MORE SET-THEORY FOR TOPOLOGISTS

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Abstract. Basic applications in topology of $\diamondsuit$, elementary submodels and forcing are illustrated. Finite and countable supported forcing iterations are reviewed. Also there are two new applications of Prikry forcing to preservation results.

This article was written for those who will be attending the 1995 Summer Conference on General Topology and Applications. During the conference I will no doubt not be able to cover all of the material that is in this article, but the article and the talks are meant to give a broad introduction of the tools of set-theory that are most commonly applied to General Topology (as judged by me of course). I have tried to illustrate the techniques in the context of proving results which are interesting in their own right. We cover some basic combinatorial tools such as $\diamondsuit$ and $\square$ and discuss mad families, gaps and coherent sequences as well as giving proofs of two interesting results from $\diamondsuit$ in section 3. In section 4 we begin a more detailed, but introductory, discussion of Logic and Model Theory, although set-theorists would say that it is just set-theory. In section 5 we prove some basic results about sets in which we use elementary submodels in the proofs for illustrative purposes. In section 6 we very briefly discuss large cardinals and finally in section 7 we begin forcing. In section 8 we do two case studies of iterated forcing, the first with finite supported ccc forcing, and the second with countable support proper forcing. In section 9, we are attempting to look to the future of set-theory in topology and we are fortunate to be able to present a result of Avraham and Todorcevic which uses Shelah's method of D-completeness. Two new results in the paper concern (the failure of) certain preservation properties in forcing. A first countable non-Lindelöf space is constructed which can be made Lindelöf with cardinal preserving forcing and a non-separable space is found which can be made separable by a cardinal preserving forcing. These answer questions from the Open Problems in General Topology book.

1. Basics

Recall that an ordinal α is equal to the set of all ordinals β which are less than α (where 0 is the empty set and, e.g. $1 = \{0\}$, $2 = \{0, \{0\}\}$ etc. . A cardinal is an initial ordinal, i.e. an ordinal α such that if β is less than α then there is no function from β onto α. However the least ordinal (in fact cardinal) δ which will map cofinally into α is called the cofinality, denoted $cf(\alpha)$, of α. A cardinal or

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ordinal is called regular if it is equal to its cofinality and singular otherwise. If an ordinal α (usually a cardinal) has uncountable cofinality then the set of closed and unbounded subsets of α generate a filter, called the cub or club filter. A subset S of α is *stationary* in α if it meets every set which is cub in α. The ideal which is dual to the cub filter is called the non-stationary ideal and, for a given α, is denoted NS_α. One of the most basic facts about stationary sets is the pressing down lemma which we postpone to Lemma 5.6. First one of the most basic facts about cubs.

Proposition 1.1. *If $cf(\alpha) > \omega$, then the cub filter on α is $cf(\alpha)$-complete, that is, the intersection any family of fewer than $cf(\alpha)$ many cubs in α is again a cub.*

Proof. Let $\delta < cf(\alpha)$ and fix a family $\{C_\beta : \beta < \delta\}$ of cubs in α. Of course the intersection of any family of closed sets is again closed hence it suffices to show that $\bigcap_{\beta<\delta} C_\beta$ is unbounded. Fix any $\gamma_0 < \alpha$. Inductively define $\gamma_{n+1} > \gamma_n$, so that $C_\beta \cap [\gamma_n, \gamma_{n+1})$ is not empty for each $\beta < \delta$. Note that, given γ_n, $C_\beta - \gamma_n$ is not empty for each $\beta < \delta$, and, since $\delta < cf(\alpha)$, $\{\min(C_\beta - \gamma_n) : \beta < \delta\}$ is not cofinal in α. Therefore γ_{n+1} can be found. Next we use the fact that $cf(\alpha) > \omega$ to deduce that $\{\gamma_n : n \in \omega\}$ is not cofinal in α. Let ξ be the supremum of the sequence $\{\gamma_n : n \in \omega\}$ (i.e. ξ is the minimum of the non-empty set of ordinals which are greater than every γ_n; in fact, ξ is equal to $\cup_n \gamma_n$). To finish the proof, we show that $\xi \in C_\beta$ for each $\beta < \delta$. Indeed, for each n, choose $\xi(\beta, n) \in C_\beta \cap [\gamma_n, \gamma_{n+1})$ and observe that $\{\xi(\beta, n) : n \in \omega\}$ is cofinal in ξ. Since $\{\xi(\beta, n) : n \in \omega\}$ is contained in the closed set C_β it follows that $\xi \in C_\beta$. $\square$

The standard notation, see [6], for the set of finite subsets of a set I, is $[I]^{<\omega}$, and similarly, $[I]^\lambda$ and $[I]^{<\lambda}$ denote the set of sets of cardinality λ, and less than λ, respectively. Kunen uses $\lim(\alpha)$ to indicate that α is a limit ordinal and we will use $\mathrm{Lim}(\alpha)$ to denote the set of limit ordinals less than α. The context should be sufficient to decide if κ^λ is used to denote cardinal exponentiation or the set of functions from λ into κ although sometimes we will use ${}^\lambda\kappa$ to denote this latter set. If f and g are in ${}^\omega\omega$, then $f <^* g$ means that $\{n \in \omega : f(n) \geq g(n)\}$ is finite. Similarly, if A, B are sets, then $A \subset^* B$ indicates that $A - B$ is finite.

Lemma 1.2. *If $g : [\kappa]^{<\omega} \to \kappa$, for a regular cardinal κ, then there is a cub C in κ such that $\alpha \in C$ implies that $g(F) < \alpha$ for all $F \in [\alpha]^{<\omega}$.*

Proof. We first reduce to a function $h : \kappa \to \kappa$. For each $\alpha < \kappa$, $[\alpha]^{<\omega}$ has cardinality less than κ, hence there is an $\xi < \kappa$ such that $g(F) < \xi$ for all $F \in [\alpha]^{<\omega}$; let $h(\alpha)$ be the minimal such ξ which is at least as large as α. Note that, by the minimality condition, if $\alpha < \beta$, then $h(\alpha) \leq h(\beta)$. We can simply define C to be the set of α such that $h(\beta) < \alpha$ for all $\beta < \alpha$. We show that C is closed and unbounded in κ. First assume that $C \cap \alpha$ is cofinal in α. Let $\beta < \alpha$ and since $C \cap \alpha$ is cofinal in α, there is a $\gamma \in C$ such that $\beta < \gamma < \alpha$. But now, γ in C implies that $g(\beta) < \gamma$. Thus, $\alpha \in C$ and C is closed. Now to show that C is unbounded. Fix any $\xi_0 \in \kappa$ and define $\xi_{n+1} = h(\xi_n) + 1$. Let α be the supremum of the increasing sequence $\{\xi_n : n \in \omega\}$ and let $\beta < \alpha$. Since ξ_n is cofinal in α, there is an n such that $\beta < \xi_n$, in which case, $h(\beta) \leq h(\xi_n) < \xi_{n+1} < \alpha$, hence α is in C and C is unbounded. $\square$

Definition 1.3. $\diamondsuit^*$ is the statement that there is a sequence $\{A_\alpha : \alpha \in \omega_1\}$ such that A_α is countable and for each $X \subset \omega_1$, $S_X = \{\alpha : X \cap \alpha \in A_\alpha\}$ is a cub.

$\diamondsuit$ is the statement that there is a sequence $\{a_\alpha : \alpha \in \omega_1\}$ such that for each X the set $S_X = \{\alpha : X \cap \alpha = a_\alpha\}$ is stationary.

Kunen has shown that $\diamondsuit^*$ implies $\diamondsuit$. See Lemma 5.10 for the consistency of $\diamondsuit^*$. Standard coding techniques (e.g. in this case, the function Φ introduced below) allow things like $\diamondsuit$ to take many forms.

Proposition 1.4. *$\diamondsuit$ implies there is a sequence $\{a_\alpha : \alpha \in Lim(\omega_1)\}$ so that*

(1) *for each function $f \in {}^{\omega_1}\omega_1$, $S_f = \{\alpha : f \restriction \alpha = a_\alpha\}$ is stationary;*
(2) *for each $n \in \omega$ and each $A \subset [\omega_1]^{<\omega}$, $S_A = \{\alpha : A \cap [\alpha]^{<\omega} = a_\alpha\}$ is stationary.*

Proof. Let Φ be any bijection from $\omega_1 \times \omega_1$ to ω_1. If $f \in {}^{\omega_1}\omega_1$, then f can be regarded as a subset of $\omega_1 \times \omega_1$; hence $\Phi(f)$ can be regarded as a subset of ω_1. The set $S_{\Phi(f)} = \{\alpha : \Phi(f) \cap \alpha = a_\alpha\}$ is stationary. We finish by showing that coding behaves well on a cub. First of all, there is a cub C_1 so that $\Phi(\alpha \times \alpha) \subset \alpha$ for each $\alpha \in C_1$ (see Lemma 1.2). In addition, for each γ, we can define $g(\gamma)$ to be minimal so that there is a pair $(\alpha, \beta) \in g(\gamma) \times g(\gamma)$ so that $\Phi(\alpha, \beta) = \gamma$ (if γ happened not to be in the range of Φ, set $g(\gamma) = 0$). So we may as well asume that $g(\gamma) < \alpha$ for all $\gamma < \alpha$ and $\alpha \in C_1$. Next, again by Lemma 1.2, there is a cub C_2 such that $f \restriction \alpha : \alpha \to \alpha$ for all $\alpha \in C_2$, i.e. $f(\gamma) < \alpha$ for all $\gamma < \alpha$. So if α is in the stationary set $C_1 \cap C_2 \cap S_{\Phi(f)}$, it follows that

$$\Phi(f \restriction \alpha) = \Phi(f \cap (\alpha \times \alpha)) = \Phi(f) \cap \alpha = a_\alpha \ .$$

Therefore we will get the desired sequence $\{f_\alpha : \alpha < \omega_1\}$ if we define it as follows. Let $I = \{\alpha \in \mathrm{Lim}(\omega_1) : \Phi^{-1}(a_\alpha)$ is a graph of a function from α to $\alpha\ \}$, and for $\alpha \in I$, let $f_\alpha \subset \alpha \times \alpha$ be such that $\Phi(f_\alpha) = a_\alpha$. For $\alpha \in \mathrm{Lim}(\omega_1) - I$, fix any function f_α from α to α. The previous paragraph shows that for each $f : \omega_1 \to \omega_1$ there is a stationary subset of I ($C_1 \cap C_2 \cap S_{\Phi(f)}$) so that for each α in this set, $f \restriction \alpha = f_\alpha$. $\square$

2. $\beta\omega$ to $\beta\omega_2$

In this section we give some applications of the previous concepts to Stone-Čech compactifications of discrete spaces as well as introduce some concepts which we will need later. Recall that $\beta\kappa$ is simply the Stone space of ultrafilters of the Boolean algebra $\mathcal{P}(\kappa)$ where the fixed ultrafilter $\{A \subset \kappa : \alpha \in A\}$ is identified with α. We let κ^* denote the remainder $\beta\kappa - \kappa$ and for a set $A \subset \kappa$, A^* is the clopen subset of κ^* consisting of those ultrafilters on κ for which A is an element.

Recall that a mad family on ω is a set $\mathcal{A}$ of infinite subsets of ω such that the members of $\mathcal{A}$ are almost disjoint from one another (i.e. have finite intersection) and are maximal in this sense (i.e. for each infinite $Y \subset \omega$, there is an $A \in \mathcal{A}$ such that $|Y \cap A| = \omega$). Recall that the cardinal $\mathfrak{a}$ is the least cardinal such that there is a mad family of that size. Other well studied cardinals are: $\mathfrak{b}$ (the least cardinality of an unbounded family in $(\omega^\omega, <^*)$), $\mathfrak{u}$ (the minimum character of an ultrafilter on ω), $\mathfrak{d}$ (the minimum cardinality of a cofinal family in $(\omega^\omega, <)$), $\mathfrak{t}$ (the least cardinal of a maximal descending mod finite family of infinite subsets of ω), and $\mathfrak{p}$ (the least cardinal of a filter base $\mathcal{B}$ of infinite subsets of ω which has no pseudo-intersection, i.e. no infinite Y such that $Y - B$ is finite for each $B \in \mathcal{B}$).

Proposition 2.1. *There is a mad family of cardinality* $\mathfrak{c}$.

Proof. A standard and useful trick is to choose some other countable set to work with instead of ω. Consider the set $\bigcup_n {}^n 2 = 2^{<\omega}$, where ${}^n 2$ denotes the set of functions from n into 2. For each $f \in {}^\omega 2$, let $A_f = \{f \restriction n : n \in \omega\}$. It is easy to see that $A_f \cap A_g$ is finite for each $f \neq g$ in 2^ω. So we can take $\mathcal{A}$ to be any maximal almost disjoint family containing $\{A_f : f \in 2^\omega\}$. □

We feel compelled to mention P. Simon's wonderful result on products of compact Fréchet spaces. Call an almost disjoint family, $\mathcal{A}$, <u>nowhere mad</u> if for each $X \subset \omega$ such that X is not in the ideal generated by $\mathcal{A} \cup [\omega]^{<\omega}$, X contains an infinite subset which is almost disjoint from every member of $\mathcal{A}$. There is a natural first countable locally compact topological space associated with an almost disjoint family $\mathcal{A}$: we put a topology on $\omega \cup \mathcal{A}$ in which ω is dense open discrete and $\{A\} \cup \{A - n : n \in \omega\}$ is a neighbourhood base for the point A. This space is pseudocompact iff $\mathcal{A}$ is a mad family and the one point compactification is Fréchet iff $\mathcal{A}$ is nowhere mad.

Proposition 2.2. *There are compact Fréchet spaces X, Y such that $X \times Y$ is not Fréchet.*

Proof. The plan is to show that there is a mad family $\mathcal{A}$ which can be partitioned into two nowhere mad families. It will suffice, of course, to find a set $X \subset \omega$ and a mad family $\mathcal{A}$ so that $\mathcal{A}_X = \{A \cap X : A \in \mathcal{A}\} - [\omega]^{<\omega}$ has such a partition.

This is remarkably simple but quite clever. Let $\mathcal{A}$ be a mad family of cardinality $\mathfrak{c}$ and let $\{A_f : f \in 2^\omega\}$ be an enumeration. For each n, we have the partition P_n, Q_n, where $A_f \in P_n$ iff $f(n) = 0$. Assuming that there is no mad family and partition as desired we inductively define infinite sets X_n for $n \in \omega$. First of all, by assumption, either P_0 or Q_0 is "somewhere" mad so fix a set X_0 witnessing this. Now either $(P_0)_{X_0}$ is an infinite mad family on X_0, in which case we set $g(0) = 0$, or $(Q_0)_{X_0}$ is one and we set $g(0) = 1$. At stage $n < \omega$, we have chosen X_{n-1}, and for each $k < n$, $g(k) = 0$ implies that $(P_k)_{X_{n-1}}$ is an infinite mad family on X_{n-1} and $g(k) = 1$ implies that $(Q_k)_{X_{n-1}}$ is such a family. Just as in the $n = 0$ step, we can find X_n an, infinite subset of X_{n-1}, and $g(n) \in \{0, 1\}$ so that $g(n) = 0$ implies $(P_n)_{X_n}$ is an infinite mad family on X_n, and similarly for $(Q_n)_{X_n}$ if $g(n)$ is set to 1. In the end, we have defined $g \in 2^\omega$ and the sequence $\{X_n : n \in \omega\}$.

Now we must prove that there is a Y such that $Y - X_n$ is finite for all n and such that Y meets infinitely many A in $\mathcal{A}$ in an infinite set. Simon attributes this fact to J. Dočkálková. For each n, choose $A_n \in \mathcal{A} - \{A_0, \ldots, A_{n-1}\}$, so that $A_n \cap X_{n+1} = \{a(n, m) : m \in \omega\}$ is infinite. Observe that if $h \in {}^\omega\omega$, then $Y_h = \{a(n, m) : m \leq h(n)\}$ has the property that $Y_h - X_n$ is finite for each n. We must simply choose an h so that Y_h meets infinitely many A_f. Let $h \in {}^\omega\omega$ be arbitrary and fix any f_0 so that $A_{f_0} \cap Y_{h_0}$ is infinite. Since the A_n's are distinct there is an $h_1 \in {}^\omega\omega$ so that, for all but at most one n, $A_{f_0} \cap \{a(n, m) : m \geq h_1(n)\}$ is empty. Find another f_1 so that $A_{f_1} \cap \{a(n, h_1(n)) : n \in \omega\}$ is infinite and choose $h_2 > h_1$ so that $(A_{f_0} \cup A_{f_1}) \cap \{a(n, m) : m \geq h_2(n)\}$ is empty for all but at most two n's. Continue in this manner until we have chosen f_n and h_n for all n. Finally, define $h(n) = \max\{h_k(n) : k \leq n\}$ and note that $Y_h \cap A_{f_k}$ is infinite for each k since

$\{a(n, h_k(n)) : k \leq n \in \omega\}$ is contained in Y_h and $A_{f_k} \cap \{a(n, h_k(n)) : k \leq n \in \omega\}$ is infinite.

Now that we have Y_h, we know there is an $f \neq g$ (g constructed above) such that $Y_h \cap A_f$ is infinite. However, let n be such that $f(n) \neq g(n)$ and note that A_g and A_f are on different sides of the partition P_n, Q_n. We leave the reader to check that A_f is therefore almost disjoint from X_{n+1}, e.g. if $g(n) = 0$ then $P_n \restriction X_{n+1}$ is a mad family in X_{n+1} but A_f is not a member of P_n and is therefore almost disjoint from A for all $A \in P_n$, from which it follows that $A_f \cap X_{n+1}$ must be finite. This of course contradicts the facts that $Y - X_{n+1}$ is finite and $A_f \cap Y$ is infinite.

Therefore there is a mad family $\mathcal{A}$ which can be partitioned into $\mathcal{B}$ and $\mathcal{C}$ each of which is nowhere mad. Let X be the one point compactification of the space constructed from $\mathcal{B}$ and let Y be that constructed from $\mathcal{C}$. We leave as an exercise to show that, in $X \times Y$, $\{(n, n) : n \in \omega\}$ has the point (∞, ∞) as a limit point but has no subsequence converging to it. $\square$

We would like to construct a special mad family for later use. We will need the assumption that $\mathfrak{p} = \mathfrak{c}$. Bell has shown that $\mathfrak{p} = \mathfrak{c}$ is equivalent to the statement that Martin's Axiom holds for σ-centered posets, MA_σ. Those familiar with Martin's Axiom will find MA_σ much easier to apply than $\mathfrak{p} = \mathfrak{c}$. However the two statements are, after all, equivalent and the statement $\mathfrak{p} = \mathfrak{c}$ is much easier to understand. In the following proof we will be able to get away with using $\mathfrak{p} = \mathfrak{c}$ directly.

Lemma 2.3. *If $\mathfrak{p} = \mathfrak{c}$, then there is a mad family $\{A_\alpha : \alpha < \mathfrak{c}\}$ such that, if $\{\{B^\alpha_n : n \in \omega\} : \alpha < \mathfrak{c}\}$ is a fixed enumeration of $[\, [\omega]^{<\omega}\,]^\omega$ and if $\mathcal{I}_\alpha$ is the ideal generated by $\{A_\beta : \beta < \alpha\} \cup [\omega]^{<\omega}$, then for each α, the following holds: If $\beta < \alpha$ is such that $|\{n : B^\beta_n \cap I = \emptyset\}| = \omega$ for all $I \in \mathcal{I}_\alpha$, then $|\{n : B^\beta_n \subset A_\alpha\}| = \omega$.*

Proof. Let $\{A_n : n \in \omega\}$ be any partition of ω into infinite sets. Suppose that $\{A_\beta : \beta < \alpha\}$ have been chosen. Rather than using ω as our countable set on which we will apply $\mathfrak{p} = \mathfrak{c}$, we will use $[\omega]^{<\omega}$. For ease of notation, we may assume that $\{n : B^\beta_n \cap I = \emptyset\}$ is infinite for all $\beta < \alpha$ and $I \in \mathcal{I}_\alpha$ (just throw away those β which do have the property). We define a filter base. For each finite sequence $\beta_0 < \cdots < \beta_n < \alpha$, let $F(\langle \beta_0, \ldots, \beta_n \rangle) = \{x \in [\omega]^{<\omega} : x \cap \bigcup_{i \leq n} A_{\beta_i} = \emptyset$ and $(\forall i \leq n)\ (\exists m > n)\ B^{\beta_i}_m \subset x\}$. It is easy to check that every finite intersection of $F(\langle \beta_0, \ldots, \beta_n \rangle)$'s is infinite, in fact, equal to another $F(\langle \beta_0, \ldots, \beta_n \rangle)$. Since this filter base has cardinality less than $\mathfrak{p}$, there is an infinite $Y \subset [\omega]^{<\omega}$ such that $Y - F(\langle \beta_0, \ldots, \beta_n \rangle)$ is finite for all finite sequences $\beta_0 < \cdots < \beta_n < \alpha$. Simply define A_α to be the union of the members of Y. To see that $A_\alpha \cap A_\beta$ is finite, notice that it is contained in the union of the members of $Y - F(\langle \beta \rangle)$. In addition, to see that $\{m : B^\beta_m \subset A_\alpha\}$ is infinite for each β (we threw away the ones for which this did not need to hold), simply find any sequence $\beta_0 < \cdots < \beta_n$ so that n is arbitrary and β is equal to one of the β_i's and let $y \in Y \cap F(\langle \beta_0, \ldots, \beta_n \rangle)$. Since y contains B^β_m for some $m > n$ and n was arbitrary, it follows that A_α contains B^β_m for infinitely many m. $\square$

Next we consider gaps. A pair of sequences $\{a_\alpha : \alpha < \omega_1\}$ and $\{b_\alpha : \alpha < \omega_1\}$ form an (ω_1, ω_1)-gap (usually these are all subsets of ω) if for all $\beta < \alpha < \omega_1$, $a_\beta \subset^* a_\alpha$, $b_\beta \subset^* b_\alpha$, and $a_\alpha \cap b_\beta =^* \emptyset$. The interest in a gap is whether or not it can be filled: a set $A \subset \omega$ fills the above gap, if $a_\alpha \subset^* A$ and $A \cap b_\alpha =^* \emptyset$ for all $\alpha < \omega_1$. In applications, one is also interested in knowing whether the gap can be filled in some larger universe of sets (see section 4). As is well known, if ω_1 is relaced by ω in the definition of gap, then every gap can be filled, but Hausdorff showed that there is a **Hausdorff gap**.

Proposition 2.4. *There is an (ω_1, ω_1)-gap which cannot be filled.*

Proof. As mentioned above, every (ω, ω)-gap can be filled, so together with the fact that there are no countably infinite mad families, one can easily inductively construct an (ω_1, ω_1)-gap. In order to make it unfillable, we add the following condition to the inductive hypotheses: first of all $a_\alpha - a_\beta$ and $b_\alpha - b_\beta$ are both infinite for $\beta < \alpha$ and

$$(\forall n)\ \{\beta < \alpha : a_\alpha \cap b_\beta \subset n\} \ \text{ is finite .}$$

Let us define a_α for a limit α. Fix β_n increasing cofinal in α. We will choose k_n inductively and then set $a'_\alpha = \bigcup_n (a_{\beta_n} - k_n)$; we will then find a set Y and define $a_\alpha = a'_\alpha \cup Y$. The purpose of a'_α is to ensure that $a_\beta \subset^* a_\alpha$ for all $\beta < \alpha$, the purpose of k_n is to ensure that $a'_\alpha \cap b_\beta$ is finite for all $\beta < \alpha$, and the purpose of Y will be to satisfy the additional inductive hypothesis. Set $k_0 = 0$ and having defined k_n, choose $k_{n+1} > k_n$ large enough so that $a_{\beta_{n+1}} \cap b_{\beta_m} \subset k_{n+1}$ for each $m \leq n$. Let $\gamma < \alpha$ and choose n so that $\gamma < \beta_n$. Since $b_\gamma - b_{\beta_n}$ is finite and $a'_\alpha \cap b_{\beta_n}$ is contained in $\bigcup_{i \leq n}(a_{\beta_i} \cap b_{\beta_n})$, it follows that $a'_\alpha \cap b_\gamma$ is finite. Now for the set Y. Fix m and suppose that $a'_\alpha \cap b_\gamma$ is contained in m. First of all, there are only finitely many $\gamma < \beta_m$ such that $(a_{\beta_m} - k_m) \cap b_\gamma$ is contained in m so forget about those. Let F_m be the finite set of $\gamma \in [\beta_m, \beta_{m+1})$ such that $(a_{\beta_{m+1}} - k_{m+1}) \cap b_\gamma \subset m$ and fix $m \leq y_m \in (\bigcap_{\gamma \in F_m} b_\gamma) - (\bigcup_{i<m} b_{\beta_i})$. Note that $Y = \{y_m : m \in \omega\}$ is almost disjoint from b_{β_n} for each n. Therefore, Y is almost disjoint from b_β for all $\beta < \alpha$. Now, fix n and let $\gamma \in [\beta_m, \beta_{m+1})$ with $m \geq n$. Then either $(a_{\beta_{m+1}} - k_{m+1}) \cap b_\gamma \not\subset m$ or $y_m \in b_\gamma$. Therefore the final inductive hypothesis holds. To choose b_α, we note that $\omega - a_\alpha$ contains, mod finite, each b_β with $\beta < \alpha$. Clearly if we were to let $b_\alpha = \omega - a_\alpha$ we could not continue the induction. So, find any infinite $X \subset \omega - a_\alpha$ which is almost disjoint from each b_β $(\beta < \alpha)$ and let $b_\alpha = \omega - (a_\alpha \cup X)$.

Now suppose that $A \subset \omega$ fills the gap. Let Y_0 be an uncountable subset of ω_1 such that there is a k_0 such that for each $\alpha \in Y_0$, $a_\alpha \cap b_\alpha \subset k_0$ (usually one assumes that a_α and b_α are disjoint). Next find $Y_1 \subset Y_0$, uncountable, so that there is a k_1 such that $a_\alpha - A \subset k_1$ and $b_\alpha \cap A \subset k_1$, for all $\alpha \in Y_1$. Let $\alpha \in Y_1$ be such that $Y_1 \cap \alpha$ is infinite. Therefore, by our final inductive hypothesis, there is a $\beta \in Y_1 \cap \alpha$ and an $m > k_1$ such that $m \in a_\alpha \cap b_\beta$. But now $m \in a_\alpha - k_1$ implies that $m \in A$ and $m \in b_\beta - k_1$ implies that $m \notin A$. $\square$

Martin's Axiom (plus the negation of CH) implies that every unfilled (ω_1, ω_1)-gap contains a Hausdorff sub-gap (in the sense that there is an uncountable subset I so that $\{a_\alpha : \alpha \in I\}$, $\{b_\alpha : \alpha \in I\}$ satisfy the condition that $\{\beta < \alpha \cap I : a_\alpha \cap b_\beta \subset n\}$ is finite for all n and $\alpha \in I$). Let us use $\diamondsuit$ to construct a gap which is unfilled but does not contain a Hausdorff subgap. Kunen has shown that, for each unfilled (ω_1, ω_1)-gap, there is a ccc poset which introduces a Hausdorff subgap (hence the above Martin's Axiom result). The interested reader can return to this section after reading later sections and prove that for the gap we are about to construct there is a ccc forcing which introduces a set which fills the gap. Combining this, Kunen's result, and the results of section 7, this same reader can prove that $\diamondsuit$ implies there are ccc posets whose product is not ccc.

Proposition 2.5. *$\diamondsuit$ implies there is an unfilled gap $\{a_\alpha : \alpha < \omega_1\}$, $\{b_\alpha : \alpha < \omega_1\}$ such that $a_\alpha \cap b_\alpha = \emptyset$ for all α and for every uncountable $\mathcal{F} \subset [\omega_1]^{<\omega}$, there are $F \neq G \in \mathcal{F}$ such that either $A_F \cap B_F$ is not empty or $A_{F\cup G} \cap B_{F\cup G}$ is empty, where $A_F = \bigcup_{\alpha\in F} a_\alpha$ and $B_F = \bigcup_{\alpha\in F} b_\alpha$.*

Proof. Let $\{c_\alpha : \alpha \in \operatorname{Lim}\omega_1\}$ enumerate the infinite subsets of ω – we will ensure that $(a_\alpha - c_\alpha) \cup (b_\alpha \cap c_\alpha)$ is infinite, hence c_α will not split the gap. Using a coding technique such as in Proposition 1.4, let $\mathcal{F}_\alpha \subset [\alpha]^{<\omega}$ for limit α be such that for all $\mathcal{F} \subset [\omega_1]^{<\omega}$ there are stationarily many α such that $\mathcal{F} \cap [\alpha]^{<\omega} = \mathcal{F}_\alpha$.

We inductively construct a_α and b_α such that

(1) a_α and b_α are disjoint and $\omega - (a_\alpha \cup b_\alpha)$ is infinite;
(2) $\beta < \alpha$ implies $a_\beta \subset^* a_\alpha$ and $b_\beta \subset^* b_\alpha$;
(3) if β is a limit and if $\mathcal{F}_\beta$ has the property that for each $G \in [\beta]^{<\omega}$ either $A_G \cap B_G$ is not empty or there is an $F \in \mathcal{F}_\beta$ such that $A_{F\cup G}$ is disjoint from $B_{F\cup G}$, then for each $G \in [\alpha]^{<\omega}$ and each finite $H, J \subset \omega$ and integer i, either $A' = A_G \cup H \cup (a_\alpha - i)$ is not disjoint from $B' = B_G \cup J \cup (b_\alpha - i)$ or there is an $F \in \mathcal{F}_\beta$ such that $A' \cap B_F$ and $B' \cap A_F$ are empty; in particular, for each $G \in [\alpha]^{<\omega}$ such that $A_G \cap B_G = \emptyset$, there is an $F \in \mathcal{F}_\beta$ such that $A_{F\cup G} \cap B_{F\cup G} = \emptyset$;
(4) if α is a limit and $\beta < \alpha$, then for each disjoint finite H, J and integer $i > \max(H \cup J)$ there is a $\xi < \alpha$ such that $H = i \cap a_\xi$, $J = i \cap b_\xi$, and $a_\xi - i = a_\beta$ and $b_\xi - i = b_\beta$.

Condition 3 is slightly stronger than it appears it needs to be because we have to anticipate the successor steps after α. Indeed, if we have picked a_α for a limit α, then we can just define $a_{\alpha+n} = H_n \cup a_\alpha - J_n$ and $b_{\alpha+k} = (J_n \cup b_\alpha) - H_n$ under some enumeration $\{(H_n, J_n) : n \in \omega\}$ of $[\omega]^{<\omega} \times [\omega]^{<\omega}$. It should be clear that all but possibly 3 of the inductive hypotheses hold for each $\alpha + k$. Since each $a_{\alpha+k}$ is almost equal to a_α and similarly for $b_{\alpha+k}$, condition 3 for $\alpha+k$ follows immediately from the fact that it holds at α.

Let $\{(\beta_k, G_k, H_k, J_k) : k \in \omega\}$ be an enumeration of $\alpha \times [\alpha]^{<\omega} \times [\omega]^{<\omega} \times [\omega]^{<\omega}$. For convenience, we assume that $\mathcal{F}_{\beta_k}$ has the property in 3 for each k, and that $H_k \cap J_k$ is empty. To construct a_α and b_α we will construct an increasing and, cofinal in α, sequence $\{\zeta_k : k \in \omega\}$ by induction and let $a_\alpha = \bigcup\{a_{\zeta_k} : k \in \omega\}$ and $b_\alpha = \bigcup\{b_{\zeta_k} : k \in \omega\}$. At stage k, we have ζ_{k-1} and we are looking to ensure that condition three holds for $\beta = \beta_k$, $G = G_k$, $H = H_k$ and $J = J_k$. We may as well assume that $(A_G \cup H) \cap (B_G \cup J)$ is empty. Fix i large enough so that, for each $j < k-1$, $a_{\zeta_j} - i \subset a_{\zeta_{k-1}}$, $b_{\zeta_j} - i \subset b_{\zeta_{k-1}}$, and, $(a_{\zeta_{k-1}} - i) \cap (B_G \cup J)$ and $(b_{\zeta_{k-1}} - i) \cap (A_G \cup J)$ are both empty. Apply condition 4, to find a ξ so that $H \subset a_\xi$, $J \subset b_\xi$, $a_\xi - i \subset a_{\zeta_{k-1}}$, and $b_\xi - i \subset b_{\zeta_{k-1}}$. Let $G' = G \cup \{\xi\}$, and note that $A_{G'}$ is disjoint from $B_{G'}$, so there is an $F \in \mathcal{F}_\beta$, which we fix, with $A_{F\cup G'} \cap B_{F\cup G'} = \emptyset$. Choose a ζ large enough so that $A_{F\cup G'} \subset^* a_\zeta$ and $B_{F\cup G'} \subset^* b_\zeta$. Next consider the set c_α; if possible find $n_1 \in \omega - (c_\alpha \cup B_{F\cup G'} \cup k)$ and $n_2 \in c_\alpha - (A_{F\cup G'} \cup k)$. Let j be large enough so that $i, n_1, n_2 < j$, $A_{F\cup G'} - j \subset a_\zeta$, and $B_{F\cup G'} - j \subset b_\zeta$. Note that so long as $a_\alpha - i \supset A_{F\cup G'}$, and $b_\alpha - i \supset B_{F\cup G'}$, we will have that $A' = A_G \cup H \cup (a_\alpha - i)$ is disjoint from B_F and $B' = B_G \cup J \cup (b_\alpha - i)$ is disjoint from A_F as required in 3. In addition, if c_α contains mod finite a_β for all $\beta < \alpha$ then we will always be able to choose n_2 and, similarly, if c_α is almost disjoint from b_β for all $\beta < \alpha$, then we can choose n_1. If we ensure that $n_1 \in a_\alpha$ and $n_2 \in b_\alpha$, then neither $a_\alpha - c_\alpha$ nor $b_\alpha \cap c_\alpha$ will be contained in k.

Now, using condition 4, there is a ζ_k such that $a_{\zeta_k} - j = a_\zeta - j$, $b_{\zeta_k} - j = b_\zeta - j$, $a_{\zeta_k} \cap [i,j) = (\{n_1\} \cup A_{F\cup G'}) \cap [i,j)$, $b_{\zeta_k} \cap [i,j) = (\{n_2\} \cup B_{F\cup G'}) \cap [i,j)$, $a_{\zeta_k} \cap i = a_{\zeta_{k-1}} \cap i$, and $b_{\zeta_k} \cap i = b_{\zeta_{k-1}} \cap i$. This completes the construction. One thing we have neglected to do is to ensure that $\omega - (a_\alpha \cup b_\alpha)$ is infinite. This can easily be done by choosing i at stage k larger than it was chosen at stage $k-1$ and large enough so that $i - (a_{\zeta_{k-1}} \cup b_{\zeta_{k-1}})$ has size at least k. By the construction and this additional assumption on i, this k sized set will be disjoint from $a_\alpha \cup b_\alpha$.

Now we show that the we have proven the lemma. Suppose that $\mathcal{F} \subset [\omega_1]^{<\omega}$ is uncountable and that $A_F \cap B_F$ is empty for all $F \in \mathcal{F}$. If $\mathcal{F}$ has the property that $A_{F\cup G} \cap B_{F\cup G}$ is not empty for all $F \neq G \in \mathcal{F}$, then we may use Zorn's Lemma to enlarge $\mathcal{F}$ to a maximal family with these two properties: i.e. $A_F \cap B_F$ is empty for each $F \in \mathcal{F}$ and, for $G \in \mathcal{F} - \{F\}$, $A_{F\cup G} \cap B_{F\cup G} \neq \emptyset$. Therefore, by the maximality, it follows that for each $G \in [\omega_1]^{<\omega}$ there is an $F \in \mathcal{F}$ such that $A_{F\cup G} \cap B_{F\cup G}$ is empty. Therefore, for each $\alpha < \omega_1$, there is a $g(\alpha) < \omega_1$ such that for each $G \in [\alpha]^{<\omega}$, there is an $F \in \mathcal{F} \cap [g(\alpha)]^{<\omega}$ such that $A_{F\cup G} \cap B_{F\cup G}$ is empty. By Lemma 1.2 there is a cub $C \subset \omega_1$ such that $g(\beta) < \alpha$ for all $\beta < \alpha$ with $\alpha \in C$. Now by $\diamondsuit$, there is a $\beta \in C$ such that $\mathcal{F}_\beta = \mathcal{F} \cap [\beta]^{<\omega}$. But now, $\mathcal{F}_\beta$ has the property in condition 3, so fix any $G \in \mathcal{F} - [\beta]^{<\omega}$ and let $\alpha = \max G$. Since $G \in \mathcal{F}$, $A_G \cap B_G$ is empty, so we can apply 3 for the case α, $H = \emptyset$, $J = \emptyset$ and $i = 0$ to obtain that there is an $F \in \mathcal{F}_\beta$ such that $A_{F\cup G} \cap B_{F\cup G}$ is empty – contradicting the assumption on $\mathcal{F}$. $\square$

Next we turn to $\mathcal{P}(\omega_1)$. Call a sequence $\{a_\alpha : \alpha < \omega_1\}$ a <u>coherent sequence</u> if $a_\alpha \subset \alpha$ and $\alpha < \beta$ implies $a_\beta \cap \alpha$ is almost equal to a_α. These coherent sequences correspond naturally to clopen subsets of $\omega_1^* - U(\omega_1)$ where $U(\omega_1)$ is the set of uniform ultrafilters on ω_1. Clearly if $\{a_\alpha : \alpha < \omega_1\}$ is a coherent sequence then so is $\{\alpha - a_\alpha : \alpha < \omega_1\}$. Say that a coherent sequence is non-trivial if, for each $\beta < \omega_1$, there is an α such that $a_\alpha - \beta$ is infinite. The pair $\{a_\alpha : \alpha < \omega_1\}$ and $\{\alpha - a_\alpha : \alpha < \omega_1\}$ form an unfillable gap iff the clopen subset of $\omega_1^* - U(\omega_1)$ given by $\bigcup_\alpha a_\alpha^*$ does not extend to a clopen subset of $\beta\omega_1$.

Proposition 2.6 (Warren). *There is a non-trivial coherent sequence $\{a_\alpha : \alpha < \omega_1\}$ such that if $A \in [\omega_1]^{\omega_1}$, then there is an α such that $A \cap (\alpha - a_\alpha)$ is infinite.*

Proof. Fix a partition $\{A_n : n \in \omega_1\}$ of ω_1 consisting of uncountable sets. We will construct a_α, by induction, so that $a_\alpha \cap A_n$ is finite for each n. If we do so then suppose that A is an uncountable subset of ω_1. There is an n so that $A \cap A_n$ is uncountable, hence there is an $\alpha < \omega_1$ so that $(A \cap \alpha) \cap A_n$ is infinite. It follows then, that $(A \cap \alpha) - a_\alpha$ is infinite as required.

We simply construct a_α by induction on α. If $\alpha = \beta + 1$, then let $a_\alpha = a_\beta$. If α is a limit then fix $\{\beta_n\}_n$ increasing cofinal in α with $\beta_0 = 0$. For each n, if $A_n \cap \alpha$ is cofinal in α, let γ_n be the minimal element of $(A_n \cap \alpha) - \beta_n$ otherwise ignore γ_n. Define

$$a_\alpha = \bigcup \{ \{\gamma_n\} \cup a_{\beta_n} - (\beta_{n-1} \cup \bigcup_{k<n} A_k) : 0 < n \in \omega \} \ .$$

By the induction hypothesis, it is easily checked that $a_\alpha \cap A_k$ is finite for each k. Similarly, $a_\alpha \cap \beta_n$ is almost equal to a_{β_n} since, $a_{\beta_n} \cap [\beta_k, \beta_{k+1})$ is almost equal to $a_{\beta_{k+1}} - \beta_k$ for each $k < n$. There are cub many α such that $A_n \cap \alpha$ is cofinal in α for each n, and for each such α, the sequence $\{\gamma_n : n \in \omega\}$ witnesses that $a_\alpha - \beta$ is infinite for each $\beta < \alpha$. $\Box$

Call a coherent sequence big if for each $A \in [\omega_1]^{\omega_1}$, there is an α so that both $A \cap a_\alpha$ and $A \cap (\alpha - a_\alpha)$ are infinite.

Proposition 2.7. *$\diamondsuit$ implies there is a big coherent sequence.*

Proof. Let $\{b_\alpha : \alpha < \omega_1\}$ denote a $\diamondsuit$-sequence. For each α such that b_α is cofinal in α, choose disjoint ω sequences S_α and T_α each of which are cofinal subsets of b_α. Again inductively construct a_α, where $a_\alpha = a_\beta$ if $\alpha = \beta + 1$ and for α a limit, let $T = \{\beta_n : n \in \omega\}$ and define $a_\alpha = S_\alpha \cup a_{\beta_0} \cup \bigcup\{a_{\beta_n} - (\beta_{n-1} + 1) : n \in \omega\}$. Notice that $T_\alpha \cap a_\alpha$ is empty, and, of course $S_\alpha \subset a_\alpha$. The proof that $\{a_\alpha : \alpha < \omega_1\}$ is a coherent sequence is again straightforward. To show that it is big, let A be an uncountable subset of ω_1. There is a cub C so that $A \cap \alpha$ is cofinal in α for each $\alpha \in C$. Therefore there is an $\alpha \in C$ such that $A \cap \alpha = b_\alpha$. But now $S_\alpha \subset a_\alpha \cap A$ and T_α is contained in $A - a_\alpha$. $\Box$

The final set-theoretic principle we would like to introduce is $\Box$.

Definition 2.8. A $\Box$-sequence on ω_2 is a sequence $\{C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\}$ such that

(1) C_α is a cub subset of α;
(2) if β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$;
(3) if α has countable cofinality, then C_α is countable.

Proposition 2.9. *$\Box$ implies there is a stationary set $E \subset \omega_2$ so that $cf(\alpha) = \omega$ for all $\alpha \in E$ and $E \cap \lambda$ is not stationary in λ for all $\lambda < \omega_2$.*

Proof. Let S^2_0 denote the set of α in ω_2 which have countable cofinality. For each $\alpha \in S^2_0$, let $g(\alpha)$ denote the order type of C_α. Since $g(\alpha)$ is countable for each α in S^2_0, there is a $\xi < \omega_1$ so that $E = \{\alpha \in S^2_0 - \omega_1 : g(\alpha) = \xi\}$ is stationary; i.e. otherwise, for each ξ fix a cub D_ξ in ω_2 so that $D_\xi \cap g^{-1}(\xi)$ is empty, and then apply Proposition 1.1, to deduce that $\bigcap_{\xi \in \omega_1} D_\xi$ is a cub missing the stationary set S^2_0. To finish, we must show that $E \cap \lambda$ is not stationary in λ for arbitrary $\lambda < \omega_2$. If $cf(\lambda)$ is countable then there is nothing to show so assume λ has uncountable cofinality. In this case, C'_λ (the limit points) is a cub in λ and since $\alpha \in C'_\lambda$ implies $C_\alpha = C_\lambda \cap \alpha$, there is at most one $\alpha \in C'_\lambda$ such that $g(\alpha) = \xi$. $\Box$

The principle "square" or box, $\Box$, is used to get inductions to go past ω_1-limits. To illustrate we give another application of $\Box$.

Proposition 2.10. *$\Box$ implies there is a sequence $\{Z_\alpha : \alpha < \omega_2\}$ such that each Z_α is a zero set of α^*, $\alpha < \beta$ implies $Z_\alpha = Z_\beta \cap \alpha^*$, and there is no zero set Z of ω_2^* such that $Z \cap \alpha^* = Z_\alpha$ for $\alpha < \omega_2$. If, in addition, GCH holds then we can ensure that each Z_α is clopen in ω_2^*.*

Proof. Define, by induction on $\alpha < \omega_2$, functions g_α from α into ω so that if $\alpha < \beta$, then there is an $n_{\alpha,\beta}$ so that $g_\alpha(\xi) < g_\beta(\xi) + n_{\alpha,\beta}$ and $g_\beta(\xi) < g_\alpha(\xi) + n_{\alpha,\beta}$ for all $\xi < \alpha$. We will define

$$Z_\alpha = \bigcap_n \{\xi < \alpha : g_\alpha(\xi) > n\}^*$$

and note that $Z_\alpha = Z_\beta \cap \alpha^*$ since $\{\{\xi < \alpha : g_\alpha(\xi) > n\} : n \in \omega\}$ and $\{\{\xi < \alpha : g_\beta(\xi) > n\} : n \in \omega\}$ generate the same filter.

If α has countable cofinality, and g_β has been chosen for $\beta < \alpha$, then there is no difficulty in defining a g_α so as to satisfy the inductive hypotheses. Indeed, choose β_n increasing cofinal in α and piece the g_{β_n} together in a manner similar to the construction of coherent sequences. However this is not always possible if $cf(\alpha) = \omega_1$ so we must make preparations at $\beta < \alpha$ with countable cofinality. This is where C_α comes in. Suppose we ensure that if $\gamma < \beta$ are both members of C'_α, then g_β is simply an extension of g_γ. If we do this, then we may let g_α be equal the union of $\{g_\beta : \beta \in C'_\alpha\}$. In actual fact, it will suffice to ensure this for all $\beta \in C'_\alpha$ above some fixed member of C'_α. This is where the stationary set E from Proposition 2.9 comes in. In 2.9, we found a $\xi < \omega_1$ so that $E = \{\beta : C_\beta$ has order type $\xi\}$ is stationary. For ordinals α in E we will do something special in order to ensure there is no Z as in the Lemma. If α is such that C_α has order type less than ξ, then let g_α be any function which satisfies the induction hypotheses. If C_α has order type greater than ξ, then let $\gamma_\alpha \in C_\alpha$ be such that $C_\alpha \cap \gamma_\alpha$ has order type ξ and define g_α to be any extension of $\bigcup\{g_\beta : \beta \in C'_\alpha - \gamma_\alpha\}$. If C'_α is cofinal in α, then g_α is uniquely defined, if C'_α is bounded in α, then C'_α has a maximum β and g_α is any extension of g_β satisfying the inductive hypothesis, i.e. pieced together from $\{g_\zeta : \zeta \in C_\alpha - \beta\}$.

Now suppose that $\alpha \in E$. In this case, we choose β_n increasing cofinal in α with $\beta_0 = 0$, and we define

$$g_\alpha = \bigcup_n \left(g_{\beta_{n+1}} \restriction [\beta_n, \beta_{n+1})\right) + n \ .$$

It is very easy to check, by the inductive hypothesis, that for each $\beta < \alpha$, there is an $n_{\beta,\alpha}$ as required. Note that if $S \subset \alpha$ is an increasing ω sequence cofinal in α, then $S^* \subset Z_\alpha$ since g_α diverges to infinity on S.

Now suppose that g is a real-valued function on $\beta(\omega_2)$ such that $Z = g^{-1}(0)$ is a zero-set such that $Z \cap \alpha^* = Z_\alpha$ for all $\alpha < \omega_2$. First of all assume that $Z \cap \omega_2$ has cardinality ω_2. There is then a cub C in ω_2 such that $Z \cap \omega_2$ is cofinal in α for each $\alpha \in C$. Fix an $\alpha \in C \cap E$ and observe that $Z \cap \alpha - \{\xi < \alpha : g_\alpha(\xi) > 0\}$ is infinite and so $(Z \cap \alpha)^* - Z_\alpha$ is not empty. Since $(Z \cap \alpha)^*$ is a subset of Z, we have that $Z \cap \omega_2$ is bounded. For each n, let $A_n = \{\xi : g(\xi) \notin (\frac{-1}{n}, \frac{1}{n})\}$, hence $\cup_n A_n$ is most of ω_2. Again, there is an $\alpha \in \omega_2$ and an n so that $A_n \cap \alpha$ is cofinal in α. Let S be any ω sequence contained in A_n which is cofinal in α. Clearly $g[S^*] \cap (\frac{-1}{n}, \frac{1}{n})$ is empty and so $S^* \cap Z$ is empty, while, as discussed above, $S^* \subset Z_\alpha$.

We will just briefly mention how we can use GCH to obtain a clopen sequence of Z_α as above. This is equivalent to finding $Z_\alpha \subset \alpha$ so that $Z_\beta =^* Z_\alpha \cap \beta$ for $\beta < \alpha$ (i.e. a coherent sequence). In order to keep the same notation, we will use $\omega_2 \times \omega$ as our underlying set and again define functions g_α as above and then we will define $Z_\alpha = \{(\xi, m) : m < g_\alpha(\xi)\}$. Now we will need that $g_\alpha \restriction \beta$ is almost equal to g_β for $\beta < \alpha$. The procedure is exactly as above except for $\alpha \in E$. We will again choose

β_n increasing cofinal in α and we will let $g_\alpha \restriction (\beta_{n-1}, \beta_n) = g_{\beta_n} \restriction (\beta_{n-1}, \beta_n)$. Except that we now change g_α on just the points β_n – we define $g_\alpha(\beta_n) = n$. In addition the sequence $\{\beta_n : n \in \omega\}$ is not just any old sequence this time but is chosen using $\diamondsuit$ on ω_2. Suprisingly GCH implies that, for any stationary set $S \subset S^2_0$, there is a sequence $\{b_\alpha : \alpha \in S\}$ such that for each $A \subset \omega_2$, there is a stationary set S_A such that $A \cap \alpha = b_\alpha$ for each $\alpha \in S_A$. So, if b_α is cofinal in α, then $\{\beta_n : n \in \omega\}$ is chosen as a subset of b_α, arbitrary otherwise. The proof that this works proceeds just as above except that we use $S = \{\beta_n : n \in \omega\}$ in the above argument. $\square$

Remarks: The failure of $\square$ implies that ω_2 is a Mahlo cardinal in L (see section 6). Therefore any statement that follows from $\square$, the consistency of its negation implies the existence of "large" cardinals. Todorcevic has proven the first statement in 2.10 follows from the hypothesis that ω_2 is not weakly compact in L.

3. Two more proofs from $\diamondsuit$

Naturally everyone should know Ostaszewski's construction so we will include the following construction.

Theorem 3.1 (Fedorcuk). *$\diamondsuit$ implies that there is a compact hereditarily separable space containing no converging sequences and no isolated points.*

Proof. Define X_α a closed subset of 2^α by induction on $\alpha \le \omega_1$ so that $\alpha < \beta$ implies X_β projects onto X_α. The plan: if $Y \subset X_{\omega_1}$ then Y has a dense subset $\{y_\beta : \beta \in \omega_1\}$. We can code $\{y_\beta : \beta < \omega_1\}$ as a subset of $\omega_1 \times \omega_1$; call it A. For a partial function φ from ω_1 into 2, let $[\varphi]$ denote the set of elements of 2^{ω_1} which contain φ. Of course if φ is finite, then $[\varphi]$ is a basic clopen set. Define a function $g \colon [\omega_1]^{<\omega} \to \omega_1$ so that for each $F \in [\omega_1]^{<\omega}$ and $\varphi \in 2^F$, if $Y \cap [\varphi] \ne \emptyset$, then there is a $\beta < g(F)$ such that $y_\beta \in [\varphi]$. Fix a cub C as in 1.2. Then there is a $\delta \in C$ so that $A \cap \delta \times \delta = a_\delta$, where a_δ is from a suitable $\diamondsuit$-sequence. Given that a_δ will be a subset of $\delta \times \delta$, let $a(\delta, \beta)$ denote the subset of δ so that $\{\beta\} \times a(\delta, \beta) = a_\delta \cap (\{\beta\} \times \delta)$. The definition of C guarantees that $\{y_\beta \restriction \delta : \beta < \delta\}$ is dense in the projection of Y into 2^δ, denoted $P^{\omega_1}_\delta[Y]$. The inductive construction will need to guarantee that $Y \subset \overline{\{y_\beta : \beta \in \delta\}}$. However, the only thing we know about $\{y_\beta : \beta \in \delta\}$ is that its projection is $\{a(\delta, \beta) : \beta < \delta\}$ (the $\diamondsuit$-sequence element), so the requirement (at first glance, too ambitious) is that Y should be contained in the closure of ANY sequence $\{z_\beta : \beta < \delta\}$ such that $z_\beta \restriction \delta = a(\delta, \beta)$ for each $\beta < \delta$. In addition, since we really don't know much about Y except its projection, this is going to have to be true about any Y with the same projection.

This translates to the following inductive assumption:
for any $y \in X_\gamma$ such that $y \restriction \delta$ is a limit point of $\{a(\delta, \beta) : \beta < \delta\}$, each neighbourhood U of y must contain, for some $\beta < \delta$, all $z \in X_\gamma$ such that $a(\beta, \delta) = z \restriction \delta$. That is, in this case, U must contain $(P^\gamma_\delta)^{-1}(a(\delta, \beta)) \cap X_\gamma$.

The other, much easier thing that we must accomplish is to guarantee that there are no converging sequences and no isolated points. For this it will suffice to ensure that $\{a(\delta, n) : n \in \omega\}$ is not the image of a converging sequence in $X_{\delta+1}$ and that $a(\delta, 0)$ is the image of at least two points in $X_{\delta+2}$. We will not use δ for this so that we can stay with Fedorcuk's original idea of using a **minimal extension** when passing from X_δ to $X_{\delta+1}$ – that is, there is a unique point of X_δ which has more than one point in $X_{\delta+1}$ mapping onto it.

Having minimal extension in mind, we know that we must choose one point $x \in X_\delta$ and partition $X_\delta - \{x\}$ into two disjoint clopen sets U_0 and U_1. Then $X_{\delta+1}$ will be defined as $(U_0 \times \{0\}) \cup (U_1 \times \{1\})$ (a moment's thought will convince the reader that this is exactly what must take place). For limit δ, we simply define $X_\delta = \bigcap\{\left(P^\delta_\gamma\right)^{-1}(X_\gamma) : \gamma < \delta\}$. The above inductive hypothesis is easily checked. The inductive hypothesis for $X_{\delta+1}$ is only a problem for the two points $(x, 0)$ and $(x, 1)$ where x is the point we chose to "double". What is required is, for each $\alpha < \delta$ such that $x \restriction \alpha$ is a limit point of $\{a(\alpha, \beta) : \beta < \alpha\}$ and each neighbourhood W of x (in X_δ) each of $U_0 \cap W$ and $U_1 \cap W$ contain a full preimage of some $a(\alpha, \beta)$. Let I be the set of $\alpha < \delta$ such that $x \restriction \alpha$ is a limit of $\{a(\alpha, \beta) : \beta \in \alpha\}$. By the inductive hypothesis, we can, for each $\alpha \in I$, fix a sequence $\{\beta(\alpha, n) : n \in \omega\} \subset \alpha$ so that $\{X_\delta \cap \left(P^\delta_\alpha\right)^{-1}(a(\alpha, \beta(\alpha, n))) : n \in \omega\}$ converges to x. Let f be a function from ω onto I so that $f^{-1}(\alpha)$ is infinite for each $\alpha \in I$. Note that X_δ is homeomorphic to a compact subset of the Cantor set hence it is a simple matter to cover $X_\delta - \{x\}$ with a pairwise disjoint sequence, W_n, of compact open subsets of $X_\delta - \{x\}$ so that both W_{2n} and W_{2n+1} contain some member of $\{X_\delta \cap \left(P^\delta_\alpha\right)^{-1}(a(\alpha, \beta(\alpha, n))) : n \in \omega\}$ where $\alpha = f(n)$. Letting $U_0 = \bigcup_n W_{2n}$ and $U_1 = \bigcup_n W_{2n+1}$ are as required.

The final detail is how to choose x. If δ is a limit, then choose x a limit point of $\{a(\delta, n) : n \in \omega\}$ and ensure that each of U_0 and U_1 contain infinitely many elements of $\{a(\delta, n) : n \in \omega\}$. This ensures that any sequence in $X_{\delta+1}$ which maps onto $\{a(\delta, n) : n \in \omega\}$ will have each of $(x, 0)$ and $(x, 1)$ as a limit point. If $\delta = \lambda + 1$ for some limit λ, then choose x so that it maps onto $a(\lambda, 0)$. For any other δ, the choice of x is arbitrary.

We check that this works. Let $x \in X_{\omega_1}$ and let us see that x is not isolated in X. If it were, then, as X_{ω_1} is simply a subspace of 2^{ω_1}, there would be a $\lambda < \omega_1$ so that x is the unique member of X extending $x \restriction \lambda$. However it is easily seen that there are stationarily many $\delta > \lambda$ so that $a(\delta, 0) = x \restriction \delta$. By construction, there are two points in $X_{\delta+2}$ which map onto $x \restriction \delta$. Since $P^{\omega_1}_\delta$ is onto, there are two points in X which extend $x \restriction \delta$ - a contradiction.

Suppose that $\{x_n : n \in \omega\}$ is a converging sequence in X. Again, there are stationarily many δ so that $\{a(\delta, n) : n \in \omega\} = \{x_n \restriction \delta : n \in \omega\}$. Apparently $\{x_n \restriction \delta + 1 : n \in \omega\}$ is a converging sequence in $X_{\delta+1}$ which maps onto $\{a(\delta, n) : n \in \omega\}$, but, by the construction there is no such sequence.

The fact that X is hereditarily separable follows from the fact that we carried out the "plan" outlined above. $\square$

Theorem 3.2 (Fleissner, Shelah, Taylor). *$\diamondsuit^*$ implies that first countable normal spaces are $\aleph_1$-collectionwise Hausdorff.*

Proof. Let ω_1 be embedded as a closed discrete subspace of a normal space X of character ω_1. Assume that there is no separation of ω_1 in X. For each $\alpha \in \omega_1$, let $\{U(\alpha, \beta) : \beta \in \omega_1\}$ enumerate a neighbourhood base at α. A potential separation of ω_1 would correspond to a function $f \in {}^{\omega_1}\omega_1$. For brevity, given a partial function f from ω_1 to ω_1, let $U_f = \bigcup\{U(\beta, f(\beta)) : \beta \in \text{dom } f\}$; $\overline{U}_f$ will denote the closure.

We will use the $\diamondsuit$-principle to define a subset E of ω_1 so that E and $\omega_1 - E$ cannot be separated by predicting all potential separations and adjusting the definition of E accordingly.

The basic fact needed for this to work is the following: for each $f \in {}^{\omega_1}\omega_1$, the set

$$S_f = \{\alpha : \overline{U}_{f\restriction\alpha} \cap \omega_1 - \alpha \neq \emptyset\}$$

is stationary.

Here is the plan which almost works. We define a chain $E_\alpha \subset \alpha$ by induction and let $E = \bigcup_\alpha E_\alpha$. For each α, suppose we are given, by $\diamondsuit$, a function $f_\alpha \in {}^\alpha\alpha$. If f_α separates E_α and $\alpha - E_\alpha$ we look to see if there is a $\delta_\alpha \geq \alpha$ which is a limit point of $U_{f\restriction\alpha}$. If there is then put δ_α in E iff it is a limit of $U_{f\restriction\alpha - E}$. Note that no f which extends this f_α will be a separation of E and $\omega_1 - E$. This defines E_β for all $\beta \leq \delta_\alpha + 1$.

Why does this almost work. Suppose that f separates E and $\omega_1 - E$. Since S_f is stationary and, by the $\diamondsuit$ property, $T_f = \{\alpha : f \restriction \alpha = f_\alpha\}$ is stationary so there is a chance that $S_f \cap T_f$ is not empty. If α were in S_f and T_f we would have a contradiction since δ_α was carefully put in the wrong side of E so that f_α would not extend to a separation of E. This is Fleissner's approach, he defines a stronger version of $\diamondsuit$ which does guarantee that $S_f \cap T_f$ is not empty.

We will take a different approach however. We follow Taylor's approach which is modelled after Kunen's proof that $\diamondsuit^*$ implies $\diamondsuit$; i.e. there is a way of choosing just one f_α when you have countably many. Before doing so however, let us note that if, for each $f \in {}^{\omega_1}\omega_1$,

$$\bar{S}_f = \{\alpha : (\omega_1 - \alpha) \cap \overline{U}_{f\restriction\alpha} \text{ is infinite}\}$$

then we could get away with $\diamondsuit^*$ directly. Therefore we may assume that there is a cub C and a sequence H_α of pairwise disjoint finite sets such that for every $f : \omega_1 \to \omega_1$ and $\alpha \in C$, $\overline{U}_{f\restriction\alpha} \cap (\omega_1 - \alpha)$ is contained in H_α.

Let $F_\alpha = \{f_{\alpha,n} : n \in \omega\}$ (for $\alpha \in \mathrm{Lim}(\omega_1)$) be a $\diamondsuit^*$ sequence. Fix a definable coding of $\omega_1 \times \omega$ in type ω_1; e.g. $\Phi(\beta, n) = \beta \cdot \omega + n$. Roughly speaking, we will assume that the sequence $\{f_{\alpha,n} : \alpha \in \mathrm{Lim}(\omega_1)\}$ is not a suitable $\diamondsuit$-sequence for each n in the sense that there there is a function f_n coding a separation of X_n, $\omega_1 - X_n$ where X_n is constructed as E was above. We use Φ to code the sequence of f_n's into one function f, and obtain a contradiction from the fact that this f should have been predicted lots of times.

For technical reasons we do not use $f_{\alpha,n}$ but rather what we will call $h_{\alpha,n}$. Looking ahead to the final step of the proof we want to think of $f_{\alpha,n}$ as coding countably many functions, so we let $h_{\alpha,n}$ be the n-th one of these countably many functions. More precisely,

$$h_{\alpha,n}(\beta) = f_{\alpha,n}(\Phi(\beta, n)) \ .$$

For each n, let X_n be the subset of ω_1 which we get by following the plan laid out above in which we use $h_{\alpha,n}$. That is, if $\omega_1 \cap \overline{U}_{h_{\alpha,n}}$ goes above α, then one of these elements is put on the wrong side of the partition $X_n, \omega_1 - X_n$ so that no extension of $h_{\alpha,n}$ is a separation of X_n and $\omega_1 - X_n$.

By the normality of X, fix, for $n \in \omega$, a separating function f_n for X_n and $\omega_1 - X_n$. Using Φ we code these countably many functions into one: $f(\alpha) = f_n(\beta)$

where $\Phi(\beta, n) = \alpha$. By $\diamondsuit^*$, there is a cub C so that $C \subset \{\alpha \in \mathrm{Lim}(\omega_1) : (\exists n)\ f \restriction \alpha = f_{\alpha,n}\}$. If we assume that $\alpha \in C$ implies that $\Phi(\beta, n) < \alpha$ for all $\beta < \alpha$, we then have that $C \subset \{\alpha \in \mathrm{Lim}(\omega_1) : (\exists n)\ f_n \restriction \alpha = h_{\alpha,n}\}$. Let $Y_n = \{\alpha \in C : f_n \restriction \alpha = h_{\alpha,n}\}$; hence $C = \bigcup_n Y_n$.

Now we have to do a little topology (because $\diamondsuit^*$ does not imply Fleissner's $\diamondsuit$ principle).

Define an ideal $I \subset \mathcal{P}(C)$ by $a \in I$ iff $(\exists g \in {}^{\omega_1}\omega_1)$ such that $S_g \cap a$ is empty (i.e. U_g does not go "up" at any member of a). Recall that $\alpha \in S_g \cap C$ iff $H_\alpha \cap \overline{U}_{g\restriction\alpha}$ is not empty. Easy topology tell us that if there is a cub in I, then there is a separation of ω_1. We need that I is countably complete. Fix $\{a_n : n \in \omega\} \subset I$ and functions g_n witnessing that $a_n \in I$. Without loss of generality the a_n's are pairwise disjoint. Fix pairwise disjoint open subsets U_n of X, so that $\bigcup_{\alpha \in a_n} H_\alpha \subset U_n$ for each n (using the normality of X). Define a function $g : \omega_1 \to \omega_1$ so that, if $\beta \in U_n$ then $g(\beta) \geq g_n(\beta)$. Assume that $\alpha \in S_g \cap a_n$; hence $\overline{U}_{g\restriction\alpha} \cap H_\alpha$ is not empty. But then $\overline{U_{g\restriction\alpha} \cap U_n}$ must meet H_α, which contradicts that g refines g_n on U_n.

We are finally ready for our contradiction. Since I does not contain a cub (i.e. C) and is countably complete, there is an n so that $Y_n \notin I$. Therefore for every $g : \omega_1 \to \omega_1$, there is an $\alpha \in Y_n$ so that $\overline{U}_{g\restriction\alpha} \cap H_\alpha$ is not empty. In particular, this is the case for $g = f_n$. But this contradicts that f_n is a separation of X_n and $\omega_1 - X_n$. $\square$

4. Some Logic

We do not intend to go far without resorting to elementary submodels. One's first reaction to elementary submodels is concern that something so mysterious cannot be understood. This is consistent with a Platonic view of the universe and the clear knowledge that we do not understand the universe. In reality, however, there is absolutely nothing mysterious about an elementary submodel. First one must be clear on "What is the language of set theory?". It is nothing but the binary relation $\in$! When specifying a model of set-theory then, one must specify a set M and then an interpretation $\in^M$ of $\in$. However we shall assume that V, the universe of sets, is a model of ZFC in which $\in$ is the usual set membership and every model M will be a subset or subclass of V and will inherit the usual set membership relation. If φ is a formula of set theory then the truth of φ refers to its truth in V. If M is contained in V, then $M \models \varphi$ denotes the satisfaction relation for M (defined recursively in any standard logic book). More exactly, we write $\varphi(v_1, \ldots, v_n)$ to mean that the free variables of φ are among $\{v_1, \ldots, v_n\}$ and we may write, let $x, y, x_1, \ldots, x_n$ be sets (or elements) and speak of $\varphi(x, y, x_1, \ldots, x_n)$ and we mean, as is standard, that φ has the form $\varphi(v_1, \ldots, v_{n+2})$ and that v_1 now denotes the set x, v_2 the set y, and v_{i+2} the set x_i.

Now consider a quantifier-free formula, $\varphi(v_1, \ldots, v_n)$ and a sequence of elements $m_1, \ldots, m_n$ of a model M. The validity of $\varphi(m_1, \ldots, m_n)$ (in V) is based simply on a Boolean combination of assertions that certain of the m_i are members (or not) of other of the m_j's. In addition, by our definition of submodel, the assertion $M \models \varphi(m_1, \ldots, m_n)$ is exactly the same, i.e. $\in^M = \in \cap M \times M$. We might say that φ is preserved when passing from M to V (and reflected when passing from V to M). Kunen ([6]) makes the definition that a formula $\varphi(v_1, \ldots, v_n)$ is absolute for

$M \subset N$ (a pair of models) if for each sequence, $m_1, \ldots, m_n$, of elements of M,

$$M \models \varphi(m_1, \ldots, m_n) \text{ iff } N \models \varphi(m_1, \ldots, m_n) \ .$$

Formulas certainly do not need to be preserved when passing from one model to the next. Consider the two very simple formulas $v_1 = v_2$ and $v_1 \subset v_2$. The first thing to ask is if they are formulas of set-theory. The informal language of set-theory is greatly expanded but one should simply adopt an acceptable translation (although there are often many defined symbols such as the empty set, ω, the Cartesian product, and ordered pairs which require certain formulas (e.g. axioms) to be valid for the formula to make sense). Usually one considers '=' to be a logical symbol hence $v_1 = v_2$ is a formula of set theory and is preserved; however $((\forall x)(x \in m_1 \leftrightarrow x \in m_2)) \leftrightarrow m_1 = m_2$ is a substitution instance of the axiom of extensionality and if m_1 and m_2 are distinct members of a model M which does not contain any elements of $m_1 - m_2 \cup m_2 - m_1$, then this axiom would not be hold in M, hence the truth value is not preserved when passing from V to M. On the other hand $v_1 \subset v_2$ is an abbreviation for $(\forall x)(x \in v_1 \rightarrow x \in v_2)$ and for M and V to agree on this formula it is sufficient (and necessary) that they agree on the formula $(\exists x)(x \in v_1 \wedge x \notin v_2)$. However, $(x \in v_1 \wedge x \notin v_2)$, being quantifier-free, is preserved, hence it is sufficient to have that

$$(\exists x)(x \in v_1 - v_2) \Rightarrow (\exists x \in M)(x \in v_1 - v_2)$$

is valid. This latter example is a special case of the Tarski-Vaught criterion (the hypothesis of Proposition 4.1) – the emphasis being on the fact that the whole assertion takes place in V and the idea that all we needed to do was to pick one witness from V and add it to M.

Proposition 4.1. *Suppose that $\varphi_1, \ldots, \varphi_m$ is a* ***sub-formula closed*** *list of formulas such that the symbol $\forall$ does not appear. Suppose that M is a set or class such that whenever φ_i is of the form $(\exists x)\varphi_j(x, v_1, \ldots, v_n)$ and $m_1, \ldots, m_n$ are members of M, the following implication holds:*

$$(\exists x)\varphi_j(x, m_1, \ldots, m_n) \Rightarrow (\exists x \in M)\varphi_j(x, m_1, \ldots, m_n) \ .$$

Then, for each φ_i and $m_1, \ldots, m_n \in M$,

$$M \models \varphi_i(m_1, \ldots, m_n) \text{ iff } V \models \varphi(m_1, \ldots, m_n) \ .$$

While it is very difficult to know which (substitution instances of) formulas an arbitrary model M may or may not satisfy, an elementary submodel is simple in the sense that its truths are the same as those for V. If the role of V in 4.1 is replaced by a model N containing M and if the list of formulas is expanded to all formulas, then M is said to be an **elementary submodel** of N, denoted $M \prec N$. Gödel's second incompleteness theorem implies that the existence of sets M which are elementary submodels of V is not provable. However interesting, this is not a practical limitation to the use of elementary submodels because of the following two facts. Levy reflection implies that for every formula (hence every finite set of formulas) there is an abitrarily large set H for which the formula is reflected. The Axiom of Choice implies that, for every set H, there are many elementary submodels M of every conceivable size (including countable!). Still further simplifications are possible if one is only worried about practical applications. Given a set X under consideration there will be *sufficiently large* θ so that the properties of X in the model $H(\theta)$, defined below, are of sufficient interest.

Definition 4.2. A set H is transitive if every one of its elements is a subset. The transitive closure of a set H exists and is defined as the smallest transitive set which contains H. For an infinite cardinal θ, $H(\theta)$ is defined as the set of all sets whose transitive closure has cardinality less than θ.

Proposition 4.3. *If θ is a regular cardinal, then $H(\theta)$ is a model of each of the axioms of ZFC except for the power set axiom - denoted ZF-P.*

We will return to the existence of elementary submodels in Section 5 but we first continue our discussion of preservation. The question of preserving formulas or properties is certainly of interest in many more instances than in defining the notion of an elementary submodel. The greatest source of confusion in using submodels is that $M \models \varphi(X)$ is not the same as $V \models \varphi(X \cap M)$. For example one can easily have an ordinal $\alpha \in M$ such that $M \models \alpha$ is the first uncountable cardinal, we would say that $\alpha = \omega_1^M$, while in fact α is a countable ordinal of V. However this does not happen in an elementary submodel (even a countable one), since the statement that asserts a certain ordinal α is the ω_1 of the model would have to be true of the same α in both models. However, if M is countable such that $\omega_1^M = \omega_1$, then $\omega_1 \cap M$ would be a countable set in V and not even an element of M.

Recall that a tree is a poset in which the set of predecessors of an element is well-ordered. Of particular interest about trees is when they have branches going through every level. Suppose that M is a model and $T \in M$ is such that $M \models T$ is a tree. Is T really a tree? In the future we will always assume that our models M satisfy enough of the axioms so that basic notions need not concern us. E.g. the separation axiom scheme (or the Comprehension Scheme in [6]) is needed to talk about the set of predecessors of $t \in T$, many axioms are needed to talk about whether this set is well-ordered, and to define the levels of T etc. However we also want certain "elementary" or "first-order" notions to have the right meaning in M such as the property of being an ordered pair, being a function, being an ordinal. If M is transitive this is true of many properties. We call $(\exists x \in y)$ and $(\forall x \in y)$ bounded quantifiers and a formula in which all quantifiers are bounded is called a Δ_0 formula. If φ is a Δ_0 formula and M is transitive, then $M \models \varphi(m_1, \ldots, m_n)$ iff $V \models \varphi(m_1, \ldots, m_n)$ for all $m_1, \ldots, m_n \in M$. Also note that if M is a transitive model of ZF-P, for example, and ZF-P$\vdash \varphi \leftrightarrow \psi$ for some Δ_0 formula ψ, then the above preservation property also hold for φ. This handles a great deal of properties.

We do not expect the property of being a cardinal to be preserved, more generally if the power set operation is involved in a formula then there is no reason to expect that it will be preserved. However some sets are considered definable (by parameters and the other axioms) – e.g. $\{s \in T : s <_T t\}$ is definable from t and $\langle T, <_T\rangle$. If M is a model of ZF-P, then there will be some set $x \in M$ such that $M \models x = \{s \in T : s <_T t\}$ (more precisely $M \models (\forall z)(z \in x \leftrightarrow (z \in T \wedge z <_T t))$). The existence of such sets follows from the separation axiom and does not require the power set axiom. The problem of whether T is really a tree simply boils down to the properties, in V, of these sets x.

We are not, however, interested in such technical questions. This type of question goes away if we assume that, either M is a transitive model of ZFC-P or M is an elementary submodel of some transitive model of ZFC-P (such as an $H(\theta)$).

The question of new branches of T is a question that does not go away and is of great interest. In the simplest case, assume that T is an Aronszajn tree and that $T \in M \prec H(\omega_2)$ with $|M| = \omega$. Since $V \models T$ has no ω_1-branches, $M \models T$ has no ω_1-branches. However let us consider the situation starting with the following useful lemma (note $\omega + 1$ is contained in every model of ZF-P).

Lemma 4.4. *If $M \prec H(\theta)$ (with θ regular) is such that $\kappa+1 \subset M$, then $M \cap \kappa^+ \in \kappa^+ + 1$. More generally, if $X \in M$ and $|X| \leq \kappa$, then $X \subset M$.*

Proof. In the case $\kappa^+ \subset M$, this is trivial, so assume that $\kappa^+ - M$ is not empty. Let δ be the minimal element of $\kappa^+ - M$ and we will show that $M \cap \kappa^+ = \delta$, i.e. $M \cap \kappa^+ \subset \delta$. Assume that $\beta \in M \cap \kappa^+$; we show that $\beta \subset M$. Since $\beta < \kappa^+$ there is a function f from κ onto β. By elementarity, $M \models (\exists f) f : \kappa \twoheadrightarrow \beta$. Therefore, by definition of "$M \models$", there is an $f \in M$ such that $M \models f : \kappa \twoheadrightarrow \beta$. Run through all the properties that f is to have in M to see that f must "really" have them: f is a set of ordered pairs, f is a function, the domain of f is κ, the range of f is β. Each of these, in turn, must be true of f or else M would possess a witness to the failure. Now it is not enough to know that $M \models$ "range $f = \beta$" in order to deduce that $\beta \subset M$. However, we have assumed that dom f is a subset of M. Fix any $\xi < \beta$ and pick $\gamma < \kappa$ so that $f(\gamma) = \xi$. We cannot yet refer to ξ as a member of M, but we can recall that $M \models (\exists \alpha) f(\gamma) = \alpha$. The α that M contains which witnesses that $f(\gamma)$ has a value must really have the (elementary) property that $f(\gamma) = \alpha$, hence α must equal ξ. $\square$

Therefore, by the previous proposition, $\delta = M \cap \omega_1$ is an element of ω_1. For each $\alpha < \omega_1$, let T_α denote the α-th level of T, i.e. all those $t \in T$ such that the order type of $\{s \in T : s < t\}$ is α. It follows that every $b \in M$ such that $M \models b \subset T$ is a branch, there is a $\beta < \delta$ such that $b \cap T_\beta = \emptyset$. Since T is Aronszajn, T_α is countable for each α, hence $T_\alpha \subset M$ for each $\alpha < \delta$. Clearly, then, the tree $T \cap M = \bigcup_{\alpha<\delta} T_\alpha$ has branches meeting each level, but $M \models T$ has no such branches. Of course this simply means that M itself does not contain any of the branches (in particular $\{s \in T : s < t\} \notin M$ for each $t \in T_\delta$).

Now consider the *transitive collapse* of M. This is defined recursively, by Mostowski, where $G(x) = \{G(y) : y \in x \cap M\}$ for each $y \in x$. The axiom of foundation guarantees that this is well-defined. In fact every set x has a <u>rank</u>, which is the least ordinal α such that each member of x has rank less than α. The axiom of foundation is essentially equivalent to the assertion that the rank function is well-defined. The function G is an $\in$-isomorphism from M to $N = G(M) = \{G(x) : x \in M\}$. Therefore N is a transitive model ZFC-P (since M was assumed to be). If we assume, without loss of generality, that T is a subtree of ${}^{<\omega_1}\omega$, then it turns out that G is the identity on $T \cap M = \bigcup_{\alpha<\delta} T_\alpha$, hence $T' = G(T) = \bigcup_{\alpha<\delta} T_\alpha$ is a member of N. Furthermore, $\omega_1^N = M \cap \omega_1$ (but $N \cap \omega_1 \neq M \cap \omega_1$, e.g. $\delta = G(\omega_1) \in N$). Now T' is a tree in the transitive model N and N is a model of "T' has no <u>uncountable</u> branches". Clearly, in V, T' has branches hitting every level of T' but, being countable, certainly no uncountable branches. Can there be a model $N' \supset N$ (of ZFC-P) such that $N' \models$ "T' has uncountable branches"? If T contains a Souslin tree, M would model this and so N would model that T' contains a Souslin tree. One can check with the method of forcing, discussed later, that, then, there is such a model N'. On the other hand, we often use a kind of relative preservation in the sense that the property is preserved in certain extensions, e.g. those that also preserve certain cardinals. For example, if T is a special Aronszajn tree then there is no such extension (T is *special* if there is an order-preserving function into the rationals). Indeed, fix such a specializing function f, which, by elementarity, may be assumed to be in M; hence $G(f) \in N$. Assume that $N' \supset N$ and $b \in N'$ is a branch of T'. It follows that $G(f) \restriction b : b \mapsto \mathbb{Q}$ is order-preserving, hence one-to-one. Therefore b is not uncountable (however it may still meet every level of T', in which case $N' \models \omega_1^N$ is countable).

5. Fun with Elementary submodels

As mentioned before, each set has an abundant supply of elementary submodels of every smaller cardinality.

Lemma 5.1. *For any set M, there are countably many Skolem functions H^n_m so that $H^n_m : M^n \to M$ and for each $N \subset M$ such that $H^n_m[N^n] \subset N$ for all $n, m \in \omega$, N is an elementary submodel of M.*

Corollary 5.2. *For each set M and each $X \in M$ and each infinite cardinal $\kappa < |M|$, there is an elementary submodel $N \prec M$ such that X is a member of N.*

The following basic result is really only a corollary if we restrict to elementary submodels of the type given by Lemma 5.1, but it holds for any chain.

Corollary 5.3. *If $\mathcal{N}$ is chain of elementary submodels of M, then $\bigcup \mathcal{N}$ is also an elementary submodel of M.*

Lemma 5.4. *If $g : [\kappa]^{<\omega} \to \kappa$, for some $n \in \omega$, and $g \in M \prec H(\kappa^+)$, then $g(\beta_1, \ldots, \beta_n) \in M$ for all $\beta_1, \ldots, \beta_n \in M \cap \kappa$.*

Proof. By elementarity, $M \models (\forall F \in [\kappa]^{<\omega})\ (\exists \xi \in \kappa)\ (F, \xi) \in g$ (i.e. $g(F) = \xi$). The definition of $M \models \varphi$ is φ^M, hence, for all $F \in [\kappa]^{<\kappa} \cap M$, there is a $\xi \in \kappa \cap M$, such that $g(F) = \xi$. $\square$

Lemma 5.5. *Elementary submodels, stationary sets and Cubs: Let S, C, X be subsets of a regular cardinal κ and let $\theta > \kappa$ be a regular cardinal.*

1. *If S is stationary then for any $A \in H(\theta)$ there is an $M \prec H(\theta)$, including A, so that $M \cap \kappa \in S$.*
2. *If $C \in M \prec H(\theta)$ and C is cub, then $M \cap \kappa \in C$.*
3. *If $X \in M \prec H(\theta)$ and $\sup(M \cap \kappa) \in X$, then X is stationary.*
4. *Let $X \subset \kappa$ such that both X and κ are members of $M \prec H(\theta)$. If $\forall \alpha \in M \cap \kappa$, $\exists \beta \in X - \alpha$, then X is cofinal in κ. Similarly, if $\exists \beta \in X$ such that $M \cap \kappa \subset \beta$, then $X \cap M$ is cofinal in $M \cap \kappa$ and X is cofinal in κ.*

A function f on a set S of ordinals is *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in S - \{0\}$. A function f on a tree T will be called *regressive* if $f(t) < t$ for all $t \in T$ which are not minimal.

Lemma 5.6 (Pressing down lemma). *If S is a stationary subset of a regular cardinal κ and if f is a regressive function on S, then there is an $\alpha \in \kappa$ so that $f^{-1}(\alpha)$ is stationary. Hence if S is written as a union of fewer than κ sets, one of them is stationary.*

Proof. This can be deduced from the previous lemma as follows. Fix f as in the statement and by 5.5.1, fix an elementary submodel $M \prec H(\theta)$ so that $\delta = M \cap \kappa \in S$. Since $\xi = f(\delta)$ is less than δ, it is a member of M. Now set $X = f^{-1}(\xi)$ and apply 5.5.3 to deduce that X is stationary in κ. $\square$

Theorem 5.7 (Todorcevic). *Suppose that f is a regressive function on a tree T so that for each $t \in T$, $f^{-1}(t)$ is special. Then T is special.*

Proof. Fix θ large enough so that $f, T, \mathcal{P}(T) \in H(\theta)$ and let $M \prec H(\theta)$ be countable and $f, T, \mathcal{P}(T) \in M$. Let $\mathcal{A} \subset \mathcal{P}(T)$ denote the family of antichains. Clearly if

$$H(\theta) \models (\exists \mathcal{A}' \in [\mathcal{A}]^{\omega})\ T = \bigcup \mathcal{A}' ,$$

then, by elementarity, there would be such a countable family $\mathcal{A}'$ in M. That is, it is necessary and sufficient to show that

$$T = \bigcup (\mathcal{A} \cap M) .$$

This gets us to thinking about which antichains of T are in M and how f will help put them there. Obviously $f^{-1}(t)$, for $t \in M \cap T$, is the union of countably many (assigned) antichains which will be in M. Maybe $f(s)$ will be a member of M for each $s \in T$. Obviously not, but a moment's reflection convinces us that, for each $s \in T$, there is an n so that $t = f^n(s)$ is in M (there's no infinite descending sequence of ordinals – hence levels of T). Now we ask if $(f^n)^{-1}(t)$ will be special. The answer is easy: since f is regressive, if $A \subset T$ is special, then so is $f^{-1}(A)$ (because if $A_a \subset f^{-1}(a)$ is an antichain for each $a \in A$, then $\bigcup\{A_a : a \in A\}$ is again an antichain). Since $M \cap T$ is countable, this completes the proof. However, we can in hindsight, recognize that $T = \bigcup_n (f^n)^{-1}(0)$ (where 0 is the root of T) and prove the theorem without using M. $\square$

Theorem 5.8 (Solovay). *If S is a stationary subset of a regular cardinal κ, then S contains κ disjoint stationary subsets.*

Proof. Without loss of generality we may assume that S contains no successor ordinals. We first show that

$$W = \{\alpha \in S : \text{either } cf(\alpha) = \omega \text{ or } S \cap \alpha \in \mathrm{NS}_\alpha\}$$

is stationary. Assume that C is cub and is disjoint from W. Let $\delta = \min(S \cap C')$ where C' is the set of limits of C. Clearly $C \cap \delta$ is closed and unbounded in δ. Since δ has uncountable cofinality, $C' \cap \delta$ is a cub in δ. Since $C' \cap (S \cap \delta)$ is empty, it follows that $S \cap \delta$ is non-stationary in δ.

For each $\alpha \in W$ choose an continuous increasing sequence $\{a^\alpha_\xi : \xi \in cf(\alpha)\}$ cofinal in α and, since $\alpha \in W$, disjoint from W (note that W contains no successor ordinals, so there's no problem if $cf(\alpha) = \omega$). Fix $M \prec H(\theta)$ (for some suitable θ) so that $\delta = M \cap \kappa \in W$ (by item 1 of Lemma 5.5).

Define, for $\xi, \beta \in \kappa$,

$$S(\xi,\beta) = \{\alpha : a^\alpha_\xi \geq \beta\} \ .$$

The plan is to show that there is a ξ so that $S(\xi,\beta)$ is stationary in κ for every $\beta < \kappa$. If ξ is such an ordinal, then it follows easily from the pressing down lemma that there are κ many ζ such that $T(\xi,\zeta) = \{\alpha : a^\alpha_\xi = \zeta\}$ is stationary; and this is the desired family of disjoint stationary sets.

Again, by elementarity, if there is such a ξ then there is one in M, so we can hope to find it there. Assume, to the contrary, that there is no such ξ. It follows then, that there is a definable function $\xi \rightarrow \beta_\xi$ where β_ξ is the minimum ordinal such that $S(\xi,\beta_\xi)$ is not stationary. Therefore, for each ξ in M, $\beta_\xi \in M$. For each $\xi \in cf(\delta)$, $a^\delta_\xi \geq a^\delta_\xi$ hence $\delta \in S(\xi, a^\delta_\xi)$ and so $S(\xi, a^\delta_\xi)$ is stationary by Lemma 5.5. Therefore $\beta_\xi > a^\delta_\xi$ for each $\xi \in cf(\delta)$.

To see that $\delta' = cf(\delta)$ must equal δ, note that otherwise $\{\beta_\xi : \xi \in \delta'\}$ (which we now know to be cofinal in δ) would be a member of M which would have to be bounded in M, since $H(\theta) \models (\exists \gamma \in \kappa)\ \{\beta_\xi : \xi < \delta'\} \subset \gamma$.

Our final contradiction will be to find an α with cofinality bigger than δ so that $a^\alpha_\delta = \delta$; the contradiction being that $\delta \in W$ and $a^\alpha_\delta \notin W$. Notice that since $\delta \in T = \{\alpha \in W : cf(\alpha) = \alpha\}$, and $T \in M$, it follows that T is stationary. Clearly then, $T - \delta$ is stationary. But now, by the definition of β_ξ, $S(\xi,\beta_\xi)$ is non-stationary for each $\xi \in M$, hence we may choose an $\alpha \in T - \bigcup\{S(\xi,\beta_\xi) : \xi \in \delta\}$ with $\alpha > \delta$. Therefore, for each $\xi \in \delta$, $a^\alpha_\xi < \delta$ and, clearly, $\{a^\alpha_\xi : \xi \in \delta\}$ is cofinal in δ. It follows that $a^\alpha_\delta = \sup\{a^\alpha_\xi : \xi \in \delta\}$ is equal to δ. $\square$

Proposition 5.9 (Baumgartner). *Suppose that T is a tree with uncountably many levels but no uncountable branches. Suppose that F_α is a finite subset of T_α for each $\alpha \in \omega_1$, then there are $\alpha < \beta$ so that $F_\alpha \cup F_\beta$ is an antichain.*

Proof. Fix a countable elementary submodel M of $H(\theta)$ with T and $\{F_\alpha : \alpha \in \omega_1\}$ in M and let $\delta = M \cap \omega_1$. We will find $\alpha \in M$ so that $F_\alpha \cup F_\delta$ is an antichain. Let $n = |F_\delta|$ and for $\alpha \in I_n = \{\alpha : |F_\alpha| = n\}$, let $\{F_\alpha(0), \ldots, F_\alpha(n-1)\} = F_\alpha$. Note that $I_m \in M$ for each m. Since δ is a limit ordinal, there is a $\beta < \delta$ so that $\vec{x}_0 = \{F(i) \restriction \beta : i < n\}$ has cardinality n. Let $\Gamma' \in M$ denote the set of $\vec{x} \in [T]^n$ such that $\{\alpha : (\exists \beta)\ \vec{x} = \{F_\alpha(i) \restriction \beta : i < n\}\}$ is uncountable and $\Gamma = \{\vec{x} \in \Gamma' : \{\alpha : (\exists \vec{x}' \in \Gamma' \cap [T_\alpha]^n)\ \vec{x} < \vec{x}' \text{ (ordered coordinatewise)}\} \text{ is uncountable}\}$. Using the elementarity of M we show that $\vec{x}_0$ is a member of Γ. Indeed, the set $\{\alpha : (\exists \vec{x} \in \Gamma' \cap [T_\alpha]^n)\ \vec{x}_0 < \vec{x}\}$ is a member of M. If it were countable it would have a bound in M. However the existence of F_δ, implies that for each $\beta < \delta$, $(F_\delta) \restriction \beta \in \Gamma'$.

Suppose that i is maximal so that there exist $\vec{x}_1 \in \Gamma \cap M$ so that $x_0(j) \leq x_1(j)$ for $j < n$ and $x_1(j)$ is incomparable with $F_\alpha(j)$ for $j < i$. We prove that $i = n$. If not, let $\Gamma_1 = \{\vec{x} \in \Gamma : \vec{x}_1 < \vec{x} \text{ coordinatewise}\}$. Observe that if $\vec{x} \in \Gamma_1$, then for every α, there is a $\vec{y} \in \Gamma \cap T_\alpha$ such that $\vec{x} < \vec{y}$; note that $\vec{y} \in \Gamma_1$. Therefore, $\{\vec{x}(i) : \vec{x} \in \Gamma_1\}$ cannot form a chain because it has members on cofinally many levels of T – in addition M will "know" this. Choose a $\beta \in M$ and $\vec{y}_1, \vec{y}_2 \in \Gamma_1 \cap T_\beta$ so that $y_1(i)$ is incomparable with $\vec{y}_2(i)$. Certainly either $y_1(i)$ or $y_2(i)$ is incomparable with $F_\alpha(i)$ so let it be $y_1(i)$. Since $x_1(j) < y_1(j), y_2(j)$, it follows that $y_1(j)$ is incomparable with $F_\alpha(j)$ for each $j < i$. Therefore, i is equal to n. Now, $x_1 \in \Gamma \cap M$, hence

$x_1 \in \Gamma' \cap M$; thus there are α, β in M such that $\vec{x}_1 = \{F_\alpha(i) \restriction \beta : i < n\}$. Clearly $F_\alpha \cup F_\delta$ is an antichain. $\square$

Elementary submodels and the preservation of formulas are is essential in the study of L and can be used to prove the following which we then use to show that $\diamondsuit^*$ is consistent.

Fact 1 (*Condensation Lemma*). If M is a transitive set model of ZF-P + $V = L$ then there is an α such that $M = L(\alpha)$.

We will certainly not attempt to give the many details leading to the previous result (see [6]), however we should say that the class $\{L(\alpha) : \alpha \text{ an ordinal }\}$ is a definable class in L such that $L(\alpha) \subset L(\beta)$ for $\alpha < \beta$, $L(\alpha) = \bigcup_{\beta<\alpha} L(\beta)$ for limit α, $|L(\alpha) \leq |\alpha \cdot \omega|$ and if $\theta > \omega$ is regular, then $L(\theta) \models$ZF-P + V=L.

Lemma 5.10. *If $\theta \geq \omega_2$ is regular, and $\{M_\alpha : \alpha \in \omega_1\}$ is a chain of countable elementary submodels of $L(\theta)$ such that $\alpha \in M_{\alpha+1}$ for each $\alpha < \omega_1$, then*

$$\{M_{\alpha+1} \cap \mathcal{P}(\alpha) : \alpha \in \omega_1\}$$

forms a $\diamondsuit^*$-sequence in L.

Proof. Fix $X \subset \omega_1$ and S a stationary subset of ω_1. We will find $\alpha \in S$ so that $X \cap \alpha \in M_{\alpha+1}$. Let C be a cub set of α such that $M_\alpha \cap \omega_1 = \alpha$. By Lemma 1, there is a countable $M \prec L(\theta)$ so that $X \in M$ and $M \cap \omega_1 \in S \cap C$. Since $L(\theta)$ models $V = L$, so does M. Let $\alpha = M \cap \omega_1$ and by the previous Fact, fix δ so that the transitive collapse of M is equal to $L(\delta)$. Since $\alpha = M \cap \omega_1$, it follows that $L(\delta) \models \alpha = \omega_1$. Also, following the transitive collapsing function, we see that $X \cap \alpha$ is in $L(\delta)$. It suffices to show that $\delta \in M_{\alpha+1}$. Now this follows from the fact that $M_{\alpha+1} \models \alpha < \omega_1$. Indeed, $L(\theta) \models$ "$(\exists \gamma < \omega_1)\ L(\gamma) \models \alpha < \omega_1$". Therefore there is a such a $\gamma \in M_{\alpha+1}$. Since $\gamma \leq \delta$ implies that $L(\gamma) \subset L(\delta)$ it follows that $\delta < \gamma$ and so $L(\delta) \subset M_{\alpha+1}$. $\square$

6. Large cardinals

A cardinal κ is said to be (strongly) inaccessible if it is a regular (strong) limit cardinal. Many inductive proofs get stuck at inaccessible cardinals. For example a proof in which one can handle singular limits and successor cardinals would be complete if there were no inaccessible cardinals. Hausdorff's proof that there are no measurable cardinals gets stuck at a strong limit since it shows that a measurable cardinal is regular and that if α is not measurable, then neither is 2^α. As is well-known the consistency of the existence of an inaccessible cardinal cannot be shown from the consistency of ZFC. This is basically because the statement

$$\kappa \text{ strongly inaccessible } \rightarrow H(\kappa) \models \text{ ZFC}$$

can be formalized and if provable in ZFC would violate Gödel's second incompleteness theorem.

A Mahlo cardinal is a cardinal in which the set of regular cardinals is stationary. Since the set of successor cardinals is not stationary in any ordinal it follows that the set of inaccessible cardinals is stationary in a Mahlo cardinal.

A cardinal κ is weakly compact if $\kappa > \omega$ and $\kappa \to (\kappa)^2_2$. This means that if the two element subsets of κ are partitioned into two sets, then there is a homogeneous set $I \subset \kappa$ of size κ, i.e. $[I]^2$ is completely contained in one of the sets.

Proposition 6.1. *If κ is weakly compact, then κ has the tree property, i.e. there are no κ-Aronszajn trees.*

Proof. A tree T is a κ-Aronszajn tree if levels and branches of T have cardinality less than κ. Suppose that κ is weakly compact and that T is a tree in which levels have cardinality less than κ. Let $\{t_\alpha : \alpha < \kappa\}$ enumerate $\bigcup\{T_\alpha : \alpha < \kappa\}$ such that $t_\alpha < t_\beta$ implies $\alpha < \beta$. For $t, s \in T$ define $\Delta(s,t) = \delta$ as follows: if $s \leq t$ or $t \leq s$, then $s \in T_\delta$ or $t \in T_\delta$ respectively, otherwise, δ is the minimum such that there are incomparable $s', t' \in T_\delta$ such that $s' \leq s$ and $t' \leq t$. Define $s \lhd t$ if either $t \leq s$ or $\xi < \eta$ where $t_\xi \leq t$ and $t_\eta \leq s$ where t_ξ and t_η both come from $T_{\Delta(s,t)}$. Define $\{\alpha, \beta\} \in P_0$ iff $\alpha < \beta$ and $t_\alpha \lhd t_\beta$. Let $I \in [\kappa]^\kappa$ be such that either $[I]^2 \subset P_0$ or $[I]^2 \cap P_0 = \emptyset$. We suppose that $[I]^2 \subset P_0$ and leave the symmetric case.
Claim: for each $\delta < \kappa$, there is a $\beta(\delta)$ such that $t_{\beta(\delta)} < t_\alpha$ for all but fewer than κ many $\alpha \in I$. Indeed, for all but fewer than κ many $\alpha \in I$, there is some β such that $t_\beta \in T_\delta$ and $t_\beta < t_\alpha$. This follows from the facts that $|T_\delta| < \kappa$ and that, since I is homogeneous, $\alpha \leq \alpha'$ implies $\beta < \beta'$ where $t_\beta < t_\alpha$ and $t_{\beta'} \leq t_{\alpha'}$ and $t_\beta, t_{\beta'} \in T_\delta$.

It follows that $\{t_{\beta(\delta)} : \delta < \kappa\}$ is a κ-chain in T. $\square$

We will skip the proofs that a weakly compact cardinal is strongly inaccessible and that the strongly inaccessible cardinals below κ are cofinal in κ (see Kunen's article in the Handbook of Logic [5]).

Proposition 6.2. *If κ is weakly compact and $S \subset \kappa$ is stationary, then there is a strongly inaccessible $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ.*

Proof. By Solovay's theorem (5.8) we may assume that the set, Y, of strongly inaccessible cardinals less than κ which are not in S is cofinal in κ. Assume that for every $\lambda \in Y$, we may fix $C_\lambda \subset \lambda$ such that $C_\lambda \cap S$ is empty – note that $\overline{C_\lambda} \cap S$ is also empty. The C_λ's form κ-tree when ordered by end-extension, hence there is a set $I \in [Y]^\kappa$ such that $\{C_\lambda : \lambda \in I\}$ forms a κ-chain. Clearly $\bigcup_{\lambda \in I} C_\lambda$ is a cub in κ which is disjoint from S. $\square$

Next up in the (my) large cardinal hierarchy is a measurable cardinal. A cardinal κ is measurable if there is a κ-complete free ultrafilter on κ. An ultrafilter $\mathcal{U}$ is said to be normal if for each regressive function f on κ there is a $\xi < \kappa$ such that $f^{-1}(\xi)$ is a member of $\mathcal{U}$. Scott showed that if κ is measurable, then κ carries a normal ultrafilter (sometimes called a normal measure). Clearly a normal ultrafilter is κ-complete.

Lemma 6.3 (Rowbottom). *If $\mathcal{U}$ is a normal ultrafilter on a measurable cardinal κ, then $\kappa \to (\mathcal{U})^{<\omega}_2$, where this means that if $f : [\kappa]^{<\omega} \to 2$, then there is a $U \in \mathcal{U}$ such that for any $n \in \omega$ and $s, t \in [U]^n$, $f(s) = f(t)$.*

Proof. Let $f : [\kappa]^{<\omega} \to 2$. We wish to find a $U \in \mathcal{U}$ so that $|s| = |t|$ implies $f(s) = f(t)$ for all $s, t \in [U]^{<\omega}$. We prove, by induction, that for each f and $n < \omega$, there is a $U_n \in \mathcal{U}$ so that $|s| = |t|$ implies $f(s) = f(t)$ for all $s, t \in [U_n]^{\leq n}$. If this is proven, then $U = \bigcap_n U_n$ is the desired set. The case $n = 1$ follows simply from the fact that $\mathcal{U}$ is an ultrafilter. For each $\alpha < \kappa$, define $f_\alpha(s) = f(s \cup \{\alpha\})$

for $s \in [\kappa - (\alpha+1)]^{<\omega}$. By inductive assumption, there is a $U_{\alpha,n}$ and a sequence $e_0^\alpha, \ldots, e_{n-1}^\alpha$ so that $f_\alpha(s) = e_i^\alpha$ for all $s \in [U_{\alpha,n}]^i$ for $i \leq n$. Since there are only finitely many choice for $e_0^\alpha, \ldots, e_{n-1}^\alpha$, there is a $U' \in \mathcal{U}$ so that $e_i^\alpha = e_i^\beta = e_i$ and, by the $n = 1$ case, $f(\{\alpha\}) = f(\{\beta\})$, for all $\alpha, \beta \in U'$ and $i < n$.

Define $\triangle\{U_{\alpha,n} : \alpha < \kappa\}$ to be the set $\{\alpha : \beta < \alpha \rightarrow \alpha \in U_{\beta,n}\}$. Since $\mathcal{U}$ is normal, $\triangle\{U_{\alpha,n} : \alpha < \kappa\}$ is a member of $\mathcal{U}$; indeed, suppose that $Y = \kappa - \triangle\{U_{\alpha,n} : \alpha < \kappa\}$ is a member of $\mathcal{U}$ and define $g(\alpha)$ to be 0 if $\alpha \notin Y$ and to be the minimum $\beta < \alpha$ such that $\alpha \notin U_\beta$ otherwise. Fix ξ such that $g^{-1}(\xi) \in \mathcal{U}$ and note that this means that $U(\xi, n) \notin \mathcal{U}$.

Let $U = U' \cap \triangle\{U_{\alpha,n} : \alpha < \kappa\}$ and fix $s \in [U]^{\leq n+1}$. We complete the inductive step by showing that $f(s) = e_{i-1}$. Let $\alpha = \min(s)$ and recall that $s - \{\alpha\} \subset U_{\alpha,n}$ and $\alpha \in U'$. Therefore, $e_{i-1} = f_\alpha(s - \{\alpha\}) = f(s \cup \{\alpha\}) = f(s)$. $\square$

If a cardinal κ is **supercompact** then for each λ and $X \in V_\lambda$, there is a $\mu \geq \lambda$ such that V_μ has an elementary submodel M such that $X \in M$, $|M| < \kappa$, $M \cap \kappa \in \kappa$, and $G(M \cap V_\lambda) = V_\beta$ for some β, where G is the Mostowski collapsing function. This characterizes the smallest supercompact cardinal (see [7]) but we will not give the definition of a supercompact cardinal. We refer the interested topologically inclined reader to [3] for more details and applications to topology. However we have not defined V_α: V_α is the set of all sets with rank less than α; it follows that if $x \in V_\alpha$, then $\mathcal{P}(x)$ is a subset of V_α and a member of $V_{\alpha+1}$.

As an example, suppose that κ is supercompact and that X is a space with character less than κ which is not metrizable. We show that X has a subspace of cardinality less than κ which is not metrizable. Find M as above with λ, hence β, a limit. We first claim that $G(X)$ is not metrizable. Assume that there is a σ-discrete base B for $G(X)$. Since $X \in V_\lambda$, $G(X)$ is in V_β, and so B is also a member of V_β. Therefore B is equal to $G(B')$ for some B' in M. Since G is an isomorphism it follows that $M \models B'$ is a base for X, in fact a σ-discrete one, but by elementarity, B' would, in fact, be a σ-discrete base for X. What we want to now show is that $X \cap M$ is non-metrizable. Here is where the character assumption comes in. For each $x \in X \cap M$, there is a local base B_x of x, such that $B_x \in M$ and $|B_x| < \kappa$. By Lemma 4.4, M contains B_x and this guarantees that $\tau \cap M$ and $\tau \restriction (X \cap M)$ are the same topology on $X \cap M$. Therefore $G(X)$ is homeomorphic to $X \cap M$.

7. Basic Forcing

Forcing divides topology – those that aren't familiar with it do not embrace the work of those that utilize it. This is very unfortunate and quite unnecessary. There are a few basic principles, one or two technical facts and forcing becomes quite understandable and it should be an integral part of set-theoretic topology. This is not to say that forcing arguments are easy but, rather, that (once beyond Cohen's fundamental theorem) the understanding of forcing itself is easy. Forcing <u>arguments</u> involve well-ordering, induction, and other combinatorial tools – the heart and soul of General Topology.

As most people know, one starts with a countable transitive model M of ZFC and add to M a P-generic filter (defined below) G of some poset $P \in M$ so as to define a model $M[G]$ (all this taking place in some larger universe V). $M[G]$ is the smallest countable transitive model of ZFC which contains M and G but this doesn't tell us much about how to construct $M[G]$. What makes forcing work is that every member of $M[G]$ can be constructed from a member of M and G by the most basic of prescriptions. For example, given a set $X \in M$, every subset of X in $M[G]$ can be obtained from some $\tau \in M$ such that $\tau \subset P \times X$ and the new set is simply equal to $\{x \in X : (\exists p \in G)\ (p,x) \in \tau\}$ (see 7.3); this latter set might be denoted τ_G and τ could be called a *name* or P-name for τ_G. Clearly all such sets *must* be added to $M[G]$ given the above requirements, but there are no more that are added. One can formalize the notion of a P-name and let M^P denote the class (contained in M) of all P-names τ and also the definition of a valuation by G, τ_G (see e.g. [6] or [10]).

It is not usually necessary to know the precise definition of a P-name but simply the existence of the class M^P and the basic forcing theorems listed below. Set-theorists ofter work with simpler more natural descriptions of the members of $M[G]$ such as the τ we mentioned above.

Definition 7.1. A set $G \subset P \in M$ is P-generic for M, if G is a filter (i.e. closed upwards and the elements of G pairwise have lower bounds in G) and if G meets every set in M which is a dense subset of P (a set $A \subset P$ is dense if every element of P is greater than some member of A).

The remarkable forcing theorem states that if G is a generic subset of a poset $P \in M$ and M is a countable transitive model of (a sufficient fragment of ZFC) then

$$M[G] = \{\tau_G : \tau \in M^P\}$$

is also a countable transitive model of (a similar fragment) of ZFC which contains M and G.

The very important thing then about $M[G]$ is that every element of $M[G]$ has a name from M and the second forcing theorem holds:

Lemma 7.2. *Suppose that M is a countable transitive model of ZFC and G is P-generic for M for some poset $P \in M$. Then for any $\tau_1, \ldots, \tau_n$ all in M^P and any formula $\varphi(v_1, \ldots, v_n)$, $M[G] \models \varphi((\tau_1)_G, \ldots, (\tau_n)_G)$ if and only if there is a $p \in G$ so that $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$.*

The notation $p \Vdash \varphi$ has two equally important meanings:
Semantic: for every generic filter G such that $p \in G$,

$$M[G] \models \varphi((\tau_1)_G, \ldots, (\tau_n)_G)\ .$$

Syntactic (and therefore subject to detailed combinatorial analysis). One of our goals is to show that it is not necessary to work with such technical definitions so we will not give it. We will only use the fact that the existence of a syntactic definition implies that the $\Vdash_P$ relation is definable in M and for everything else we will use the semantic definition. For example, it follows that $\tau_G = \{\sigma_G : (\exists p \in G)\ p \Vdash \sigma \in \tau\}$.

Most forcing arguments involve subsets of members of M or sets of subsets of members of M and for these we introduce very simple descriptions or what we might call unofficial names. We need names to do combinatorial analysis and to make statements in the forcing language, i.e. $p \Vdash \varphi(\tau)$. Our first unofficial name convention is that we will use members of M as names for themselves. If there is a potential for confusion then it is common to use $\check{x}$ as a P-name for the set $x \in M$.

When a set $A \in M[G]$ is being discussed it is also common to let $\dot{A}$ denote a P-name for A. Now suppose that $\tau \in M^P$ and $p \in P$ are such that $p \Vdash \tau \subset M$. Let us show that there is an $X \in M$ and (unofficial name) $\sigma \in M$ such that $\sigma \subset P \times X$ and, for every P-generic G with $p \in G$,

$$\tau_G = \{x : (\exists p \in G)\ (p, x) \in \sigma\} .$$

This latter set is just the projection of σ along G. Let us agree that if σ is a subset of $P \times X$ (for any set X), then σ_G denotes this projection. We can treat σ as a name and note that we are claiming that $p \Vdash \tau = \sigma$. To find X and σ simply use the forcing relation in M. For each $q \in P$, $\{x : q \Vdash x \in \tau\}$ is a set in M (it is definable in M and it is a set since if $q \in G$, then this set is a subset of τ_G). Let $X = \bigcup_{q<p}\{x : q \Vdash x \in \tau\}$ and define $\sigma = \{(q,x) : q < p, q \in P, x \in X, \text{ and } q \Vdash x \in \tau\}$. The forcing theorem 7.2, implies directly that σ_G is equal to τ_G for all generic G which contain p. We will consider sets of subsets of X in 7.3.

When it is necessary to refer to G in a a forcing sentence, e.g. $p \Vdash G \subset P$, it is common to use Γ to denote a P-name for G (we cannot use G itself because names must be in M).

Before stating the next result let us make some remarks on some common conventions which reduce the verbiage in stating forcing results. We have been using M to denote the original (ground) model and thus $M[G]$ to denote the extension. The implicit assumption is that M is a countable transitive model of ZFC in the "real" universe of sets, hence for each $P \in M$ there is a P-generic filter for M, and thus $M[G]$ is another countable transitive model of ZFC in V (and the process could continue). However, in practice, it is common to let V denote the ground model and to not mention that V is a countable transitive submodel of the real universe. We will now switch to using V to denote the ground model so that we will have the letter M available to denote submodels of V. With this understanding we will also assume that given any poset $P \in V$, there is a P-generic filter for V.

The key combinatorial notion for forcing is when a set is dense in the forcing poset P. Related notions are dense below p and predense below p. A set A is dense below p if for every $q \leq p$, there is an $a \in A$ such that $a \leq q$, and A is predense below p, if $\{q \in P : (\exists a \in A)\ q \leq a\}$ is dense below p. Also the notion of **incompatible** elements is important (but incomparable is not): $p \perp q$ if there is no $r \in P$ such that $r \leq p$ and $r \leq q$. A set $A \subset P$ is an antichain if it consists of pairwise incompatible elements. A maximal antichain is dense below each element and every dense set contains maximal (in P) antichains. Similarly if $p \in G$ and D is predense below p then $G \cap D$ is not empty (i.e. $p \Vdash G \cap D \neq \emptyset$). Another simplification we will make is that we will assume that all our posets have a largest element denoted by 1 and that poset means **separative** poset, i.e. if $p \not\leq q$ then there is an $r \leq p$ such that $r \perp q$.

We return to our unofficial names. This is as technical as we will have to get with names.

Proposition 7.3. *If τ is a P-name and $p \in P$ is such that $p \Vdash$ "$\emptyset \neq \tau \subset \mathcal{P}(X)$" for some set $X \in V$, then there is a family $S \in V$ such that $\sigma \subset P \times X$ for all $\sigma \in S$ and for every P-generic G containing p, $V[G] \models \tau_G = \{\sigma_G : \sigma \in S\}$.*

Proof. Assume that ρ is a P-name for B and let $p \in G$ be such that $p \Vdash$ "$\rho \neq \emptyset$ and $(\forall y)(y \in \rho \rightarrow y \subset X)$" (since $B \subset \mathcal{P}(X)$).

Let S be the set of all (unofficial names) $\sigma \subset P \times X$ such that $p \Vdash \sigma \in \tau$. It is clear that p forces that $\{\sigma_G : \sigma \in S\}$ is a subset of τ_G but it is not clear that p forces the two sets are equal. To see this, suppose that G is P-generic ($p \in G$) and that A is a member of τ_G (in $V[G]$ of course). We have already shown that there is a $\sigma_0 \subset P \times X$ such that $(\sigma_0)_G = A$; hence there is some $q_0 \in G$ such that $q_0 \Vdash \sigma_0 \in \tau$. However, it does not follow that $p \Vdash \sigma_0 \in \tau$ (thus $\sigma_0 \in S$). We need what is called the maximum principle (see 7.4) but we will prove it in this special case. We may assume that we chose $q_0 \leq p$ and suppose that we have chosen $q_\beta < p$ (for $\beta < \alpha$) together with σ_β such that $q_\beta \Vdash \sigma_\beta \in \tau$ and $\{q_\beta : \beta < \alpha\}$ is an antichain. If $\{q_\beta : \beta < \alpha\}$ is a maximal (below p) antichain, we stop. Otherwise we find some q' below p which is incomparable with each q_β. Since $q' < p$, $q' \Vdash \tau \neq \emptyset$, hence there is some $q_\alpha < q'$ and a σ_α such that $q_\alpha \Vdash \sigma_\alpha \in \tau$. There is some $\alpha < |P|^+$ such that $\{q_\beta : \beta < \alpha\}$ is maximal. Recall that if G' is a generic filter containing p, then there is exactly one $\beta < \alpha$ such that $q_\beta \in G'$. To get our $\sigma \in S$ (which names A) we "glue" all the σ_α together as follows:

$$\sigma = \bigcup_{\beta<\alpha} \{(q, x) : q \leq q_\beta \text{ and } q \Vdash x \in \sigma_\beta\} \ .$$

The main thing to show (which we skip because it is easy) is that if q_β is in a generic filter G', then $\sigma_{G'} = (\sigma_\beta)_{G'}$. Therefore, $p \Vdash \sigma \in \tau$ and $q_0 \Vdash \sigma_0 = \sigma$. From this latter fact, it follows that $\sigma_G = A$. Thus $A \in \{\sigma_G : \sigma \in S\}$. $\square$

Another fundamental forcing theorem is called the maximum principle.

Lemma 7.4. *If $\tau_1, \ldots, \tau_n$ are P-names and $p \Vdash (\exists x)\ \varphi(x, \tau_1, \ldots, \tau_n)$, then there is a name τ, such that $p \Vdash \varphi(\tau, \tau_1, \ldots, \tau_n)$. Furthermore, if $\{p_i : i \in I\} \subset P$ is a maximal antichain, and if $\{\tau_i : i \in I\}$ is a family of P-names, then there is a name τ such that for every P-generic G, if $p_i \in G$, then $\tau_G = (\tau_i)_G$.*

A lot of forcing can be learned by forcing with $\mathrm{Fn}(I, 2)$, defined below, which adds what are called Cohen reals (and they are by far the simplest). Another thing one needs to do is to get some practice with the kind of combinatorics on posets that is most relevant to forcing (as in Kunen's book doing MA first). Recall that P satisfies the κ-chain condition (κ-cc) if every antichain in P has cardinality less than κ. A poset is ccc if it satisfies the $\aleph_1$-cc. A poset is said to be κ-directed closed, or usually just κ-closed for short, if every directed set of size less than κ has a lower bound.

Examples: If I, J are sets then $\mathrm{Fn}(I, J)$ denotes the set of finite partial functions, ordered by reverse inclusion, whose domain are contained in I and whose range are contained in J. If κ is a cardinal then $\mathrm{Fn}(I, J, \kappa)$ is the same except that "finite" is replace by "cardinality less than κ".

Proposition 7.5. *If $\lambda^{<\kappa} \leq \kappa$ for each $\lambda < \kappa$, then $Fn(I, 2, \kappa)$ satisfies the κ^+-cc for each set I.*

We can make use of the following useful strengthening of Lemma 5.2

Lemma 7.6. *Given a set X in a set model H and a regular cardinal κ, there is an elementary submodel $M \prec H$ such that $X \in M$, $|M| \leq \kappa^{<\kappa}$, and $M^{<\kappa} \cap H \subset M$.*

Proof. Inductively use Lemma 5.2 to find an increasing sequence $\{M_\alpha : \alpha < \kappa\}$ of elementary submodels of H so that $X \in M_0$ and for each $\alpha < \kappa$, $|M_\alpha| \leq \kappa$ and $M_\alpha^{<\kappa} \cap H \subset M_{\alpha+1}$. By Lemma 5.3, $M = \bigcup_{\alpha<\kappa} M_\alpha$ is the desired model. □

Proof. (of Lemma 7.5) Let $\mathcal{A}$ be a maximal antichain of $\mathrm{Fn}(I,2,\kappa)$ and fix a regular θ much larger than $|I \cup \kappa|$ so that $H(\theta)$ is closed under κ sequences and $I \in H(\theta)$. Use the previous lemma to find $M \prec H(\theta)$ so that I, κ are both in M, $|M| \leq \kappa$, and $M^{<\kappa} \subset M$. We finish by proving that $\mathcal{A} \cap M$ is a maximal antichain of $\mathrm{Fn}(I,2,\kappa)$ (hence $\mathcal{A} \subset M$). One can, in fact, prove that $M \cap \mathrm{Fn}(I,2,\kappa)$ is a complete subalgebra of $\mathrm{Fn}(I,2,\kappa)$, see Definition 7.21. Let $f \in \mathrm{Fn}(I,2,\kappa)$ be arbitrary and let $g = f \cap M = f \restriction (M \cap I)$. Since $M^{<\kappa} \subset M$, $g \in M$. Now since $M \models \mathcal{A}$ is a maximal antichain, there is an $a \in \mathcal{A} \cap M$ such that $g \cup a \in \mathrm{Fn}(I,2,\kappa)$. Since dom $(f-g) \cap$ dom (a) is empty, it follows that $f \cup a$ is also a function, hence f is compatible with $\mathcal{A}$. Therefore $\mathcal{A} \cap M$ is a maximal antichain. □

We return to the discussion of our example. If $G \subset \mathrm{Fn}(\omega, 2)$, then $\bigcup G$ is a partial function from ω to 2.

Exercise: if G is P-generic over V, then dom $(\bigcup G) = \omega$.

Lemma 7.7. *$\forall y \in V$, $\bigcup G \neq y$*

Proof. The simplest method is to observe that for each $y \in V \cap {}^\omega\omega$, the set $D_y = \{p \in \mathrm{Fn}(\omega,2) : (\exists n \in \mathrm{dom}\, p)\ p(n) \neq y(n)\}$ is a dense subset of $\mathrm{Fn}(\omega, 2)$. Since $D_y \in V$, G will meet D_y and so $\bigcup G$ will not equal y. □

Lemma 7.8. *For all $y \in V \cap 2^\omega$, the set $\{n : \bigcup G \restriction [2^n, 2^{2n}) = y \restriction [2^n, 2^{2n})\}$ is infinite.*

Proof. We could proceed by defining a set of dense sets as in the previous proof but we'll use this as an opportunity to introduce another common method. Frequently to show that a set in the forcing extension is infinite, it is best to show that it is not finite. Assume that $y \in V \cap 2^\omega$ is such that $V[G] \models$ there are only finitely many n such that $y \restriction [2^n, 2^{2n})$ agrees with $\bigcup G \restriction [2^n, 2^{2n})$. Use the forcing lemma 7.2 to obtain a $p \in \mathrm{Fn}(\omega,2)$ so that $p \Vdash (\exists m)(n > m \rightarrow y \restriction [2^n, 2^{2n}) \neq \bigcup \Gamma \restriction [2^n, 2^{2n})$. Next use the maximum principle to find τ so that $p \Vdash \tau \in \omega$ and $p \Vdash$ for each $n \geq \tau$, $y \restriction [2^n, 2^{2n}) \neq \bigcup \Gamma \restriction [2^n, 2^{2n})$. The next step is to note that there is a $q < p$ and an $m \in \omega$ so that $q \Vdash \tau = m$; indeed, since $V[G] \models \tau_G = m =$ for some m, there is a $q \in G$ as desired by Lemma 7.2. But now, there is an $n \geq m$ so that the domain of q is contained in n. Let $r = q \cup y \restriction [2^n, 2^{2n})$ and check that r is an extension of q in $\mathrm{Fn}(\omega, 2)$ (i.e. $r < q$). Clearly r forces that $y \restriction [2^n, 2^{2n})$ is equal to $\bigcup \Gamma \restriction [2^n, 2^{2n})$. This is a contradiction since r being less than q will also forces, as does q, that $y \restriction [2^n, 2^{2n})$ does not equal $\bigcup \Gamma \restriction [2^n, 2^{2n})$. □

Corollary 7.9. *$V[G] \models V \cap 2^\omega$ has outer measure zero.*

Exercise (after reading the next section): If G is $\mathrm{Fn}(\omega_2, 2)$-generic over V, then $V[G] \models$ every set of reals of size $\aleph_1$ has measure zero.

The main application of "everything has a name" is to the preservation of ground model properties. These results usually depend on the combinatorial properties of the forcing notion (poset).

Lemma 7.10. *If G is $Fn(\omega, 2)$-generic over V, then ω_1 is preserved, that is, $1 \Vdash \check{\omega}_1$ is a cardinal.*

Lemma 7.11. *If P is ccc then every cardinal of V is preserved by forcing with P. That is, for each cardinal $\kappa \in V$, $p \Vdash \check{\kappa}$ is a cardinal.*

See Lemma 7.14 for proofs of the previous two Lemmas. In the case of $\mathrm{Fn}(\omega, 2)$ we can prove a stronger result.

Lemma 7.12. *If G is $Fn(\omega, 2)$-generic over V then $V[G] \models$ for each cofinal $I \subset \omega_1^V$ there is an $I' \in V$ such that $V \models |I'| = \omega_1$ and $V[G] \models I' \subset I$. Therefore $V[G] \models$ "there is no ω-sequence cofinal in ω_1^V.*

Proof. Let $\tau \in V$ be a name of a subset of ω_1 and, in $V[G]$, τ_G is cofinal in ω_1^V. Further suppose that τ_G does not contain any uncountable ground model set I'. Fix $p \in G$ such that $p \Vdash \tau$ is cofinal in $\check{\omega}_1$ and that τ does not contain any uncountable ground model set. It is common to consider the following set in V: let τ_p denote the set of all α such that there is some $q < p$ such that $q \Vdash \alpha \in \tau$. τ_p is the smallest set Y in V with the property that τ_G is contained in Y for every generic G which contains p. Another way to say this is that τ_p is the smallest set in V with the property that $p \Vdash \tau \subset \tau_p$. Clearly then τ_p is uncountable. For each $\alpha \in \tau_p$, fix a $p_\alpha < p$ such that $p_\alpha \Vdash \alpha \in \tau$. Since $\mathrm{Fn}(\omega, 2)$ is countable, there is a q so that $I' = \{\alpha : p_\alpha = q\}$ is uncountable (usually the combinatorics on P is more difficult than this). Therefore $q \Vdash I' \subset \tau$, which is a contradiction. □

By contrast consider the following "collapsing" poset.

Proposition 7.13. *Assume that G is $Fn(\omega, \omega_1)$-generic over V, then $V[G] \models \omega_1^V$ is countable; i.e. $1 \Vdash |\check{\omega}_1| = \omega$ (and $\check{\omega}_2 = \omega_1$).*

Proof. Again, $\bigcup G$ is a function, this time the domain is ω and the range is a subset of ω_1^V. We can show that the range is in fact all of ω_1^V by showing that

$$D_\alpha = \{p \in \mathrm{Fn}(\omega, \omega_1) : \alpha \in \mathrm{range}\ p\}$$

is dense for each $\alpha < \omega_1$. This is trivial: given any p, choose $n \notin p$ and extend p by attaching (n, α); this extension is in D_α. To prove that

$$1 \Vdash \check{\omega}_2 = \omega_1$$

it suffices to show that ω_2^V is a cardinal in $V[G]$. This follows from the next lemma. □

Lemma 7.14. *If P satisfies the κ-cc and if $\tau \in V$ is a P-name of a function with domain I such that $p \Vdash \tau(i) \in V$ for each $i \in I$, then there is a function $F \in V$ such that $F(i)$ is a subset of V of cardinality less than κ, and,*

$$p \Vdash \tau(i) \in \breve{F(i)} \text{ for each } i \in I .$$

Therefore if κ is regular and $\dot{a}$ is a P-name such that $p \Vdash \dot{a} \in [V]^{<\kappa}$, then there is an $A \in V \cap [V]^{<\kappa}$ such that $p \Vdash \dot{a} \subset \check{A}$.

Proof. Fix an i; we define $F(i) = \{a \in V : (\exists q \in P)\ q < p \text{ and } q \Vdash \tau(\check{i}) = \check{a}\}$, since the a associated with a q is unique, $F(i)$ is a set in V by the axiom of replacement. For each $a \in F(i)$, let $P_a = \{q \in P : q < p \text{ and } q \Vdash \tau(i) = a\}$. Since $p \Vdash \tau(i) \in V$, it follows that $\bigcup_{a\in F(i)} P_a$ is predense below p (i.e. given any $p' < p$, $p' \Vdash \tau(i) \in V$, hence there are $q < p'$ and a so that $q \Vdash \tau(i) = a$; thus $q \in P_a$). Therefore $p \Vdash \tau(i) \in F(i)$. Now we must show that $|F(i)| < \kappa$. For each $a \in F(i)$, fix any $p_a \in P_a$. We show that $F(i)$ has cardinality less than κ by showing that $\{p_a : a \in F(i)\}$ is an antichain of P. Fix $a \neq b \in F(i)$ and observe that $p_a \Vdash \tau(\check{i}) = \check{a}$ and similarly for p_b. Now if q were below both p_a and p_b, q would simultaneously force that $\tau(\check{i})$ was equal to $\check{a}$ and to $\check{b}$. This contradicts that p (and hence $q < p_a < p$) forces that τ is a function. $\square$

A poset P is said to be **proper** if forcing with P preserves stationary subsets of $[\theta]^\omega$ for all θ. One defines cub subsets of $[\theta]^\omega$ in a natural way: $C \subset [\theta]^\omega$ is unbounded if for every $b \in [\theta]^\omega$, there is a $c \in C$ such that $b \subset c$, and C is closed if the union of every countable chain from C is again in C. Hence a set is stationary if it meets every cub. However the working definition of proper is usually the following (see [10]):

Definition 7.15. A condition p in a poset P is said to be (M, P)-generic if p forces the generic filter to meet $A\cap M$ for each $A \in M$ such that A is a dense subset of P. This is equivalent to saying that for each dense $A \subset P$ $(A \in M)$ and each $q < p$, there is an $r \in A \cap M$ such that r is compatible with q. A poset P is proper if there is a λ such that for every regular $\theta \geq \lambda$ and every countable $M \prec H(\theta)$ with $P \in M$, there is, for each $p \in P \cap M$, a $q < p$ which is (M, P)-generic.

Lemma 7.16.
1. *A ccc poset is proper.*
2. *An $\aleph_1$-closed poset is proper and adds no new countable sets of ordinals.*
3. *A proper poset preserves ω_1, but not necessarily ω_2.*
4. *If G is P-generic for a proper poset P and if $a \in V[G]$ is a countable subset of V, then there is a countable $b \in V$ such that $a \subset b$.*

Proof. A ccc poset P is proper because any $p \in P$ is (M, P)-generic for any $M \prec H(\theta)$ with $P \in M$. This is simply because every dense set in M contains maximal antichains which are in M. If A is a maximal antichain in M, then, by Lemma 4.4, A is contained in M since it is countable. Since A is predense below p, p forces that G meets $A = A \cap M$.

Now suppose that P is $\aleph_1$-closed and that $P \in M \prec H(\theta)$ with $|M| = \omega$. Let $\{D_n : n \in \omega\}$ be an enumeration of the dense subsets of P which are members of M (of course this enumeration is not in M unless P is finite). Fix any $p_0 \in P \cap M$ and recursively choose $p_{n+1} < p_n$ so that $p_{n+1} \in D_n \cap M$. To choose p_{n+1} one must note that $p_n \in M$ and $M \models (\forall q \in P)\ (\exists r \in D_n)\ r < q$. In the end, use that P is $\aleph_1$-closed to find a $q \in P$ such that $q < p_n$ for all n. The condition q is (M, P)-generic since if q is a member of a filter G, then $p_n \in G \cap (D_n \cap M)$ for each n. To see that P adds no new countable sets of ordinals, assume that $\tau \in M$ is the name of a function into the ordinals with domain ω. We can assume that for each $p \in D_n$, there is an ordinal $\alpha(p, n)$ such that $p \Vdash \tau(n) = \alpha(p, n)$. Therefore q forces that $\tau(n) = \alpha(p_n, n)$ for each n, i.e. $q \Vdash \{\tau(n) : n \in \omega\}$ is the ground model set $\{\alpha(p_n, n) : n \in \omega\}$. The key is that q decides the values $\tau(n)$ for each n, since for any name τ and any condition q, the set $\{x : q \Vdash x \in \tau\}$ is a ground model set.

The fact that a proper poset preserves ω_1 follows from the fourth item. To see that a proper poset need not preserve ω_2, note that the poset ${}^{<\omega_1}\omega_2$ (sometimes called the countable condition collapse of ω_2) is countably closed, hence proper. One easily shows that the union of the generic filter is a function from ω_1 onto ω_2 (this is nearly identical to Proposition 7.13).

Let τ be a name and $p \in P$ be such that p forces that τ_G is a function from ω onto a and that each member of a is from V (as in the semantic definition of forcing). Fix a suitably large θ and let M be a countable elementary submodel of $H(\theta)$ such that p, P, τ are all in M. Let $q < p$ be (M, P)-generic. If we show that $q \Vdash \tau(i) \in M$ for each $i \in \omega$, then we will have that q forces that a is contained in the countable set M. Define

$$D_i = \{r \in P : (\exists x)\ r \Vdash \tau(i) = x\}\ .$$

Since all the parameters of D_i are in M it follows that D_i is in M. We would like to say that D_i is dense in P and therefore that q forces that $G \cap (D_i \cap M)$ is not empty. However all we have said is that p forces that $\tau(i)$ is a member of V – hence it follows that D_i is predense below p. Since $q < p$ it does follow that $G \cap (D_i \cap M)$ is not empty. That is, let $\bar{D}_i = D_i \cup \{r \in P : r \perp p\}$. Again $\bar{D}_i$ is in M and, this time, is dense. Assume that q (hence p) is in G and fix $r \in G \cap (\bar{D}_i \cap M)$. Since p and r are both in G, they must be compatible. Therefore $r \in D_i$. Now, $r \in D_i \cap M$, so there is an $x \in M$, such that $r \Vdash \tau(i) = x$. This completes the proof. $\square$

There are posets which preserve ω_1 but which do not preserve stationary subsets of ω_1 – hence being stationary is not preserved by forcings which preserve cardinals (but being a cub is preserved).

Proposition 7.17. *Let S be a stationary subset of ω_1 which does not contain a cub. Then there is a poset P which preserves ω_1 and introduces a cub which is disjoint from S.*

Proof. Let P be the set of closed subsets of ω_1 which are disjoint from S. Order P by reverse end extension, i.e. $b < a$ if $b \cap (\max a) + 1 = a$. For each $\alpha < \omega_1$, $E_\alpha = \{a \in P : a - \alpha \neq \emptyset\}$ is easily seen to be dense. Therefore, if G is a generic subset of P, $\bigcup G$ is a cofinal subset of ω_1 which is disjoint from S. To see that $\bigcup G$ is closed in ω_1, assume that β is a limit point. Fix any $a \in G \cap E_\alpha$ and check that β is in fact a limit point of a, i.e. $\bigcup G \cap \beta + 1 = b$ because the ordering was end extension. The main thing to show is that ω_1 is preserved. We can actually mimick the proof that ω_1-closed posets are proper. Suppose that $\dot{f}$ is the name a function from ω into ω_1. We must show that if $a \in P$ is arbitrary, then there is a $b < a$ which forces that the range of $\dot{f}$ is bounded. Since S does not contain a cub, its complement is stationary, hence there is a countable $M \prec H(\theta)$ such that $a, \dot{f} \in M$ (i.e. $H(\theta)$ is large enough) and $\delta = M \cap \omega_1 \notin S$ (see Lemma 5.5). As above, $D_n = \{r \in P : (\exists \alpha \in \omega_1)\ \dot{f}(n) = \alpha\}$ is dense (we may assume, by the maximum principle that $1 \Vdash \dot{f}$ is a function from ω to ω_1). The plan is to choose a decreasing sequence $a_n \in D_n \cap M$ (hence a_n forces that $\dot{f}(n)$ is below δ) so that there is a $b \in P$ so that $b < a_n$ for each n (and $a_0 < a$). In other words we need to ensure that the closure of $\cup_n a_n$ is disjoint from S. We do so by ensuring that $\max a_n$ converges to δ. Since δ will be the only new limit point of $\cup_n a_n$ and $\delta \notin S$, $\{\delta\} \cup \cup_n a_n$ is a member of P which forces that the range of $\dot{f}$ is bounded. To ensure this, let β_n be any sequence which is cofinal in δ. When a_n is chosen, be sure that it is a member of $D_n \cap E_{\beta_n}$. $\square$

Another interesting and basic forcing is Prikry forcing. This forcing preserves cardinals but not cofinalities.

Proposition 7.18 (Prikry). *Assume that $\mathcal{U}$ is a normal ultrafilter on a measurable cardinal κ and define $p \in P$ iff there is a finite $t_p \in [\kappa]^{<\omega}$ and a $U_p \in \mathcal{U}$ such that $p = \langle t_p, U_p \rangle$ and $U_p \cap \max(t_p) + 1 = \emptyset$. P is ordered by $p < q$ if $t_p \supset t_q$, $t_p - t_q \subset U_q$, and $U_p \subset U_q$. If G is P-generic, then in $V[G]$ the cofinality of κ is ω, but all cardinals are preserved.*

Proof. It is easily seen that if G is P-generic filter, then $\cup\{t_p : p \in G\}$ is a countable sequence which is cofinal in κ. The key property of this forcing is known as the Prikry property which is used in several other forcings as well. Fix an ordinal α and a P-name τ. For each $p \in P$, there is a condition $q < p$ so that $t_q = t_p$ and either $q \Vdash \tau = \alpha$ or $q \Vdash \tau \neq \alpha$. That is, one can decide whether a formula in the forcing language is true or false without extending the finite part of the condition. This follows easily from Lemma 6.3. Define a function $f : [\kappa]^{<\omega} \to 2$ as follows: $f(s) = 1$ if and only if there is a $q < p$ such that $t_q = t_p \cup s$ and $q \Vdash \tau = \alpha$. Apply Lemma 6.3 to obtain a $U \in \mathcal{U}$ such that $U \subset U_p$ and $f(s)$ depends only on $|s|$ for all $s \in [U]^{<\omega}$. The condition we want is (t_p, U). If (t_p, U) does not decide whether τ is equal to α or not, then there are q_1 and q_2 both less than (t_p, U) such that $q_1 \Vdash \tau = \alpha$ and $q_2 \Vdash \tau \neq \alpha$. We may extend either of them so as to ensure that $|t_{q_1}| = |t_{q_2}|$. Clearly, by definition, $f(t_{q_1} - t_p) = 1$, hence $f(t_{q_2} - t_p)$ is also equal to 1. But this leads to a contradiction, for if $q < p$ is such that $t_q = t_{q_2}$ and $q \Vdash \tau \neq \alpha$, then q and q_2 would have to be incompatible. However they are not, since $(t_{q_2}, U_q \cap U_{q_2})$ is below both of them.

Note that the fact that two conditions are compatible if they have the same first coordinate implies that P has the κ^+-cc, hence all cardinals above κ are preserved. If we show that all cardinals below κ are also preserved then we have that all cardinals are preserved since κ is a limit cardinal. Not only are cardinals below κ preserved but P adds no new subsets of any $\lambda < \kappa$. Indeed assume that $\dot{a}$ is a name and $p \in P$. We will find a $q < p$ and $b \in V$ so that $q \Vdash \dot{a} \cap \lambda = b$. We will even have that $t_q = t_p$. For each $\alpha < \lambda$ we can find $U_\alpha \subset U_p$ so that either $(t_p, U_\alpha) \Vdash \alpha \in \dot{a}$ or $(t_p, U_\alpha) \Vdash \alpha \notin \dot{a}$. Let $U = \bigcap\{U_\alpha : \alpha < \lambda\}$ and $b = \{\alpha < \lambda : (t_p, U) \Vdash \alpha \in \dot{a}\} = \dot{a}_{(t_p, U)}$. Since $U \in \mathcal{U}$, we let $q = (t_p, U)$ and we are done. $\square$

Lemma 7.19. *A standard ccc trick: if A is a subset of a ccc poset P and if $|A|$ has uncountable cofinality, then there is a generic G which meets A in cardinality $|A|$.*

Proof. One can prove this by an indirect method or in a more combinatorial fashion as follows. For each $B \subset A$, let D_B be the set of all $p \in P$ such that B is predense below p. If B is not empty then D_B is not empty and if $q \in D_B$ and $r < q$

then $r \in D_B$. Note that if $B \subset B'$, then $D_B \subset D_{B'}$. Let us also note that $E_B = \{p : (\exists a \in B)\ p < a\}$ is a dense subset of D_B.

Claim: there is an $S \subset A$ with $|S| < \kappa$ such that for all T such that $S \subset T \subset A$ and $|T| < \kappa$, $D_{A-T} = D_{A-S}$. (One should note that this does not hold for the family of E_{A-S}'s).

Indeed, if not we could choose an increasing sequence, $\{S_\xi : \xi < \omega_1\}$, of less than κ-sized subsets of A so that $\xi < \eta < \omega_1$ implies that D_{A-S_η} is strictly smaller than D_{A-S_ξ}. For each $\xi < \omega_1$, there is some $p_\xi \in D_{A-S_\xi}$ which is not in $D_{A-S_{\xi+1}}$. Since $p_\xi \notin D_{A-S_{\xi+1}}$ there is a $q_\xi < p_\xi$ which is incompatible with each member of $A - S_{\xi+1}$. It follows that if $\xi < \eta < \omega_1$, then q_η is incompatible with q_ξ. Indeed, if r were below q_ξ, then r would be incompatible with every member of $A - S_\eta$ and so r cannot be below p_η since this would make r a member of D_{A-S_η}. Since such an uncountable antichain cannot exit in a ccc poset, the Claim is proved.

Fix an $S \in [A]^{<\kappa}$ as in the previous claim. Now let $\mathcal{B}$ be any collection of pairwise disjoint countable subsets of $A - S$ such that each $B \in \mathcal{B}$ is predense below each member of D_{A-S} (equivalently, $D_B = D_{A-S}$) and such that $\mathcal{B}$ is maximal. We claim that $\mathcal{B}$ has cardinality κ. Indeed, if $|\mathcal{B}| < \kappa$, then let $T = S \cup \bigcup \mathcal{B}$ and note that $D_{A-T} = D_{A-S}$. Let C be any maximal antichain of E_{A-T}. Since E_{A-T} is dense in D_{A-T}, C is predense below each member of D_{A-T}. For each $c \in C$ fix an $a_c \in A - T$ such that $c \leq a_c$ (which we may do since $c \in E_{A-T}$). Let $B = \{a_c : c \in C\}$; since P is ccc, C, and thus B, are countable. B is predense below each element of C and so is predense below every element of $D_{A-T} = D_{A-S}$. This contradicts the maximality of $\mathcal{B}$. To finish the proof of the Proposition, we just note that if $p \in D_{A-S}$ is arbitrary and if G is P-generic and contains p, then $G \cap B$ is not empty for each $B \in \mathcal{B}$. $\square$

Here is a much more subtle and difficult preservation argument. First we define what is often called Hechler forcing. An element of P_H is a pair $p = (s, f)$, where, for some n, $s \in {}^n\omega$ and $f \in {}^\omega\omega$. The ordering on P_H is given by $(s, f) < (t, g)$ if $s \supset t$, $f(n) \geq g(n)$ for all n, and $s(n) > g(n)$ for all $n \in \text{dom}(s) - \text{dom}(t)$. If G is P_H-generic, then $\bigcup\{s : (\exists f)\ (s, f) \in G\}$ is a function in ω^ω which dominates (mod finite) the family $\omega^\omega \cap V$.

Lemma 7.20 (Baumgartner). *If $\{x_\xi : \xi \in \lambda\}$ (λ regular) is a tower in $[\omega]^\omega$ (ordered mod finite), and if G is P_H-generic, then $\{x_\xi : \xi \in \lambda\}$ remains a tower in $V[G]$.*

Proof. Let $\dot{x}$ be a name of a subset of ω and let $\dot{e}$ be the name of the enumerating function for $\dot{x}$, i.e.

$$1 \Vdash \{\dot{e}(k) : k \in \omega\} \text{ enumerates } \dot{x} \text{ in increasing order .}$$

Let SEQ denote the set $\bigcup_n {}^n\omega$ and, for each $k \in \omega$, define the following well-founded ranking on SEQ .

(1) $D^k_0 = \{s \in \text{SEQ} : (\exists f \in {}^\omega\omega)\ (\exists j)\ \langle s, f\rangle \Vdash \dot{e}(k) = j\}$;
(2) $D^k_{\alpha+1} = \{s \in \text{SEQ} : (\exists n \geq |s|)(\forall r \in \omega)(\exists t \in D^k_\alpha \cap {}^n\omega)\ t \supset s \text{ and } t(i) \geq r \text{ for } i \geq |s|\}$;
(3) $D^k_\alpha = \bigcup_{\beta<\alpha} D^k_\beta$ when α is a limit ordinal.

It follows that $D^k_\alpha \subset D^k_\beta$ for $\alpha < \beta$.

Claim 1. For all k, SEQ $= \bigcup_\alpha D^k_\alpha$, i.e. every node in SEQ gets a rank.

Assume otherwise and let s be a member of SEQ which is not a member of $D^k = \bigcup_\alpha D^k_\alpha$. Call $t \supset s$ a minimal extension if t is minimal with respect to being a member of D^k. For each $n \geq |s|$, there are only finitely many minimal $t \in {}^n\omega$. Indeed, otherwise there would be some maximal j so that $\{t(j) : t \in {}^n\omega$ is a minimal extension of $s\}$ is bounded. Then some $t' \in {}^{j+1}\omega$ would be contained in infinitely many of these t and it would follow that t' is in D^k – contradicting minimality. So we can define an $f \in {}^\omega\omega$ so that, for each $n \geq |s|$, $f(n) > t(n)$ for each mimimal extension $t \in {}^{n+1}\omega$. But now, there is $\langle t, g\rangle < \langle s, f\rangle$ such that there is a $j \in \omega$ so that $\langle t, g\rangle \Vdash \dot{e}(k) = j$. So t is in D^k_0, hence there must be an n so that $t \restriction n+1$ is a minimal extension of s. But this means that $f(n) > t(n)$ contradicting the definition of $\langle t, g\rangle < \langle s, f\rangle$.

Now fix a countable $M \prec H(\omega_2)$ so that $\dot{e} \in M$. Also find $\xi < \lambda$ so that $a - x_\xi$ is infinite for each $a \in M \cap [\omega]^\omega$. We prove that

$$\Vdash \text{range } \dot{e} - x_\xi \text{ is infinite .}$$

Otherwise, fix $\langle s, f\rangle$ which forces that $\dot{e}(k) \in x_\xi$ for all k greater than some fixed n. Now make the following definition:

$Z_k(t) = \{j : (\forall g \in {}^\omega\omega)\ \langle t, g\rangle \not\Vdash \dot{e}(k) \neq j\}$, i.e. $Z_k(t)$ is the set of j such that, regardless of g, $\langle t, g\rangle$ has an extension which forces that $\dot{e}(k) = j$.

Surprisingly, for all $t \in$ SEQ and $k > n$, $Z_k(t)$ is not empty. This is proven by induction on α, where $t \in D^k_{\alpha+1}$. Since $t \in D^k_{\alpha+1}$, there is an m such that there are, for each $r \in \omega$, $t_r \in D^k_\alpha \cap {}^m\omega$ extending t so that $t_r(i) \geq r$ for $m > i \geq |t|$. Let $Z_r = \bigcup_r Z_k(t_r)$. First of all, Z_r is finite. For otherwise, there is a $j \in Z_k(t_r) - x_\xi$, but notice that $\langle t_r, f\rangle$ forces that $\dot{e}(k)$ is in x_ξ and so not equal to j, contradicting that $j \in Z_k(t_r)$. Therefore there are $j \in Z_r$ which are in $Z_k(t_r)$ for infinitely many r. To see that $j \in Z_k(t)$, fix any g. Let r be one of these infinitely many r which is larger than $g(i)$ for each $i < m$ and observe that $\langle t_r, g\rangle < \langle t, g\rangle$. Since $\langle t_r, g\rangle$ has an extension forcing $\dot{e}(k)$ to equal j, so does $\langle t, g\rangle$.

Finally we finish the proof, by observing that since $Z_k(s) - k$ is not empty for each k (since $\dot{e}(k) \geq k$), it follows that there is a $j \in \bigcup_k Z_k(s) - x_\xi$. This contradicts that $\langle s, f\rangle \Vdash \dot{e}(k)$ cannot equal j since $j \notin x_\xi$. $\square$

There are two major basic notions remaining. Iterated forcing and the factoring of forcing. These are really two sides of the same coin. One thinks of iterated forcing as the process of extending the model in many stages (finite or infinite). However it turns out that it is really a process of defining new posets and then forcing with these. More fundamental is the process of factoring a forcing, very roughly, if G' is a subset of G, we can consider $V[G']$ and then work on how $V[G]$ is obtained from $V[G']$. Most of the time G' is equal $G \cap P'$ for some poset $P' \subset P$ (which is **completely embedded**).

Definition 7.21. A poset P is a **complete** subposet of Q (denoted $P <\circ Q$) if $<_P = <_Q \cap P \times P$, $(p \perp_P q) \leftrightarrow (p \perp_Q q)$, and maximal antichains of P are again maximal in Q. We would say that P can be completely embedded in Q if there is an isomorphism j from P into Q such $j(P)$ is a complete subposet of Q.

For example, if P and Q are posets with 1 then both $P \times \{1_Q\}$ and $\{1_P\} \times Q$ are complete subposets of $P \times Q$. $P \times Q$ is called the product forcing and one usually identifies $P \times \{1_Q\}$ with P and $\{1_P\} \times Q$ with Q.

If a poset P is densely embedded in Q then P is completely embedded. Two posets are *forcing equivalent* if there is a single poset in which they can both be densely embedded. In fact, given a poset P, there is a unique complete Boolean algebra B such that P is completely embedded in $B - \{0\}$. Therefore two posets are forcing equivalent if these completions are isomorphic. One might guess why they are called forcing equivalent – they give the same generic extensions. If P is dense in Q and if $G \subset Q$ is Q-generic, then certainly $G \cap P$ is P-generic. Similarly, if $G \subset P$ is P-generic, then $G_Q = \{q : (\exists p \in G)\ p < q\}$ is Q-generic. In addition, e.g. in the latter case, $V[G]$ is equal to $V[G_Q]$ (where in the first case you use P-names and the second Q-names).

Exercise: $\mathrm{Fn}(\omega, 2)$ is forcing equivalent to any countable atomless (separative) poset.

Lemma 7.22. *A generic filter for $P \times Q$ has the form $G \times H$ where G is V-generic for P and H is $V[G]$-generic for Q. In addition, given generic G and Q-generic H over $V[G]$, H is certainly V-generic for Q, but also G is $V[H]$-generic for P.*

Proof. Assume that $K \subset P \times Q$ is a generic filter and let $G = \{p \in P : (\exists q \in Q)\ (p, q) \in K\}$. We first show that G is P-generic for V. Indeed, if $D \subset P$ is dense, then $D' = D \times Q$ is easily seen to be dense in $P \times Q$. Since $K \cap D'$ is not empty it follows that $G \cap D$ is not empty. Similarly, let $H = \{q \in Q : (\exists p \in P)\ (p, q) \in K\}$. We will now show that H is Q-generic for $V[G]$. Let $\tau \subset P \times Q$ be such that $\tau_G = \{q \in Q : (\exists p \in G)\ (p, q) \in \tau\}$ is a dense subset of Q. Fix $p \in G$ such that $p \Vdash \tau$ is a dense subset of Q. We first show, in V, that τ (as a subset $P \times Q$) is predense below (p, q) for each $q \in Q$. Indeed, let $(p', q') < (p, q)$ and note that p' also forces that τ (as a P-name now) is a dense subset of Q. Therefore, by the definition of τ_G, there is a p'' compatible with p' and a q'' such that $(p'', q'') \in \tau$ and $p'' \Vdash q'' \not\perp q'$. The ordering on Q is the same in $V[G]$ as in V, hence q'' and q' are compatible elements of Q. Since $p' < p$, p'' is also compatible with p. Therefore (p', q') is compatible with (p'', q'').

Now suppose that G is P-generic over V and H is Q-generic over $V[G]$. We want to show that $G \times H$ is $P \times Q$-generic over V (it is easily seen to be a filter). Let $D \subset P \times Q$ be dense and define $D^* = \{q : (\exists p \in G)\ (p, q) \in D\}$. Clearly D^* is a member of $V[G]$ and if $D^* \cap H$ is not empty, then $D^* \cap (G \times H)$ is not empty. Therefore we must show that D^* is dense in Q. Fix any $q \in Q$ and define, in V, $D_q = \{p \in P : (\exists q' \in Q)\ (q' \leq q \text{ and } (p, q') \in D)\}$. Since D is dense in $P \times Q$, D_q is dense in P. Choose any $p \in G \cap D$; the $q' \in Q$ which witnesses that $p \in D_q$ is a member of D^* which completes the proof that D^* is dense.

To prove the last statement, we use the standard trick that product forcing can be done in either order. Indeed, suppose that G is P-generic and H is Q-generic over $V[G]$. Then by the first part of the Lemma, $H \times G$ is $Q \times P$-generic over V and therefore G is $V[H]$-generic for P. $\square$

Corollary 7.23. *If P and Q are posets then forcing with $P \times Q$ is the same as forcing first with P and then, in the resulting extension, forcing with Q. Of course $P \times Q$ is also forcing equivalent to $Q \times P$.*

Proof. Let $G\times H$ be $P\times Q$-generic over V. We must show that $V[G\times H] = V[G][H]$. However both $V[G\times H] \subset V[G][H]$ and $V[G][H] \subset V[G\times H]$ follow from the previous Lemma and the minimality of forcing extensions. $\square$

Let us illustrate with some examples.

Proposition 7.24. *If G is a generic filter for $Fn(\omega_2, 2)$, then in $V[G]$ every ultrafilter on ω has character at least ω_2.*

Proof. By Lemma 7.5, $\mathrm{Fn}(\omega_2, 2)$ is ccc, hence ω_2^V is equal to ω_2 in the extension. Assume that $\{\tau_\xi : \xi < \omega_1\}$ is a family of $\mathrm{Fn}(\omega_2, 2)$-names of subsets of ω. Assume that $p \Vdash \{\tau_\xi : \xi < \omega_1\}$ is an ultrafilter. A standard technique is to introduce equivalent small or canonical names. For each ξ and n, fix an antichain $A(\xi, n) \subset \mathrm{Fn}(\omega_2, 2)$ (possibly empty) which is maximal with respect to the property that $q \Vdash n \in \tau_\xi$ for each $q \in A(\xi, n)$. Define

$$\sigma_\xi = \{(q, n) : q \in A(\xi, n)\} \ .$$

Claim: $p \Vdash \sigma_\xi = \tau_\xi$. If this were not the case, then there would be some $q < p$ and an n such that $q \Vdash n \in \tau_\xi$ and $q \Vdash n \notin \sigma_\xi$ (or vice versa). Since $q \Vdash n \notin \sigma_\xi$, q must be incompatible with each element of $A(\xi, n)$. However, by the maximality condition on $A(\xi, n)$ and the fact that $q \Vdash n \in \tau_\xi$, there must be some member of $A(\xi, n)$ which is compatible with q. The "vice-versa" case is even easier.

The advantage of the names σ_ξ is that they are small, in fact, they are each countable. Therefore there is an $\aleph_1$-sized set $I \subset \omega_2$ such that $A(\xi, n) \subset \mathrm{Fn}(I, 2)$ for each ξ, n. We may assume, in addition, that $p \in \mathrm{Fn}(I, 2)$. Now the technique is to note that $\mathrm{Fn}(\omega_2, 2)$ is forcing isomorphic to $\mathrm{Fn}(I, 2)\times \mathrm{Fn}(\omega_2 - I, 2)$ (which is left as an exercise). Let $G\times H \subset \mathrm{Fn}(I, 2)\times \mathrm{Fn}(\omega_2 - I, 2)$ be generic with $p \in G$. By Proposition 7.23, we know that $V[G\times H] = (V[G])[H]$ where H is $\mathrm{Fn}(\omega_2 - I, 2)$-generic over $V[G]$. Here's what happens now. Clearly $\{(\sigma_\xi)_G : \xi < \omega_1\}$ is a member of $V[G]$. We show that no filter on ω in $V[G]$ is still an ultrafilter in $V[G][H]$ and this contradicts the assertion that p forces $\{(\tau_\xi)_{(G\times H)} : \xi < \omega_1\}$ is an ultrafilter. The key step is getting the small filter into the inner model $V[G]$ and then checking how the further forcing affects it.

We leave as an exercise that Lemma 7.8 finishes the proof. $\square$

Exercise: Use the technique of small names to show that if G is $\mathrm{Fn}(\omega_2, 2)$-generic over V (a model of CH), then $\mathfrak{c} = \omega_2$ in $V[G]$.

We illustrate the factoring idea further with the following technical result which also begins to introduce the concept of iterated forcing. It is important to realize that $\mathrm{Fn}(I, J, \kappa)$ is intended as a *defined notion* which is absolute if $\kappa = \omega$ but need not be otherwise. That is, $\mathrm{Fn}(\omega_1, 2, \omega_1)$ in $V[G]$ need not agree with $\mathrm{Fn}(\omega_1, 2, \omega_1)^V$. In this proof we also introduce a key idea of Silver.

Proposition 7.25. *Assume that V is a model of GCH and let G be $Fn(\omega_1, 2)$-generic over V. Next, suppose that H is $Fn(\omega_3, 2, \omega_1)$-generic over $V[G]$. Finally let K be $Fn(\omega_1, 2)\times Fn(\omega_3, 2, \omega_1)$-generic over V. Then each of $V[G][H]$ and $V[K]$ are models of $\omega_1 = \omega_1^V$, CH, and $2^{\omega_1} = \omega_3 = \omega_3^V$ but they are not the same models.*

Proof. Without loss of generality, K traces the same generic filter on $\mathrm{Fn}(\omega_1, 2)$ as that given by G, i.e. $G \times \{1\} \subset K$. However despite our discussion in Lemma 7.23, we cannot assert that K has the form $G \times H'$ where H' is $\mathrm{Fn}(\omega_3, 2, \omega_1)$-generic over $V[G]$. This is because, as mentioned above, this notation refers to a defined notion rather than simply a set (e.g. $[0, 1]$ always refers to the unit interval, whether in V or $V[G]$). We first have to worry about the values of ω_3 and ω_1, but these are unchanged in this case. Therefore we can say that K has the form $G \times H'$ where H' is $(\mathrm{Fn}(\omega_3, 2, \omega_1) \cap V)$-generic over $V[G]$. The two sets $\mathrm{Fn}(\omega_3, 2, \omega_1)$ and $\mathrm{Fn}(\omega_3, 2, \omega_1) \cap V$ are certainly different. If G had been $\mathrm{Fn}(\omega_4, 2)$-generic they would not even have the same cardinality; if it had even been $\mathrm{Fn}(\omega_2, 2)$-generic only one would have had the ω_2-cc.

Let us first discuss the model $V[K]$ – why, for example, does it preserve ω_1? We can consider $V[H'][G]$ or $V[G][H']$ – this is easy the first way and educational the second.

By Lemma 7.5, $\mathrm{Fn}(\omega_3, 2, \omega_1)$ satisfies the $\aleph_2$-cc in both the models V and $V[G]$ (in addition, $\mathrm{Fn}(\omega_3, 2, \omega_1) \cap V$ is $\aleph_2$-cc in $V[G]$ since it is a subposet of $\mathrm{Fn}(\omega_3, 2, \omega_1)$). Therefore by Lemma 7.14, it preserves all cardinals greater than ω_1. Exactly analogous to the case of $\mathrm{Fn}(\omega_2, 2)$, one can check that it adds ω_3 new subsets of ω_1. By Lemma 7.16, it adds no new countable sets, hence CH is preserved and so is ω_1. The passage from $V[H']$ to $V[H'][G]$ has been discussed so we skip the details.

On the other hand, $V[G][H]$ yields the same extension so, in $V[G]$, $\mathrm{Fn}(\omega_3, 2, \omega_1) \cap V$ preserves cardinals and adds no new countable sets of ordinals etc. and yet it is not $\aleph_1$-closed. It is what is called $\aleph_1$-distributive or $\aleph_1$-Baire, which we shall not get into here. Since the poset $\mathrm{Fn}(\omega_2, 2)$ is $\aleph_1$-cc and $\mathrm{Fn}(\omega_3, 2, \omega_1)$ is $\aleph_1$-closed (in V), one can show that every maximal antichain of $\mathrm{Fn}(\omega_3, 2, \omega_1) \cap V$ of $V[G]$ is refined by a maximal antichain from V. One can then show directly that $\mathrm{Fn}(\omega_3, 2, \omega_1) \cap V$ does not add new countable sets of ordinals. This is due to Easton.

Let us now turn to $V[G][H]$. In this case we cannot turn this around and take $V[H][G]$ since H is not a subset of a poset in V. However the cardinal arithmetic is quite easy to check. We have already covered how to analyze the cardinal arithmetic in $V[G]$ and one can check that it gives us a model of CH. Therefore in passing from $V[G]$ to $V[G][H]$ we are forcing with an $\aleph_1$-closed $\aleph_2$-cc poset – these preserve cardinals and do not add new countable sets of ordinals.

Finally we want to show that the two models are not the same. In V, let $T = {}^{<\omega_1}2$ be the usual complete binary tree of height ω_1. Clearly in $V[H'][G]$, T acquires many new branches (indeed ω_3 of them). We will show that in the model $V[G][H]$, T gets no new branches. This is a modification of a result of Silver which is due to Mitchell ([9]). They are both very fundamental ideas of set theory. We put these results in their own respective lemmas. □

Lemma 7.26 (Silver). *Assume that T is a tree whose levels have cardinality less than $\mathfrak{c}$ and that P is an $\aleph_1$-closed poset. Then P adds no new branches to T.*

Proof. Let $\dot{b}$ be the name of a branch of T and choose λ minimal so that there is a $p \in P$ such that $p \Vdash \dot{b} \cap \check{T}_\lambda$ is empty. Therefore, for each $q < p$ and $\alpha < \lambda$ there is an $r < q$ and a $t \in T_\alpha$ such that $r \Vdash \check{t} \in \dot{b}$. Assume there is a $q < p$ such that $b_q = \{t \in T : (\exists r < q)\ r \Vdash \check{t} \in \dot{b}\}$ forms a chain. Then $q \Vdash \dot{b} = \check{b}_q$ and $\dot{b}$ is not a new branch of T. Therefore we may assume that b_q is never a chain. Let $p_0 = p$ and inductively define ordinals $\alpha_n < \lambda$ and conditions $\{p_s : s \in {}^n2\}$ as follows. Let

α_0 be such that there are incomparable $t_0, t_1 \in b_{p_0} \cap T_{\alpha_0}$ and fix $p_{\langle 0\rangle}, p_{\langle 1\rangle}$ in P so that $p_{\langle i\rangle} \Vdash t_i \in \dot{b}$. Having defined p_s for each $s \in {}^{n-1}2$ choose α_n large enough so that for each $s \in {}^{n-1}2$, there are incomparable $t_{s\frown 0}$ and $t_{s\frown 1}$ in $b_{p_s} \cap T_{\alpha_n}$ (one can easily check that if there are incomparable t, t' in b_{p_s} then there are incomparable members of $b_{p_s} \cap T_\beta$ for all $\beta < \lambda$ above the maximum of the levels of t, t'). For each $s \in {}^{n-1}2$ and $i < 2$, fix $p_{s\frown i} < p_s$ so that $p_{s\frown i} \Vdash t_{s\frown i} \in \dot{b}$ (where $t_{s\frown i}$ was chosen above). Now let α be the supremum of the α_n for $n \in \omega$. Using the fact that P is countably closed, choose, for $f \in 2^\omega$, $p_f \in P$ so that $p_f < p_{f\restriction n}$ for each n. Note that p_f forces that $\dot{b}$ contains the chain $\{t_{f\restriction n} : n \in \omega\}$. If α were equal to λ then each p_f would have completely determined $\dot{b}$, contrary to our assumption on $\dot{b}$. Therefore $\alpha < \lambda$, and for each $f \in 2^\omega$ we may assume that p_f has been extended so that there is $t_f \in T_\alpha$ so that $p_f \Vdash t_f \in \dot{b}$. However, if $f \neq g$, then $t_f \neq t_g$ for if n is minimal such that $f \restriction n \neq g \restriction n$, then $t_{f\restriction n} < t_f$ was chosen to be incompatible with $t_{g\restriction n} < t_g$. This, of course, contradicts that $|T_\alpha| < 2^\omega$. □

Mitchell's argument is more subtle. It can actually be improved to include the case where $|I| = \omega$ but the argument is more complicated.

Lemma 7.27 (Mitchell). *Assume that T is a tree in V and that G is $Fn(I,2)$-generic over V with $|I| > \omega$. If P is a poset in $V[G]$ which is countably closed and H is P-generic over $V[G]$, then every branch of T which is in $V[G][H]$ is already in $V[G]$.*

Proof. Work in $V[G]$ and carry out the exact same construction of the sequence $\{\alpha_n : n \in \omega\}$ together with the sets $\{p_s : s \in 2^{<\omega}\}$ and $\{t_s : s \in 2^{<\omega}\}$ and let α be the supremum of the α_n. In addition we can choose the p_f's for $f \in 2^\omega$ as above. However we do not assume that $|T_\alpha| < 2^\omega$ so at this point we take a different tact. We will choose an $f \in 2^\omega$ which is "too" generic for there to be a $t_f \in T_\alpha$ which is below $t_{f\restriction n}$ for each n. Here's how we choose f. This argument has taken place in the model $V[G]$, hence we can fix $\mathrm{Fn}(I,2)$-names for the α_n as well as the p_s and t_s for all $s \in 2^{<\omega}$. These are countably many names. We repeat the standard trick which shows that we can assume that the names for the t_s are countable. Fix $s \in 2^{<\omega}$, and let $T(s)$ denote the set of all $t \in T$ such that there is a $r_t \in \mathrm{Fn}(I,2)$ such that $r_t \Vdash t = t_s$. Clearly if $t \neq t'$ are both in $T(s)$ then r_t is incompatible with $r_{t'}$. Therefore $T(s)$ is countable. In addition, for each $t \in T(s)$ there is an antichain, $A(s,t) \subset \mathrm{Fn}(I,2)$, maximal with respect to the property that $r \Vdash t = t_s$ for each $r \in A(s,t)$. It is easy to check that the name $\bigcup\{(r,t) : r \in A(s,t) \text{ and } t \in T(s)\}$ is a name for t_s. Now let I' be a countable subset of I so that $A(s,t) \subset \mathrm{Fn}(I',2)$ for each $s \in 2^{<\omega}$ and $t \in T(s)$. Next, let $J = \{j_n : n \in \omega\}$ be any infinite subset of $I - I'$ and let $f(n) = (\bigcup G)(j_n)$ for each n. We can view $G \cap \mathrm{Fn}(J,2)$ as being generic over $V[G \cap \mathrm{Fn}(I',2)]$ (by Lemma 7.23) and so, by Lemma 7.7, f will not equal y for any $y \in 2^\omega \cap V[G \cap \mathrm{Fn}(I',2)]$. Well then, working in $V[G \cap \mathrm{Fn}(I',2)]$, for each $t \in T_\alpha$ let $y_t = \bigcup\{s \in 2^{<\omega} : t_s < t\}$. We know that f will be distinct from y_t for each $t \in T_\alpha$ (some may not be in 2^ω but f will still not equal them). Return to $V[G]$ and consider p_f, which we may assume, for some $t \in T_\alpha$, forces that $t \in \dot{b}$. But then, as before, it follows that $t_{f\restriction n} < t$ for each n. By the construction of the t_s, we also have that $t_s \not< t$ for each s which is incomparable with f. Therefore y_t is equal to f – a contradiction. □

We finish this section with two new results which are examples of non-preservation results in topology. These results give negative answers to questions posed in [4].

Theorem 7.28. *It is possible to lower the density of a space without collapsing cardinals*

Proof. Let κ be a measurable cardinal and let $\mathcal{U}$ be a normal measure on κ. We will use the product forcing $\mathrm{Pr}\times\mathrm{Fn}(\kappa,2)$ where Pr is Prikry forcing using the measure $\mathcal{U}$. Let $G\times H$ be $\mathrm{Pr}\times\mathrm{Fn}(\kappa,2)$-generic over V and work in the model $V[H]$. In this model we define a space X whose density will be κ but which will be separable in $V[G\times H]$. Fix a bijection $f:\kappa\times\kappa\to\kappa$ and define $x_\alpha(\gamma)$ to be equal $H(f(\alpha,\gamma))$ if $\gamma<\alpha$ and to be 0 for $\gamma\geq\alpha$. It is obvious that the density of $X=\{x_\alpha:\alpha\in\kappa\}$ (as a subspace of 2^κ) is no more than the cofinality of κ.

Now we prove that X is separable in $V[G\times H]$. Let $\{\alpha_n:n\in\omega\}$ denote the Prikry sequence added by Pr. Recall that conditions in Pr are of the form $(<\alpha_0,\ldots,\alpha_n>,U)$ where $U\in\mathcal{U}$ and $\alpha_0<\cdots<\alpha_n<\kappa$. The ordering on Pr is end-extension of the α_i's with the restriction that the new α_i's must come from U and the usual ordering on the members of $\mathcal{U}$. We will show that $\{x_{\alpha_n}:n\in\omega\}$ is dense in 2^κ, hence in X. Let $F\subset\kappa$ be finite and let $\varphi\in 2^F$ and let $((<\alpha_0,\ldots,\alpha_n>,U),p)$ be an arbitrary member of $\mathrm{Pr}\times\mathrm{Fn}(\kappa,2)$. It suffices to find an extension $((<\alpha_0,\ldots,\alpha_n,\alpha>,U'),p')$ which forces that $\varphi\subset x_\alpha$. Choose any $U'\subset U$ ($U'\in\mathcal{U}$) such that $\mathrm{dom}(p)\cap U'$ is empty. Fix any $\alpha\in U'$ and define $p'\supset p$ so that for each $\gamma\in F$, $p'(f(\alpha,\gamma))=\varphi(\gamma)$. It should be clear that p' forces that $x_\alpha\supset\varphi$. □

Theorem 7.29. *It is possible to make a regular non-Lindelof space Lindelöf with cardinal preserving forcing. Similarly, the non-Lindelöf property need not be preserved for first countable Hausdorff spaces.*

Proof. As in the previous result let $G\times H$ be $\mathrm{Pr}\times\mathrm{Fn}(\kappa,2)$-generic, where κ is a measurable cardinal and $\mathcal{U}$ is a normal measure on κ. Also fix $f:\kappa\times\kappa\to\kappa$ as above. For each $\alpha<\kappa$, define $x_\alpha\in 2^\kappa$ by $x_\alpha(\gamma)=0$ if $\gamma+\omega<\alpha$ and otherwise, define $x_\alpha(\gamma)=H(f(\alpha,\gamma))$. Clearly, in $V[H]$, $X=\{x_\alpha:\alpha<\kappa\}$ is not Lindelof, but we will show that it is Lindelof in $V[G\times H]$.

Any open cover of X can be refined by a cover by basic open sets so fix a sequence of $\mathrm{Pr}\times$ $\mathrm{Fn}(\kappa,2)$-names, $\{\dot F_\alpha:\alpha<\kappa\}$ of finite subsets of κ. Letting, for $x\in 2^\kappa$, $[x\restriction F]$ denote the usual basic clopen subset of 2^κ consisting of all those y which agree with x on F, we have that $1\Vdash\{[x_\alpha\restriction\dot F_\alpha]:\alpha<\kappa\}$ is a typical cover of X. It suffices to show that $1\Vdash\{[x_\alpha\restriction\dot F_\alpha]:\alpha<\kappa\}$ has a countable subcover (where the forcing relation refers to $\mathrm{Pr}\times$ $\mathrm{Fn}(\kappa,2)$). For each α, let $\dot s_\alpha$ denote the finite partial function $x_\alpha\restriction\dot F_\alpha$ and let $[\dot s_\alpha]$ denote the corresponding basic clopen set.

Working in V, recursively fix countable elementary submodels M_F for $F\in\kappa^{<\omega}$ so that s_α, M_F are both in $M_{F^\frown\alpha}$. We will show that $V[G\times H]\models\mathcal{S}=\{[s_\alpha]:\alpha\in\cup_n M_{<\alpha_0,\ldots,\alpha_n>}\}$ is a cover of X.

Assume that $(t,U)\in G$ and $p\in\mathrm{Fn}(\kappa,2)$ are such that $((t,U),p)\Vdash x_\beta\notin\bigcup\mathcal{S}$. By extending $t\in\kappa^{<\omega}$ we may assume that $\beta<\max t$. Let $\lambda=\min M_t-\beta$ (of course we assume that $\beta\notin M_t$). We will need to apply the Prikry Lemma. First choose k large enough so that $f(\{\lambda+k\}\times\kappa)\cap\mathrm{dom}\,(p)=\emptyset$ and let J be such that

$f(\{\beta\} \times J) = \mathrm{dom}\,(p) \cap f(\{\beta\} \times \kappa)$ (i.e. p says nothing about $x_{\lambda+k}$ and decides x_β exactly on J). Extend p to p_0 so that p_0 forces that $x_{\lambda+k}$ and x_β agree on J (keeping $\mathrm{dom}\,(p_0) \cap f(\{\beta\} \times \kappa) = \{\beta\} \times J$). Recall that G is Pr-generic and consider $\dot{s}_{\lambda+k}$; we define a $\mathrm{Fn}(\kappa, 2)$-name $\dot{s}$ in $V[G]$ such that $(\dot{s})_H = (\dot{s}_{\lambda+k})_{G\times H}$. For each $\alpha \in \kappa$ and $e \in 2$, let $A^e_\alpha \subset \mathrm{Fn}(\kappa, 2)$ be an antichain which is maximal with respect to the property that, for each $q \in A_\alpha$, there is $(t', U') \in G$ such that $(\,(t', U'),\ q) \Vdash \dot{s}_{\lambda+k}(\check{\alpha}) = \check{e}$. Define $\dot{s} = \{(\,(\alpha, e)\,, q) : \alpha < \kappa, e \in 2,$ and $q \in A^e_\alpha\}$. It is easy to check that $\dot{s}$ is as desired. Now, still working in $V[G]$, $\dot{s}$ is a $\mathrm{Fn}(\kappa, 2)$-name of a finite partial function on κ, therefore, by Lemma 7.19, there is a countable set $E \subset \kappa$ such that $A^0_\alpha \cup A^1_\alpha = \emptyset$ for all $\alpha \notin E$. Since each A_α is at most countable, there is a countable set $Y_0 \subset \kappa$ such that $q \in \mathrm{Fn}(Y_0, 2)$ for all $q \in \bigcup_{\alpha\in E} A^0_\alpha \cup A^1_\alpha$. Hence there is a countable set $Y \supset E$ such that $Y_0 \cap f(\{\alpha\} \times \kappa) = \emptyset$ for all $\alpha \notin Y$. Let $\dot{Y} \in V$ be a Pr-name for Y; note that since $\dot{Y}$ is definable from $\dot{s}_{\lambda+k}$, we may assume that $\dot{Y}$ is in M_t. By the Prikry property, there are $\bar{U}$ and $\bar{Y}$ in M_t such that $(t, \bar{U}) \Vdash \dot{Y} \cap \check{\lambda} = \bar{Y}$. Since $\bar{Y} \in M_t$ and $M_t \models \bar{Y} \subset \lambda$ and λ is the minimal element of $M_t - \beta$, it follows that $\bar{Y} \subset \beta$. By the definition of $\dot{Y}$, we $(\,(t', U'),\ p') < (\,(t, \bar{U}),\ p_0)$ such that there is an s such that $(\,(t', U'),\ p') \Vdash \dot{s} = \check{s}$ and $\mathrm{dom}\,(p' - p_0) \cap f(\{\beta\} \times \kappa) = \emptyset$. Therefore we can extend p' to some $\bar{p}$ such that $\bar{p} \Vdash \check{s} \subset x_\beta$ (note that $\mathrm{dom}\,(s) \cap \lambda \subset \bar{Y}$, hence $0 = s \restriction \lambda = s \restriction \beta \subset x_\beta \restriction \beta$).

Now we turn to the first countable example. For each $\alpha < \kappa$, let $y_\alpha = x_\alpha \restriction \omega$, where $x_\alpha \in V[G]$ is defined as above. A neighbourhood base for y_α will be $\{[y_\alpha \restriction n] \cap \{y_\beta : \beta \le \alpha\} : n \in \omega\}$, where $[y \restriction n] = \{z \in 2^\omega : y \restriction n \subset z\}$. One easily checks that in $V[G]$, the space $Y = \{y_\alpha : \alpha < \kappa\}$ with the above topology is Hausdorff and not Lindelöf (using the fact that κ is regular uncountable). In the model $V[G][H]$, κ has countable cofinality of course and we will just use the fact that $\{y_\alpha : \alpha < \kappa\}$ as a subspace of 2^ω, is a Luzin set in $V[G]$. We leave as exercises this fact together with the fact that, for each $\alpha < \kappa$, $\{y_\beta : \alpha \le \beta < \alpha + \omega\}$ is a dense subset of 2^ω. Now suppose, in $V[G][H]$, that $\mathcal{U}$ is an open cover of Y (in the new topology). For each α let $n_\alpha < \omega$ be chosen so that $U_\alpha = [y_\alpha \restriction n_\alpha] \cap \{y_\beta : \beta \le \alpha\}$ is contained in some $U \in \mathcal{U}$. Also let γ_n be a cofinal sequence in κ. For each n,

$$W_n = \bigcup\{[y_\beta \restriction n_\beta] : \gamma_n \le \beta < \gamma_n + \omega\}$$

is a dense open subset of 2^ω which is a member of $V[G]$ (since Prikry forcing does not add any new small sets). Since Y is a Luzin set in 2^ω, $Y - W_n$ is countable. Therefore,

$$\{y_\xi : \xi < \gamma_n\} - \bigcup\{U_\beta : \gamma_n \le \beta < \gamma_n + \omega\}$$

is countable. It follows, then that $\mathcal{U}$ has a countable subcover. $\square$

8. Iterations – Case Studies

Suppose that α is any ordinal, $\mathcal{I} \subset \mathcal{P}(\alpha)$ is a (not necessarily proper) ideal on α containing $[\alpha]^{<\omega}$. An α-staged iterated forcing sequence with supports in $\mathcal{I}$ is an object of the form P_α together with a sequence

$$\langle P_\xi, \dot{Q}_\xi : \xi < \alpha\rangle$$

which satisfies the following conditions:

(1) elements of P_ξ ($\xi \le \alpha$) are sequences of length ξ;

(2) each P_ξ ($\xi \le \alpha$) is a poset ordered as below;
(3) $\dot{Q}_\xi$ is a P_ξ-name of a poset;
(4) if $\xi < \eta \le \alpha$ and $p \in P_\eta$, then $p \restriction \xi \in P_\xi$;
(5) for each $\xi \le \alpha$, $p \in P_\xi$ and $\beta < \xi$, $p \restriction \beta \Vdash p(\beta) \in \dot{Q}_\xi$;
(6) the support of $p \in P_\xi$, i.e. the set $\{\beta \in \xi : p(\beta) \ne 1_\beta\}$, is a member of $\mathcal{I}$;
(7) if $p, p' \in P_{\xi+1}$, then $p \le p'$ iff $p \restriction \xi \le p' \restriction \xi$ and $p \restriction \xi \Vdash p(\xi) \le p'(\xi)$ (in $\dot{Q}_\xi$);
(8) if $\xi \le \alpha$ is a limit ordinal then $p \le p'$ iff $p \restriction \eta \le p' \restriction \eta$ for all $\eta < \xi$.

The above definition actually defines a proper class but the easy way out of this is to restrict $p(\beta)$ to be a P_β-name which is an element of $H(\theta)$ for some suitably large θ.

Theorem 8.1. *If $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \kappa\rangle$ is a finite support iteration such that, for each $\alpha < \kappa$, $\Vdash_{P_\alpha} \dot{Q}_\alpha$ is ccc, then P_κ is ccc.*

The situation for chain conditions in other iterations is not so simple. However the following result is frequently sufficient.

Theorem 8.2 ([10], P96). *Suppose that $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \kappa\rangle$ is a countable support iteration such that $\Vdash_{P_\alpha} \dot{Q}_\alpha$ is proper and has cardinality less than κ, and suppose that κ is regular such that $\mu^\omega < \kappa$ for all $\mu < \kappa$, then P_κ satisfies the κ-cc and $\Vdash_{P_\alpha} 2^\omega < \kappa$ for all $\alpha < \kappa$.*

8.1. Finite support. Let κ be an uncountable regular cardinal such that $\lambda^\omega < \kappa$ for all $\lambda < \kappa$ and, for $\alpha \le \kappa$, let P_α be the finite support iteration where we take $\dot{Q}_\beta$ to be the P_β-name of P_H as defined in Lemma 7.20. If G is P_κ-generic over V, let us see what we get in the model $V[G]$ and what we do not get.

Proposition 8.3. *MA(countable) holds, i.e. we add κ Cohen reals.*

Proof. Finite support iterations of (non-trivial) posets always add Cohen reals. For each $\alpha \in \kappa$, fix P_α-names $\{q(\alpha, n) : n \in \omega\}$ so that $\Vdash \{q(\alpha, n) : n \in \omega\}$ is a maximal antichain of Q_α. For each $s \in Fn(\kappa, \omega)$, define $p_s \in P_\kappa$ so that the support of p_s is the same as the domain of s and for each α in the domain of s, let $p_s(\alpha) = q(\alpha, s(\alpha))$. The collection $R = \{p_s : s \in Fn(\kappa, \omega)\}$ is completely embedded in P_κ.

We could also show that P_H itself adds a Cohen real. This can be seen as follows. If $G \subset P_H$ is the generic filter then let $g \in {}^\omega\omega$ be defined as the union of $\{t \in {}^{<\omega}\omega : (\exists f \in {}^\omega\omega)\ \langle t, f\rangle \in G\}$. Let $h(n) = 0$ if and only if $g(n)$ is even. We claim that h is a Cohen real, i.e. if A is a dense subset of $Fn(\omega, 2)$ (in V) and is an element of V then there is an $a \in A$ such that $a \subset h$. Assume, to the contrary, that there is no such a. By the Forcing Theorem, there is $\langle t, f\rangle \in P_H$ which forces there is no such A. However we first decode t so as to find $a' \in Fn(\omega, 2)$ so that $a'(n) = 0$ iff $t(n)$ is even and $a'(n) = 1$ iff $t(n)$ is odd (hence the domain of a' is the same as the domain of t). Now, since A is dense, there is an $a \in A$ so that $a' \subset a$. Extend t (staying above f) to some t' so that $t'(n)$ is even iff $a(n) = 0$ and the domain of t' is the same as the domain of a. Now if $\langle t', f\rangle \in G$, then $a \subset h$ and we have finished the proof. $\square$

Proposition 8.4. *P_κ does not fill towers. More generally if P_κ is a finite support iteration of posets which do not fill towers, then P_κ does not fill towers.*

Proof. By Lemma 7.20, P_H does not fill any towers. Fix a tower $\{x_\xi : \xi \in \lambda\} \in V$ (with λ regular) and proceed by induction on $\mu \leq \kappa$ to show that P_μ does not fill this tower. The successor case follows immediately from Lemma 7.20 so assume that μ is a limit. Suppose that $\dot{x}$ is a P_μ-name so that some p forces that $\dot{x} - x_\xi$ is finite for all $\xi < \lambda$. For each ξ, fix $p_\xi < p$ and n_ξ so that $p_\xi \Vdash \dot{x} - x_\xi \subset n_\xi$. Since $\{x_\xi : \xi < \lambda\}$ is a tower, λ will be uncountable. Therefore, there is an n so that $n = n_\xi$ for all $\xi \in L$ with L cofinal in λ. Now find an $I \subset L$ of cardinality λ so that $\{\operatorname{dom} p_\xi : \xi \in I\}$ forms a Δ-system and let $\delta < \mu$ be the maximum element of the root. By Lemma 7.19, there is a $P_{\delta+1}$-generic filter G so that $I' = \{\xi \in I : p_\xi \restriction \delta + 1 \in G\}$ has cardinality λ (note, if this set does not have cardinality λ, then it would suffice to put just one element in G). Now we work in $V[G]$. Consider $X = \{m : (\exists q \in P_\mu)\ q \restriction \delta + 1 \in G \text{ and } q \Vdash m \in \dot{x}\}$. This X is clearly an infinite set, so there is a $\xi \in I'$ such that $X - x_\zeta$ is infinite for all $\zeta > \xi$ (recall that $\{x_\xi : \xi \in \lambda\}$ is still a tower in $V[G]$ by the induction hypothesis). Fix an $m > n$ so that $m \in X - x_\zeta$ for infinitely many $\zeta \in I'$ (i.e. for each $\zeta \in I' - \xi$ there is an $m_\zeta \in X - x_\zeta$ and one such m will be used λ many times). Since $m \in X$, there is a q such that $q \restriction \delta + 1 \in G$ and $q \Vdash m \in \dot{x}$. The support of q is finite so there is a $\zeta \in I'$ such that $m \notin x_\zeta$ and the intersection of the support of p_ζ with the support of q is contained in $\delta + 1$. It follows that p_ζ and q are compatible. But this contradicts that p_ζ forces that $\dot{x} - x_\zeta$ is contained in n. □

Corollary 8.5. *$P(\mathfrak{c})$ fails in $V[G]$, but $\mathfrak{b} = \mathfrak{c}$ holds.*

Proposition 8.6. *If $\{x_\xi : \xi < \lambda\}$ is a tower in $V[G] \cap [\omega]^\omega$, then $\lambda < \kappa$.*

Proof. This is an example of a reflection result. Assume that $\{x_\xi : \xi < \lambda\}$ is a tower in $V[G]$. By Lemma 7.3, there is a sequence $\{\tau_\xi : \xi < \lambda\}$ in V so that, for each $\xi < \lambda$, $(\tau_\xi)_G = x_\xi$, and $p \Vdash \{\tau_\xi : \xi < \lambda\}$ is a tower. Next, fix an elementary submodel $M \prec H(\kappa^+)$ such that $\{\tau_\xi : \xi < \lambda\}$ and p are elements of M, $\mu = M \cap \kappa \in \kappa$, and $M^\omega \subset M$ (see 7.6). If $\lambda < \kappa$ then there is nothing to show, so assume otherwise and let δ be the supremum of $M \cap \lambda$. The procedure now is to note that $G_\mu = G \cap P_\mu$ is a P_μ-generic filter and, since $M^\omega \subset M$, $V[G_\mu] \models \{x_\xi : \xi \in \delta \cap M\}$ is a maximal tower of $[\omega]^\omega$. We also want to note that $x_\xi = (\tau_\xi)_G$ is the same as $(\tau_\xi)_{G_\mu}$.

But now if we treat $V[G_\mu]$ as the ground model. Then $V[G]$ is obtained by forcing with $P_\kappa/G_\mu = \{p \in P_\kappa : p \restriction \mu \in G_\mu\}$. It is easy to prove, by induction on $\alpha \leq \kappa$, that P_α/G_μ is forcing isomorphic to the finite support iteration of length α' (where $\mu + \alpha' = \alpha$) in which each iterand is P_H, i.e. P_κ/G_μ is P_κ from the point of view of $V[G_\mu]$. However since we know that P_κ does not fill towers, this contradicts that τ_δ is supposed to fill the tower. □

Proposition 8.7. *If $\{x_\xi : \xi < \lambda\}$ is a tower in $V[G] \cap [\omega]^\omega$, then $\lambda = \omega_1$. Left as an exercise.*

Proposition 8.8. *If $T \in V$ is a Souslin tree, then T is still Souslin in $V[G]$.*

Proof. P_κ has what is called property K – given any uncountably many members of P_κ there are uncountably many that are pairwise compatible. Kunen and Tall have shown that Souslin trees are preserved by property K forcings. Indeed, assume that $\{\dot{t}_\alpha : \alpha < \omega_1\}$ are P_κ names and that $p \Vdash \{\dot{t}_\alpha : \alpha < \omega_1\}$ is an antichain of T. For each α, choose some $p_\alpha < p$ and $t_\alpha \in T$ so that $p_\alpha \Vdash \dot{t}_\alpha = t_\alpha$. Because P_κ has property K, there is an uncountable $I \subset \omega_1$ such that $\{p_\alpha : \alpha \in I\}$ are pairwise compatible. Since T is Souslin, there are α, β in I so that $t_\alpha \leq t_\beta$. But if q is less than p_α and p_β then $q \leq p$ and it forces that $\dot{t}_\alpha$ is compatible with $\dot{t}_\beta$ contradicting the assumption on p. $\square$

Question 8.9. *Does the Normal Moore Space Conjecture hold in $V[G]$ if κ is supercompact in V?*

8.2. Countable support.

Definition 8.10. Miller forcing or rational perfect set forcing is defined as follows. For a tree $p \subset {}^{<\omega}\omega$ (which we always assume is closed downwards, i.e. if $s \in p$, then $\{s \restriction j : j \in \text{dom } s\} \subset p$), let us say that s is a splitting node of s if the set $X^p_s = \{m : s \frown m \in p\}$ is infinite. Miller forcing, P_M, then is the set of subtrees of ${}^{<\omega}\omega$ with the property that for each $t \in p$, there is a splitting node $s \in p$ such that $t \subset s$. Let $split(p)$ denote the set of splitting nodes of p. The poset is ordered by inclusion. The set of p with the property that each $s \in p$ is either splitting or has a unique immediate successor is dense so we shall assume that each condition has this property. Therefore $split(p)$ has a minimal element called the root of p. A node $s \in split(p)$ is said to be at splitting level n if there are precisely n members of $split(p)$ below it. A final useful bit a notation: if $s \in p$ then $p_s = \{t \in p : s \subset t \text{ or } t \subset s\}$ is a condition of P_M such that $p_s \leq p$ and, if $s \in split(s)$, s is the root of p_s.

Let P_{ω_2} denote the countable support iteration of Miller forcing.

The key to proving properties of Miller forcing and many forcing notions like it (beginning with Sacks and Laver forcings) is the idea of a fusion sequence. Beginning with a condition p_0, one builds a descending sequence of conditions, $\{p_n : n \in \omega\}$, with the idea that the intersection $p_\omega = \bigcap_n p_n$ will be a condition. The intersection need not, in general, be a condition since it may not have enough splitting nodes. The method requires that at stage n, certain members of $split(p_{n-1})$ are chosen (e.g. the nodes at level n) and a commitment is made that they will be in p_m for all m. In order that these chosen nodes be splitting nodes in p_ω one must be sure that, for each of them, for infinitely many m, the nodes committed at stage m will include an increasingly large (finite is sufficient) set of splitting successors of the given node. The idea is given in the following standard proof. Of course one could take $p_m = p_0$ for all m, but the idea is that there are infinitely many properties that we would like p_ω to have – if we are able to find $p_{n+1} < p_n$ with the n-th property (while keeping the promise on keeping certain nodes) then p_ω (being less than p_n for each n) will have all of these properties. Usually it is $(p_n)_s$, for s among the nodes newly committed at the n-th stage that have the special propery. This is best illustrated with examples.

Proposition 8.11. *Miller forcing is proper.*

Proof. Let $p \in P_M$ and fix a countable elementary submodel M of $H(\omega_2)$ such that $p, P_M \in M$. Let $\{D_n : n \in \omega\}$ enumerate all the dense subsets of P_M which are members of M. We build a fusion sequence p_n with $p_0 = p$ so that $p_\omega = \bigcap_n p_n$ is (M, P_M)-generic. The proof when P_M is countably closed is to ensure that $p_n \in D_n \cap M$ however this is simply impossible (e.g. $q \in D_n$ may imply that the root of q has domain at least n). Rather, here is what is done. If q is any extension of p_ω, then the root of q will be a splitting node of p_ω. The sequence p_n can easily be constructed so that the splitting nodes of p_ω are precisely the specially saved nodes in the fusion sequence. We must show that, for a fixed n, q is compatible with some member of $D_n \cap M$. This is where p_{n+1} comes in: the root of q is compatible with one of the nodes, s, on the n-th level of p_{n+1} (the lower levels are for D_k $k < n$). In the construction we ensure that $(p_{n+1})_s$ is in $D_n \cap M$; hence q is compatible with a member of $D_n \cap M$. It is worth noting that while $(p_{n+1})_s$ is in $D_n \cap M$, hence in M, p_{n+1} itself is not expected to be in M let alone D_n. We leave the rest of the proof as an exercise after proving the following lemma (to apply the lemma here, choose each p^s to be a member of $D_n \cap M$). $\square$

Lemma 8.12. *Suppose $p \in P_M$ and that S is a set of pairwise incomparable elements of p. For each $s \in S$, let p^s be any extension of p_s and let*

$$q = (p - \bigcup\{p_s : s \in S\}) \cup (\bigcup\{p^s : s \in S\}\ .)$$

Then q is a member of P_M and $q \le p$. In words, q is obtained from p by replacing each p_s by p^s.

As an example let us prove that Miller forcing preserves that $(\omega^\omega)^V$ is unbounded with respect to $<^*$. We prove something slightly stronger. This is what Shelah calls the property of being **almost ω^ω-bounding**. The stronger property is needed in order to apply the iteration lemmas cited later.

Lemma 8.13. *If τ is a P_M-name of a function from ω to ω, and if $p \in P_M$ then there is a $g \in \omega^\omega$, such that for every $A \in [\omega]^\omega$, there is a $q < p$ such that for each $s \in split(q)$ and $m \in X^q_s$, there is an $n \in A - m$ so that $q_{s^\frown m} \Vdash \tau(n) < g(n)$. Therefore, for each $A \in [\omega]^\omega$ there is a $q < p$ such that $q \Vdash \{n \in A : \tau(n) < g(n)\}$ is infinite.*

Proof. We say that q forces a value on $\tau(n)$ if there is some $m \in \omega$ such that $q \Vdash \tau(n) = m$.

Step 1: there is a $p' < p$ so that for each n and each s at splitting level n of p', and for each $m < n$, $(p')_s$ forces a value on $\tau(m)$.

Perform the following fusion. Let $p_0 = p$. At stage $n+1$, let S_n be the set of all $s \in p_n$ which are at splitting level n (note that the members of S_n are pairwise incompatible). For each $s \in S_n$, choose $p^s < (p_n)_s$ so that p^s forces a value on $\tau(n)$ (by induction, $(p_n)_s$ already forces a value on $\tau(m)$ for all $m < n$). Let p_{n+1} be obtained by replacing each $(p_n)_s$ $(s \in S_n)$ by p^s as in Lemma 8.12. Then p' is just equal to $\bigcap_n p_n$. Note that at stage $n+1$, the members of S_n did not necessarily remain splitting nodes but S_n is a subset of p' and this guarantees that $p' \in P$.

For ease of notation, suppose that p has the property that we have shown that p' will have. Now for each branch $b \subset p$, there is a function $f \in \omega^\omega$ such that, for each n, there is an $s \in b$ such that $p_s \Vdash f \restriction n = \tau \restriction n$. For each $s \in p$, fix a branch b_s containing s and let f_s denote the corresponding function on ω. Let $g \in \omega^\omega$ be such that $f_s <^* g$ for all $s \in p$. Given A, we perform another fusion to find our q.

By induction we will have that if s is at splitting level n in p_n then $(p_n)_s = p_s$ (if we work inside a countable elementary submodel M and choose g so that $f <^* g$ for all $f \in M \cap \omega^\omega$ we wouldn't have to be this careful). Again, let $p_0 = p$ and suppose we have chosen p_n so that for each s at splitting level less than n in p_n and each $m \in X^{p_n}_s$, there is a $k \in A - m$ such that $(p_n)_{s^\frown m} \Vdash \tau(k) < g(k)$. Let S_{n+1}

denote the nodes in p_n which are at splitting level $n+1$. For each $s \in S_{n+1}$ there is a unique m_s and s' such that s' is at splitting level n, and $s' ^\frown m_s \subset s$. It follows that $(p_n)_{s'^\frown m_s} = (p_n)_s$. We must ensure that $(p_{n+1})_s \Vdash \tau(k) < g(k)$ for some $k \in A - m_s$ for each $s \in S_{n+1}$. By our inductive assumption, $(p_n)_s = p_s$, hence b_s is a subset of p_n. Choose a $k \in A - m_s$ so that $f_s(k) < g(k)$ and let $s'' \in b$ be such that $p_{s''} \Vdash \tau(k) = f_s(k)$. Without loss of generality $s \leq s''$ and we set $p^s = (p_n)_{s''}$ for the application of Lemma 8.12. Now if $s' < s$ is the node at splitting level n, then $(p_{n+1})_{s'^\frown m_s} = p^s$ and is greater than $q_{s'^\frown m_s}$ (where $q = \bigcap_n p_n$) hence we will have that there is a $k \in A - m_s$ such that $q_{s'^\frown m_s} \Vdash \tau(k) < g(k)$ as desired.

Fix any $n \in \omega$. To show that $q \Vdash (\exists k \in A - n)\ \tau(k) < g(k)$, fix an $r < q$. It suffices to show that each r has an extension r' which forces $\tau(k) < g(k)$ for some $k \in A - n$. The root of r is a splitting node, s, of q. Choose any $m > n$ such that $s ^\frown m \in r \subset q$, and fix $k \in A - m$ such that $q_{s^\frown m} \Vdash \tau(k) < g(k)$. Therefore $r_{s^\frown m} \Vdash \tau(k) < g(k)$. □

Lemma 8.14. *Let $p \in P_M$ and let $f : split(p) \to 2$ be arbitrary, then there is a $q < p$ so that $f(s) = f(s')$ for all $s, s' \in split(q)$.*

Proof. Assume that for each $s \in split(p)$ there is an $s' \supset s$ so that $f(s') = 0$. Otherwise, there is $s \in split(p)$ so that $q = p_s$ is such that $f(s') = 1$ for all $s \in split(q)$. Recursively build a condition q by choosing the members of $split(p)$ which will also be in $split(q)$. Let s_0 be the root of p and let $\{(i_k, j_k) : k \in \omega\}$ be an enumeration of $\omega \times \omega$ so that $i_k, j_k \leq k$. At stage k, fix any $m \in X^p_{s_{i_k}}$ so that no s_ℓ $(\ell < k)$ extends $s_{i_k} ^\frown m$ and let s_k be any extension of $s_{i_k} ^\frown m$ so that $f(s_k) = 0$. Then $q = \bigcup\{\{s_k \upharpoonright j : j \in \mathrm{dom}\ s_k\} : k \in \omega\}$ is the desired condition. We need to check that $split(q) = \{s_k : k \in \omega\}$. □

Proposition 8.15 ([8]). *Miller forcing preserves P-points of ω^*, i.e. if $\mathcal{U}$ is a P-point ultrafilter on ω and if G is P_M-generic, then, in $V[G]$, $\mathcal{U}$ generates an ultrafilter.*

Proof. Let $\mathcal{U}$ be a P-point of ω^* and suppose that $p \Vdash \dot{Y} \subset \omega$. We will show that there is an $X \in \mathcal{U}$ and a $q < p$ so that either $q \Vdash X \subset Y$ or $q \Vdash X \cap Y = \emptyset$.

By an argument similar to the first part of the proof of Propostion 8.13, we may assume that if s is a level n splitting node of q, then q_s decides $\dot{Y} \cap n$. In a similar fashion, we can assume that if m is in $X^q_s = \{k \in \omega : s ^\frown k \in q\}$, then $q_{s^\frown m}$ decides $Y \cap m$. Consider yet one more fusion. We want to show that we may assume that there is a $Y_s \subset \omega$ such that for each $n \in Y_s$, the set of $m \in X^q_s$ such that $q_{s^\frown m} \Vdash n \notin Y_s$ is finite, and for each $n \notin Y_s$, the set of $m \in X^q_s$ such that $q_{s^\frown m} \Vdash n \in Y_s$ is also finite. One gets this by performing the following fusion at each splitting node s (by induction on the splitting level). Let us just state one step in the fusion and leave the rest as an exercise. Suppose we have fixed integers $m_0 < \cdots < m_{n-1}$ all in X^q_s and have decided on $Y_s \cap n$. Consider the integer n: if there are infinitely many $m \in X^q_s$ such that $q_{s^\frown m} \Vdash n \in \dot{Y}$ then put n in Y_s and let $m_n > m_{n-1}$ be the minimal such m and throw away all $m > m_{n-1}$ such that $q_{s^\frown m} \Vdash n \notin \dot{Y}$. On the other hand if there are not infinitely many such m, then do not put n in Y_s and throw away all $m > m_{n-1}$ such that $q_{s^\frown m} \Vdash n \in \dot{Y}$ and let let $m_n > m_{n-1}$ be the minimal of those that remain.

By Lemma 8.14, we may assume that either $Y_s \in \mathcal{U}$ for all $s \in split(q)$ or that $Y_s \notin \mathcal{U}$ for all $s \in split(q)$. In the first case fix $X \in \mathcal{U}$ so that $X - Y_s$ is finite for each $s \in split(q)$ (in the latter case, $X \cap Y_s$ is finite for each $s \in split(q)$).

Next we perform another fusion building $\{s_n : n \in \omega\} \subset split(q)$ together with an increasing sequence $\{k_n : n \in \omega\}$ so that $\bigcup\{\{s_n \restriction k : k \in \mathrm{dom}\ s_n\} : n \text{ even}\}$ and $\bigcup\{\{s_n \restriction k : k \in \mathrm{dom}\ s_n\} : n \text{ odd}\}$ will both be conditions. To begin let $s_0 = s_1$ be the minimal member of $split(q)$ and let $k_0 = 0$.

If n is even, fix some even m so that s_m needs another splitting successor (i.e. using an enumeration as in the proof of Lemma 8.14). Choose some s_n which is a minimal new member of $split(q) \cap q_{s_m}$ and so that $Y_{s_n} \cap k_{n-1} = Y_{s_m} \cap k_{n-1}$. Why can we do this? Choose ℓ large enough so that $q_{s_m \frown \ell'} \Vdash Y_{s_m} \cap k_{n-1} = \dot{Y} \cap k_{n-1}$ for all $\ell' \in X^q_{s_m} - \ell$. Suppose that $s_n \supset s_m{}^\frown \ell'$ for some $\ell' \geq \ell$. By the definition of Y_{s_n}, there are infinitely many j so that $q_{s_n \frown j} \Vdash Y_{s_n} \cap k_{n-1} = \dot{Y} \cap k_{n-1}$ – fix such a j. Since $q_{s_n \frown j} < q_{s_m \frown \ell'}$, it follows that $q_{s_n \frown j} \Vdash Y_{s_m} \cap k_{n-1} = \dot{Y} \cap k_{n-1} = Y_{s_n} \cap k_{n-1}$ – hence $Y_{s_n} \cap k_{n-1} = Y_{s_m} \cap k_{n-1}$. Next let $k_n > k_{n-1}$ be large enough so that $X - k_n \subset Y_{s_n}$.

For odd n, do the same thing except be sure to choose odd m.

Assume without loss of generality that $X_0 = X \cap \bigcup\{[k_n, k_{n+1}) : n \text{ even}\}$ is in $\mathcal{U}$. Let $q_0 = \bigcup\{\{s_n \restriction j : j \in \mathrm{dom}\ s_n\} : n \text{ even}\}$. We claim that $q_0 \Vdash X_0 \subset \dot{Y}$ (wlog $X \subset Y_{s_0}$). Suppose that $q' \leq q_0$ and $j \in X_0$ are such that $q' \Vdash j \notin \dot{Y}$. Let s_n be the first splitting node of q'. Since $q' \leq q_0$, n is even. Fix ℓ large enough so that $\ell' \in X^q_{s_n}$ implies that $q_{s_n \frown \ell'} \Vdash Y_{s_n} \cap j+1 = \dot{Y} \cap j+1$. Since $s_n \in split(q')$, there are $\ell' \geq \ell$ so that $s{}^\frown \ell' \in q'$ and so, $q'_{s_n \frown \ell'} \leq q_{s_n \frown \ell'}$. It follows, then, that $j \notin Y_{s_n}$. Therefore $j < k_n$ since $X - k_n \subset Y_{s_n}$. But now j is actually in X_0, hence $j < k_{n-1}$. Fix $m < n$ so that s_n was constructed to extend s_m. By the construction $Y_{s_n} \cap k_{n-1} = Y_{s_m} \cap k_{n-1}$, hence $j \notin Y_{s_m}$. We can repeat the above steps: j must be less than k_m but m is even, hence $j < k_{m-1}$ and we recursively find a descending sequence $m = m_0 > m_1 > \cdots$ so that $j \notin Y_{s_{m_i}}$ and $j < k_{m_i - 1}$. This stops when m_i gets to 0 and $j \in X - Y_{s_0}$ is a contradiction. □

Having proven that the single stage forcing is proper, almost ω^ω-bounding, and preserves P-point ultrafilters we turn to the iteration and to several of Shelah's preservation results. We treat them as Axioms, however someone who intends to delve deeply into the subject should certainly learn the proofs of Lemmas 8.18 to 8.20. Our proof of Theorem 8.24 illustrates these techniques. Using Lemmas 8.19 and 8.2, we have the following result.

Proposition 8.16. *P_{ω_2} is proper and satisfies the ω_2-cc.*

In these iteration results one often need the following result. Let us point out that $M[G]$ is defined as $\{\tau_G : \tau \in M \cap V^P\}$.

Lemma 8.17. *If P is a poset and $M \prec H(\theta)$ with $P \in M$ then $M[G] \prec H(\theta)[G]$. If, in addition, there is an (M,P)-generic condition q in G, then $M[G] \cap H(\theta)^V = M$.*

Proof. If $\tau_1, \ldots, \tau_n$ are P-names in M and there is a $p \in G$, such that $H(\theta) \models p \Vdash (\exists x)\ \varphi(\sigma, \tau_1, \ldots, \tau_n)$. Now, $p, P, \tau_1, \ldots, \tau_n$ are all in M, and $H(\theta) \models (\exists\sigma)\ p \Vdash \varphi(\sigma, \tau_1, \ldots, \tau_n)$, hence, there is such a $\sigma \in M$. This verifies that, in $V[G]$, the Tarski-Vaught criteria holds between $H(\theta)[G]$ and $M[G]$, hence $M[G] \prec H(\theta)[G]$.

Now suppose that $q \in G$ is (M, P)-generic and that $h \in M[G]$. Since $h \in M[G]$, there is a P-name $\tau \in M$ such that $\tau_G = h$. Let $D_\tau = \{p \in P : \text{either } (\exists x)\ (p \Vdash \tau = \check{x}) \text{ or } (\forall x) p \Vdash \tau \neq \check{x}\}$. Clearly D_τ is in M and it is easily seen to be a dense subset of P. Since $q \in G$ is (M, P)-generic, there is a p in $G \cap D_\tau \cap M$. Since $p \in G$ and $\tau_G = h$, p does not force that $\tau \neq \check{h}$. Therefore, by elementarity, there is an $x \in M$ such that $p \Vdash \tau = \check{x}$. Clearly this x must be h, showing that $h \in M$. □

Lemma 8.18. *If $M \prec H(\theta)$, p is (M, P)-generic, and $p \Vdash q$ is $(M[G], Q)$-generic for a P-name of a poset Q (and G is the canonical name of the P-generic filter), then $p * q$ is $P * Q$-generic.*

Lemma 8.19. *A countable support iteration of proper posets is proper.*

Lemma 8.20. *If $\langle P_\beta, \dot{Q}_\beta : \beta < \alpha\rangle$ is a countable support iterated forcing system with limit P_α, then for all countable $M \prec H(\theta)$, with θ suitably large and $\langle P_\beta, \dot{Q}_\beta : \beta < \alpha\rangle \in M$, and for all $\beta < \alpha$ with $\beta \in M$ and for all $p \in M \cap P_\alpha$, and $q \in P_\beta$, if q is (M, P_β)-generic and $q \leq p \restriction \beta$, there is an $r \in P_\alpha$ such that r is (M, P_α)-generic, $r \leq q$ and $r \restriction \beta = q$.*

Proposition 8.21 ([11]). *For any countable support proper forcing iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \kappa\rangle$, if*

$$\Vdash_{P_\alpha} \dot{Q}_\alpha \text{ is almost } \omega^\omega\text{-bounding}$$

then

$$\Vdash_{P_\kappa} \omega^\omega \cap V \text{ is unbounded w.r.t. } <^* \ .$$

Proposition 8.22 ([1]). *For any countable support proper forcing iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \kappa\rangle$ and any P-point $\mathcal{U}$, if, for each $\alpha < \kappa$,*

$$\Vdash_{P_\alpha} \mathcal{U} \text{ generates a } P\text{-point}$$

then also

$$\Vdash_{P_\kappa} \mathcal{U} \text{ generates a } P\text{-point} .$$

Corollary 8.23. *Miller forcing does not add a Cohen real.*

One can deduce several other of the cardinal functions from 8.22. For example $\mathfrak{u} = \omega_1$, hence $\mathfrak{b} = \mathfrak{s} = \mathfrak{t} = \omega_1$. However the proof of Proposition 8.22 is quite difficult, not to mention only applicable to cases where P-points are preserved. There is a quite general preservation scheme available in [11] which applies to things like $\mathfrak{b}$, $\mathfrak{s}$ and $\mathfrak{t}$. These take the form that if certain kinds of families are splitting (see [2]) or form a tower [11], and if each iterand preserve that this for the family, then the countable support iteration does as well.

Proposition 8.24. $V[G] \models \mathfrak{s} = \mathfrak{a} = \mathfrak{b} < \mathfrak{d} = \omega_2$.

Proof. We treat the above discussion as having taken care of the cardinals $\mathfrak{s}$ and $\mathfrak{b}$. We first show that $\mathfrak{d} = \omega_2$. As mentioned earlier Miller forcing adds a function $f \in {}^\omega\omega$ which is not bounded by any ground model function. Let $\{\dot{f}_\alpha : \alpha < \omega_1\}$ be a set of P_{ω_2}-names of functions from ω into ω. For each $\alpha < \omega_1$ and $n \in \omega$, there is a maximal antichain $A(\alpha, n) \subset P_{\omega_2}$ so that for each $p \in A(\alpha, n)$ there is an m so that $p \Vdash \dot{f}_\alpha(n) = m$ (i.e. p forces a value on $\dot{f}_\alpha(n)$). Choose a $\xi < \omega_2$ so that the support of p is contained in ξ for each p in $\bigcup\{A(\alpha,n) : \alpha < \omega_1\ ,\ n \in \omega\}$. Let G be P_{ω_2}-generic and let $G_\xi = G \cap P_\xi$. Note that, for each $\alpha < \omega_1$,

$$(\dot{f}_\alpha)_{G_\xi} = (\dot{f}_\alpha)_G\ .$$

Therefore $\{(\dot{f}_\alpha)_{G_\xi} : \alpha < \omega_1\}$ is a set of functions in $V[G_\xi]$. Naturally the forcing Q_ξ is just Miller forcing and adds a function which is not less than any of the functions in $V[G_\xi] \cap {}^\omega\omega$.

Now we turn to the much harder task of showing that $\mathfrak{a} = \omega_1$. It doesn't appear that any of the preservation results mentioned above can be used directly but the proof we give (also essentially taken from [11]) gives the flavour of how these results are proven.

We first show that the family $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ constructed in Lemma 2.3 has a special combinatorial property which we can preserve (note that not every mad family is preserved). Let $\mathcal{B} = \{\ \{B^\beta_n : n \in \omega\} : \alpha < \omega_1\}$ enumerate all countable families of finite subsets of ω. Recall that $\mathcal{A}$ has the property that for each $\alpha < \omega_1$ and for each $\beta < \alpha$, either there are infinitely many n such that $B^\beta_n \subset a_\alpha$ or there is an $I \in \mathcal{I}_\alpha$ (the ideal generated by $\{a_\beta : \beta < \alpha\} \cup [\omega]^{<\omega}$) such that $B^\beta_n \cap I \neq \emptyset$ for all n.

Claim: If $M \prec H(\theta)$ is countable and includes $\mathcal{A}$ and $\mathcal{B}$, then for each family $\{B_n : n \in \omega\} \in M$ of finite subsets of ω, either there is an I in $\mathcal{I}_\delta$ with $B_n \cap I$ not empty for all n, or there are infinitely many n with $B_n \subset a_\delta$, where $\delta = M \cap \omega_1$. The proof is simply that $\{B_n : n \in \omega\}$ is equal to $\{B^\beta_n : n \in \omega\}$ for some $\beta \in M$.

We finish the proof (but we have a long way to go) by showing that the above claim still holds in $V[G_\alpha]$ for $\alpha \le \omega_2$. Just as we had to construct the family $\mathcal{A}$ with special properties we have to strengthen this inductive assumption. Call a model M satisfying both the hypothesis and the conclusion of the claim <u>good</u>. Recall that P_α/G_β (for $\beta < \alpha$ is equal to $\{p \in P_\alpha : p \restriction \beta \in G_\beta\}$.

ST_α is the statement: for each $\beta < \alpha$ and countable $M \prec H(\theta)$ such that P_α, β are both in M and for generic $G_\beta \subset P_\beta$, if $M[G_\beta] \cap \omega_1 = M \cap \omega_1$ and $M[G_\beta]$ is good and if $p \in M[G_\beta] \cap P_\alpha$ is such that $p \restriction \beta \in G_\beta$, then there is a $q \in P_\alpha$ such that $q \restriction \beta \in G_\beta$, $q \le p$, q is $(M[G_\beta], P_\alpha/G_\beta)$-generic, and, if $G_\beta \cup \{q\}$ is contained in a generic G_α, then $M[G_\alpha]$ is also good.

The successor step involves combinatorics on the iterand, in this case Miller forcing, while the limit stages involve arguments similar to those of 8.19, 8.20, and 8.21.

Assume ST_α holds and let us show that $\mathrm{ST}_{\alpha+1}$ holds. By 8.20 it suffices to let $\beta = \alpha$ (and "α" is $\alpha+1$) in the statement ST_α. We may treat $V[G_\alpha]$ as the ground model in which we use M to denote $M[G_\alpha]$ (which is assumed to be good) and $p = p(\alpha) \in M$ is a member of the Miller poset, Q. We must produce a $q < p$ which is (M, Q)-generic and which forces that $M[G]$ is good. Let $\delta = M \cap \omega_1$ and let $\{\{\tau^m_n : n \in \omega\} : m \in \omega\}$ enumerate all the Q-names in M of members of $[\ [\omega]^{<\omega}\ ]^\omega$ and let $\{D_n : n \in \omega\}$ enumerate the dense subsets of Q which are members of M. Set $p_0 = p$ and we'll perform a usual fusion so that $q = \bigcup_n p_n$ will be M-generic. Assume that p_n has been chosen and that we are "freezing" the nodes in p_n which are at splitting level less than or equal n. Let t be a level n splitting node of p_n. Recursively, for $m \leq n$, refine $(p_n)_t$ so that if it has a root preserving extension in D_m, then assume that $(p_n)_t$ is already in D_m. Now we turn to the τ^m_k's. We repeat the following procedure for each $m \leq n$. If $(p_n)_t$ has a root preserving extension which forces that every member of $\{\tau^m_n : n \in \omega\}$ meets a fixed $I \in \mathcal{I}_\delta$ then assume that $(p_n)_t$ is such a condition. Otherwise for each j such that $t ^\frown j \in p_n$, find an extension q_j of $(p_n)_{t^\frown j}$ which forces some $B^m_\ell \subset a_\delta$ for some $\ell \geq n$. To see that this is possible, let $\{B_k : k \in \omega\} = \{B : (\exists q < (p_n)_{t^\frown j}\ q \Vdash B = \tau^m_\ell$ for some $\ell \geq n\}$. By assumption, for each $I \in \mathcal{I}_\delta$, there are infinitely many k such that B_k is disjoint from I. In addition $\{B_k : k \in \omega\}$ is a member of M (the indexing is not important), therefore there is a k so that $B_k \subset a_\delta$. Choose q_j to be a condition in M associated with this B_k. Finally let $(p_n)_t$ be replaced by $\cup\{q_j : t ^\frown j \in (p_n)_t\}$. We let $(p_{n+1})_t$ be the result when we finish $m = n$ and p_{n+1} when we finish each t.

To see that $q = \cup_n p_n$ is (M, Q)-generic, let $r \leq q$ be in $D_m \cap G$ for any m. Let t be the root of r and note that t is a branching node of q so we may assume, by extending if necessary, that t is at the n-th branching level of q for some $n \geq m$. In the inductive construction we came to the node t at stage n and we see that we obtained a "yes" answer to the question of whether $(p_n)_t$ has a root preserving extension in D_m (i.e. r is such an extension). Therefore, by the construction $(p_{n+1})_t$ is a member of D_m. Since $r \leq p_{n+1}$ it follows that $(p_{n+1})_t$ is a member of $D_m \cap G$ as was to be shown.

Now we check that $M[G]$ is good. Since q is (M, Q)-generic, it follows that $M[G] \prec H(\theta)[G]$. By definition, if $\{B_n : n \in \omega\}$ is in $M[G]$ then there is an m, so that $\{(\tau^m_n)_G : n \in \omega\} = \{B_n : n \in \omega\}$. Assume that, for each $I \in \mathcal{I}_\delta$, there are infinitely many n so that $B_n \cap I$ is empty and that B_n is not contained in a_δ for all $n > \ell$. Fix $r \in G$ which forces the previous statement and let t be the root of r. Again without loss of generality, t is an n-th level splitting node of q for some $n > \max\{\ell, m\}$. So when we came to the node t in the fusion, we discovered that $(p_n)_t$ did <u>not</u> force that τ^m_k met I for all k and some fixed I. Therefore, for each appropriate j, q_j was chosen. In particular, choose any j so that $t ^\frown j \in r$ and note that $(p_{n+1})_{t^\frown j}$ forces that τ^m_k is contained in a_δ for some $k > n$. This contradicts that r forces the opposite and is compatible with $(p_{n+1})_{t^\frown j}$.

Finally we turn to the limit case. The case where α has uncountable cofinality is essentially vacuous because the iteration is proper and so any new countable subset of $[\omega]^{<\omega}$ will occur at some countable cofinality stage. This deserves a little more explanation. Let τ be a P_α-name $(cf(\alpha) > \omega)$ and let $p \Vdash_{P_\alpha} \tau \subset \omega$. We produce a $q < p$ and a P_β-name σ such that $\beta < \alpha$ and $q \Vdash_{P_\alpha} \sigma = \tau$. Let M be any countable elementary submodel of a suitably large $H(\theta)$ such that p, P_α and τ are all in M. Let $q < p$ be (M, P_α)-generic and let $\sigma = \tau \cap M$, i.e. $\tau \subset P_\alpha \times \omega$ and so $\sigma \subset (P_\alpha \cap M) \times \omega$. Since q is (M, P_α)-generic, it follows that, for each n, and each generic G containing q, $G \cap \{p : (p, n) \in \tau\} \neq \emptyset$ iff $G \cap \{p : (p, n) \in (\tau \cap M)\} \neq \emptyset$. The only thing remaining to prove is that σ is a P_β-name for some $\beta < \alpha$. This follows from the fact that M is a countable elementary submodel. Indeed, if $p \in P \cap M$, then the support of p is a countable subset of α which, by 4.4, is a subset of M. Therefore each $p \in P \cap M$ is a member of P_β where $\beta = \sup(M \cap \alpha)$.

For $cf(\alpha) = \omega$ we fix an increasing sequence $\{\beta_n : n \in \omega\}$ cofinal in α and work with the iteration $\langle P_n, \dot{Q}'_n : n \in \omega\rangle$ where P_n is isomorphic to what was P_{β_n} and $\dot{Q}'_n$ is the P_{β_n}-name of the iteration $\langle P'_\gamma, \dot{Q}_\gamma : \gamma < \beta'_{n+1}\rangle$ where, $\beta_n + \beta'_{n+1} = \beta_{n+1}$ and $\dot{Q}_\gamma$ is the P'_γ-name of the Miller poset. By induction on γ, $P_{\beta_n} * P'_\gamma$ is forcing isomorphic to $P_{\beta_n+\gamma}$. The poset $\dot{Q}'_n$ is often denoted by $P_{\beta_{n+1}}/P_{\beta_n}$ (see [10] 3.3). However let us remark that in some circumstances one has to be careful about the set of P'_n-names that $p(n)$ can equal for $p \in P_\omega$. We refer the interested reader to 8.20 for the details.

Again let $\{\{\tau^m_n : n \in \omega\} : m \in \omega\}$ enumerate all the P_ω-names in M of members of $[\,[\omega]^{<\omega}\,]^\omega$ and let $\{D_n : n \in \omega\}$ enumerate the dense subsets of P_ω which are members of M. Define by induction on $k \in \omega$, $q_k, p_k, \{\dot{n}(k,m) : m \leq k\}$ such that

(1) $q_k \in P_k$ is (M, P_k)-generic and forces that $M[G_k]$ is good,
(2) $q_k \restriction n = q_n$ for $n < k$,
(3) $p_k \in P_\omega$
(4) $q_k \leq p_k \restriction k$
(5) $p_{k+1} \restriction k = p_k \restriction k$ and $p_{k+1} \leq p_k$
(6) $q_k \Vdash_{P_k}$ "$p_k \in M \cap D_{k-1}$"
(7) $\dot{n}(k,m)$ is a P_k-name and $p_{k+1} \Vdash$ "$\tau^m_{\dot{n}(k,m)} \subset a_\delta$ or $(\exists I \in \mathcal{I}_\delta)\ \tau^m_\ell \cap I$ is not empty for all ℓ".

For $k+1$, we work in $V[G_k]$, G_k a generic subset of P_k with $q_k \in G_k$. So $p_k \in M[G_k]$ and $p_k \restriction k \in G_k$. Similar to Lemma 7.22, we can find $p_{k+1} \in M[G_k] \cap D_k$ so that $p_{k+1} \leq p_k$ and $p_{k+1} \restriction k \in G_k$. Indeed, let $D_k/G_k = \{p \in D_k : p \restriction k \in G_k\}$. One more experienced with iterations would say that D_k/G_k is a member of $M[G_k]$ and is a dense subset of $P_\omega/G_k = \{p \in P_\omega : p \restriction k \in G_k\}$. Then, since $M[G_k] \prec H(\theta)[G_k]$, there is a $p_{k+1} < p_k$ in $M[G_k] \cap D_k/G_k$. More directly, we argue as follows. Since D_k is dense, $D^*_k = \{p \restriction k : p \in D_k \text{ and } p < p_k\}$ is predense below $p_k \restriction k$ in P_k. Now D^*_k is a member of M and so there is some $p' \in G_k \cap (M \cap D^*_k)$. By elementarity, there is a $p_{k+1} \in M \cap D_k$ such that $p_{k+1} \restriction k = p'$ and $p_{k+1} < p_k$.

In addition, we may use the induction hypothesis that $M[G_k]$ is good to justify the assumption that, for each $m \leq k$, p_{k+1} forces that there is an $n(k,m) \geq k$ so that $\tau^m_{n(k,m)} \subset a_\delta$ or that there is some $I \in \mathcal{I}_\delta$ such that $\tau^m_n \cap I$ is not empty for all n. Indeed, just as in the successor case, let $\mathcal{B}_0 = \{B : (\exists r < p_{n+1})\ r \Vdash B = \tau^0_\ell$ for some $\ell > k\ \}$. If possible, find a $B \in \mathcal{B}_0$ which is contained in a_δ and then choose an extension, $p^0_{n+1} \in M[G_k]$ of p_{n+1} which forces that this B to equal $\tau^0_{n(k,0)}$ for the appropriate value of $n(k,0)$. Repeat this for $m = 1, \ldots, k$, and let the new p_{n+1} be equal p^n_{n+1}. Since all of this was done in $V[G_k]$ (with G_k arbitrary generic with $q_k \in G_k$), there are P_k names for the $n(k,m)$ and p_{k+1} such that q_k forces them to have the above properties. In actual fact, p_{k+1} is not a member of M but q_k forces that there is some $p \in M$ such that $p = p_{k+1}$. Similarly, by the inductive hypothesis ST_{k+1} there is a P_k-name $\dot{q}$ which is forced to be less than $p_{k+1}(k)$ and which, itself, forces that $M[G_{k+1}]$ is good. Define q_{k+1} to be $q_k * \dot{q}$.

In the end, we let $q = \cup_n q_n$. This q is (M, P_ω)-generic since $q \Vdash p_{n+1} \in D_n \cap M$. Let us check that $M[G]$ is good. Fix $m \in \omega$ and assume that some $r \in G$ forces that $\{n : \tau^m_n \cap I = \emptyset\}$ is infinite for each $I \in \mathcal{I}_\delta$. If $k > m$ is arbitrary, then, since w.l.o.g. $r < q_k$, we see that $r \Vdash \tau^m_{\dot{n}(k,m)} \subset a_\delta$. $\square$

9. $\mathbb{D}$-COMPLETENESS

In this section we introduce (by example only) Shelah's notion of $\mathbb{D}$-completeness in forcings. This technique is useful for producing models of CH in which diamond fails. This should be useful for the Moore-Mrowka question in the presence of CH. As a discipline we have spent a lot of time in studying ccc extensions and ultimately PFA whch resulted in many beautiful results concerning the structure of compact sequential spaces (inconsistent with $\diamondsuit$). In this section we reproduce the key new step of Avraham and Todorcevic's new result that it is consistent with CH that there are no big coherent sequences (they have proven a much more general result). Foolishly, and so as to not alarm the reader, we do not give the definition of a $\mathbb{D}$-completeness system until we need it (in the proof of Lemma 9.4) and, in fact, we give the definition of a simple $\mathbb{D}$-completeness system ("simple" here does not mean simpler to understand).

Definition 9.1. Suppose that $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ is a coherent sequence in ω_1. Define the poset $P_{\mathcal{A}}$, by $p = (a_p, \alpha_p, \mathcal{S}_p)$ where $a_p \subset \alpha_p$ and $\mathcal{S}_p$ is a countable subset of $\mathrm{NS}^+ \times \alpha_p + 1$ such that for each $(S,\gamma) \in \mathcal{S}_p$ there is an $S' \subset S$ such that (S',γ) is in $\mathcal{S}_p^+$ where

$$\mathcal{S}_p^+ = \{(S,\gamma) \in \mathcal{S}_p : a_p - \gamma \subset a_\beta \text{ for all } \beta \in S\} .$$

$p < q$ is defined by: $a_p \cap \alpha_q = a_q$, $\alpha_p \geq \alpha_q$, and $\mathcal{S}_q \subset \mathcal{S}_p$.

Forcing with $P_{\mathcal{A}}$ should give us $A = \bigcup\{a_p : p \in G\}$, an uncountable subset of ω_1 such that $A \cap \alpha$ is almost contained in a_α for each $\alpha < \omega_1$. The reason for introducing $\mathbb{D}$-completeness is so that this can be iterated and to not add any new reals, hence preserving CH. For the remainder, assume that $\mathcal{A}$ has the property that every uncountable subset of ω_1 meets some member of $\mathcal{A}$ in an infinite set. We will now use P for $P_{\mathcal{A}}$.

Lemma 9.2 (Density). *For each $p \in P$ and $\gamma \in \omega_1$, there is a $q \in P$ with $q < p$ and $a_q - \gamma \neq \emptyset$.*

Proof. Let $\{(S_n,\gamma_n) : n \in \omega\}$ enumerate $\mathcal{S}_p$ and for each $\xi \geq \gamma$, assume there is an $n_\xi \in \omega$ so that $\{\beta \in S_{n_\xi} : \xi \in a_\beta\}$ is not stationary. Clearly, there is an uncountable set $I \subset \omega_1$ and an n so that $n_\xi = n$ for all $\xi \in I$. Now find any α so that $a_\alpha \cap I$ is infinite. Then there is a stationary $S \subset S_n$ so that there is some finite b so that $a_\alpha - a_\beta = b$ for all $\beta \in S$. This is a contradiction, since if $\xi \in (a_\alpha - b) \cap I$ then $\xi \in a_\beta$ for all $\beta \in S \subset S_{n_\xi}$ – contradicting the definition of n_ξ. $\square$

Lemma 9.3 (Extension). *Let M be a countable elementary submodel of $H(\mu)$ such that $P \in M$. Then for each $p \in M \cap P$ and dense open subset of P, there is a $q \in D \cap M$, such that $a_q - \alpha_p \subset a_\delta$, where $\delta = M \cap \omega_1$. More generally, given any $x \subset \delta$ such that $a_\beta - x$ is finite for all $\beta < \delta$, there is a $q \in D \cap M$ such that $q < p$ and $a_q - a_p \subset x$.*

Proof. Let $N = \{\beta : (\exists b \in [\beta]^{<\omega})\ ((q \in D \text{ and } q < p) \rightarrow a_q - a_p \not\subset (a_\beta - b))\}$ and observe that $N \in M$. If $N \notin \mathrm{NS}$, then there is a stationary $N' \subset N$ and (by the pressing down lemma) a finite set b such that ($q \in D$ and $q < p$) implies $a_q - a_p \notin\subset (a_\beta - b)$ for each $\beta \in N'$. By elementary, b and N' may be chosen in M.

Let $p' = (a_p, \max(\alpha_p, \max(b)\ ), \mathcal{S}_p \cup \{(N', \alpha_p)\})$ and note that $p' \in M \cap P$. Choose $q \in D \cap M$ with $q < p'$ and fix $(\bar{N}, \alpha_p) \in \mathcal{S}_q^+$ so that $\bar{N} \subset N'$. If $\beta \in \bar{N}$, then $a_q - \alpha_{p'} \subset a_\beta$ since $(\bar{N}, \alpha_p)$ is in $\mathcal{S}_q^+$. But now, notice that $a_q \cap \alpha_{p'} = a_{p'} = a_p$ and so $a_q - \alpha_p \subset (a_\beta - b)$ contradicting the definition of N'. Therefore $N \in \mathrm{NS} \cap M$ and so we may choose $\beta \in \delta - N$. Set $b = a_\beta - x$ and, since $\beta \notin N$, there is a $q \in D$ such that $q < p$ and $a_q - a_p \subset (a_\beta - b) \subset x$. By elementarity there is such a $q \in D \cap M$. The statement for a_δ now follows since obviously a_δ is such an x. □

Lemma 9.4. *There is an $\aleph_1$-completeness system $\mathbb{D}$, such that for every coherent sequence $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ there is a $\mathbb{D}$-complete forcing notion $P_{\mathcal{A}}$ which introduces an $I \in [\omega_1]^{\omega_1}$ such that either $I \cap a_\alpha$ is finite for all α or $I \cap \alpha - a_\alpha$ is finite for all α. Also $P_{\mathcal{A}}$ is α-proper for every countable ordinal α.*

Proof. A completeness system is called simple if there is a first order formula ψ such that

$$\mathbb{D}(M, P, p) = \{A_x : x \text{ a finitary relation on } M\}$$

where

$$A_x = \{G \in \mathrm{Gen}(M, P) : \langle M \cup \mathcal{P}(M), \in\rangle \models \psi[G, x, p, P, M]\}\ .$$

$\mathbb{D}$ is $\aleph_1$-complete if each suitable $\mathbb{D}(M, P, p)$ has the property that any countable intersection is not empty. Finally P is $\mathbb{D}$-complete if

$$\mathrm{Gen}^+(M, P, p) = \{G \in \mathrm{Gen}(M, P) : p \in G \text{ and there is a lower bound for } G \text{ in} P\}$$

contains some member of $\mathbb{D}(M, P, p)$.

We will define a suitable ψ after we have shown how it would be used. Our primary concern is the completeness. However let us remark that x will code a_δ (where $\delta = M \cap \omega_1$) and a function from NS^+ to ω_1 where $x(S) = \gamma$ will mean that S contains a stationary set, which we will denote, $T(S)$ such that $a_\delta - a_\beta \subset \gamma$ for all $\beta \in T(S)$. Note the coherent sequence $\mathcal{A} = \{a_\gamma : \gamma \in M\}$ can be recovered from P. One of the properties of ψ will be to assert that the statement of Lemma 9.3 holds with a_δ replaced by x_0. If there is an uncountable set I which meets every member of $\mathcal{A}$ in a finite set then simply let $P_{\mathcal{A}} = {}^{<\omega_1}2$. Therefore in the remainder assume that there is no such uncountable set and let P denote $P_{\mathcal{A}}$.

Claim. P is $\mathbb{D}$-complete.

Indeed, $\mathrm{Gen}^+(M, P, p)$ (for $M \prec H(\theta)$ and $P \in M$) contains A_x where x is the code as described above – we use x_0 to denote the subset of δ coded by x and recall that, for $S \in \mathrm{NS}^+ \cap M$, $x(S) = \gamma < \delta$ means there is a stationary set $T(S)$ (not in M) contained in S such that $a_\delta - a_\beta \subset x(S)$ for all $\beta \in T(S)$.

Then A_x will be all $G \in \mathrm{Gen}(M, P, p)$ which contain a subsequence $\{p_n : n \in \omega\}$ which satisfy the following induction.

Assume that for $q \in P$ there is a fixed enumeration of $\mathcal{S}_q$ in order type ω. Also let $\{(i_n, j_n) : n \in \omega\}$ be a fixed enumeration of $\omega \times \omega$ chosen so that $(i_n, j_n) \in n + 1 \times n + 1$. Let $\{D_n : n \in \omega\}$ enumerate all elements of M which are dense open subsets of P. When p_n is chosen, let $\{(S_{n,m}, \gamma_{n,m}) : m \in \omega\}$ be the fixed enumeration of $\mathcal{S}_{p_n}$. The sequence p_n is chosen so that

(1) $p_0 \leq p$,
(2) $p_{n+1} \in D_n$,

(3) $a_{p_{n+1}} - a_{p_n} \subset x_0 - x(S'_n)$ for some $(S'_n, \gamma_{i_n,j_n}) \in \mathcal{S}^+_{p_n}$ such that $S'_n \subset S_{i_n,j_n}$,
(4) $\alpha_{p_{n+1}} \geq x(S'_n)$.

Suppose that p_n has been chosen and fix $(S'_n, \gamma_{i_n,j_n}) \in \mathcal{S}^+_{p_n}$ so that $S'_n \subset S_{i_n,j_n}$. Let $\gamma \geq \max\{\gamma_{i_n,j_n}, x(S'_n)\}$ and define $p' = (a_{p_n}, \gamma, \mathcal{S}_{p_n})$; clearly $p' < p_n$. Apply Lemma 9.3 to obtain $p_{n+1} \in D_n$ with $p_{n+1} < p'$ and $a_{p_{n+1}} - \gamma \subset x_0$.

Now we claim that there is a suitable x such that $\{p_n : n \in \omega\}$, defined above, has a lower bound in P. Of course the point is to show that there is a suitable $\mathcal{S} \supset \bigcup_n \mathcal{S}_{p_n}$ so that $(a = \cup a_{p_n}, \delta, \mathcal{S})$ is a member of P. Define x_0 to be a_δ and, for each $S \in \mathrm{NS}^+$, $a_\delta - a_\beta \subset x(S) < \delta$ for all $\beta \in T(S) \in \mathcal{P}(S) \cap \mathrm{NS}^+$. Let S'_n be defined as in the inductive statement 3. Define $\mathcal{S}$ to be $\bigcup_n \mathcal{S}_{p_n}$ together with $\{(T(S'_n), \gamma_{i_n,j_n}) : n \in \omega\}$. Clearly $q = (a, \delta, \mathcal{S})$ is below each p_n so it suffices to show that $q \in P$ and, it should be clear that, it suffices to show that $\mathcal{S}^+_q$ is large enough. For this, it should be clear that it suffices to show that $(T(S'_n), \gamma_{i_n,j_n}) \in \mathcal{S}^+_q$ for each n. Fix n and note that since $(S'_n, \gamma_{i_n,j_n}) \in \mathcal{S}_{p_n}$, it follows that $a_{p_n} - \gamma_{i_n,j_n} \subset a_\beta$ for each $\beta \in T(S'_n)$. By the definition of $T(S'_n)$ and $x(S'_n)$ it follows that $a_\delta - a_\beta \subset x(S'_n)$ for all $\beta \in T(S'_n)$. By construction, $a_{p_{n+1}} - a_{p_n} \subset x_0 - x(S'_n)$ and for $m > n$ $a_{p_{m+1}} - a_{p_m} \subset x_0 - \alpha_{p_m} \subset x_0 - \alpha_{p_{n+1}} \subset x_0 - x(S'_n)$. Therefore $\bigcup_m a_{p_m} - a_{p_n} \subset a_\beta$ for all $\beta \in T(S'_n)$. This completes the proof that $a - \gamma_{i_n,j_n} \subset a_\beta$ for all $\beta \in T(S'_n)$.

Claim. If $M \prec H(\mu)$ for a suitable μ and $P \in M$, then $\mathbb{D}(M, P, p)$ is countably complete for each $p \in P \cap M$.

Fix any sequence $\{x^n : n \in \omega\}$ such that wlog $a_\gamma - (x^n)_0$ is finite for all $\gamma \in (M \cap \omega_1)$ and $n \in \omega$. The plan is to choose a descending sequence p_n so that $p_n \in D_{n+1} \cap M$ for each n. At stage n, the only thing that needs to be modified from the above construction is to replace $x(S'_n)$ by the maximum of $\{x^k(S'_n) : k \leq n\}$. The sequence which shows that the resulting generic filter G is in A_{x^n} is $\{p_k : n \leq k < \omega\}$.

Now we must establish that P is α-proper where $\alpha \in \omega_1$. Fix a continuous chain of countable elementary submodels of a suitable $H(\theta)$, $\{M_\gamma : \gamma \in \alpha + 1\}$, such that $P \in M_0$, $\{M_\gamma : \gamma \leq \xi\} \in M_{\xi+1}$ for $\xi \leq \alpha$, and a condition $p \in M_0$; we must find a condition $q < p$ which is (M_γ, P)-generic for each $\gamma \leq \alpha$. Inductive assumption: for any continuous chain $\{M'_\gamma : \gamma \leq \beta\}$ and $p \in P \cap M'_0$, and finite set $b \subset \omega_1$, there is a $q < p$ such that $a_q - a_p \subset (a_{\delta'} - b)$ (where $\delta' = M_\beta \cap \omega_1$) and $\{r \in P : r < q\} \cap M_\gamma$ is P-generic for M_γ for each $\gamma \leq \beta$. The case where α is a successor is handled by the proof that P is $\mathbb{D}$-complete, so assume that α is a limit and let $\{\beta_n : n \in \omega\}$ be increasing and cofinal in α.

Recursively choose $p_n \in M_{\beta_n+1}$ together with stationary $S_n \subset S_{i_n,j_n}$ (use the enumeration as above) such that $S_n \subset S'_n$ for some $(S'_n, \gamma_{i_n,j_n}) \in \mathcal{S}^+_{p_n}$ and there is a finite set b_n such that $a_\delta - a_\beta = b_n$ for all $\beta \in S_n$. Apply the induction hypothesis to choose $p_{n+1} \in M_{\beta_{n+1}}$ such that $p_{n+1} < p_n$, $a_{p_{n+1}} - a_{p_n}$ is contained in $a_{\delta_{n+1}} - \left(b \cup (\bigcup_{k \leq n} b_k) \cup (a_{\delta_{n+1}} - a_\delta)\right)$, $\alpha_{p_{n+1}} = \delta_{n+1}$, and $\{r \in P : r < p_{n+1}\} \cap M_{\beta_{n+1}}$ is (M_γ, P)-generic for all $\gamma \in [\beta_n + 1, \beta_{n+1})$. $\square$

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