A *NOT TOO DENSE* SUBSET OF N^*

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In honour of P. S. Alexandroff

ABSTRACT. We use independant matrices to inductively construct a dense subset, Y, of N^*. The space $N \cup Y$ has a compactification in which no subset of N is C^*-embedded. This is then used to answer a question of Finn, Martinez, and McGovern on the cardinality of C^*-embedded cellular families of open sets.

1. INTRODUCTION

A new cardinal invariant, the *separated cellularity* for Tychonoff spaces is defined in [FMM2] . We simplify slightly by saying that they were interested in the cardinalities of C^*-embedded families of pairwise disjoint open subsets of a space. They observed [FMM2, Lemma 1.4] that this cardinal invariant was monotone decreasing for regular closed subsets in normal spaces ([AU24]) but indicated that they did not know if the normality hypothesis was necessary. We first noticed that this question gave rise to a quite natural question concerning dense subsets of N^* which we pose and answer in this note.

2. DEFINITIONS

Recall that a subspace Y in a space X is C^*-embedded if every bounded continuous real-valued function defined on Y can be continuously extended to all of X . A set A is *completely separated* from a set B (in a space X) if there is a continuous real-valued function f such that $f[A] = 0$ and $f[B] = 1$. A space is compact ([AU23]) if every open cover has a finite subcover and a compact space K is a compactification of X if X is dense in K .

A family of subsets of X is *cellular* if the members (usually open) are pairwise disjoint. Such families, $\mathcal{A}$, are called *open cells* in [FMM2] . An open cell, $\mathcal{A}$, is called a *separated cell* if, for each disjoint $\mathcal{A}_0, \mathcal{A}_1 \subset \mathcal{A}$, $\bigcup \mathcal{A}_0$ is completely separated from $\mathcal{A}_1$. A space is said to be an *fs*-space (for finite separation) if every separated cell is finite, and, more generally, $sc(X)$ is defined as the supremum of the cardinallities of the separated cells in X . The notion of separated cellularity arose because of its relationships with the global dimensions of Boolean algebras (see [F2, FM, FMM1, FMM2])

Of course N is the countable discrete space, N^* denotes $\beta N \setminus N$, and βX is the Stone-Čech compactification of the space X. If B is a Boolean algebra, then $S(B)$ denotes the Stone space of ultrafilters on B. If spaces X and Y each contain a homeomorphic copy of a space K, we let $X \cup_K Y$ denote the quotient ([Ale27]) of

1991 *Mathematics Subject Classification.* 54D35,54G05,54C45.

Key words and phrases. compactification, C^*-embedding, βN, independant matrix.

the space $X \oplus Y$ in which each point of one copy of K is identified with its twin in the other.

A $\kappa \times \lambda$-independant matrix, introduced by Kunen [Kun83], on a set A is a doubly indexed family

$$\{a(\alpha,\beta) : (\alpha,\beta) \in \kappa \times \lambda\}$$

of subsets of A such that

1. for each $\alpha \in \kappa$ and $\beta < \gamma \in \lambda$, $a(\alpha,\beta) \cap a(\alpha,\gamma)$ is finite;
2. for each finite $H \subset \kappa$ and each function $\varphi : H \to \lambda$, the set $\bigcap\{a(\alpha,\varphi(\alpha)) : \alpha \in H\}$ is infinite.

This notion is generalized to the notion of *independant mod a filter base* $\mathcal{F}$, if the "is infinite" in the second condition is strengthened to "meets every member of $\mathcal{F}$". In addition, there is no mention of well-ordering or cardinality in the definition, so the meaning of an $I \times J$-independant matrix, for arbitrary index sets I and J, is unambiguous.

The following folklore result is a simple illustration of their use and should obviate the need for unnecessarily cluttering our main construction with a repetition of these details. It is essentially taken from Bell and Kunen's construction, [BK81]), of a point in N^* which has π-character equal to the cofinality of $\mathfrak{c}$.

Proposition 2.1. *If Y is a subset of N^* with cardinality at most $\mathfrak{c}$, then there is a point $x \in N^*$ which is not a limit point of any subset of Y with cardinality less than the cofinality of $\mathfrak{c}$.*

Proof. Enumerate $Y = \{y_\alpha : \alpha \in \mathfrak{c}\}$, $\mathcal{P}(N) = \{b_\alpha : \alpha \in \mathfrak{c}\}$, and fix a $\mathfrak{c} \times \mathfrak{c}$-independant matrix on N. Set $x_0 = \{N\}$. We construct an increasing chain of filter bases, $\{x_\alpha : \alpha \in \mathfrak{c}\}$, together with finite subsets L_α of $\mathfrak{c}$, so that, for each $\alpha \in \mathfrak{c}$,

1. one of $\{b_\alpha, N \setminus b_\alpha\}$ is a member of $x_{\alpha+1}$;
2. the matrix $\{a(\xi,\beta) : \xi \in \mathfrak{c} \setminus \bigcup_{\gamma \leq \alpha} L_\gamma\ , \beta \in \mathfrak{c}\}$ is independant mod $x_{\alpha+1}$;
3. there is a set $c_\alpha \in x_{\alpha+1}$ such that c_α is not a member of y_γ for all $\gamma \leq \alpha$.

There are two key ideas involved. The first is that, at stage α, we may choose any finite subset, L_α, of $\mathfrak{c} \setminus \bigcup_{\gamma<\alpha} L_\gamma$ and any function $\varphi : L_\alpha \to \mathfrak{c}$ and set $x_{\alpha+1}$ to be the filter base generated by $x_\alpha \cup \{\bigcap_{\xi \in L_\alpha} a(\xi,\varphi(\xi))\}$. It is easily checked, but herein lies the power of the method, that this will preserve the second inductive hypothesis. The second main idea is what is needed to ensure the first inductive hypothesis (while preserving the second). Either we can set $x_{\alpha+1}$ to be the filter base generated by $x_\alpha \cup \{b_\alpha\}$ (and set L_α to be anything) or there is a choice for L_α and $\varphi : L_\alpha \to \mathfrak{c}$ such that $N \setminus a_\alpha$ contains some member of the filter generated by $x_\alpha \cup \{\bigcap_{\xi \in L_\alpha} a(\xi,\varphi(\xi))\}$. The choice for L_α and φ in the second case is simply any witness to the failure of "independence mod $x_{\alpha+1}$" if the first case fails. At this stage, we have constructed a, possibly empty, finite subset L_α of $\mathfrak{c}$ and a set, say S_α, which we will add to $x_{\alpha+1}$.

Finally here is how the third condition in the inductive hypotheses is (easily) satisfied. There is nothing lost in adding one new ordinal to L_α . Select any $\xi \in \mathfrak{c} \setminus \bigcup_{\gamma \leq \alpha} L_\gamma$ (and add this to L_α). Observe that, for each $\gamma \leq \alpha$, there is at most one $j_\gamma \in \mathfrak{c}$ such that $a(\xi, j_\gamma) \in y_\gamma$. Fix any $\eta \in \mathfrak{c} \setminus \{j_\gamma : \gamma \leq \alpha\}$ and also add $c_\alpha = a(\xi,\eta)$ to $x_{\alpha+1}$. For the same reasons as above, this will preserve the first and second inductive hypotheses and clearly ensures the third. $\square$

3. The requirement and application

We pose the question, is there a dense subspace Y of N^* such that that $N \cup Y$ has a compactification so that no (infinite) subset of N is C^*-embedded? That is, each infinite subset a of N has a subset b such that b and $a \setminus b$ do not have disjoint closures in the compactification.

The motivation for this question is the following. Assume that X is a space in which there are no infinite separated cells Further suppose that the countable discrete space N is a regular open subset of X. Is it possible to have that the closure of N (hence a regular closed subset of X) does have a infinite C^*-embedded family of open subsets, namely $\{\{n\} : n \in N\}$ itself? It appeared that this is the essence of the the question about non-normal *fs*-spaces. If there is such an example and we examine the set, Y, of limit points of N in X, we will find that $N \cup Y$ is a subspace of βN (simply because N is dense and C^*-embedded). Furthermore, since X itself did not have any infinite C^*-embedded families of open sets, it must be the case that Y is dense in N^* (a subset A of N with no limits would certainly be C^*-embedded). Carrying this one step further, we see that the closure of N in βX is a compactification of $N \cup Y$ in which no subset of N is C^*-embedded.

Let us check that the converse is also true, the existence of a subset Y and its compactification will entail the existence of an *fs*-space X with a regular closed subset which is not an *fs*-space.

Begin with a simple Lemma about C^*-embedded subsets.

Lemma 3.1. *Assume that A, B are subsets of a space X such that A is C^*-embedded in $\overline{A} \cap (A \cup B)$ and B is C^*-embedded in X. If B contains a relatively dense subset of $\overline{A} \setminus A$, then $A \cup B$ is C^*-embedded in X.*

Proof. Suppose that $f : A \cup B \mapsto [0,1]$ is continuous and fix a point $x \in \overline{A \cup B}$. It suffices to show that x is not a common limit point of $f^{-1}(0)$ and $f^{-1}(1)$. Naturally we assume otherwise and come to a contradiction. Since f is continuous on A and B is C^*-embedded in X, x must be a member of $\overline{A} \setminus A$. Thus x is a limit point of B and there is a value t which the continuous extension of $f \restriction B$ will assign to x.

Fix r, s such that $0 < r < s < 1$ and for definiteness, let us assume that $t \in (s, 1]$. Working in $\overline{B}$, there is a neighbourhood W of x such that $f[W] \subset [s, 1]$. Find a cozero subset U of $\overline{A}$ so that $x \in U \cap \overline{B} \subset W$. Now $U \cap \left(A \cap f^{-1}(0)\right)$ has x in its closure. Consider $f \restriction A$ and recall that A is C^*-embedded in $\overline{A} \cap (A \cup B)$. From this it follows that the closure (in $A \cup B$) of $f^{-1}([0, r])$ when intersected with W has interior relative to $(B \cap \overline{A}) \setminus A$. Since B is dense in $\overline{A} \setminus A$, there is a point $b \in B$ in this set. Clearly this situation implies that f is not continuous at b. $\square$

Now suppose we have our dense subspace Y of N^* such that $N \cup Y$ has a compactification $N \cup K$ in which no infinite subset of N is C^*-embedded. Recall that the space $[0,1]^{\mathfrak{c}}$ is universal for spaces of weight $\mathfrak{c}$ (see [Tyc30, Ale36]). Thus we may choose a copy of K in $[0,1]^{\mathfrak{c}}$ such that there is an uncountable subset L of $\mathfrak{c}$ such that each $k \in K$ is constantly equal to 1 on L. Our desired space X is a subset of the adjunction space $(N \cup K) \cup_K [0,1]^{\mathfrak{c}}$ which is obtained by removing $K \setminus Y$. Since the closure of N in X is equal to $N \cup Y$, which is a subspace of βN, it follows N is C^*-embedded. To see that X itself has no infinite C^*-embedded cellular families we will first apply Lemma 3.1 with $A = N$ and $B = X \setminus N$ to see that βX is simply $(N \cup K) \cup_K [0,1]^{\mathfrak{c}}$. For this, we just have to recall that the sigma product on $[0,1]^{\mathfrak{c}}$ is dense and C^*-embedded (and disjoint from K). Finally, βX

has no infinite C^*-embedded family of pairwise disjoint open sets. Clearly those that trace an infinite family on N will not be C^*-embedded by the hypothesis on K. It is very well known that $[0,1]^{\mathfrak{c}}$ contains no such families.

4. The subspace of N^*

Inductively define filter bases, $p(\alpha,\beta,i)$ for $\alpha \le \beta < \mathfrak{c}$ and $i \in 3$, on N so that each has cardinality at most $\omega + |\beta|$. Let $\{a_\alpha : \alpha \in \mathfrak{c}\}$ be an enumeration of the infinite subsets of N. We demand that a_α is in the filter generated by each of $p(\alpha,\alpha,i)$ for $i < 3$. The filter bases $p(\alpha,\beta,0)$ have the property that each member x has the property that for each $\gamma \le \beta$, x is in either in both $p(\gamma,\beta,1)$ and $p(\gamma,\beta,2)$ or $N \setminus x$ is in both (i.e. x is in neither).

We will take B to be the Boolean subalgebra of $\mathcal{P}(N)$ which is generated by the union of all the filter bases

$$\{p(\alpha,\beta,0) : \alpha < \beta < \mathfrak{c}\}$$

together with the finite sets. Then our space K will be such that $N \cup K$ will be the Stone space of B. Finally, the space Y will be the ultrafilters on N generated by each of the filter bases in $\{p(\alpha,\mathfrak{c},0) : \alpha < \mathfrak{c}\}$. This is a dense subset of N^* . Now recall that each $p(\alpha,\mathfrak{c},0)$ has a base consisting of sets which do not distinguish between $p(\beta,\mathfrak{c},1)$ and $p(\beta,\mathfrak{c},2)$ for any β and so this is true of every member of B . So the role of these two ultrafilters on N in this construction is to serve as witnesses to the fact that the set a_β is not C^*-embedded in $S(B)$.

At stage α, we choose a $\mathfrak{c} \times \mathfrak{c}$-independent matrix $\mathcal{A}_\alpha$ on $\mathcal{P}(a_\alpha)$ and as we go we ensure that some large piece, $\mathcal{A}_{\alpha,\beta}$ of $\mathcal{A}_\alpha$ is still independent mod $p(\alpha,\beta,0)$.

Our typical problem is that we have a set $x \subset N$ and some $\alpha \le \beta$ and we must pick a set y such that $y \subset x$ or $y \cap x = \emptyset$ and then put $y \in p(\alpha,\beta,0)$. The special requirement for y is that for each $\gamma \le \beta$ we must either put $y \in p(\gamma,\beta,1) \cap p(\gamma,\beta,2)$, or we must put $N \setminus y$ in each of these filter bases. We will worry about choosing $p(\beta,\beta,i)$ in a moment.

We can ask if we can remove only finitely many rows from $\mathcal{A}_{\alpha,\beta}$ and have the resulting matrix be independent modulo $p(\alpha,\beta,0) \cup \{x\}$. If not, there is some a which is a finite intersection of members of $p(\alpha,\beta,0) \cup \mathcal{A}_{\alpha,\beta}$ and which is disjoint from x. So by either working with x (in the first case) or a in the second, we can assume that $\mathcal{A}_{\alpha,\beta}$ is independent modulo the filter generated by $p(\alpha,\beta,0) \cup \{x\}$. Now let $\{y_\xi : \xi \in \mathfrak{c}\}$ be a row of (the remaining submatrix of) $\mathcal{A}_{\alpha,\beta}$. We will pick a ξ such that y_ξ does not distinguish between any $p(\gamma,\beta,1)$ and $p(\gamma,\beta,2)$ for $\gamma < \beta$. More precisely, in this context, it means that we cannot extend the two filters uniformly with respect to y_ξ. It follows that this only happens if there is an $r \in p(\gamma,\beta,j) \setminus p(\gamma,\beta,3-j)$ such $r \subset y_\xi$. Since distinct y_ξ cannot mess up with distinct (γ,j) (i.e. they cannot both contain such an r), there is a ξ such that $N \setminus y_\xi$ is a member of $p(\gamma,\beta,j)^+$ for all $\gamma < \beta$ and $j \in \{1,2\}$. This is what we do: we put this $y_\xi \cap x$ in $p(\alpha,\beta+1,0)$ and we put $N \setminus y_\xi$ in $p(\gamma,\beta+1,1) \cap p(\gamma,\beta+1,2)$ for all $\gamma < \beta$.

The next step is to pick $p(\beta,\beta,i)$ for $i \in 3$. Picking $p(\beta,\beta,0)$ is really no different than the above step. In fact we can just pick $\mathcal{A}_\beta$ in $\mathcal{P}(a_\beta)$ and let $p(\beta,\beta,0) = \{N\}$ (the construction of $\mathcal{A}_\beta$ already ensures that a_β will be in the filter generated by $p(\beta,\mathfrak{c},0)$. To pick the filters $p(\beta,\beta,1)$ and $p(\beta,\beta,2)$. We want to ensure that they will be distinct from each other. Let B_β be the Boolean subalgebra generated by

$\bigcup\{p(\gamma,\beta,0):\gamma<\beta\}$ and let $\mathcal{U}$ be any ultrafilter containing a_β which has character $\mathfrak{c}$. Since character is equal to pseudocharacter in compact spaces ([Ale24]) we can extend $\mathcal{U}\cap B_\beta$ to two distinct filter bases $p(\beta,\beta,1)$ and $p(\beta,\beta,2)$.

This completes the construction.

5. Larger cardinals

We have still not produced a space X with a regular closed subset R such that $sc(R) < sc(X)$. This is a sup=max distinction caused by the fact that $sc(X)$ is defined as the supremum of the cardinalities of separated cells while our example and its regular closed subset, both have this cardinal equal to ω . We offer the following remarks in response.

Clearly, for any cardinal κ for which there is a $2^\kappa\times 2^\kappa$-independant matrix in $\mathcal{P}(\kappa)$, $\beta\kappa$ will have a subset Y which is dense in the uniform ultrafilters on κ and such that $\kappa\cup Y$ has a compactification in which no subset of κ with cardinality κ will be C^*-embedded. Thus there will be a space X which has no C^*-embedded families of cardinality κ and has a regular closed subset which does have such families.

These examples will suffer from the same sup=max problems as above unless κ is a successor cardinal. The problem though is that it is not known if any successor cardinals carry (in ZFC) a suitable independant matrix. It is known that a regular cardinal κ carries a $2^\kappa\times\kappa^+$-independant matrix and, if $2^{<\kappa}=\kappa$, then it carries a $2^\kappa\times 2^\kappa$-independant matrix.

We can however, solve, rather simply, the sup=max problem. With no changes to our construction, we can choose a subset Y of $\beta\mathfrak{c}$ which is dense in the remainder (but still consisting of ultrafilters with a base of countable sets) such that $\mathfrak{c}\cup Y$ has a compactification $\mathfrak{c}\cup K$ in which there is no infinite cellular C^*-embedded family.

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