such that $i(p)$ and q are compatible

We prove the following easy lemma.

LEMMA 5.1.5. (1) *If P is an almost homogeneous poset, then for any formula $\varphi(x_1, \ldots, x_n)$ with all free variables shown, for any $u_1, \ldots, u_n$ in the ground model, either $\mathbf{1} \Vdash_P$ "$\varphi(\check{u}_1, \ldots, \check{u}_n)$" or $\mathbf{1} \Vdash_P$ "$\neg\varphi(\check{u}_1, \ldots, \check{u}_n)$".*

(2) *Suppose $\mathcal{A} = \{(c_\alpha, d_\alpha) : \alpha < \lambda\}$ is a coherent sequence and each d_α is co-infinite. Then $\mathbb{P}_{\mathcal{A}}$ is an almost homogeneous poset.*

Proof: (1) is well-known. We prove (2). Let p and q be in $\mathbb{P}_{\mathcal{A}}$. There are $\alpha, \beta < \lambda$ and $p', q' \in \mathrm{Fn}(\omega, 2)$ and $m \in \omega$ such that $p = p' \cup \mathrm{Ch}^{c_\alpha \setminus m}_{d_\alpha \setminus m}$ and $q = q' \cup \mathrm{Ch}^{c_\beta \setminus m}_{d_\beta \setminus m}$. Without loss of generality, we can assume that $\alpha \leq \beta$, $c_\alpha \setminus m \subseteq c_\beta$, $d_\alpha \setminus m \subseteq d_\beta$ and $\mathrm{dom}\, p' \cup \mathrm{dom}\, q' = m$. Take a finite subset H of

$$\omega \setminus (\mathrm{dom}\, p \cup \mathrm{dom}\, q)$$

such that H has cardinality m. Let $j : m \to H$ be a bijection. Now let $i : \mathbb{P}_{\mathcal{A}} \to \mathbb{P}_{\mathcal{A}}$ be defined as follows. For any $r \in \mathbb{P}_{\mathcal{A}}$, $\mathrm{dom}\, i(r) = (\mathrm{dom}\, r \setminus (m \cup H)) \cup \{n \in m \cup H : j(n) \in \mathrm{dom}\, r \text{ or } j^{-1}(n) \in \mathrm{dom}\, r\}$. And

$$i(r)(n) = \begin{cases} r(n) & \text{if } n \in \mathrm{dom}\, r \setminus (m \cup H) \\ r(j(n)) & \text{if } n \in m \text{ and } j(n) \in \mathrm{dom}\, r \\ r(j^{-1}(n)) & \text{if } n \in H \text{ and } j^{-1}(n) \in \mathrm{dom}\, r \end{cases}$$

Obviously, $i \circ i = \mathrm{id}_P$, hence i is a bijection. It is also easy to see that i is order-preserving . Hence i is an automorphism of P. Observe that $\mathrm{dom}\, i(p) \cap q = d_\alpha \setminus m$ and the restrictions of $i(p)$ and q on $d_\alpha \setminus m$ are all equal to $\mathrm{Ch}^{c_\alpha \setminus m}_{d_\alpha \setminus m}$. Hence $i(p)$ and q are compatible. $\square$

5.2. Forcing a Good Coherent Sequence

We list some notations and properties in the next lemma to make the proof of Theorem 5.1.1 shorter.

LEMMA 5.2.1. *Suppose $\mathcal{A}_\alpha \prec \mathcal{A}_\beta$ $(\alpha < \beta)$ are coherent sequences, $\mathcal{R} = \{R_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ and τ is a nice $\mathbb{P}_{\mathcal{A}_\alpha}$-name. Then the following are true:*

(1) *Each of the following are predense subsets of $\mathbb{P}_{\mathcal{A}_\alpha}$,*

(i): $D_n(\mathcal{A}_\alpha, \mathcal{R}, \tau) = D^0_n(\mathcal{A}_\alpha, \mathcal{R}, \tau) \cup (D^0_n(\mathcal{A}_\alpha, \mathcal{R}, \tau))^\perp$, *where* $D^0_n(\mathcal{A}_\alpha, \mathcal{R}, \tau) = \{q \in \mathbb{P}_{\mathcal{A}_\alpha} : \exists i > n,\ q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}} \text{“}\tau \cap R_i \neq^* \emptyset\text{”}\}$.

(ii): $E_{i,n}(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ *for each* i, n, *where,*

$$E_{i,n}(\mathcal{A}_\alpha, \mathcal{R}, \tau) = \{q \in \mathbb{P}_{\mathcal{A}_\alpha} : q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}} \text{“}\tau \cap R_i \subseteq n\text{”} \\ \text{or } \exists k \in R_i \setminus n,\ q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}} \text{“}k \in \tau\text{”}\}\ .$$

(iii): $H^\tau_n = A^\tau_n \cup (A^\tau_n)^\perp$, *where* $A^\tau_n = \{p \in \mathbb{P}_{\mathcal{A}_\alpha} \colon (n, p) \in \tau\}$.

(2) *If $p \in \mathbb{P}_{\mathcal{A}_\alpha}$ and*

$$p \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}} \text{“}\exists^\infty i\, R_i \cap \tau \neq^* \emptyset\text{”}\ ,$$

then, for all n in ω, $D^0_n(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ is dense below p.

(3) *If $D_n(\mathcal{A}_\alpha, \mathcal{R}, \tau)$, $E_{i,n}(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ and H^τ_n are predense in $\mathbb{P}_{\mathcal{A}_\beta}$, and*

$$p \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}} \text{“}\exists^\infty i\, R_i \cap \tau \neq^* \emptyset\text{”}\ ,$$

then

$$p \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}} \text{“}\exists^\infty i\, R_i \cap \tau \neq^* \emptyset\text{”}\ .$$

(4) *If H^τ_n are predense in $\mathbb{P}_{\mathcal{A}_\beta}$, then for any subsets A, B of ω,*

$$\mathbf{1} \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}} \text{“}\tau' = (\tau \cap A) \triangle B\text{”}\ ,$$

where τ' is the $\mathbb{P}_{\mathcal{A}_\alpha}$-name defined as follows,

$$\tau' = \{\langle n,p\rangle \in \tau, n \in A \setminus B\} \cup \{\langle n, \mathbf{1}\rangle : n \in B \setminus A\} \\ \cup \{\{n\} \times (A^\tau_n)^\perp : n \in A \cap B\}.$$

Proof: (1) and (2) are trivial. (4) is also straightforward. We prove (3) by contradiction. Take $p' \in \mathbb{P}_{\mathcal{A}_\beta}$ and $n \in \omega$ such that

$$\forall i > n \; p' \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}} \text{“} R_i \cap \tau =^* \emptyset\text{”} .$$

Since $D_n(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ is predense in $\mathbb{P}_{\mathcal{A}_\beta}$, there exists $q \in D_n(\mathcal{A}_\alpha, \mathcal{R}, \tau)$ such that p' and q are compatible. Since $p \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$\exists^\infty i\, R_i \cap \tau \neq^* \emptyset$” and q is compatible with p, q must be in $D^0_n(\mathcal{A}_\alpha, \mathcal{R}, \tau)$. Therefore, there is $i > n$ such that $q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$\tau \cap R_i \neq^* \emptyset$”. By the choice of p', we can find $r \in \mathbb{P}_{\mathcal{A}_\beta}$ and $m \in \omega$ such that $r \leq q \cup p'$ and $r \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}}$ “$\tau \cap R_i \subseteq m$”. Since $E_{i,m}$ is predense in $\mathbb{P}_{\mathcal{A}_\beta}$, there is some $s \in E_{i,m}$ which is compatible with r. Since s is compatible with q and $q \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$\tau \cap R_i \neq^* \emptyset$”, $s \nVdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$\tau \cap R_i \subseteq m$”. Therefore there is a $k \in R_i \setminus m$ such that $s \Vdash_{\mathbb{P}_{\mathcal{A}_\alpha}}$ “$k \in \tau$”. It follows that A^τ_k is predense below s in $\mathbb{P}_{\mathcal{A}_\alpha}$. We claim that A^τ_k is also predense below s in $\mathbb{P}_{\mathcal{A}_\beta}$. Indeed, since H^τ_k is predense in $\mathbb{P}_{\mathcal{A}_\beta}$, it is sufficient to see that any element in $(A^\tau_k)^\perp$ is incompatible with s. But this is obvious from the definition of $(A^\tau_k)^\perp$. Therefore, we have $s \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}}$ “$k \in \tau$”. However, from the choice of r , $s \cup r \Vdash_{\mathbb{P}_{\mathcal{A}_\beta}}$ “$\tau \cap R_i \subseteq m$”. Contradiction. □

In the sequel, we will use $\mathcal{A}_\alpha$ to denote $\{(b_\xi, a_{\mu(\xi)}) : \xi < \alpha\} \in \mathfrak{M}_{\mu(\alpha)}$. Let $\mathcal{D}$ denote the set of all the countable partial functions from ω to $\{0,1\}$. Recall that $|\mathcal{D}|$ is ω_2. Since $\mathbb{P}$ is ω_1-closed, the definition of $\mathcal{D}$ is absolute between V and $V^{\mathbb{P}}$. Since $\mathbb{P}$ collapses ω_2^V to ω_1, we can enumerate, in $V^{\mathbb{P}}$, $\mathcal{D}$ as $\{f_\alpha : \alpha < \omega_1\}$.

Proof of Theorem 5.1.1: Let G be a $\mathbb{P}$-generic over V and let g be $\bigcup G$. Recall we use δ_α to denote $\sup\{g(\beta) : \beta < \alpha\}$. Fix a $\diamondsuit$-sequence satisfying

Corollary 5.1.3. Since $\mathbb{P}$ collapses ω_2^V to ω_1, we can enumerate all the τ of the form

$$\bigcup_{n\in\omega}\{n\}\times A_n$$

with $A_n \in [\mathcal{D}]^{\leq\omega}$ as $\{\tau_\alpha : \alpha < \omega_1\}$ such that for each τ, $\{\alpha \in \lim\omega_1 : \tau = \tau_\alpha\}$ is uncountable.

Note that since $\mathbb{P}$ is ω_1-closed, there is no new countable coherent sequence in $V^{\mathbb{P}}$ and for any countable coherent sequence $\mathcal{A}'$, there is no new countable nice $\mathbb{P}_{\mathcal{A}'}$-name.

In $V[G]$, we will define, by induction on α, $\mathcal{A}_\alpha$, $\mathcal{D}_\alpha$, $\mathcal{R}^\alpha = \{R_n^\alpha : n \in \omega\}$, ρ'_α, ρ''_α, p_α, τ'_α and g_α such that $\mathcal{A}_\alpha = \{(b_\beta, a_{\mu(\beta)}) : \beta < \alpha\}$ is a good coherent sequence. For each infinite countable ordinal α, $\mathcal{D}_\alpha$ is a family of predense subsets of $\mathbb{P}_{\mathcal{A}_\alpha}$ which is closed. Moreover, for each infinite ordinal α and β the following are true.

(1) if $\alpha < \beta$, then $\mathcal{A}_\alpha$ is an initial segment of $\mathcal{A}_\beta$.

(2) $\mathcal{A}_{\alpha+1} \in \mathfrak{M}_{\mu(\alpha)}$, and $b_\alpha = g_\alpha^{-1}(1) \cap a_{\mu(\alpha)}$; if $\alpha = \beta + 1$, then $g_\alpha \in \mathfrak{M}_{\mu(\beta)+1}$.

(3) if $\alpha \in \lim\omega_1$ and $\tau_\alpha \in \mathfrak{M}_{\mu(\alpha)}$ is a $\mathbb{P}_{\mathcal{A}_{\alpha+1}}$-name, then $\mu(\alpha) + 2 < \rho''_\alpha < \rho'_\alpha < \mu(\alpha+1)$ and the following are true,

  (i): $F(a_{\mu(\alpha)})$, $F(a_{\rho'_\alpha})$, $F(a_{\rho''_\alpha})$, $F(g_{\alpha+1}^{-1}(1) \cap a_{\rho'_\alpha})$, $F(g_{\alpha+1}^{-1}(1) \cap a_{\rho''_\alpha})$, $F(b_\alpha)$, p_α, τ'_α and $\{R_n^\alpha : n \in \omega\}$ are elements of $\mathfrak{M}_{\mu(\alpha+1)}$ and τ'_α is a $\mathbb{P}_{\mathcal{A}_{\alpha+2}}$ name for a subset of ω defined as in Lemma 5.2.1 with $A = F(a_{\rho'_\alpha}) \setminus F(a_{\rho''_\alpha})$ and $B = F(g_{\alpha+1}^{-1}(1) \cap a_{\rho'_\alpha}) \setminus F(g_{\alpha+1}^{-1}(1) \cap a_{\rho''_\alpha})$. Thus **1** forces that

$$\tau'_\alpha = (\tau_\alpha \cap (F(a_{\rho'_\alpha}) \setminus F(a_{\rho''_\alpha})))\triangle(F(g_{\alpha+1}^{-1}(1) \cap a_{\rho'_\alpha}) \setminus F(g_{\alpha+1}^{-1}(1) \cap a_{\rho''_\alpha}))\ .$$

  Moreover, $p_\alpha \in \mathbb{P}_{\mathcal{A}_{\alpha+1}}$ is such that $p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \in \mathbb{P}_{\mathcal{A}_{\alpha+2}}$ and

$$p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_{\mathcal{A}_{\alpha+2}}} \text{“}\,\exists^\infty i, R_i^\alpha \cap \tau'_\alpha \neq^* \emptyset\text{”}\ .$$

  (ii): For all $A \subseteq \omega$, if $\exists^\infty i$, $A \cap R_i^\alpha \neq^* \emptyset$, then $A \notin \mathcal{I}_{\mathcal{N}}$.

(4) (i): $\omega < \alpha \leq \beta \to \mathcal{D}_\alpha \subseteq \mathcal{D}_\beta$; if β is a limit, then $\mathcal{D}_\beta = \bigcup_{\alpha<\beta} \mathcal{D}_\alpha$.

(ii): $\mathcal{D}_{\alpha+1} \subseteq \mathfrak{M}_{\mu(\alpha)}$.

(iii): If α,τ'_α, τ_α, ρ'_α, ρ''_α and $\mathcal{R}^\alpha$ and $g_{\alpha+1}$ are as in (3), then for all n and i in ω, $D_n(\mathcal{A}_{\alpha+2}, \mathcal{R}^{\alpha+1}, \tau'_\alpha)$, $E_{i,n}(\mathcal{A}_{\alpha+2}, \mathcal{R}^{\alpha+1}, \tau'_\alpha)$ and $H_n^{\tau'_\alpha}$ are in $\mathcal{D}_{\alpha+2}$.

(iv): If α is a limit and $S_\alpha, T_\alpha \in \mathfrak{M}_{\mu+1}$, where $\mu = \sup\{\mu(\xi) : \xi < \alpha\}$ and $\mathcal{A}_\alpha, \{f_\xi : \xi \in T_\alpha\} \in \mathfrak{M}_{\mu+1}$ and $\{f_\xi : \xi \in T_\alpha\}$ is predense in $\mathbb{P}_{\mathcal{A}_\alpha}$, then

$$\{f_\xi : \xi \in T_\alpha\} \in \mathcal{D}_{\alpha+1} \ .$$

(v): For $\alpha > \omega$, g_α is a $\mathcal{D}_\alpha$-generic.

(5) $\mu(\alpha) > \delta_\alpha$.

The basic idea involved in (3) and (4) is to keep various predense subsets of $\mathbb{P}_{\mathcal{A}_\alpha}$ still predense in $\mathbb{P}_{\mathcal{A}_\beta}$ when $\alpha < \beta$. It is not difficult to see that (4) will guarantee that the final poset $\mathbb{P}_{\mathcal{A}}$ is ccc. To achieve (b), we handle each Y at stage $\alpha + 2$ when τ_α is a name for Y. In (3) we pose a condition which ensures that $\mathbb{P}_{\mathcal{A}_{\alpha+2}}$ will force that $Y \cap F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)})$ is not equal to $F(b_{\alpha+1}) \setminus F(b_\alpha)\}$ mod $\mathcal{I}_{\mathcal{N}}$. We then preserve this condition by preserving certain predense subsets of $\mathbb{P}_{\mathcal{A}_{\alpha+2}}$.

First of all, we show that if $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha < \omega_1\}$ satisfies (1) to (5), then $\mathcal{A}$ satisfies (a) to (c). To check that $\mathbb{P}_{\mathcal{A}}$ is ccc, suppose K is an maximal antichain in $\mathbb{P}_{\mathcal{A}}$, and $K = \{f_\alpha : \alpha \in \Gamma\}$. Let $\Sigma = \{\alpha : f_\alpha \in \mathbb{P}_{\mathcal{A}}\}$. Recall $\mathcal{D} = \{f_\alpha \colon \alpha < \omega_1\}$ is an enumeration of all countable partial functions from ω to $\{0, 1\}$ (see the discussion after Lemma 5.2.1). Let $f : \omega_1 \to \mathcal{D}$ defined by $f(\alpha) = f_\alpha$. Then by Corollary 5.1.3, $E = \{\alpha : \Sigma \cap \alpha = S_\alpha, \Gamma \cap \alpha = T_\alpha, f \restriction S_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$ and $f \restriction T_\alpha \in \mathfrak{M}_{\delta_\alpha+1}\}$ is stationary in ω_1. Let C_1 be the set of α for which $\mathbb{P}_{\mathcal{A}_\alpha} = \{f_\xi : \xi \in \Sigma \cap \alpha\}$ and for which $K \cap \mathbb{P}_{\mathcal{A}_\alpha}$ is a maximal antichain in $\mathbb{P}_{\mathcal{A}}$. $C_2 = \{\alpha \in \omega_1 : \delta_\alpha = \sup\{\mu(\beta) : \beta < \alpha\}\}$. Obviously C_1 and C_2 are cubs on ω_1.

Take $\alpha \in E \cap C_1 \cap C_2$, then $\Sigma \cap \alpha = S_\alpha$ and $\Gamma \cap \alpha = T_\alpha$. Moreover $\mathbb{P}_{\mathcal{A}_\alpha} = \{f_\xi : \xi \in \Sigma \cap \alpha\} = \operatorname{ran} f \restriction S_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$. Similarly, $K \cap \mathbb{P}_{\mathcal{A}_\alpha} = \operatorname{ran} f \restriction T_\alpha \in \mathfrak{M}_{\delta_\alpha+1}$. By (i) and (iv) of (4) $K \cap \mathbb{P}_{\mathcal{A}_\alpha}$ is predense in $\mathbb{P}_{\mathcal{A}_\beta}$ for all $\beta > \alpha$. Therefore $K \cap \mathbb{P}_{\mathcal{A}_\alpha}$ is predense in $\mathbb{P}_{\mathcal{A}}$. This means $K = K \cap \mathbb{P}_{\mathcal{A}_\alpha}$ and so is countable. Condition (c) is immediate by (2).

Now we show, by contradiction, that (b) is true. By Lemma 5.1.5, we have **1** forces that

$$\exists x \subseteq \omega \{\alpha : x \cap F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) \neq_{\mathcal{I}_\mathcal{N}} F(b_{\alpha+1}) \setminus F(b_\alpha)\} \text{ is countable } .$$

Since $\mathbb{P}_{\mathcal{A}}$ is ccc, there is a nice $\mathbb{P}_{\mathcal{A}}$-name τ for subsets of ω and $\eta \in \omega_1$ such that, for all $\alpha > \eta$,

$$\mathbf{1} \Vdash \text{“}\tau \cap F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) =_{\mathcal{I}_\mathcal{N}} F(b_{\alpha+1}) \setminus F(b_\alpha)\text{”} .$$

Since $\mathbb{P}$ is ccc, τ is a countable name. Therefore τ is in V. Therefore there are uncountably many limit α such that $\tau = \tau_\alpha$. By (5), $\{\mu(\alpha) : \alpha < \omega_1\}$ is cofinal in ω_2^V. Therefore, there is a limit α such that $\tau = \tau_\alpha \in \mathfrak{M}_{\mu(\alpha)}$ and $\alpha > \eta$.

We claim that $p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \in \mathbb{P}_{\mathcal{A}_{\alpha+2}}$ and

$$p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}\tau \cap (F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)}) \neq_{\mathcal{I}_\mathcal{N}} F(b_{\alpha+1}) \setminus F(b_\alpha))\text{”}$$

This will contradict the fact $\alpha > \eta$.

We have to prove that

$$p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}(\tau \cap (F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)})))\triangle(F(b_{\alpha+1}) \setminus F(b_\alpha)) \notin \mathcal{I}_\mathcal{N}\text{”} .$$

By (4) of Lemma 5.2.1 and (i) and (iii) of (4) in the above construction, we have

$$\mathbf{1} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“}\tau'_\alpha = (\tau_\alpha \cap (F(a_{\rho'_\alpha}) \setminus F(a_{\rho''_\alpha})))\triangle(F(g^{-1}_{\alpha+1}(1) \cap a_{\rho'_\alpha}) \setminus F(g^{-1}_{\alpha+1}(1) \cap a_{\rho''_\alpha}))\text{”} .$$

Since $\mathcal{A}$ is a good coherent sequence, it is easy to see that **1** forces that τ'_α is contained mod $\mathcal{I}_\mathcal{N}$ in $(\tau_\alpha \cap (F(a_{\mu(\alpha+1)}) \setminus F(a_{\mu(\alpha)})))\triangle(F(b_{\alpha+1}) \setminus F(b_\alpha))$. Therefore it is sufficient to prove

$$p_\alpha \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“ } \tau'_\alpha \notin \mathcal{I}_\mathcal{N} \text{” }.$$

It is obvious that property (ii) of (3) is absolute. Therefore, it suffices to prove that

$$p \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_\mathcal{A}} \text{“ } \exists^\infty i\, \tau'_\alpha \cap R_i \neq^* \emptyset \text{” }.$$

Since

$$p \cup \mathrm{Ch}^{b_{\alpha+1}}_{a_{\mu(\alpha+1)}} \Vdash_{\mathbb{P}_{\mathcal{A}_{\alpha+2}}} \text{“ } \exists^\infty i\, \tau'_\alpha \cap R_i \neq^* \emptyset \text{” },$$

by Lemma 5.2.1, it is sufficient to see that $D_n(\mathcal{A}_{\alpha+2}, \mathcal{R}^\alpha, \tau'_\alpha)$, $E_{i,n}(\mathcal{A}_{\alpha+2}, \mathcal{R}^\alpha, \tau'_\alpha)$ and $H^{\tau'}_n$ are predense in $\mathbb{P}_\mathcal{A} = \mathbb{P}_{\mathcal{A}_{\omega_1}}$. However by (i) and (iii) of (4), for all $\beta > \alpha + 2$, $D_n(\mathcal{A}_{\alpha+2}, \mathcal{R}^\alpha, \tau'_\alpha)$, $E_{i,n}(\mathcal{A}_{\alpha+2}, \mathcal{R}^\alpha, \tau'_\alpha)$ and $H^{\tau'_\alpha}_n$ are predense in $\mathbb{P}_{\mathcal{A}_\beta}$, therefore they are predense in $\mathbb{P}_{\mathcal{A}_{\omega_1}}$.

Next we construct the sequences satisfying (1) to (5). We will proceed by induction. Notice that for $\mathcal{R}^\alpha$, ρ'_α, ρ''_α and τ'_α we only have to define them when α is of the form $\beta + 1$ for β a limit ordinal. For simplicity, we will not mention them in any other case in the following construction. For g_α, we will define it at stage $\alpha + 1$. We will construct p_α at stage $\alpha + 2$.

We let $\mathcal{A}_0$ be $\emptyset$. At stage α we proceed according to the following cases.

Case 1 α is a limit.

We let $\mathcal{A}_\alpha = \bigcup_{\xi<\alpha} \mathcal{A}_\xi = \{(b_\xi, a_{\mu(\xi)}) : \xi < \alpha\}$. Let μ be the least ordinal bigger than $\sup\{\mu(\xi) : \xi < \alpha\}$ such that $\mathcal{A}_\alpha \in \mathfrak{M}_\mu$. Let $\mathcal{D}_\alpha = \bigcup_{\beta<\alpha} \mathcal{D}_\beta$. Obviously, $\mathcal{D}_\alpha$ is closed. For all $\xi < \alpha$ and $D \in \mathcal{D}_\xi$, we prove that D is predense in $\mathbb{P}_{\mathcal{A}_\alpha}$. Take any $p \in \mathbb{P}_{\mathcal{A}_\alpha}$, we can find $\eta < \alpha$ such that $\xi < \eta$ and $p \in \mathbb{P}_{\mathcal{A}_\eta}$. But by the induction hypothesis, D is predense in $\mathbb{P}_{\mathcal{A}_\eta}$, therefore there is $q \in D$ such that p and q are compatible.

Case 2 $\alpha = \beta + 1$ and $\mathcal{A}_\beta \in \mathfrak{M}_{\mu+1}$, where β is a limit and $\mu = \sup\{\mu(\xi) \colon \xi \in \beta\}$.

By the construction of $\{\mathfrak{M}_\alpha : \alpha \in \omega_2\}$ in chapter2, there is a g in $\mathfrak{M}_{\mu+2}$ which is $\mathrm{Fn}(\omega, 2)$-generic over $\mathfrak{M}_{\mu+1}$. Since $\mathbb{P}_{\mathcal{A}_\beta}$ and $\mathrm{Fn}(\omega, 2)$ are both countable and atomless, they have isomorphic dense subsets. Therefore, there is a $\mathbb{P}_{\mathcal{A}_\beta}$-generic g_β over $\mathfrak{M}_{\mu+1}$ such that $g_\beta \in \mathfrak{M}_{\mu+2}$. Let $\mu(\beta) = \mu + 2$ and $b_\beta = g_\beta^{-1}(1) \cap a_{\mu(\beta)}$. Observe that for any $M \in \mathcal{M} \cap \mathfrak{M}_{\mu+2} = \mathcal{M}_\mu$, there is a $\xi < \alpha$ such that $M \subseteq^* a_{\mu(\xi)}$ and that for such ξ, $g_\beta^{-1}(1) \cap a_{\mu(\xi)} =^* b_\xi$. Since b_ξ is a partitioner of $\mathcal{M}$, $g_\beta^{-1}(1)$ does not split M. Therefore $g_\beta^{-1}(1)$ does not split any member of $\mathcal{M} \cap \mathfrak{M}_{\mu+2}$. On the other hand, $g_\beta^{-1}(1)$ is in $\mathfrak{M}_{\mu+2}$. By Fact 2.3.3, $g_\beta^{-1}(1)$ does not split any member of $\mathcal{M} \setminus \mathfrak{M}_{\mu+2}$. Therefore, $g_\beta^{-1}(1)$ is a partitioner of $\mathcal{M}$. Thus b_β is a partitioner of $\mathcal{M}$. Since g_β is generic over $\mathfrak{M}_{\mu+1}$, by Lemma 4.2.4, every set D in $\mathfrak{M}_{\mu+1}$ which is predense in $\mathbb{P}_{\mathcal{A}_\beta}$ is still predense in $\mathbb{P}_{\mathcal{A}_{\beta+1}}$. However, by the induction hypothesis, $\mathcal{D}_\beta \subseteq \mathfrak{M}_{\mu+1}$. Therefore, every member D of $\mathcal{D}_\beta$ is still predense in $\mathbb{P}_{\mathcal{A}_{\beta+1}}$. Now let $\mathcal{D}_{\beta+1}$ be all the predense subsets of $\mathbb{P}_{\mathcal{A}_{\beta+1}}$ which are in $\mathfrak{M}_{\mu(\beta)} = \mathfrak{M}_{\mu+2}$. Obviously, $\mathcal{D}_{\beta+1}$ is closed and $\mathcal{D}_\beta \subseteq \mathcal{D}_{\beta+1}$. It is also clear that $\{f_\xi : \xi \in T_\beta\} \in \mathcal{D}_{\beta+1}$ provided $\{f_\xi : \xi \in T_\beta\}$ is predense in $\mathbb{P}_{\mathcal{A}_\beta}$ and it is in $\mathfrak{M}_{\mu+1}$.

Case 3 $\alpha = \beta + 1$, β is a limit and $\mathcal{A}_\beta \notin \mathfrak{M}_{\mu+1}$, where $\mu = \sup\{\mu(\xi) : \xi < \beta\}$.

Note $\xi < \beta$ implies that $\mathcal{A}_\xi$ is in $\mathfrak{M}_\mu$. Let g in $\mathfrak{M}_{\mu+1}$ be a Cohen generic real over $\mathfrak{M}_\mu$. Let λ be big enough such that $\{\mathcal{A}_\beta, \mathcal{D}_\beta\} \subseteq \mathfrak{M}_\lambda$ and $\lambda > \delta_\beta$. Choose a function $f \in {}^\omega\omega \cap \mathfrak{M}_{\lambda+1}$ which dominates $\mathfrak{M}_\lambda$. Also, in $\mathfrak{M}_\lambda$, choose $\{\beta_n : n \in \omega\}$ cofinal in β. We define $h \in {}^\omega\omega$ such that $h(n+1) = f(h(n)+1)$. Since $f \in \mathfrak{M}_{\lambda+1}$, $h \in \mathfrak{M}_{\lambda+1}$. Observe that for each increasing f', if $f' <^* f$ then, for almost all n, $h(n) < f'(h(n)) < f'(h(n)+1) < h(n+1)$. Let $x_n = a_{\mu(\beta_n)} \setminus \bigcup_{j<n} a_{\mu(\beta_j)}$ and $x = \bigcup_{n\in\omega} x_n \setminus h(n)$. Let $g_\beta \in {}^\omega\omega$ be such that g_β agrees with g on $\omega \setminus x$ and, for each n, agrees with g_{β_n} on $x_n \setminus h(n)$. Clearly g_β is in $\mathfrak{M}_{\lambda+1}$. Let $\mu(\beta)$ be $\lambda + 1$

and b_β be $a_{\lambda+1} \cap g_\beta^{-1}(1)$. Let $\mathcal{D}_{\beta+1} = \mathcal{D}_\beta$. We check that $\mathcal{A}_{\beta+1}$ is a good coherent sequence and g_β is a $\mathcal{D}_\beta$-generic. It will then follow that $E \in \mathcal{D}_\beta$, E will still be predense in $P_{\mathcal{A}_{\beta+1}}$. Let us use g_n to denote the restriction of the function g_{β_n} on x_n.

First we prove that $\mathcal{A}_{\beta+1}$ is a good coherent sequence. It suffices to prove that $g_\beta^{-1}(1)$ is a partitioner of $\mathcal{M}$. Let N be in $\mathcal{M}$. We need to show that $g_\beta^{-1}(1)$ does not split N. Let ξ be such that N first appears in $M_{\xi+1}$. Since $g_\beta \in M_{\lambda+1}$, by Fact 2.3.3, $g_\beta^{-1}(1)$ does not split any member of $\mathcal{M}$ which is not in $\mathfrak{M}_{\lambda+1}$. Hence we can assume that $\xi < \lambda$. We may assume that N is not contained mod finite in a_β for any $\beta < \mu$ since then N is contained in $a_{\mu(\beta_n)}$ for some n and g_β almost agrees with g_n on $a_{\mu(\beta_n)}$. Therefore, N is almost disjoint with each x_n, in which case $\xi > \mu$. Consequently, $\xi \geq \mu+2$. Now f and h are dominating over $\mathfrak{M}_\lambda$. Thus N is almost disjoint from x. On the other hand , since $g^{-1}(1)$ is in $\mathfrak{M}_{\mu+1}$ and N is not in $\mathfrak{M}_{\mu+1}$, $g^{-1}(1)$ does not split N. Finally, noting that the restrictions of g and g_β on $\omega \setminus x$ are the same, we deduce that $g_\beta^{-1}(1)$ does not split N.

Next we prove that g_β is a $\mathcal{D}_\beta$-generic.

Given E in $\mathcal{D}_\beta$, we must prove that there exists an e in E, such that $e \subseteq g_\beta$. Let $\xi \in \beta$ such that $E \in \mathcal{D}_\xi$. Take an $m \in \omega$ such that $\beta_m > \xi$. Then, for all $e \in E$, $\mathrm{dom}(e)$ is almost contained in $a_{\mu(\beta_m)}$. Define, in $\mathfrak{M}_\lambda$, a strictly increasing function F_E from ω to ω such that

(1) $F_E(0) = m$,

(2) If $F_E(n) = l$, then for all $s \in Fn(l,2)$ and k such that $m \leq k \leq l$, there is an $e \in E$ such that

$$e \subseteq s \,\overline{\cup} \bigcup_{j \leq k} g_j \,\overline{\cup}\, g \ \text{ and } \ \mathrm{dom}(e) \setminus \bigcup_{j \leq m} x_j \subseteq F_E(n+1)\ .$$

We first show that if we have F_E in $\mathfrak{M}_\lambda$ with above property, then we can find an e such that $e \subseteq g_\beta$. Indeed, since $F_E <^* f$, for almost all n, there is a k in ω

such that $h(n) \leq F_E(k) < F_E(k+1) < h(n+1)$. Choose such $n > m$ and k. Let $s = g \restriction \bigcup_{j \leq n} h(j) \cap x_j$, then $s \in \mathrm{Fn}(F_E(k), 2)$ and

$$g_\beta \restriction \bigcup_{j \leq n} x_j \,\overline{\cup}\, g = s \,\overline{\cup} \bigcup_{j \leq n} g_j \,\overline{\cup}\, g \ .$$

Therefore there is an $e \in E$ such that

$$e \subseteq g_\beta \restriction \bigcup_{j \leq n} x_j \,\overline{\cup}\, g$$

and

$$\mathrm{dom}(e) \setminus \bigcup_{j \leq m} x_j \subseteq F_E(k+1) < h(n+1) \ .$$

Since $\mathrm{dom}(e) \setminus \bigcup_{j \leq m} x_j \subseteq h(n+1)$ and $g_\beta \restriction (h(n+1) \setminus \bigcup_{j \leq n} x_j) = g \restriction (h(n+1) \setminus \bigcup_{j \leq n} x_j)$, it is clear that $e \subseteq g_\beta$.

The construction of F_E is straightforward. Suppose we have defined $F_E(n) = l$. For each $s \in \mathrm{Fn}(l, 2)$ and k such that $m \leq k \leq l$, since $s \,\overline{\cup} \bigcup_{j \leq k} g_j \,\overline{\cup}\, g$ is a $P_{\mathcal{A}_{\beta_k+1}}$ generic over $\mathfrak{M}_{\mu(\beta_j)}$, we can find an $e(s,k) \in E$ such that $e(s,k) \subseteq s \,\overline{\cup} \bigcup_{j \leq k} g_j \,\overline{\cup}\, g$. Recall that $\mathrm{dom}(e(s,k)) \subseteq^* a_{\mu_{\beta_m}} =^* \bigcup_{j \leq m} x_j$. Let $F_E(n+1)$ be bigger than l and large enough such that for all $s \in \mathrm{Fn}(l, 2)$ and k with $m \leq k \leq l$, $F_E(n+1)$ contains $\mathrm{dom}(e(s,k)) \setminus \bigcup_{j \leq m} x_j$.

Case 4 $\alpha = \beta + 2$ and β is a limit ordinal

We have to define $\mu(\beta+1)$, $g_{\beta+1}$, p_β, ρ'_β, ρ''_β and $\mathcal{R}^\beta$ satisfying the required properties. Let τ be a nice $\mathbb{P}_{\mathcal{A}_{\beta+1}}$-name for a subset of ω such that if τ_β is a nice $\mathbb{P}_{\mathcal{A}_{\beta+1}}$-name for a subset of ω and $\tau_\beta \in \mathfrak{M}_{\mu(\beta)}$, then τ is τ_β.

Let μ be $\mu(\beta)$. By Lemma 4.1.1, there are $p \in \mathbb{P}_{\mathcal{A}_{\beta+1}}$ and g in $\mathfrak{M}_{\mu+1}$ such that g is a $\mathbb{P}_{\mathcal{A}_{\beta+1}}$-generic over $\mathfrak{M}_\mu$ and there are μ', λ' and λ such that $\mu+2 < \mu' < \lambda' < \lambda$ and $\delta_{\beta+1} < \lambda$ and $\mathcal{A}_{\beta+1} \in \mathfrak{M}_\lambda$. There are also $\{R_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ such that

(i): For all $A \subseteq \omega$, if $\exists^\infty n\ A \cap R_n \neq^* \emptyset$, then $A \notin \mathcal{I}_{\mathcal{N}}$;

(ii): $p \cup g \restriction a_\lambda \Vdash_{\mathbb{P}_{\mathcal{A}'}}$ "$\forall^\infty n\ R_n \cap ((\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda'}) \setminus F(a_{\mu'}))) \notin \mathcal{I}_{\mathcal{N}}$", where $\mathcal{A}' = \mathcal{A}_{\beta+1}(g, a_\lambda)$.

(iii): $\{R_n : n \in \omega\} \cup \{F(g^{-1}(1)), F(a_{\mu'}), F(a_{\lambda'})\} \subseteq \mathfrak{M}_\lambda$.

As argued in the previous cases, $g^{-1}(1)$ is a partitioner of $\mathcal{M}$. Now let $\mu(\beta+1) = \lambda$, $g_{\beta+1} = g$, $b_{\beta+1} = g^{-1}(1) \cap a_\lambda$, $\rho'_\beta = \lambda'$ and $\rho''_\beta = \mu'$. Let $p_\alpha = p$ and for each n, $R^\beta_n = R_n$. If $\tau_\beta \in \mathfrak{M}_{\mu(\beta)}$ is nice $\mathbb{P}_{\mathcal{A}_\beta}$-name, then τ'_α is defined as in Lemma 5.2.1. It is easy to see that

$$\mathbf{1} \Vdash_{\mathbb{P}_{\mathcal{A}'}} \text{“} \tau'_\beta =_{\mathcal{I}_{\mathcal{N}}} (\tau \triangle F(g^{-1}(1))) \cap (F(a_{\lambda'}) \setminus F(a_{\mu'})) \text{”} .$$

Otherwise, let $\tau'_\beta = \emptyset$. Since τ'_β is definable from the sets in $\mathfrak{M}_{\mu(\beta+1)}$, τ'_β is also in $\mathfrak{M}_{\mu(\beta+1)}$. For the same reason, all the following dense subsets of $\mathbb{P}_{\mathcal{A}_{\beta+2}}$ are in $\mathfrak{M}_{\mu(\beta+1)}$: $D_n(\mathcal{A}_{\beta+2}, \mathcal{R}^\beta, \tau'_\beta)$, $E_{i,n}(\mathcal{A}_{\beta+2}, \mathcal{R}^\beta, \tau'_\beta)$ and $H^{\tau'_\beta}_n$.

Since $\mathcal{D}_{\beta+1} \subseteq \mathfrak{M}_{\mu(\beta)}$, $g_{\beta+1}$ is a $\mathcal{D}_{\beta+1}$-generic. Thus each member of $\mathcal{D}_{\beta+1}$ is still predense in $\mathbb{P}_{\mathcal{A}_{\beta+2}}$. Let $\mathcal{D}_{\beta+2}$ be the set of all of predense subsets of $\mathbb{P}_{\mathcal{A}_{\beta+2}}$ in $\mathfrak{M}_{\mu(\beta+1)}$.

Case 5 $\quad \alpha = \beta + n + 1$, where β is a limit ordinal and $n \geq 2$.

Since $\mathcal{A}_{\beta+n} \in \mathfrak{M}_{\mu(\beta+n-1)}$, there is a g in $\mathfrak{M}_{\mu(\beta+n-1)+1}$ which is $\mathbb{P}_{\mathcal{A}_{\beta+n}}$-generic over $\mathfrak{M}_{\mu(\beta+n-1)}$. Take any limit λ such that λ is bigger than both $\delta_{\beta+n}$ and $\mu(\beta+n-1)$ and $\mathfrak{M}_\lambda$ contains $\mathcal{A}_{\beta+n}$ and $\mathcal{D}_{\beta+n}$. Let $\mu(\beta+n)$ be λ, $g_{\beta+n}$ be g, and let $\mathcal{D}_{\beta+n+1}$ be the set of all predense subsets of $\mathbb{P}_{\mathcal{A}_{\beta+n+2}}$ which are in $\mathfrak{M}_\lambda$. It is easily checked that all the conditions from (1) to (5) are satisfied. $\square$

By Theorem 5.1.1, if we take the poset $\mathbb{P} * \mathbb{P}_{\mathcal{A}}$ and the good coherent sequence $\mathcal{A}$, we are almost done provided $\mathcal{A}$ is in V. We try to use PFA, via a dense-subset argument, to show that such $\mathcal{A}$ can exist in V. However, the property (b) can not be decided by $\aleph_1$ many dense subsets of $\mathbb{P} * \mathbb{P}_{\mathcal{A}}$. There are actually $\aleph_2$ many reals in the forcing extension. We fix this in the rest of the Chapter.

In the next section, we work on an improved good coherent sequence which will be given by the next Theorem. We will basically force, by a ccc forcing notion $\mathbb{Q}$, countably many Hausdorff-Luzin type gaps simultaneously which will ensure (b) is true. However, any Hausdorff-Luzin type gap can be decided by $\aleph_1$ many dense subsets. The next theorem produce a better good coherent sequence which enable us to find the ccc forcing notion $\mathbb{Q}$.

THEOREM 5.2.2. *Let G be a $\mathbb{P}$ generic filter over V. Suppose $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \omega_1\}$ satisfies the conclusion of Theorem 5.1.1. Then in $V[G]$, there is a subsequence $\{\delta(\alpha) : \alpha < \omega_1\}$ of ω_1 such that*

(i): *$\forall \alpha, \widetilde{\mathcal{A}}_\alpha = \{(b_{\delta(\xi)}, a_{\mu(\delta(\xi))}) : \xi < \alpha\}$ is in $\mathfrak{M}_{\mu(\delta(\alpha))}$,*

(ii): *$\mathbb{P}_{\widetilde{\mathcal{A}}_{\omega_1}}$ forces that for all $Y \subseteq \omega$ there exists β in ω_1 such that either $\forall \xi > \eta > \beta$,*

$$Y \cap [F(a_{\mu(\delta(\xi))}) \setminus (F(a_{\mu(\delta(\eta))}) \cup F(b_{\mu(\delta(\xi))}))] \notin \mathcal{I}_{\mathcal{N}} \ ;$$

or $\forall \xi > \eta > \beta$,

$$(F(b_{\mu(\delta(\xi))}) \setminus F(b_{\mu(\delta(\eta))})) \setminus Y \notin \mathcal{I}_{\mathcal{N}} \ .$$

We denote the least such β, by β_Y.

(iii): *$\mathbb{P}_{\widetilde{\mathcal{A}}_{\omega_1}}$ is ccc.*

Proof: Let H be a $\mathbb{P}_{\mathcal{A}}$-generic filter over $V[G]$. Since CH holds in $V[G]$ and $\mathbb{P}_{\mathcal{A}}$ is a ccc poset with size ω_1, CH still holds in $V[G][H]$. In $V[G][H]$, let $\{Y_\alpha : \alpha \in \omega_1\}$ enumerate $\mathcal{P}(\omega)$. For $\alpha \in \omega_1$, let

$$\Gamma_\alpha = \{\xi \in \omega_1 \colon Y_\alpha \cap (F(a_{\mu(\xi+1)}) \setminus F(a_{\mu(\xi)})) \neq_{\mathcal{I}_{\mathcal{N}}} F(b_{\xi+1}) \setminus F(b_\xi)\} \ ,$$

$$\Gamma^0_\alpha = \{\xi \in \Gamma_\alpha : [Y_\alpha \cap (F(a_{\mu(\xi+1)}) \setminus F(a_{\mu(\xi)}))] \setminus [F(b_{\xi+1}) \setminus F(b_\xi)] \notin \mathcal{I}_{\mathcal{N}}\} \ ,$$

$$\Gamma^1_\alpha = \{\xi \in \Gamma_\alpha : [F(b_{\xi+1}) \setminus F(b_\xi)] \setminus [Y_\alpha \cap (F(a_{\xi+1}) \setminus F(a_\xi))] \notin \mathcal{I}_\mathcal{N}\} ,$$

Obviously,

$$\Gamma^0_\alpha = \{\xi \in \Gamma_\alpha : Y_\alpha \cap [F(a_{\mu(\xi+1)}) \setminus (F(a_{\mu(\xi)}) \cup F(b_{\xi+1}))] \notin \mathcal{I}_\mathcal{N}\} ,$$

and

$$\Gamma^1_\alpha = \{\xi \in \Gamma_\alpha : (F(b_{\xi+1}) \setminus F(b_\xi)) \setminus Y_\alpha \notin \mathcal{I}_\mathcal{N}\} ,$$

Clearly $\Gamma_\alpha = \Gamma^0_\alpha \cup \Gamma^1_\alpha$. Therefore, for each α, there is $i(\alpha) \in \{0,1\}$ such that $\Gamma^{i(\alpha)}_\alpha$ is uncountable. Let C_α be the set of all the limit points of $\Gamma^{i(\alpha)}_\alpha$. Then C_α is a cub in ω_1 and there is an element of $\Gamma^{i(\alpha)}_\alpha$ between each two elements of C_α. If $i(\alpha) = 0$. Then for any $\eta < \xi \in C_\alpha$, take $\beta \in \Gamma^0_\alpha$ such that $\eta < \beta < \xi$, we have

$$Y_\alpha \cap [F(a_{\mu(\beta+1)}) \setminus (F(a_{\mu(\beta)}) \cup F(b_{\beta+1}))] \notin \mathcal{I}_\mathcal{N} .$$

Since $F(a_{\mu(\eta)}) \subseteq_{\mathcal{I}_\mathcal{N}} F(a_{\mu(\beta)})$, $F(a_{\mu(\beta+1)}) \subseteq_{\mathcal{I}_\mathcal{N}} F(a_{\mu(\xi)})$ and
$F(b_{\beta+1}) =_{\mathcal{I}_\mathcal{N}} F(a_{\mu(\beta+1)} \cap b_\xi)$. We have

$$\begin{aligned} F(a_{\mu(\beta+1)})\setminus&(F(a_{\mu(\beta)}) \cup F(b_{\beta+1})) \\ &\subseteq_{\mathcal{I}_\mathcal{N}} F(a_{\mu(\beta+1)}) \setminus (F(a_{\mu(\eta)}) \cup F(a_{\mu(\beta+1)} \cap b_\xi)) \\ &=_{\mathcal{I}_\mathcal{N}} F(a_{\mu(\beta+1)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \\ &\subseteq_{\mathcal{I}_\mathcal{N}} F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \end{aligned}$$

Therefore,

$$Y_\alpha \cap [F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi))] \notin \mathcal{I}_\mathcal{N} .$$

Similarly, if $i(\alpha) = 1$, then for all $\eta < \xi \in C_\alpha$,

$$(F(b_\xi) \setminus F(b_\eta)) \setminus Y_\alpha \notin \mathcal{I}_\mathcal{N} .$$

Set $C = \mathop{\triangle}\limits_{\delta<\omega_1} C_\delta = \{\xi : \forall \delta < \xi, \xi \in C_\delta\}$. For all $\alpha < \omega_1$, if $\eta < \xi \in C$ and $\alpha < \eta$ then

$$i(\alpha) = 0 \to Y_\alpha \cap F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \notin \mathcal{I}_\mathcal{N} ,$$

and

$$i(\alpha) = 1 \to (F(b_\xi) \setminus F(b_\eta)) \setminus Y_\alpha \notin \mathcal{I}_{\mathcal{N}} .$$

Since C is a cub in $V[G][H]$ and $\mathbb{P}_{\mathcal{A}}$ is ccc, C contains a cub C^0 in $V[G]$. Furthermore, since $\mathbb{P}$ is a countably closed forcing, any countable subset of $\mathcal{A}$ is actually in V. By the construction of $\mathcal{A}$, $\{\mu(\alpha) : \alpha < \omega_1\}$ is cofinal in ω_2^V. Therefore we can find a subsequence $C^1 = \{\delta(\alpha) : \alpha < \omega_1\}$ of C^0 such that for all $\alpha \in \omega_1$,

$$\{(b_{\delta(\xi)}, a_{\mu(\delta(\xi))}) : \xi \in \alpha\} \in \mathfrak{M}_{\mu(\delta(\alpha))} .$$

Obviously $\mathbb{P}_{\mathcal{A}} = \mathbb{P}_{\widetilde{\mathcal{A}}}$ and $\mathbb{P}_{\widetilde{\mathcal{A}}}$ is ccc. We only need to prove (ii). Since $\mathbb{P}_{\mathcal{A}}$ is almost homogeneous we only need to prove that property stated in (ii) holds in $V[G][H]$.

In $V[G][H]$, let $Y \subseteq \omega$ and choose a β in ω_1 such that $Y = Y_\beta$. Obviously if $i(\beta) = 0$ then for all $\xi > \eta > \beta$ both $\mu(\delta(\eta))$ and $\mu(\delta(\xi))$ are in $C \setminus \beta$, hence

$$Y \cap [F(a_{\mu(\delta(\xi))}) \setminus (F(a_{\mu(\delta(\eta))}) \cup F(b_{\delta(\eta)}))] \notin \mathcal{I}_{\mathcal{N}} .$$

Similarly if $i(\beta) = 1$ then for all $\eta > \xi$ both in $C \setminus \beta$,

$$(F(b_{\delta(\xi)}) \setminus F(b_{\delta(\eta)})) \setminus Y \notin \mathcal{I}_{\mathcal{N}} .$$

□

From this point on we will only use the sequence $\widetilde{\mathcal{A}}$ in Theorem 5.2.2, but for convenience we simply assume $\mathcal{A} = \widetilde{\mathcal{A}}$.

Therefore we have the following

THEOREM 5.2.3. *In $V^{\mathbb{P}}$, there is a good coherent sequence $\mathcal{A} = \{(b_\alpha, a_{\mu(\alpha)}) : \alpha \in \omega_1\}$ such that*

(i): $\forall \alpha\ \mathcal{A}_\alpha \in \mathrm{RCSQ}$,

(ii): $\mathbb{P}_{\mathcal{A}}$ *is ccc,*

(iii): $\mathbb{P}_{\mathcal{A}}$ *forces that, for all* $Y \subseteq \omega$*, there exists* β *such that either*

$$\forall \eta, \xi > \beta, (\eta < \xi \to Y \cap F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \notin \mathcal{I}_{\mathcal{N}}),$$

or

$$\forall \eta, \xi > \beta, (\eta < \xi \to (F(b_\xi) \setminus F(b_\eta)) \setminus Y \notin \mathcal{I}_{\mathcal{N}}).$$

□

For the good coherent sequence $\mathcal{A}$, $\mathbb{P}_{\mathcal{A}}$ adds an $X \subseteq \omega$ such that $X \cap a_{\mu(\alpha)} =^* b_\alpha$ for all $\alpha \in \omega_1$. We will show that there is a ccc forcing $\mathbb{Q}$ in $V^{\mathbb{P} * \mathbb{P}_{\mathcal{A}}}$ such that $\mathbb{Q}$ adds some Hausdorff-Luzin type gap which will imply that there is no $Y \subseteq \omega$ such that $F(X) = Y$. We intend to apply PFA as follows. Since $\mathbb{P} * \mathbb{P}_{\mathcal{A}} * \mathbb{Q}$ is proper and we only need $\aleph_1$ many dense subsets to determine the above $\mathcal{A}$, X and these Hausdorff-Luzin gaps as well as the related properties, we can get them in V.

We need more notions to define the poset $\mathbb{Q}$. In the next section we will modify the usual techniques in order to add what might be called countably many simultaneous Hausdorff-Luzin type gaps.

5.3. Hausdorff-Luzin type gaps

DEFINITION 5.3.1. Suppose $I = \{c_\alpha : \alpha \in \omega_1\}$ and $J = \{e_\alpha : \alpha \in \omega_1\}$ are two sequences of infinite subsets of ω. We say that I is *near to* J if for any subset Y of ω, either $\{\alpha : c_\alpha \subseteq^* Y\}$ is countable or $\{\alpha : Y \cap e_\alpha$ is finite $\}$ is countable. We say I and J are tight or $\{(c_\alpha, e_\alpha) : \alpha \in \omega_1\}$ is tight if, for any pair Σ and Γ of uncountable subsets of ω_1 and $n \in \omega$, there exist uncountable subsets Σ' and Γ' of Σ and Γ respectively such that

$$\left| \bigcap_{\alpha \in \Sigma'} c_\alpha \cap \bigcap_{\alpha \in \Gamma'} e_\alpha \right| \geq n \ .$$

FACT 5.3.2. *If* $\{c_\alpha : \alpha \in \omega_1\}$ *is near to* $\{e_\alpha : \alpha \in \omega_1\}$*, then* $\{c_\alpha : \alpha \in \omega_1\}$ *and* $\{e_\alpha : \alpha \in \omega_1\}$ *are tight .*

Proof: Suppose Σ and Γ are uncountable subsets of ω_1 and $n \in \omega$. We will define uncountable sets $\{\Sigma_i, \Gamma_i : i \leq n\}$ and integers $\{m_i : i < n\}$ such that

(1) $\Sigma_n \subseteq \cdots \subseteq \Sigma_0 \subseteq \Sigma$, $\Gamma_n \subseteq \cdots \subseteq \Gamma_0 \subseteq \Gamma$ and $m_0 < \cdots < m_{n-1}$.

(2) $m_i \in \bigcap_{\alpha\in\Sigma_{i+1}} c_\alpha \cap \bigcap_{\alpha\in\Gamma_{i+1}} e_\alpha$

Obviously, if we have the above sequence and we let , $\Sigma' = \Sigma_n$ and $\Gamma' = \Gamma_n$ then

$$\{m_0, \ldots, m_{n-1}\} \subseteq \bigcap_{\alpha\in\Sigma'} c_\alpha \cap \bigcap_{\alpha\in\Gamma'} e_\alpha \ .$$

Let $\Sigma_0 = \Sigma$ and $\Gamma_0 = \Gamma$. At stage $i > 0$, let Y be the set of $n \in \omega$ which is in c_ξ for uncountably many ξ in Σ_i. Then the set of $\xi \in \Sigma_i$ for which $a_\xi \not\subseteq^* Y$ is countable. Since $\{c_\alpha : \alpha \in \omega_1\}$ is near to $\{e_\alpha : \alpha \in \omega_1\}$, for all but countably many ξ in Γ_i, $Y \cap e_\xi$ is infinite. Let $\Gamma'_i = \{\xi \in \Gamma_i : Y \cap e_\xi \neq^* \emptyset\}$. Γ'_i is uncountable. For each $\xi \in \Gamma'_i$, take $n_\xi \in Y \cap e_\xi \setminus \max\{m_j : j < i\}$. There is a n^* such that $n^* = n_\xi$ for uncountably many ξ. Let $m_i = n^*$ and $\Sigma_{i+1} = \{\xi \in \Sigma_i : n \in c_\xi\}$ and $\Gamma_{i+1} = \{\xi \in \Gamma'_i : n^* = n_\xi\}$. $\square$

FACT 5.3.3. *If* $\{(c_\alpha, e_\alpha) : \alpha \in \omega_1\}$ *is tight , then there is a ccc forcing* $\mathbb{Q}$ *with size* ω_1 *such that* **1** *forces*

$\exists I \in [\omega_1]^{\omega_1}, \forall k \in \omega, \{(c_{\alpha+k}\, e_{\alpha+k}) : \alpha \in I\}$ *is a Hausdorff-Luzin type family .*

Proof: Let $\mathbb{R} = \{(H, n) : H \in [\omega_1]^{<\omega}, n \in \omega\}$. Define $(H, n) < (K, m)$ if the following holds

(1) $K \subseteq H$ and $m \leq n$;

(2) if $\beta < \alpha$, $\alpha \in K$, and $\beta \in H \setminus K$ then for all $k < m$, $|c_{\alpha+k} \cap e_{\beta+k}| > m$.

It is easy to see that $(\mathbb{R}, <)$ is a poset.

Claim 1 $(\mathbb{R}, <)$ is ccc.

Suppose $\mathbb{R}$ has an uncountable antichain A. By the $\triangle$-system Lemma, we can assume A has the form $\{(H_\alpha, n) : \alpha \in \omega_1\}$ for some n and that $\{H_\alpha : \alpha \in \omega_1\}$ forms a $\triangle$-system with root H. We can also assume $H_\alpha = H \cup \{\xi_1^\alpha, \ldots, \xi_m^\alpha\}$ for all α and that, for $\alpha < \beta$, $H < \{\xi_1^\alpha, \ldots, \xi_m^\alpha\} < \{\xi_1^\beta, \ldots, \xi_m^\beta\}$. By applying the fact that $\{(c_\alpha, e_\alpha) : \alpha \in \omega\}$ is tight finitely many times, we can find uncountable subsets Γ_1 and Γ_2 of ω_1 such that for each $k < n$ and for all $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$ and $i, j \in \{1, \ldots, m\}$,

$$|c_{\xi_i^\alpha + k} \cap e_{\xi_j^\beta + k}| > n \text{ and } |e_{\xi_i^\alpha + k} \cap c_{\xi_j^\beta + k}| > n \ .$$

Now it is easy to see that for any $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$, $(H_\alpha \cup H_\beta, n)$ is stronger than both (H_α, n) and (H_β, n).

Next we prove that forcing with a subset of $\mathbb{R}$ will add a lot of Hausdorff-Luzin gaps as described. Let $\mathbb{R}_\alpha = \{(H, n) \in \mathbb{R}, H \cap \alpha = \emptyset\}$.

We need the following

Claim 2 There exits α such that for all $\beta > \alpha$, $D_\beta^\alpha = \{(H, n) \in \mathbb{R}_\alpha : H \not\subseteq \beta\}$ is dense in $\mathbb{R}_\alpha$.

Suppose the Claim is false. Then for any $\alpha \in \omega_1$, there is a $g(\alpha) > \alpha$ such that $D_{g(\alpha)}^\alpha$ is not dense in $\mathbb{R}_\alpha$. We can define an increasing sequence $\{\xi_\alpha : \alpha \in \omega_1\}$ and $\{(H_\alpha, n_\alpha) : \alpha \in \omega_1\}$ such that $\xi_{\alpha+1} = g(\xi_\alpha)$ and (H_α, n_α) witnesses that $D_{\xi_{\alpha+1}}^{\xi_\alpha}$ is not dense in $\mathbb{R}_{\xi_\alpha}$.

Since $\mathbb{R}$ is ccc, there exists $\alpha < \beta$ such that (H_α, n_α) and (H_β, n_β) are compatible. By the choice of (H_α, n_α), this will imply that $H_\beta \subseteq \xi_{\alpha+1} \leq \xi_\beta$. However, $H_\beta \cap \xi_\beta$ is empty which is contradiction. Thus we finish proof the Claim 2.

Fix an α^* which satisfies Claim 2. Let $\mathbb{Q}$ be $\mathbb{R}_{\alpha^*}$. Obviously $\mathbb{Q}$ is ccc. We check that $\mathbb{Q}$ also adds appropriate subsets I of ω_1. Suppose G be a $\mathbb{Q}$-generic. Let $I = \bigcup\{H : \exists n\, (H, n) \in G\}$. Since for all $\beta > \alpha$, $D_\beta^{\alpha^*}$ is dense in $\mathbb{Q}$, I is uncountable. Let $k \in \omega$, we prove that $\{(c_{\alpha+k}, e_{\alpha+k}) : \alpha \in I\}$ is a Hausdorff-Luzin type gap. Let $\alpha \in I$ and $m \in \omega$. We have to show that $\{\beta \in I : \beta < \alpha$ and $|c_{\alpha+k} \cap e_{\beta+k}| < m\}$ is finite. Pick $(H, n) \in G$ such that $\alpha \in H$ and $n > m + k$. Take any $\beta \in I \cap \alpha \setminus H$. There is $(K, l) \in G$ such that $\beta \in K$ and $(K, l) < (H, n)$ and so that $|c_{\alpha+k} \cap e_{\alpha+k}| > n \geq m$ Thus $\{\beta \in I : \beta < \alpha$ and $|c_{\alpha+n} \cap e_{\beta+n}| < m\} \subseteq H$. We are done. $\square$

Working in $V^{\mathbb{P}}$, for each limit $\alpha \in \omega_1$ and $k \in \mathbb{N}$, let

$$
\begin{aligned}
d_{\alpha+\omega} &= F(a_{\mu(\alpha+\omega)}) \\
c_{\alpha+\omega} &= F(b_{\alpha+\omega}) \cap d_{\alpha+\omega}, \\
e_{\alpha+\omega} &= d_{\alpha+\omega} \setminus c_{\alpha+\omega} \\
d_{\alpha+k} &= (F(a_{\mu(\alpha+k)}) \setminus \bigcup_{i<k} F(a_{\mu(\alpha+i)})) \cap d_{\alpha+\omega} \ , \\
c_{\alpha+k} &= c_{\alpha+\omega} \cap d_{\alpha+k} \\
e_{\alpha+k} &= d_{\alpha+k} \setminus c_{\alpha+k}
\end{aligned}
$$

Recall we call sequence $\{(c_{\alpha+k}, e_{\alpha+k}) \colon \alpha \in \lim\omega_1$ and $0 < k \leq \omega\}$ a derived sequence of F and $\mathcal{A}$ and it is clear that for all $\alpha \in \lim\omega_1$ and $k \in \mathbb{N}$, $c_{\alpha+k} \subseteq c_{\alpha+\omega}$ and $e_{\alpha+k} \subseteq e_{\alpha+\omega}$ and also that $k < l \in \mathbb{N}$ implies $c_{\alpha+k}$, $e_{\alpha+k}$, $c_{\alpha+l}$ and $e_{\alpha+l}$ are mutually disjoint. Furthermore, each of the following holds:

$$
\begin{aligned}
c_{\alpha+\omega} &=_{\mathcal{I}_{\mathcal{N}}} F(b_{\alpha+\omega}) \\
e_{\alpha+\omega} &=_{\mathcal{I}_{\mathcal{N}}} F(a_{\mu(\alpha+\omega)}) \setminus F(b_{\alpha+\omega}) \\
c_{\alpha+k} &=_{\mathcal{I}_{\mathcal{N}}} F(b_{\alpha+k}) \setminus F(b_{\alpha+k-1}) \\
e_{\alpha+k} &=_{\mathcal{I}_{\mathcal{N}}} F(a_{\mu(\alpha+k)}) \setminus [F(a_{\mu(\alpha+k)}) \cup F(b_{\alpha+k})]
\end{aligned}
$$

LEMMA 5.3.4. *In $V^{\mathbb{P} * \dot{\mathbb{P}}_{\mathcal{A}}}$, $\{c_\alpha : \alpha$ is a successor$\}$ is near to $\{e_\alpha : \alpha$ is a successor$\}$.*

Proof: Let $Y \subseteq \omega$. Suppose there are uncountably many successor ordinals

α such that $c_\alpha \subseteq Y$. Recall that β_Y is the minimal ordinal β such that either

$$\forall \eta, \xi > \beta, (\eta < \xi \rightarrow Y \cap F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \notin \mathcal{I}_\mathcal{N}),$$

or

$$\forall \eta, \xi > \beta, (\eta < \xi \rightarrow (F(b_\xi) \setminus F(b_\eta)) \setminus Y \notin \mathcal{I}_\mathcal{N}).$$

We will show that for any successor $\alpha > \beta_Y$, $e_\alpha \cap Y$ is infinite. Take $\alpha > \beta_Y$ such that $c_\alpha \subseteq Y$. Since $c_\alpha =_{\mathcal{I}_\mathcal{N}} F(b_\alpha) \setminus F(b_{\alpha-1})$, it is not true that

$$\forall \eta, \xi > \beta_Y, (\eta < \xi \rightarrow (F(b_\xi) \setminus F(b_\eta)) \setminus Y \notin \mathcal{I}_\mathcal{N}).$$

Therefore

$$\forall \eta, \xi > \beta_Y, (\eta < \xi \rightarrow Y \cap F(a_{\mu(\xi)}) \setminus (F(a_{\mu(\eta)}) \cup F(b_\xi)) \notin \mathcal{I}_\mathcal{N}),$$

and so for all successor $\alpha > \beta_Y$,

$$Y \cap F(a_{\mu(\alpha)}) \setminus (F(a_{\mu(\alpha-1)}) \cup F(b_\alpha)) \notin \mathcal{I}_\mathcal{N}.$$

But

$$e_\alpha =_{\mathcal{I}_\mathcal{N}} F(a_{\mu(\alpha)}) \setminus (F(a_{\mu(\alpha-1)}) \cup F(b_\alpha)) \ ,$$

hence $Y \cap e_\alpha$ is infinite since it is not in $\mathcal{I}_\mathcal{N}$. □

By Fact 5.3.3, we have

THEOREM 5.3.5. *In $V^{\mathbb{P}*\mathbb{P}'_\mathcal{A}}$, there is a ccc forcing $\mathbb{Q}$ with size ω_1 such that*

$\mathbf{1} \Vdash_\mathbb{Q}$ "$\exists I \in [\omega_1]^{\omega_1}$ *such that* $\forall k \in \mathbb{N}$,

$\{(c_{\alpha+k}, e_{\alpha+k}) : \alpha \in I\}$ *is a Hausdorff-Luzin type family"*

□

By Theorem 5.2.3 and Theorem 5.3.5, we have now are ready to prove Theorem 2.4.4.

Proof of Theorem 2.4.4: Let $\dot{Q}$ be the $\mathbb{P} * \mathbb{P}_\mathcal{A}$-name for the poset in Theorem 5.3.5 and let $\mathbb{D}$ be the the poset $\mathbb{P} * \mathbb{P}_\mathcal{A} * \dot{Q}$. Obviously in $V^\mathbb{P}$, $\mathbb{P}_\mathcal{A}$ adds an X which uniformizes $\mathcal{A}$. Let $\{(c_{\alpha+k}, e_{\alpha+k})\colon \alpha \in \lim\omega_1$ and $k \in \mathbb{N}$ or $k = \omega\}$ be the derived sequence obtained before Lemma 5.3.4 and Σ be the sequence $I \in V^{\mathbb{D}}$ obtained by Theorem 5.3.5. We enumerate I by $\{\sigma_\alpha : \alpha \in \omega_1\}$. It is easily checked that these sequences and sets satisfy the statement of the Theorem. $\square$

This completes the proof that PFA$^+$ $+\diamondsuit(S^2_0)$ implies there is a subalgebra of a representable algebra which is not representable. We should remark that we have no information as to whether the hypotheses can be weakened. The most interesting questions are to ask if PFA$^+$ can be weakened to PFA and, still more important, if the consistency can be established relative to ZFC without any large cardinal assumption.

CHAPTER 6

Pseudoradialness under CH

As indicated in the introduction, we will now be discussing the relationship between the class of regular pseudoradial spaces and the spaces which are of the form $\omega \cup \{p\}$ for some ultrafilter p on ω. Specifically the question of whether these spaces are sub pseudoradial.

In this chapterwe introduce the concept of pseudoradial space and discuss the sub pseudoradialness of space $\omega \cup \{p\}$ under the assumption CH or $\mathfrak{p} = \mathfrak{c}$.

6.1. Definition

DEFINITION 6.1.1. A sequence $\{x_\alpha : \alpha < \lambda\}$ of points in X is said to be *convergent to* x provided that for each neighbourhood U of x, there exists $\alpha < \lambda$ such that $\{x_\beta : \beta > \alpha\} \subseteq U$. (We will call any transfinite sequence a sequence).

A space X is *radial* if for each subset A of X and x in $\bar{A}$, there is a sequence in A convergent to x. The pseudoradial spaces are characterized by a natural relaxation of this property. This is analogous to, and motivated, by the relationship between sequential spaces and Frechet spaces.

DEFINITION 6.1.2. A subset A of a space of X is said to be *radially closed* if no sequence in A is convergent to a point outside of A. A space X is said to be *pseudoradial* provided that any radially closed subset of X is closed in X.

In [AIG86] the authors asked if $\omega \cup \{p\}$ is a subspace of a pseudoradial space for $p \in \beta\omega \setminus \omega$, where $\omega \cup \{p\}$ takes the subspace topology in $\beta\omega$, i.e. p is the only non-isolated point and a neighbourhood of p is $A \cup \{p\}$ for $A \in p$. It is proved in [Zho94] that under the assumption $\mathfrak{p} = \mathfrak{c}$, each space with countable tightness is a subspace of some pseudoradial space. It is essentially proved that, under the assumption $\mathfrak{p} = \mathfrak{c}$, $\omega \cup \{p\}$ is a subspace of a Hausdorff pseudoradial space for $p \in \beta\omega \setminus \omega$. However the pseudoradial space constructed in [Zho94] is not regular. In a communication with the author, J. Vaughan asked if it is possible to make the pseudoradial space in [Zho94] regular. In [Nyi92] P. Nyikos asked whether each topological space can be embedded into a pseudoradial space (Problem 6.22). Our discussion will be focused on the class of regular spaces. We first prove in the next section that, under CH, $\omega \cup \{p\}$ is a subspace of a regular pseudoradial space for each ultrafilter p on ω. However, the question of whether CH can be replaced by MA remains open.

6.2. Under CH

If p is a $P_{\mathfrak{c}}$ point on ω , let $\{A_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ be a strictly decreasing modulo finite base for p and let B be the Boolean Algebra generated by $\{A_\alpha : \alpha \in \mathfrak{c}\} \cup [\omega]^{<\omega}$, then it is easy to see that the Stone space of B is just $\omega \cup [\omega, \mathfrak{c}+1]$ where $[\omega, \mathfrak{c}+1]$ has the usual ordinal topology and $\omega \cup \{\mathfrak{c}\}$ is homomorphic to $\omega \cup \{p\}$. An immediate corollary is that , under CH, if p is a P-point, then $\omega \cup \{p\}$ can be embedded into a zero-dimensional Hausdorff compact pseudoradial space.

However, we have the following result about non P-points.

LEMMA 6.2.1 ($\mathrm{P}=\mathfrak{c}$). *If p is not a P-point, then $\omega \cup \{p\}$ can be embedded in a zero-dimensional pseudoradial space.*

Proof: Let $p = \{A_\alpha : \alpha < \mathfrak{c}\}$ such that $\{A_i : i < \omega\}$ witnesses that p is not a P-point. First of all, we will construct a base $\{B_\alpha : \alpha < \mathfrak{c}\}$ of p and an almost disjoint family $\{C_\alpha : \alpha < \mathfrak{c}\}$ such that, for $\xi, \alpha \in \mathfrak{c}$, the following are true:

$(i)_\alpha$ $B_\alpha \subseteq A_\alpha$,

$(ii)_{\xi\alpha}$ $\xi \leq \alpha \to C_\alpha \subseteq^* B_\xi$,

$(iii)_{\xi\alpha}$ $\alpha < \xi \to C_\alpha \cap B_\xi =^* \emptyset$.

Without loss of generality, we can assume that $\{A_n : n \in \omega\}$ is strictly decreasing with empty intersection. We also assume that $A_0 = \omega$. It is easy to construct $\{B_\xi, C_\xi : \xi \leq \omega\}$ with $B_n = A_n$ for $n \in \omega$. Suppose we have constructed $\{B_\xi, C_\xi : \xi < \alpha\}$ with $\alpha \geq \omega$ and such that $(i)_\xi$, $(ii)_{\xi\eta}$ and $(iii)_{\xi\eta}$ are true for each $\xi, \eta < \alpha$. We have to define B_α and C_α. For $\xi \in \alpha$ and $n \in \omega$, if $\xi \geq \omega$, then $C_\xi \subseteq^* A_n$. By $\mathfrak{p} = \mathfrak{c}$, there is a $D \subseteq \omega$ such that D is a pseudo-intersection of $\{A_n : n \in \omega\}$ and, for $\omega \leq \xi < \alpha$, $C_\xi \subseteq^* D$. Indeed, we can take, for $\omega \leq \xi < \alpha$, an $f_\xi \in {}^\omega\omega$ such that for each n, $C_\xi \setminus A_n \subseteq f_\xi(n)$. Since $\mathfrak{p} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{c}$, hence we can choose f which is an upper bound of $\{f_\xi : \omega \leq \xi < \alpha\}$ in $({}^\omega\omega, <^*)$. We assume f is strictly increasing. Let $D = \bigcup_{n\in\omega} A_n \cap f(n)$. It is not difficult to see that D works. By the choice of D, we know that D is not a member of p. Let $B_\alpha = A_\alpha \cap B_\omega \setminus D$. Let C_α be any pseudo-intersection of $\{B_\xi : \xi \leq \alpha\}$. It is easy to check that B_α and C_α satisfies the requirements.

Next we will use the family $\{B_\alpha, C_\alpha : \alpha \in \mathfrak{c}\}$ to construct a topology τ on $X = \omega \cup ((\mathfrak{c}+1) \times \{0\})$ by assigning a neighbourhood system for each point. First of all, ω is an open discrete dense subset of $\langle X, \tau\rangle$. For $\alpha < \mathfrak{c}$, $\{(C_\alpha - n) \cup \{\langle \alpha, 0\rangle\} : n \in \omega\}$ is a neighbourhood base for $\langle \alpha, 0\rangle$. Let $\{(B_\alpha - n) \cup \{\langle \xi, 0\rangle : \alpha \leq \xi \leq \mathfrak{c}\} : \alpha \in \mathfrak{c}, n \in \omega\}$ be a neighbourhood base for $\langle \mathfrak{c}, 0\rangle$. It is easy to see that $\langle X, \tau\rangle$ is a zero-dimensional Hausdorff pseudoradial space and the subspace $\omega \cup \{\langle \mathfrak{c}, 0\rangle\}$ is homomorphic to $\omega \cup \{p\}$. $\square$

Thus we proved

THEOREM 6.2.2 (CH). *For every point p in ω^*, $\omega \cup \{p\}$ can be embedded in a zero-dimensional pseudoradial space.* $\square$

By Theorem 7.1.1 in the next chapter, the above lemma is not a consequence of ZFC.

CHAPTER 7

Pseudoradialness in the Cohen model

7.1. Main Theorem

In this chapter we prove the following

THEOREM 7.1.1 (CH). *If $\mathbb{P} = \mathrm{Fn}(\omega_2, 2)$, then in $V^{\mathbb{P}}$, for every P-point p , the space $\omega \cup \{p\}$ can not be embedded in any regular pseudoradial space.*

Let us just remark that the Theorem remains valid if CH is dropped and κ many Cohen reals are added for any $\kappa > \mathfrak{c}$.

Suppose $\dot{p}$ and $\dot{\tau}$ are $\mathbb{P}$-names, X is a ground model set and $q \in \mathbb{P}$ such that

$$q \Vdash \text{“}\dot{p} \text{ is an ultrafilter on } \omega \text{ and } \omega \cup \{\dot{p}\} \text{ is a subspace of regular pseudoradial space } \langle X, \dot{\tau}\rangle\text{”},$$

We will prove

$$(*) \qquad q \not\Vdash \text{“}\dot{p} \text{ is a P-point ”}$$

We will prove $(*)$ by contradiction. We will work in $V^{\mathrm{Fn}(\lambda,2)}$, where λ is $M \cap \omega_2$ for some elementary submodel of H_θ for certain regular cardinal θ. We prove some preliminary facts in next sectionand give the proof of $(*)$ in section 7.3. In next two sections, we assume that CH is true.

7.2. Some facts

Suppose we are given $\mathbb{P}$-names $\dot{p}$, $\dot{\tau}$ and a set X in the ground model such that

$$\mathbf{1} \Vdash \text{“}\dot{p} \text{ is an ultrafilter on } \omega \text{ and } \omega \cup \{\dot{p}\} \text{ is a subspace of regular pseudoradial space } \langle X, \dot{\tau}\rangle\text{”} ,$$

We furthermore assume that $\dot{p}$ has the form $\bigcup\{\{\pi\} \times A_\pi : \pi \in \mathrm{dom}(\dot{p})\}$, where π is a nice name for a subset of ω. Let $\dot{\mathcal{I}}$ be a nice $\mathbb{P}$-name such that

$$\mathbf{1} \Vdash \text{“}\dot{\mathcal{I}} = \{I \in [\omega]^\omega : I \text{ converges in } \langle X, \dot{\tau}\rangle\}\text{”}$$

We also fix a sufficiently large regular cardinal θ and an elementary submodel M of H_θ such that the following are true:

(i) $M^\omega \subseteq M$;

(ii) $\{\mathbb{P}, X, \dot{p}, \dot{\tau}, \dot{\mathcal{I}}\} \subset M$; and

(iii) M has cardinality $\aleph_1$.

Let $\mathbb{P}_M = \mathbb{P} \cap M$, $X_M = X \cap M$ and $\lambda = M \cap \omega_2$. For each $\mathbb{P}$-name $\dot{Y} \in M$, if $\mathbf{1} \Vdash_{\mathbb{P}}$ “$\dot{Y} \subseteq X$”, we can define a $\mathbb{P}_M$-name $\dot{Y}_M$ as follows. For each $x \in X_M$, let A_x be a maximal antichain in $\{q \in \mathbb{P}_M : q \Vdash_{\mathbb{P}}$ “$x \in \dot{Y}$”$\}$. $\mathbb{P}_M$ is actually $\mathrm{Fn}(\lambda, 2)$, therefore $\mathbb{P}_M$ is completely embedded in $\mathbb{P}$. Hence A_x is also a maximal antichain in $\{q \in \mathbb{P} : q \Vdash_{\mathbb{P}}$ “$x \in \dot{Y}$”$\}$. Since $M^\omega \subseteq M$ and $\mathbb{P}$ is *ccc*, $A_x \in M$. Let $\dot{Y}_M$ be $\bigcup\{\{x\} \times A_x \colon x \in X_M\}$. If, on the other hand, $\dot{\mathcal{Y}}$ is a $\mathbb{P}$-name such that $\mathbf{1} \Vdash_{\mathbb{P}}$ “$\dot{\mathcal{Y}} \subseteq \mathcal{P}(X)$”, we define a $\mathbb{P}_M$-name $\dot{\mathcal{Y}}_M$ to be

$$\{\langle \dot{Y}_M, \mathbf{1}\rangle : \dot{Y} \in M \text{ and } \mathbf{1} \Vdash_{\mathbb{P}} \text{“}\dot{Y} \in \dot{\mathcal{Y}}\text{”}\}$$

Fact 7.2.1. *Suppose G is a $\mathbb{P}$-generic over V, and $\sigma \in M$ such that $\mathbf{1} \Vdash_{\mathbb{P}}$ “$\sigma \subseteq X$”, then the following hold:*

(i) *if $\mathbf{1} \Vdash_{\mathbb{P}}$ “$\sigma \subseteq \omega$”, then $\sigma_M \in M$, moreover, if σ is a nice name we can define σ_M to be σ;*

(ii) $V[G] \models$ "$\mathrm{val}(\sigma_M, G) = \mathrm{val}(\sigma_M, M \cap G) = \mathrm{val}(\sigma, G) \cap M$"; *and*

(iii) *if* $\mathbf{1} \Vdash_{\mathbb{P}}$ "*$\dot{p}$ is a P-point* ", *then,* $\mathbf{1} \Vdash_{\mathbb{P}_M}$ "*$\dot{p}_M$ is an P-point* ".

Proof: (i) and (ii) are straightforward.
For (iii), suppose that $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{p}$ is a P-point ". It is easy to see that $\mathbf{1} \Vdash_{\mathbb{P}_M}$ "$\dot{p}_M$ is an ultrafilter on ω". We prove that $\mathbf{1} \Vdash_{\mathbb{P}_M}$ "$\dot{p}_M$ is countably complete ". Suppose $\dot{f}$ is a $\mathbb{P}_M$-name and $q \in \mathbb{P}_M$ such that

$$q \Vdash_{\mathbb{P}_M} \text{“}\dot{f} \text{ is a function from } \omega \text{ into } \dot{p}_M\text{”}$$

For each $n \in \omega$, we can choose an antichain $B_n = \{q_i^n : i \in \omega\}$ which is maximal in $\{r \in \mathbb{P}_M : r \leq q\}$ and an associated set $\{\pi_i^n : i \in \omega\} \subseteq \mathrm{dom}(\dot{p}_M)$ such that

$$q_i^n \Vdash_{\mathbb{P}_M} \text{“}\dot{f}(n) = \pi_i^n\text{”}$$

for each $i \in \omega$. By (i) , $\mathrm{dom}(\dot{p}_M) \subseteq M$. Therefore the $\mathbb{P}_M$-name

$$\rho = \{\langle \mathrm{op}\langle \check{n}, \pi_i^n\rangle, q_i^n\rangle : n,\ i \in \omega\}$$

is in M, where $\mathrm{op}\langle \check{n}, \pi_i^n\rangle$ is a $\mathbb{P}_M$-name such that $\mathrm{val}(\mathrm{op}\langle \check{n}, \pi_i^n\rangle, G)$ is $\langle n, \mathrm{val}(\pi_i^n, G)\rangle$ (See [Kun80]). It is obvious that $q \Vdash$ "$\dot{f} = \rho$".

Since $q \Vdash_{\mathbb{P}}$ "$\dot{p}$ is countably complete ", we can choose a nice $\mathbb{P}$-name $\dot{Y} \in M$ such that $q \Vdash_{\mathbb{P}}$ "$\dot{Y}$ is a pseudo-intersection of $\mathrm{ran}(\rho)$" and $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{Y} \in \dot{p}$". By (i), we can define $\dot{Y}_M$ to be $\dot{Y}$. It is clear that $q \Vdash_{\mathbb{P}_M}$ "$\dot{Y}_M = \dot{Y}$", hence $q \Vdash_{\mathbb{P}_M}$ "$\dot{Y}_M$ is a pseudo-intersection of $\mathrm{ran}(\dot{f}) = \mathrm{ran}(\rho)$ and $\dot{Y}_M \in \dot{p}_M$" □

Note that $\mathbb{P}_M$ is actually $\mathrm{Fn}(\lambda, 2)$. Let $\mathbb{P}_1$ be $\mathbb{P}_M$ and $\mathbb{P}_2$ be $\mathrm{Fn}(\omega_2 \setminus \lambda, 2)$, then $\mathbb{P}$ is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_2$. For a $\mathbb{P}$-generic filter G, we will also refer to $G \cap M$ as G_M or as G_1 and G_2 will be $G \cap \mathbb{P}_2$ while V_1 will be $V[G_1]$. To further simplify the notation, we will use Y to denote $\mathrm{val}(\dot{Y}, G)$ for a $\mathbb{P}$-name $\dot{Y}$. If $\dot{\mathcal{Y}}$ and $\dot{Y}$ are in M, we will use Y_M and $\mathcal{Y}_M$ to denote $\mathrm{val}(\dot{Y}_M, G_M)$ and $\mathrm{val}(\dot{\mathcal{Y}}_M, G_M)$ respectively.

Thus p, τ, p_M and τ_M etc. are well-defined when it is clear from context which $\mathbb{P}$-generic filter G is being used.

FACT 7.2.2. *The following are true:*

(i) $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{p}_M = \dot{p} \cap V_1$, $\dot{\mathcal{I}}_M = \dot{\mathcal{I}} \cap V_1$";

(ii) *if* $\rho \in M$ *is a* $\mathbb{P}$*-name such that* $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\rho \subseteq \mathcal{P}(\omega)$", *then* $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\rho_M = \rho \cap V_1$"; *and*

(iii) $\mathbf{1} \Vdash_{\mathbb{P}_M}$ "$\dot{\mathcal{I}}_M = \{a \in [\omega]^\omega : a$ *converges in* $\langle X_M, \dot{\tau}_M\rangle\}$".

Proof: (i) is a corollary of (ii). To prove (ii) , let ρ satisfy the conditions of (ii) and let G be a $\mathbb{P}$-generic filter over V. In $V[G]$, take arbitrary $a \in \mathrm{val}(\rho, G) \cap V_1$. Let $\pi \in \mathrm{dom}(\rho)$ and σ be a nice $\mathbb{P}_M$-name such that $a = \mathrm{val}(\pi, G) = \mathrm{val}(\sigma, G)$. Take a q in G which forces that $\pi = \sigma \in \rho$. Since σ, ρ are in M, $q \cap M \Vdash_{\mathbb{P}}$ "$\sigma \in \rho$". Let $\dot{Y} \in M$ such that $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{Y} \in \rho$ and $(\sigma \in \rho \rightarrow \dot{Y} = \sigma)$", then $\mathrm{val}(\dot{Y}_M, G) \in \mathrm{val}(\rho_M, G)$. But $q \Vdash_{\mathbb{P}}$ "$\pi = \sigma = \dot{Y}$". Hence $a = \mathrm{val}(\sigma, G) = \mathrm{val}(\dot{Y}, G) = \mathrm{val}(\dot{Y}_M, G) \in \mathrm{val}(\rho_M, G)$ Thus we proved $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\rho_M \supseteq \rho \cap V_1$". By the definition of ρ_M , it is obvious that

$$\mathbf{1} \Vdash_{\mathbb{P}} \text{"} \rho_M \subseteq \rho \cap V_1 \text{"}.$$

Therefore $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\rho_M = \rho \cap V_1$". We are left to prove (iii)

Let H be a $\mathbb{P}_M$-generic filter over V, there exists a $\mathbb{P}$-generic filter G over V such that $H = G \cap \mathbb{P}_M$. We have to prove that, in $V_1 = V[G_M]$,

$$\mathcal{I}_M = \{a \in [\omega]^\omega : a \text{ converges in } \langle X_M, \tau_M\rangle\}$$

" $\subseteq$ ": Let $a \in \mathcal{I}_M$. We have to prove that a converges in $\langle X_M, \tau_M\rangle$. There exits $\dot{Y}_M \in \mathrm{dom}(\dot{\mathcal{I}}_M)$, such that $a = Y_M$. Since $a \subseteq \omega$, by (i) of Fact 7.2.1, $Y_M = Y$, hence $a \in M[G]$. In $V[G]$, there exists $x \in X$, such that a converges to x in $\langle X, \tau\rangle$. According to [She82], $M[G]$ is an elementary submodel of $H_\theta^{V[G]}$.

By elementarity, we can assume x in X_M. Since $a \subseteq \omega \subseteq X_M$ and $\langle X_M, \tau_M \rangle$ is a weaker topology than $\langle X_M, \tau \upharpoonright X_M \rangle$, a obviously converges to x in $\langle X_M, \tau_M \rangle$.

" $\supseteq$ ": Let $a \in V_1 \cap [\omega]^\omega$ and for some $x \in X_M$, a converges to x in $\langle X_M, \tau_M \rangle$. We have to show that a is in $\mathcal{I}_M$. By (i), we only have to prove that a is in $\mathcal{I}$, i.e.

$$V[G] \models \text{“}a \text{ converges in } \langle X, \tau \rangle\text{”}.$$

By the facts that $M^\omega \subseteq M$ and that $\mathbb{P}$ is *ccc* , any nice $\mathbb{P}_M$-name for a subset of ω is in M. In particular, a is in $M[G]$. By the choice of θ and the elementarity, it is sufficient to prove that

$$M[G] \models \text{“}a \to x \text{ in } \langle X, \tau \rangle\text{”} .$$

Let U be in $\tau \cap M[G]$ containing x. We have to show that a is almost contained in U. There exists $\dot{Y} \in M$ such that $U = \mathrm{val}(\dot{Y}, G)$, we can assume $\mathbf{1} \Vdash_{\mathbb{P}}$ "$\dot{Y} \in \tau$", hence $\dot{Y}_M \in \mathrm{dom}(\dot{\tau}_M)$. Notice that $U \cap X_M = U \cap M = Y_M$ is in τ_M. Therefore a is almost contained in $U \cap X_M$ therefore in U. $\square$

7.3. Proof

Assume that $q \Vdash$ "$\dot{p}$ is a P-point". We can assume that q is the empty function, i.e. $q = \mathbf{1}$ (otherwise we can work instead with $\mathbb{P}_q = \{r \in \mathbb{P} : r \leq q\}$ which is isomorphic to $\mathbb{P}$). Therefore we have

$$\mathbf{1} \Vdash \text{“}\dot{p} \text{ is a P-point and } \omega \cup \{\dot{p}\} \text{ is a subspace of regular pseudoradial space } \langle X, \dot{\tau} \rangle\text{”}.$$

Suppose θ, M, and $\mathcal{I}$ are obtained as in the previous discussions and G is a $\mathbb{P}$-generic filter over V. Let $a \in V[G]$ be a Cohen real over V_1 such that a is in p. Since, $\langle X, \tau \rangle$ is regular, we can take two open neighbourhoods of p , say W and U such that $\mathrm{cl}(W) \subseteq U$ and $a = U \cap \omega$. Let $b = W \cap \omega$. Given any I in $\mathcal{I}_M$, let x be in X_M such that I converges to x. By (i) of the Fact 7.2.2, I also converges to x in $\langle X, \tau\rangle$. If $b \cap I$ is infinite , then x is in $\mathrm{cl}(W) \subseteq U$. Therefore I is almost contained in a. This contradicts the fact that a is a Cohen real over V_1. Thus we proved that for each I in $\mathcal{I}_M$, $b \cap I$ is finite. Let $\dot{\tau}, \dot{\mathcal{I}}, \dot{a}, \dot{U}, \dot{W}, \dot{b}$ and $\dot{p}$ denote both the $\mathbb{P}$-names and $\mathbb{P}_2$-names for the corresponding objects in $V[G]$. Choose an $r \in G_2$, such that r forces the above fact over V_1.

Now work in V_1, let κ be a sufficiently large regular cardinal, let N be a countable submodel of H_κ such that N contains $\{\dot{\tau}, \dot{\mathcal{I}}, \dot{a}, \dot{U}, \dot{W}, \dot{b}, \dot{p}, \lambda\}$. By (iii) of Fact 7.2.1, p_M is a P-point. We can take A in p_M such that for each $B \in N \cap p_M$, $A \subseteq^* B$. Since in $V[G]$, A is in p and $\langle X, \tau\rangle$ is pseudoradial, there exists $I \in \mathcal{I}$ such that $I \subset A$. Notice that $A \in M[G]$, and , according to [She82], $M[G]$ is an elementary submodel of $H_\theta^{V[G]}$. By elementarity, we can assume $I \in M[G]$. Hence $I \in V_1$. Therefore , by (i) of Fact 7.2.2. $I \in \mathcal{I}_M$. By the choice of r, $r \Vdash_{\mathbb{P}_2}$ "$\dot{b} \cap I$ is finite ", let $s \in \mathbb{P}_2$ and $m \in \omega$, such that $r \geq s \Vdash_{\mathbb{P}_2}$ "$\dot{b} \cap I \subseteq m$". Claim: $s \cap N \Vdash_{\mathbb{P}_2}$ "$\dot{b} \cap I \subseteq m$". Otherwise, let $k \in I \setminus m$, such that, there is $t \leq s \cap N$, $t \Vdash_{\mathbb{P}_2}$ "$k \in \dot{b}$". Since k, $s \cap N$ and $\dot{b}$ are in N, we can assume $t \in N$. But then t and s are compatible while they force the contradicting statements. Therefore we can assume $s \in N$. Let $Z = \{n : \exists t \leq s\, t \Vdash_{\mathbb{P}_2} \text{"} n \in \dot{b}\text{"}\} \in N$. Since $Z \supseteq b$, $Z \in p_M$. Now $I \subseteq A \subseteq^* Z$, take $k \in I \cap Z \setminus m$ and $t \leq s$ such that $t \Vdash_{\mathbb{P}_2}$ "$k \in \dot{b}$". This contradicts the fact that $s \Vdash_{\mathbb{P}_2}$ "$\dot{b} \cap I \subseteq m$". $\square$

Remark: In a similar way, we can prove that in $V^{\mathrm{Fn}(\omega_2, 2)}$, for every ultrafilter p on ω, if p is a P-point limit of a sequence of P-points, then the space $\omega \cup \{p\}$ can not be embedded in any regular pseudoradial space.

CHAPTER 8

Pseudoradialness for a Special Ultrafilter

In this chapter, we will show that if there exists a reflecting self-indexed stationary set on $[\omega_2]^\omega$ and $\mathfrak{c} = \omega_2$, then for a special ultrafilter p constructed in chapter 9, $\omega \cup \{p\}$ can be embedded into a zero-dimensional Hausdorff pseudoradial space. We will also prove that it is consistent that $\mathfrak{c} = \omega_2$ while there is such a stationary set. We conclude that there is a forcing extension in which, for every P-point p, $\omega \cup \{p\}$ can not be embedded in any regular pseudoradial space while there is an ultrafilter q on ω such that $\omega \cup \{q\}$ is a subspace of a zero-dimensional Hausdorff pseudoradial space. We will have a detailed discussion for the special ultrafilter in the next chapter.

8.1. Reflecting Self-Indexed Stationary Set

A sequence $\{S_\alpha : \alpha < \omega_2\}$ is a *reflecting self-indexed stationary set* on $[\omega_2]^\omega$ if for any $\lambda < \omega_2$ with $\mathrm{cf}(\lambda) = \omega_1$, $\{S_\xi : \xi < \lambda \text{ and } \sup S_\xi = \xi\}$ is stationary on $[\lambda]^\omega$.

The reflecting self-indexed stationary set is very similar to the notion of stationary coding set defined in [Zwi84].

LEMMA 8.1.1. *Let $V \models CH$ and $\mathbb{P} = \{p : p$ is countable functions such that $\mathrm{dom}(p) \subseteq \omega_2$ and $p(\alpha)$ is a countable subset of α for $\alpha < \omega_2\}$. Let $\mathbb{P}$ have the* *inverse inclusion as its partial order. Then in $V^{\mathbb{P}}$, there is a reflecting self-indexed stationary set on $[\omega_2]^\omega$ and CH is true.*

Proof: First of all, $\mathbb{P}$ is countably closed, hence CH preserves in the forcing extension. $\mathbb{P}$ is also $\aleph_2$-cc, whence $\aleph_2$ is preserved.

If we replace ω_2 by ω_1 in the definition of $\mathbb{P}$, then it is well-known that we get a diamond sequence. This is not much different.

Indeed, for $\mathbb{P}$-generic G, let $S_\alpha = G(\alpha)$. Suppose that $p_0 \in \mathbb{P}$, $\lambda < \omega_2$ has uncountable cofinality and $\dot{\mathcal{C}}$ be a name of a cub subset of $[\lambda]^\omega$ such that $p_0 \Vdash$ "$\dot{\mathcal{C}}$ is cub on $[\lambda]^\omega$". Since $\mathbb{P}$ is countably closed, it is straightforward to define an increasing sequence $\{C_n : n \in \omega\}$ in $[\lambda]^\omega$ and a descending sequence $\{p_n : n \in \omega\}$ in $\mathbb{P}$ such that

(1) for each n, $p_{n+1} \Vdash$ "$C_n \in \dot{\mathcal{C}}$",
(2) for each n, $\operatorname{dom} p_n \cap \lambda \subseteq C_n$ and $C_n \cup \{\sup C_n + 1\} \subseteq \operatorname{dom} p_{n+1}$.

Let $p_\omega = \bigcup_{n\in\omega} p_n$ and $C = \bigcup_{n\in\omega}$. Obviously $p_\omega \Vdash$ "$C \in \dot{\mathcal{C}}$" and $\alpha = \sup C$ is a limit ordinal which is not in $\operatorname{dom} p_\omega$. Extend p_ω to some p such that $p(\alpha) = C$. Then $p \Vdash$ "$S_\alpha = C \in \dot{\mathcal{C}}$ and $\alpha = \sup S_\alpha$". □

LEMMA 8.1.2. *Reflecting self-indexed stationary set is preserved by ccc forcing*

Proof: It is well known that after ccc forcing every cub subset of ω_1 contains a ground model cub. The same proof gives that this also holds for cub subsets of $[\lambda]^\omega$ for each $\lambda < \omega_2$. □

Remark: S. Todorcěvić informed us that the existence of a reflecting self-indexed stationary set follows from the existence of the ρ-function in [Bek91]. Indeed, one can easily produce such a stationary set directly from a $\square$-sequence (See [Dow]).

In the next section we show that with the presence of a reflecting self-indexed stationary set one can embed $\omega \cup \{p\}$ into a zero-dimensional pseudoradial space for a special ultrafilter p which will be constructed in chapter 9.

8.2. A Special Space $\omega \cup \{p\}$

DEFINITION 8.2.1 ([BS89]). Given a family $\mathcal{F}$ of subsets of some set S, a family $\mathcal{I}$ is called a refinement of $\mathcal{F}$ if for each F in $\mathcal{F}$, there is an I in $\mathcal{I}$ such that $I \subseteq F$. If $\mathcal{I}$ is an almost disjoint family we say $\mathcal{I}$ is an almost disjoint refinement of $\mathcal{F}$.

The hypothesized ultrafilter in the following proposition is interesting enough for a separate discussion. We will give the proof in chapter 9.

PROPOSITION 8.2.2. *There is an ultrafilter p on ω which has an almost disjoint refinement $\mathcal{I}$ and a base $\mathcal{B}$ such that*

(i) *for each subset $\mathcal{I}' \in [\mathcal{I}]^{<\mathfrak{c}}$ there is a member of p which is almost disjoint with each member of $\mathcal{I}'$, and*
(ii) *for each $B \in \mathcal{B}$ and $I \in \mathcal{I}$ either $B \cap I$ is finite or B almost contains I and I is almost disjoint with all but finitely many members of $\mathcal{B}$.*

We have the following results about this ultrafilter.

THEOREM 8.2.3. *Suppose p is the ultrafilter stated in the Proposition* 8.2.2. *If $\mathfrak{c} = \omega_2$ and there is a reflecting self-indexed stationary set in $[\omega_2]^\omega$, then the space $\omega \cup \{p\}$ can be embedded in a zero-dimensional pseudoradial space.*

Before we start to prove above theorem, we prove the following corollary which is one of our main results about the pseudoradial spaces.

COROLLARY 8.2.4. *It is consistent that, for every P-point p, $\omega \cup \{p\}$ can not be embedded into any regular pseudoradial space while there is an ultrafilter p on ω such that $\omega \cup \{p\}$ is a subspace of a zero-dimensional pseudoradial space.*

Proof: Let V be the model obtained through Lemma 8.1.1. Let $Q = \mathrm{Fn}(\omega_2, 2)$. By Theorem 7.1.1, in V^Q, for every P-point p, $\omega \cup \{p\}$ can not be embedded into any regular pseudoradial space.

On the other hand, by Lemma 8.1.2, in V^Q, there is a reflecting self-indexed stationary set on $[\omega_2]^\omega$ and $\mathfrak{c}$ is ω_2. Therefore, by Theorem 8.2.3, there is an ultrafilter p on ω such that $\omega \cup \{p\}$ is a subspace of a zero-dimensional pseudoradial space. $\square$

We will need the following Lemma in the proof of Theorem 8.2.3.

LEMMA 8.2.5. *Suppose R is a subset of space X, $x \in X$, and $\mathcal{U}$ is a local subbase at x, i.e. the intersections of finitely many members of $\mathcal{U}$ constitute a neighbourhood base at x. Let κ be any cardinal number such that the following are true,*

(i) *for all $\mathcal{U}' \in [\mathcal{U}]^{<\kappa}$, $R \cap \bigcap \mathcal{U}'$ is not empty ;*

(ii) *for all $y \in R$, $\{U \in \mathcal{U} : y \notin U\}$ has cardinality at most κ,*

Then there is a sequence $\{y_\alpha : \alpha < \kappa\}$ in R convergent to x.

Proof: For each $y \in R$, let $\{U^y_\alpha : \alpha < \kappa\}$ enumerate, with possible repetition, the set $\{U \in \mathcal{U} : y \notin U\}$. It is straightforward to define, by induction, a sequence $\{y_\alpha : \alpha < \kappa\}$ such that for all $\alpha < \kappa$

$$(*) \qquad y_\alpha \in R \cap \bigcap \{U^{y_\xi}_\eta : \eta < \alpha, \xi < \alpha\}$$

We show that $\{y_\alpha\}_{\alpha<\kappa}$ is convergent to x by showing that each member of $\mathcal{U}$ contains a tail of the sequence. Suppose $U \in \mathcal{U}$. We have to show that there is a

$\alpha < \kappa$, such that , for all $\beta \in \kappa$, $\beta > \alpha$ implies $y_\beta \in U$. If $\{y_\alpha : \alpha < \kappa\}$ is contained in U, we are done. Otherwise, choose any ξ such that $y_\xi \notin U$. Take $\eta < \kappa$ such that $U = U_\eta^{y_\xi}$. Let $\alpha = \max\{\eta, \xi\}$, then, for any $\beta > \alpha$, by $(*)$, $y_\beta \in U$. We are done. $\square$

The rest of the section is to prove Theorem 8.2.3

Proof of Theorem 8.2.3: We first fix a reflecting self-indexed stationary set $\{S_\alpha \colon \alpha \in \omega_2\}$ in $[\omega_2]^\omega$. Let $p = \{A_\alpha \colon \alpha < \omega_2\}$ be the ultrafilter in Proposition 8.2.2. Let $\{I_A \colon A \in p\}$ be the almost disjoint refinement of p such that $I_A \subseteq A$ and $\mathcal{B}$ is the base with the properties in Proposition 8.2.2. By induction we can define an almost disjoint family $\{a_\alpha \colon \alpha < \omega_2\}$ and a base $\{F_\alpha \colon \alpha < \omega_2\}$ of p such that

(i) for all α, $a_\alpha \subseteq F_\alpha \subseteq A_\alpha$,

(ii) for all $\alpha\,, \beta$ $\alpha < \beta$ implies $a_\alpha \cap F_\beta$ is finite,

(iii) for all α there are only finitely many β such that $a_\alpha \cap F_\beta$ is infinite and if $a_\alpha \cap F_\beta$ is infinite, then $a_\alpha \subseteq F_\beta$,

(iv) for all λ if $\mathrm{cf}(\lambda)$ is uncountable, then $\{F_\alpha \colon \alpha < \lambda\}$ is a filter base.

We construct $\{a_\alpha, F_\alpha \colon \alpha < \omega_1\eta\}$ by induction on η. At stage η, the only non trivial case is when η is a successor ordinal. Suppose $\eta = \xi + 1$. We define $a_{\omega_1\xi+\alpha}$ and $F_{\omega_1\xi+\alpha}$ for $\alpha \in \omega_1$ by induction on ω_1. Let $f \colon \omega_1 \to [\omega_1\xi + \omega_1]^{<\omega}$ be an onto mapping such that $f(\alpha) \subseteq \omega_1\xi + \alpha$ for each $\alpha \in \omega_1$. At stage α, take a B in $\mathcal{B}$ such that B is almost disjoint with a_ζ for each $\zeta < \omega_1\xi + \alpha$. Now simply define $F_{\omega_1\xi+\alpha}$ be $A_{\omega_1\xi+\alpha} \cap B \cap \bigcap\{F_\zeta \colon \zeta \in f(\alpha)\}$ and let $a_{\omega_1\xi+\alpha}$ to be $I_{F_{\omega_1\xi+\alpha}}$.

Let $\mathcal{H} = \{b \subseteq \omega \colon (\forall\alpha \in \omega_2)(a_\alpha \subseteq^* b)\}$. Let $\mathcal{A}$ be the subalgebra of $\mathcal{P}(\omega)$ generated by $[\omega]^{<\omega} \cup \{a_\alpha, F_\alpha \colon \alpha \in \omega_2\} \cup \mathcal{H}$

We will define a subspace X of the Stone space $\mathrm{St}(\mathcal{A})$ such that X contains a copy of $\omega \cup \{p\}$ and X is pseudoradial. X will be of the form $\omega \cup \{x_\alpha : \alpha \in \Gamma\}$, where Γ is a subset of $\omega_2 + 1$ and each x_α is an ultrafilter of $\mathcal{A}$. We specify the set Γ and x_α for $\alpha \in \Gamma$ as follows.

For convenience, we define a subset x^+_α of $\{F_\gamma : \gamma < \omega_2\}$ for each $\alpha \in \omega_2$. For successor ordinal $\alpha + 1$, $x^+_{\alpha+1}$ is the finite of F_β's which contains a_α. For limit α with $\mathrm{cf}(\alpha) = \omega$, x^+_α is $\{F_\gamma \colon \gamma \in S_\alpha\}$, where S_α is from the reflecting self-indexed stationary set. For limit α with $\mathrm{cf}(\alpha) = \omega_1$, x^+_α is $\{F_\gamma \colon \gamma < \alpha\}$. We denote the set $\{\omega \setminus F_\beta : F_\beta \notin x^+_\alpha\}$ by x^-_α.

First of all, Γ contains all the successor ordinals and the ordinals with uncountable cofinality.

For each α, since no member of $\mathcal{A}$ splits a_α, the cofinite subsets of a_α generate an ultrafilter on $\mathcal{A}$. Let $x_{\alpha+1}$ be this ultrafilter. The point x_{ω_2} will simply be the ultrafilter $p \cap \mathcal{A}$. Before we handle other ordinals, we prove the following.

Fact 8.2.6. *If $\lambda < \omega_2$ has cofinality ω_1, then $x^+_\lambda \cup x^-_\lambda \cup \mathcal{H}$ is actually $\{F_\alpha, \omega \setminus F_\beta \colon \alpha < \lambda$ and $\beta \geq \lambda\} \cup \mathcal{H}$ and generates an ultrafilter x_λ on $\mathcal{A}$. If $\alpha < \omega_2$ has cofinality ω and $x^+_\alpha \cup x^-_\alpha \cup \mathcal{H}$ has the finite intersection property, then it generates an ultrafilter x_α on $\mathcal{A}$.*

Proof of Fact 8.2.6 If $\lambda < \omega_2$ has cofinality ω_1, then $\{F_\alpha \colon \alpha < \lambda\}$ is a filter base and it is easy to see that $\{F_\alpha, \omega \setminus F_\beta \colon \alpha < \lambda \text{ and } \beta \geq \lambda\} \cup \mathcal{H}$ has the finite intersection property. To prove that it generates an ultrafilter, it is sufficient to show that for each $\beta \in \omega_2$, $\omega \setminus a_\beta$ contains some member of the family. Indeed, if $\beta < \lambda$, then $F_{\beta+1}$ is almost disjoint with a_β hence $F_{\beta+1} \subseteq^* \omega \setminus a_\beta$. If $\beta \geq \lambda$, then $\omega \setminus F_\beta$ is almost contained in $\omega \setminus a_\beta$. The proof for the case when α has countable cofinality is virtually the same.

We resume the proof of the Theorem. For λ with cofinality ω_1, let λ be in Γ and x_λ be the ultrafilter defined Fact in 8.2.6. For an ordinal α with $\mathrm{cf}(\alpha) = \omega$,

$\alpha \in \Gamma$ if $x_\alpha^+ \cup x_\alpha^- \cup \mathcal{H}$ generate a filter, we let x_α be the ultrafilter defined in Fact 8.2.6

By above discussion Γ is well defined and, for each $\alpha \in \Gamma$, x_α is a point in the Stone space $\mathrm{St}(\mathcal{A})$. The following claim is trivial.

CLAIM 8.2.1. $\omega \cup \{x_{\omega_2}\}$ is homomorphic to $\omega \cup \{p\}$

For simplicity we will abuse the notation of Stone duality. If $F \subset \omega$ is a member of $\mathcal{A}$, we also let F denote the set of the points of X which as an ultrafilter on $\mathcal{A}$ contain F as a member. Of course, F is clopen. Hence, to say $x_\alpha \in F$ is equivalent to saying $F \in x_\alpha$.

Let X_0 be all x_α with α a successor and X_1 be all x_α with $\mathrm{cf}(\alpha) = \omega$. Recall that $x_{\alpha+1}$ corresponds to a_α. Let X_2 be all x_α such that α has uncountable cofinality.

FACT 8.2.7. *The sets $X_1 \cup X_2$ and X_2 are closed in X. X_2 is homomorphic to a subspace of $\omega_2 + 1$, namely $\{\alpha \leq \omega_2 \colon \mathrm{cf}(\alpha)$ is uncountable $\}$ and so X_2 is radial.*

Proof of Fact 8.2.7*:* For the first statement of the claim, the only case that requires a proof is that a point x_α of X_1 is not in the closure of X_2. If α is countable, then $\omega \setminus F_{\alpha+1}$ is disjoint from X_2. On the other hand, if α is uncountable, then we can find $\gamma < \beta < \alpha$, so that $F_\beta \in x_\alpha^+$ and $F_\gamma \notin x_\alpha^+$. Note that $F_\beta \setminus F_\gamma$ is not a member of x_λ for any λ with uncountable cofinality. For the second statement, it is sufficient to see that, for any $\lambda < \omega_2$ with uncountably cofinality, the typical neighhourhood of x_λ in the subspace X_2 is $(F_\alpha \setminus F_\lambda) \cap X_2 = \{x_\xi \in X_2 \colon \alpha < \xi \leq \lambda\}$ for some $\alpha < \lambda$. Similarly, a typical neighbourhood of x_{ω_2} is $\{x_\xi \in X_2 \colon \alpha < \xi \leq \omega_2\}$.

The following claim will finish the proof of Theorem 8.2.3.

CLAIM 8.2.2. X is a zero-dimensional pseudoradial space.

Proof of Claim 8.2.2*:* Since X is a subspace of a $\mathrm{St}(\mathcal{A})$, X is zero-dimensional. Suppose X is not pseudoradial. Let R be a radially closed subset of X which is not closed. Let $x_\alpha \in \operatorname{cl} R \setminus R$ and α is the minimal such index. Obviously $x_\alpha \in X_1 \cup X_2$ because other points have countable neighbourhood bases. We will prove $x_\alpha \in R$ to produce a contradiction. We will need the following fact.

FACT 8.2.8. *$x_\alpha \in \operatorname{cl}(R \setminus \omega)$.*

Proof of Fact 8.2.8*:* Suppose $x_\alpha \notin \operatorname{cl}(R \setminus \omega)$. Then $x_\alpha \in \operatorname{cl}(R \cap \omega) \setminus \operatorname{cl}(R \setminus \omega)$. Let U be a clopen neighbourhood of x_α such that $U \cap (R \setminus \omega) = \emptyset$. Therefore $b = R \cap U$ is a subset of ω. Since R is radially closed and U is closed , b is also radially closed. For any $\beta \in \omega_2$, if $a_\beta \cap b$ is infinite, then $x_{\beta+1}$ would be in b contradicting that $b \subseteq \omega$. Hence, b is almost disjoint with each a_β, i.e. $\omega \setminus b$ is in $\mathcal{H}$. Thus $\omega \setminus b$ is in the ultrafilter x_α, which contradicts that x_α is in the closure of b.

Since $R \setminus \omega$ is also radially closed in X, we can now assume $R \cap \omega = \emptyset$. By Fact 8.2.7, X_2 is radial and closed so we can assume that $R \cap X_2 = \emptyset$ by again restricting to a neighbourhood of x_α. For the rest of the proof we assume that $R \subseteq X_0 \cup X_1$. Recall that $x_\alpha \in X_1 \cup X_2$.

We claim that $\alpha \neq \omega_2$. Indeed fix any $M \prec H(\omega_3)$ such that $|M| = \omega_1$ and M contains all relevant sets such as R and X, and so that the cofinality of $\delta = M \cap \omega_2$ is uncountable. We show that x_δ is in the closure of R which will contradicts that R is assumed to be disjoint from X_2 and that α is the minimal index of a limit point which is not in R. A typical neighbourhood of x_δ has the form $H \cap \bigcap\{F_\xi : \xi \in f\} \setminus \bigcup\{F_\gamma : \gamma \in g\}$, where $f \in [\delta]^{<\omega}$, $g \in [\omega_2 \setminus \delta]^{<\omega}$ and $H \in \mathcal{H}$. However, $\bigcap\{F_\xi : \xi \in f\}$ is a neighbourhood of x_{ω_2} and is a member of M. Choose any $\beta \in M$ such that $x_\beta \in R \cap \bigcap\{F_\xi : \xi \in f\}$. Clearly $x_\beta \notin F_\gamma$ for all $\gamma > \delta$, hence x_α is in the above neighbourhood of x_δ.

Next we apply Lemma 8.2.5 to prove that there exists a sequence in R convergent to x_α and thus finish the proof.

Case 1. $\mathrm{cf}(\alpha) = \omega$

To apply Lemma 8.2.5, let κ be ω, $\mathcal{U}$ be $x_\alpha^+ \cup x_\alpha^- \cup \mathcal{H}$, and x be x_α. $\mathcal{U}$ is a local subbase of x. We show x, R and $\mathcal{U}$ satisfies the conditions in Lemma 8.2.5. Since $x \in \mathrm{cl}(R)$, the condition (i) is trivial. To verify the condition (ii), let $x_\xi \in R$. Then $\mathrm{cf}(\xi) \leq \omega$. It is easy to see that

$$\{U \in \mathcal{U} \colon x_\xi \notin U\} \subseteq x_\alpha^+ \cup \{\omega \setminus F \colon F \in x_\xi^+\}.$$

But both x_α^+ and x_β^+ are countable. Therefore $\{U \in \mathcal{U} \colon x_\xi \notin U\}$ is countable.

Case 2. $cf(\alpha) = \omega_1$.

We let κ be ω_1, and $\mathcal{U}$ be $x_\alpha^+ \cup x_\alpha^- \cup \mathcal{H}$, and x be x_α. Again $\mathcal{U}$ is a local subbase at x. The second condition of Lemma 8.2.5 is verified as in the Case 1. We are left to show that for any $\mathcal{U}' \in [\mathcal{U}]^\omega$,

$$R \cap \bigcap \mathcal{U}' \neq \emptyset.$$

Since for each H in $\mathcal{H}$, $R \subseteq X \setminus \omega \subseteq H$, we can assume $\mathcal{U}' = \{F_{\alpha_i}, \omega \setminus F_{\beta_i} : i \in \omega\}$. We take a countable elementary submodel M of $H(\omega_3)$, such that $\{X,\ R,\ \mathcal{U}',\ \alpha\} \subseteq M$ and $M \cap \alpha = S_\gamma$ and $\gamma = \sup(M \cap \alpha)$, where S_γ is from the the reflecting self-indexed stationary set. Since $S_\gamma = M \cap \alpha$ and $\{F_\gamma : \gamma < \alpha\}$ is a filter base, x_γ^+ is a filter base. It follows that $x_\gamma^+ \cup x_\gamma^- \cup \mathcal{H}$ has the finite intersection property. To see this, it suffices to check that for $H \in \mathcal{H}$, $\beta \in S_\gamma$ and a finite subset g disjoint from S_γ the set $H \cap F_\beta \setminus \bigcup_{\delta \in g} F_\delta$ is infinite. However, $\beta \in M$, the finite subset f of δ's with $a_\beta \subseteq F_\delta$ is contained in $M \cap \alpha = S_\gamma$. Therefore, $a_\beta \subseteq H \cap F_\beta \setminus \bigcup_{\delta \in g} F_\delta$. Therefore it generates an ultrafilter on $\mathcal{A}$. Therefore $\gamma \in \Gamma$, i.e. x_γ is a point of our X. Next we show that $x_\gamma \in R \cap \bigcap \mathcal{U}'$.

For $i \in \omega$, since, $\mathcal{U}' \in M$, $\alpha_i \in M \cap \alpha = S_\gamma$. Thus, $F_{\alpha_i} \in x_\gamma^+$ while $F_{\beta_i} \notin x_\gamma^+$ because $\beta_i \geq \alpha > \gamma$. This proves that $x_\gamma \in \bigcap \mathcal{U}'$.

To prove $x_\gamma \in R$, it is sufficient to prove that $x_\gamma \in \mathrm{cl}(R)$ because α is the minimal index such that $x_\alpha \in \mathrm{cl}(R) \setminus R$. Take a neighbourhood W of x_γ, we have to show $R \cap W \neq \emptyset$. We can assume $W = \bigcap \mathcal{F} \setminus \bigcup \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are finite subsets of x_γ^+ and $\{F_\gamma : \gamma \in \omega_2\} \setminus x_\gamma^+$ respectively. Obviously $\mathcal{F}$ and $\mathcal{G} \cap M$ are members of M. If $F_\xi \in \mathcal{G} \cap M$, then by the definition of x_γ^+ and the fact $S_\gamma = M \cap \alpha$, $\xi \geq \alpha$. Therefore, $(R \cap \bigcap \mathcal{F}) \setminus \bigcup(\mathcal{G} \cap M)$ is a neighbourhood of x_α and it is in M. Take a $x_\xi \in M \cap R \cap \bigcap \mathcal{F} \setminus \bigcup(\mathcal{G} \cap M)$. Since x_ξ^+ is a countable member of M, $x_\xi^+ \subseteq M$, therefore $\mathcal{G} \setminus M$ is disjoint with x_ξ^+. Therefore $x_\xi \notin \bigcup(\mathcal{G} \setminus M)$ and $x_\xi \in R \cap \bigcap \mathcal{F} \setminus \bigcup \mathcal{G} = R \cap W$. We are done. □

We finish this chapterwith the following question.

QUESTION 8.2.9. *Does MA or* $\mathfrak{p} = \mathfrak{c}$ *implies that for each ultrafilter* p *on* ω, $\omega \cup \{p\}$ *is a subspace of a regular pseudo-radial space?*

By Lemma 6.2.1, the answer is yes for the non P-point p.

CHAPTER 9

Another Ultrafilter On ω

In this chapter, we will give the proof of Proposition 8.2.2. We will restate here Proposition 8.2.2 as Proposition 9.0.1. It will follow that the ultrafilter in Proposition 9.0.1 has a strong Noetherian base. Therefore we give a positive answer to Malykin's question.

Recall the following definition of *refinement* for some family (see page 101). Given a family $\mathcal{F}$ of subsets of some set S. A family $\mathcal{I}$ is called a refinement of $\mathcal{F}$ if for each F in $\mathcal{F}$, there is an I in $\mathcal{I}$ such that $I \subseteq F$. If $\mathcal{I}$ is an almost disjoint family we say $\mathcal{I}$ is an almost disjoint refinement of $\mathcal{F}$.

There has been many interesting results about almost disjoint refinements of filters on ω, see section 4 of [BS89] for a survey. Proposition 9.0.1 gives an ultrafilter which has an almost disjoint refinement that does not follow from the constructions in [BS89].

A Noetherian space is a space with a base $\mathcal{B}$ such that for each $U \in \mathcal{B}$, the set $\{V \in \mathcal{B}\colon U \subseteq V\}$ is finite (see [Mal81]). A base $\mathcal{B}$ for an ultrafilter p on ω is a *Noetherian base (strong Noetherian base)* if for each $A \in \mathcal{B}$, the set $\{B \in \mathcal{B}\colon A \subseteq B\}$ ($\{B \in \mathcal{B}\colon A \subseteq^* B\}$) is finite. Malykin has asked if there exists an ultrafilter with a Noetherian base. The base $\mathcal{B}$ in the ultrafilter p in the following proposition is actually strong Noetherian. Thus we give a positive answer to Malykin's question. In addition the property (i) of the filter p is quite unexpected.

PROPOSITION 9.0.1. *There is an ultrafilter p on ω which has an almost disjoint refinement $\mathcal{I}$ and a base $\mathcal{B}$ such that*

(*i*) *for each subset $\mathcal{I}' \in [\mathcal{I}]^{<\mathfrak{c}}$ there is a member of p which is almost disjoint with each member of $\mathcal{I}'$, and*

(*ii*) *for each $B \in \mathcal{B}$ and $I \in \mathcal{I}$ either $B \cap I$ is finite or B almost contains I and I is almost disjoint with all but finitely many members of $\mathcal{B}$.*

REMARK 9.0.2. It is straightforward to see that the property (ii) in Proposition 9.0.1 implies the base $\mathcal{B}$ is a strong Noetherian base. A careful analysis of the construction in [BS89] will reveal that the almost disjoint refinements constructed by that method will fail to satisfy property (ii) for any base, i.e. for any base, there will be some member of the refinement almost contained in infinitely many members of the base.

The following Lemmas give the proof of the Proposition 9.0.1.

LEMMA 9.0.3. *If $\mathcal{F}$ is a filter base on ω such that, for each $F \in \mathcal{F}$, there is an infinite $I \subseteq F$ such that $\{F' \in \mathcal{F} : I \cap F' is\ infinite\}$ is finite, then there is an ultrafilter p on ω which extends $\mathcal{F}$ and satisfies the following property*

$$(\dagger) \qquad For\ each\ F \in p\ , there\ exists\ I \in [F]^{\omega}$$
$$such\ that\ \ \{F' \in \mathcal{F} : I \cap F' is\ infinite\}\ is\ finite$$

Proof: Let $[\omega]^{\omega} = \{Y_\alpha : \alpha < \mathfrak{c}\}$. We will build inductively an increasing sequence $\{\mathcal{F}_\alpha : \alpha \leq \mathfrak{c}\}$ of filters on ω such that

(i) $\mathcal{F} \subseteq \mathcal{F}_0$

(ii) For each $\alpha \leq \mathfrak{c}, \mathcal{F}_\alpha$ satisfies $(\dagger)$(with p taken to be $\mathcal{F}_\alpha$).

(iii) For each $\alpha \in \mathfrak{c}$, either $Y_\alpha \in \mathcal{F}_{\alpha+1}$ or $\omega \setminus Y_\alpha \in \mathcal{F}_{\alpha+1}$,

Let $\mathcal{F}_0$ be the filter generated by $\mathcal{F}$. Suppose we have constructed $\{\mathcal{F}_\beta : \beta < \alpha\}$ which satisfies (i)-(iii). If α is a limit ordinal, let $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. If α is the successor of β, we have to build $\mathcal{F}_{\beta+1}$ which extends $\mathcal{F}_\beta$ and satisfies (ii) and (iii).

We say a set $Y \subseteq \omega$ is $\mathcal{F}_\beta$-acceptable if for each $F \in \mathcal{F}_\beta$ there exists an $I \in [F \cap Y]^\omega$ such that $\{F' \in \mathcal{F} : F' \cap I \text{ is infinite}\}$ is finite.

Claim 9.0.3. There exists an $Y \in \{Y_\beta,\ \omega \setminus Y_\beta\}$ which is $\mathcal{F}_\beta$-acceptable.

Proof of Claim 9.0.3*:* Suppose not. Then there exist F_1 and F_2 in $\mathcal{F}_\beta$ such that for each $I \in [\omega]^\omega$, if $I \subseteq F_1 \cap Y_\beta$ or $I \subseteq F_2 \setminus Y_\beta$, then $\{F' \in \mathcal{F} : F' \cap I \text{ is infinite}\}$ is infinite. Since $\mathcal{F}_\beta$ satisfies $(\dagger)$, there exists an $I_0 \subseteq F_1 \cap F_2$ such that $\{F' \in \mathcal{F} : F' \cap I_0 \text{ is infinite}\}$ is finite. Without loss of generality, we can assume $I_0 \cap Y_\beta$ is infinite. Then $I_0 \cap Y_\beta \in [\omega]^\omega$ and $I_0 \cap Y_\beta \subseteq F_1 \cap Y_\beta$ and $\{F' \in \mathcal{F} : F' \cap I \cap Y_\beta \text{ is infinite}\}$ is finite. This contradicts the choice of F_1.

Now take an $Y \in \{Y_\beta,\ \omega \setminus Y_\beta\}$ which is $\mathcal{F}_\beta$-acceptable. Let $\mathcal{F}_{\beta+1}$ be the filter generated by $\mathcal{F}_\beta \cup \{Y\}$. Obviously, $\mathcal{F}_{\beta+1}$ satisfies (ii) and (iii). Finally, we let $p = \mathcal{F}_\mathfrak{c}$. It is easy see that p works. $\square$

Lemma 9.0.4. *Under the condition of the Lemma* 9.0.3*, if* $|\mathcal{F}| = \mathfrak{c}$*, and* p *is obtained from Lemma* 9.0.3*, then Proposition* 9.0.1 *holds for* p*.*

Proof: Let $p = \{a_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{F} = \{d_\alpha : \alpha < \mathfrak{c}\}$, such that if $\alpha \neq \beta$, then $d_\alpha \neq d_\beta$. For each $F \in [\omega]^\omega$, denote by $\mathcal{I}_F$ the set $\{I \in [F]^\omega : \{F' \in \mathcal{F} : F' \cap I \text{ is infinite}\} \text{ is finite}\}$. Thus Lemma 9.0.3 says that for each member F of p, $\mathcal{I}_F$ is not empty. We will construct, by induction, $\{b_\alpha : \alpha < \mathfrak{c}\} \subseteq p$ and $\{I_\alpha : \alpha < \mathfrak{c}\}$ such that

(i) $I_\alpha \subseteq b_\alpha \subseteq a_\alpha \cap d_\alpha$,

(ii) $\forall \xi < \alpha$, $I_\xi \cap b_\alpha$ is finite,

(iii) $\forall \alpha < \mathfrak{c}$, $\{\xi < \alpha : I_\alpha \subseteq b_\xi\} = \{\xi < \alpha : I_\alpha \cap b_\xi$ is infinite $\}$ is finite,

(iv) $\forall \alpha < \mathfrak{c}$, $\{F \in \mathcal{F} : I_\alpha \cap F$ is infinite $\}$ is finite.

Suppose first that we have constructed the above sequences. Let $\mathcal{I} = \{I_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{B} = \{b_\alpha : \alpha < \mathfrak{c}\}$. Obviously, condition (i) and (ii) ensures that $\mathcal{B}$ is a base of p and $\mathcal{I}$ is an almost disjoint family. It is also straightforward to see that $\mathcal{I}$ and $\mathcal{B}$ satisfies the rest properties of Proposition 9.0.1. Let b_0 be $a_0 \cap d_0$ and I_0 be any element of $\mathcal{I}_{b_0}$. At stage α, we have to find b_α and I_α which satisfies (i) to (iv). By condition (iv) and the fact that $|\mathcal{F}| = \mathfrak{c}$, we can find $F \in \mathcal{F}$ such that $F \cap I_\xi$ is finite for each $\xi < \alpha$. Let $b_\alpha = a_\alpha \cap d_\alpha \cap F$. Take any $I \in \mathcal{I}_{b_\alpha}$. Since for each $\xi \in \alpha$, $b_\xi \cap I \subseteq d_\xi \cap I$, there are only finitely many ξ such that $b_\xi \cap I$ is infinite. It is easy to find a subset I_α of I such that, for each of those finitely many b_ξ's which has infinite intersection with I, either it contains I_α or it is disjoint from I_α. $\square$

LEMMA 9.0.5. *There exists a filter base, $\mathcal{F}$, on ω such that $|\mathcal{F}| = \mathfrak{c}$ and for each $F \in \mathcal{F}$ there exists an infinite set $I \subseteq F$ such that $\{F' \in \mathcal{F} : F' \cap I$ is infinite$\}$ is finite.*

Proof: We will identify ω with the set $P = \bigcup_{n\in\omega} \mathcal{P}({}^n 2)$. For each $E \in [{}^\omega 2]^{<\omega}$, let $X_E = \bigcup_{n\in\omega}\{A \subseteq {}^n 2 \colon E\restriction n \subseteq A\}$ and $I_E = \{E\restriction n \colon n \in \omega\}$.

For finite subsets E and E', it is easy to see that $I_E \subseteq X_E$ and $X_{E\cup E'} = X_E \cap X_{E'}$. Moreover, if $E' \nsubseteq E$, then, there is an m such that for all $n > m$, $E'\restriction n \nsubseteq E\restriction n$. Therefore, $I_E \cap X_{E'} \subseteq \{E\restriction n \colon n \leq m\}$ is finite. Let $\mathcal{F} = \{X_E \subseteq P : E \in [{}^\omega 2]^{<\omega}\}$. Clearly $\mathcal{F}$ is a filter base on P with cardinality $\mathfrak{c}$ and for each X_E in $\mathcal{F}$, I_E is an infinite subset of X_E which is almost disjoint from all but finitely many members of $\mathcal{F}$. $\square$

REMARK 9.0.6. If we take p as in Proposition 9.0.1, the space $\omega \cup \mathcal{I} \cup \{p\}$ with the usual topology is quite interesting: the first property ensures that it has tightness $\mathfrak{c}$ while the second property says that it is a 0-dimensional scattered space.

CHAPTER 10

Conclusions

In this chapter, we will try to give a summary of our results. As we have almost listed all the results in Chapter 1, we will rather focus on some aspects of our work which we think deserve further mention. We will also discuss the potential topics arising from the present work for further research. Finally we will give a list of open questions at the end of this chapter.

In the study of the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$. One important technique frequently used in the iterative construction of models is that one creates a gap in $\mathcal{P}(\omega)/\operatorname{Fin}$ in some intermediate stage and then freezes the gap in the further construction, i.e. no set will allowed to fill the gap in any extension of. For example, Shelah, used this technique in [She82] to construct a model in which there are only trivial automorphisms of the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$. This result but even more so, the techniques and the idea involved have been widely and numerously applied. There are also other standard techniques in handling gaps in $\mathcal{P}(\omega)/\operatorname{Fin}$ which have been widely exploited in both set theory and set-theoretic Topology. But progress has been much slower for gaps in $\mathcal{P}(\omega)/\mathcal{I}$ when $\mathcal{I}$ is a more general ideal. The gap structure is simply much more complicated. We believe that this is one of the reasons that there has been no single progress in solving Baumgartner and Weese's problem in twelve years.

We invented a technique to produce simultaneously infinitely many gaps in $\mathcal{P}(\omega)/\operatorname{Fin}$ which we could relate to an unfilled gap in $\mathcal{P}(\omega)/\mathcal{I}_{\mathcal{M}}$. Our technique will ensure that these gaps will remained unfilled in any meaningful extension of the present model. In the construction of the counter example to Baumgartner and Weese's question, we were adopting Shelah's idea to prove that the algebra $\mathfrak{B}$ is not representable. But all the argument involve gaps in Shelah's construction had to be carefully modified. . This technique helped us to overcome the difficulty to find a gap in our partitioner algebra. We think it is because of this technique that our proof is not merely an improvement of Shelah's proof rather it in some sense shows how to transplant Shelah's argument to a much broader area which is was not done before. We believe that this is the first technique in producing an unfilled gap in $\mathcal{P}(\omega)/\mathcal{I}$ when $\mathcal{I}$ is as complicated as our $\mathcal{I}_{\mathcal{M}}$ (See Chapter 2). More to the point, this technique should have more applications in the study of representable algebras as well as the Boolean algebra of the form $\mathcal{P}(\omega)/\mathcal{I}$. We will investigate if our technique can lead to the generalizations of Shelah's results to many more Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ which is an important, interesting but largely unexplored area. In particular, we are optimistic that we can construct representable algebra which has only trivial automorphisms.

An immediate question following from our proof is that if the large cardinal is needed. In actual fact, the only place it is now necessary is the application of Chang's conjecture in the theorem about strong chains in $[\omega_1]^{\omega_1}$. At other places the real strength we are able to restrict the application PFA to the posets with cardinality not bigger than $\aleph_2$ which is consistent relatively to ZFC. However, we currently do not know how to prove the fundamental theorem (Theorem 3.0.1) without using Chang's Conjecture.

As pointed out in the introduction the spaces of the form $\omega \cup \{p\}$ are important in the study of sub pseudoradialness. Our results about the pseudoradial spaces shows that one can not expect to decide generally that, for ultrafilters $p \in \omega^*$, whether or not the space $\omega \cup \{p\}$ is sub pseudoradial. By Theorem 6.2.2, we know it is impossible to find an ultrafilter p on ω, in ZFC, such that $\omega \cup \{p\}$ is not a subspace of any regular pseudoradial spaces. It is still not known if, in ZFC, there exists an ultrafilter p on ω such that $\omega \cup \{p\}$ is a subspace of a regular pseudoradial space. However, we think it is very likely that the ultrafilter constructed in Chapter 9 could be such an ultrafilter. At least our proof in Chapter 8 suggests that $\mathfrak{c} = \omega_2$ could imply that $\omega \cup \{p\}$ will be a subspace of a pseudoradial spaces. We wish we can resolve this problem.

Two other interesting things arising from our study of pseudoradial spaces are Lemma 8.2.5 and the notion of *reflecting self-indexed stationary set* on ω_2.

Two extreme cases for the conditions in the Lemma 8.2.5 are: (1) when $\mathcal{U} \cap R$ is the family of co-finite subsets of R; (2) $\mathcal{U} \cap R$ are well-ordered non empty set. These two cases are actually most frequently used to guarantee a convergent sequence to x. Lemma 8.2.5 offers an easy tool to find a convergent sequence in the situation which is neither as good as (1) nor as good as (2) but somehow a combination of them. It is quite surprising that Lemma 8.2.5 is so easy to prove but we believe the idea is quite new. We hope it will be really helpful in the studies of topological spaces where the question of existence of certain convergent sequence is involved.

The notion of a reflecting self-indexed stationary set on ω_2 has been very interesting to quite a number of set theorists after we introduced it. Other applications have already been found (see [Dow]). We believe that it will also be an useful tool when the argument involves reflecting certain properties of a structure on ω_2 to lower ordinals. It may be a theorem of ZFC that there exists a reflecting self-indexed stationary set on ω_2. Certainly we can prove the existence in a very wide variety of models and that a large cardinal must be involved in any non-existence result. Of course we do not know if, in the Cohen model, we can embed $\omega \cup \{p\}$ into a regular pseudoradial space, for the ultrafilter p constructed in Chapter 9, we need that there is a reflecting self-indexed stationary set on ω_2 in the ground model. We hope we can answer whether ZFC ensures that there is a a reflecting self-indexed stationary set on ω_2.

Finally we give a list of problems

1. Is the large cardinal strength is really necessary in our construction of the non representable algebra $\mathfrak{B}$?

2. Does PFA or the existence of a supercompact cardinal implies that, for each Boolean algebra of the form $\mathcal{P}(\omega)/\mathcal{I}$, each automorphism is basically induced by a bijection from a co-finite subset of ω to a co-finite subset of ω?

3. Does the reflecting self-indexed stationary set on ω_2 exists in ZFC?

4. Does MA imply that, for each $p \in \omega^*$, the space $\omega \cup \{p\}$ is a subspace of regular pseudoradial space?

5. Is there a model of MA and $\neg$CH in which for each $p \in \omega^*$, the space $\omega \cup \{p\}$ is a subspace of regular pseudoradial space.
We expect the answer to this question is yes.

6. Does there exist, in ZFC, an ultrafilter p on ω such that $\omega \cup \{p\}$ is a subspace of regular pseudoradial spaces?

7. Does $\mathfrak{c} = \omega_2$ implies that for the ultrafilter p constructed in Chapter 9, $\omega \cup \{p\}$ is a subspace of a regular pseudoradial spaces?

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