A NEW SUBCONTINUUM OF $\beta\mathbb{R}\setminus\mathbb{R}$

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ABSTRACT. We present a method for describing subcontinua of $\beta\mathbb{R}\setminus\mathbb{R}$. This method enables us to construct in ZFC a new subcontinuum of $\beta\mathbb{R}\setminus\mathbb{R}$.

We also show that the nontrivial layers of standard subcontinua can be described by our method. This allows us to construct a layer with a proper dense F_σ-subset and bring the number of (known) nonhomeomorphic subcontinua of $\beta\mathbb{R}\setminus\mathbb{R}$ to 14.

INTRODUCTION

It is fairly well-known that the Čech-Stone remainder $\mathbb{H}^*$ of the half line $\mathbb{H} = [0,\infty)$ is an indecomposable continuum — a fact that was proved by WOODS in [1968] and BELLAMY in [1971]. Much of the structure of $\mathbb{H}^*$ is known but many questions remain.

In this paper we continue the investigation into the number of topologically different subcontinua of $\mathbb{H}^*$. So far through the efforts of VAN DOUWEN [1977], SMITH [1986] and ZHU [1993] nine different subcontinua of $\mathbb{H}^*$ have been discovered, in ZFC. We shall describe these continua briefly in Subsection 1.6.

The purpose of this paper is to construct, in ZFC, a new (tenth) subcontinuum of $\mathbb{H}^*$ and show how to use this continuum to raise, again in ZFC, the number of different subcontinua of $\mathbb{H}^*$ to fourteen. We do this by showing how, under CH, our continuum can be seen as a layer in a standard subcontinuum; because we have essentially four different layers in standard subcontinua (so far) this gives us four extra decomposable subcontinua. Under $\neg$CH we appeal to DOW [1984] for at least six more decomposable subcontinua.

As a prelibation of things to come we give an outline of the rest of the paper. In Section 2 we show how, with the aid of certain subsets of ω^*, one can parameterize the indecomposable subcontinua of $\mathbb{H}^*$. In Section 3 we use this parameterization with a minimal closed and σ-invariant subset of ω^* as input to construct our new continuum. Here σ is the selfmap of ω^* determined by the shift on ω. We also show that our continuum has $2^{\mathfrak{c}}$ composants, thus providing a 'naturally occurring' continuum of this type. In Section 4 we prove in ZFC that $\mathbb{H}^*$ has at least fourteen topologically different subcontinua although the families differ according to whether CH holds or not. Finally, in Section 5 we make some remarks and pose some questions.

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1. Preliminaria

In this section we establish some notation and recall some facts that will be needed later on. We refer to Hart [1992] for proofs and proper references.

1.1. The shift on ω^*. The shift σ on ω^* is the map determined by the usual shift on ω: $\sigma(n) = n+1$. For a point u of ω^* its image under σ is generated by the family $\{A+1 : A \in u\}$; because of this we sometimes write $u+1$ for $\sigma(u)$. Likewise we write $u-1$ for $\sigma^{\leftarrow}(u)$.

We will adopt the slightly nonstandard convention of calling a subset F of ω^* σ-invariant if $\sigma[F] \subseteq F$ *and* $\sigma^{\leftarrow}[F] \subseteq F$ (nonstandard because normally one takes σ-invariant to mean that $\sigma[F] \subseteq F$ only). The reason for this will become clear in Section 2; the closed and σ-invariant subsets occur naturally when we parameterize indecomposable subcontinua of $\mathbb{H}^*$.

We note that, with our interpretation, the set of accumulation points of a forward orbit is always σ-invariant. For let q be an accumulation point of $\mathrm{cl}\{p+n : n \in \omega\}$; there is an ultrafilter $u \in \omega^*$ such that $q = u\text{-}\lim(p+n)$. But then $q-1 = u\text{-}\lim\bigl(p+(n-1)\bigr)$ and $q+1 = u\text{-}\lim\bigl(p+(n+1)\bigr)$. Here, for a sequence $\langle x_n\rangle_n$ and $u \in \omega^*$, we say $x = u\text{-}\lim x_n$ iff for every neighbourhood U of x the set $\{n : x_n \in U\}$ belongs to u.

1.2. The space $\mathbb{M}$. The remainder $\mathbb{M}^*$ of the space $\mathbb{M} = \omega \times \mathbb{I}$ determines the structure of $\mathbb{H}^*$ completely. Before we show this we fix some notation. We abbreviate $\{n\} \times \mathbb{I}$ by $\mathbb{I}_n$ and we let $\pi : \beta\mathbb{M} \to \beta\omega$ be the map determined by $\pi(n,x) = n$. For $u \in \omega^*$ the fiber $\pi^{\leftarrow}(u)$ is denoted $\mathbb{I}_u$ (it is in a natural sense the u-th term of the sequence $\langle \mathbb{I}_n : n \in \omega\rangle$: in the hyperspace of closed subsets of $\beta\mathbb{M}$ one has $\mathbb{I}_u = u\text{-}\lim \mathbb{I}_n$). Similarly, if $x \in {}^\omega\mathbb{I}$ then we use x_u to denote the point $u\text{-}\lim\langle n, x(n)\rangle$ of $\mathbb{I}_u$.

The set $\mathbb{I}_u$ is a continuum that is irreducible between the two points $0_u = u\text{-}\lim\langle n, 0\rangle$ and $1_u = u\text{-}\lim\langle n, 1\rangle$. That $\mathbb{I}_u$ is irreducible between 0_u and 1_u follows from the fact that $\{x : x \in {}^\omega\mathbb{I}\}$ is a dense set of cut points of $\mathbb{I}_u$. There is a natural quasi-order on $\mathbb{I}_u$: one says $x \leqslant_u y$ if every subcontinuum of $\mathbb{I}_u$ that contains 0_u and y also contains x. We shall employ the usual interval notation for this ordering. We note that if $x, y \in {}^\omega\mathbb{I}$ then $x_u \leqslant_u y_u$ iff $\bigl\{n : x(n) \leqslant y(n)\bigr\} \in u$.

This ordering $\leqslant_u$ divides $\mathbb{I}_u$ into layers: the equivalence classes under the relation

$$x \equiv y \qquad \text{iff} \qquad x \leqslant_u y \text{ and } y \leqslant_u x.$$

With a layer L one associates two subsets of ${}^\omega\mathbb{I}$:

$$A_L = \{a \in {}^\omega\mathbb{I} : L \cap \mathrm{cl}[0,a] = \varnothing\}$$

and

$$B_L = \{b \in {}^\omega\mathbb{I} : L \cap \mathrm{cl}[b,1] = \varnothing\}.$$

Here, for $x, y \in {}^\omega\mathbb{I}$, the symbol $[x,y]$ abbreviates $\bigcup_n \{n\} \times \bigl[x(n), y(n)\bigr]$; observe that $[x_u, y_u] = \mathrm{cl}[x,y] \cap \mathbb{I}_u$.

The pair $\langle A_L, B_L\rangle$ represents a gap in the linearly ordered set ${}^\omega\mathbb{I}/u$. The gap is filled iff $L = \{x_u\}$ for some $x \in {}^\omega\mathbb{I}$. If the gap is unfilled then one can show that L consists of more than one point iff there is a function $f : \omega \to \mathbb{N}$ such that $\bigl\{n : b(n) - a(n) > 1/f(n)\bigr\}$ belongs to u whenever $a \in A_L$ and $b \in B_L$.

If L is such that $[0_u, L)$ has countable cofinality then L does consist of more than one point. For take a cofinal sequence $\langle a_n\rangle_n$ in $[0_u, L)$; its set of accumulation points D is contained in L and, because $\mathbb{M}^*$ is an F-space, homeomorphic to ω^*. We also note that in this case the interval $(L, 1_u]$ necessarily has uncountable coinitiality. In other words, the gap $\langle A_L, B_L\rangle$ is not of type $\langle\omega,\omega\rangle$.

1.3. Parameterizing $\mathbb{H}^*$. The map $q : \beta\mathbb{M} \to \beta\mathbb{H}$ determined by $q(n, x) = n + x$ maps $\mathbb{M}^*$ onto $\mathbb{H}^*$. Because q is one-to-one on every $\mathbb{I}_n$ it is also one-to-one on every $\mathbb{I}_u$ and because it only identifies $(1, n)$ and $(0, n + 1)$ for every n it also only identifies the points 1_u and 0_{u+1} for every $u \in \omega^*$. One can readily check these statements using standard properties of the Čech-Stone compactification.

One can define a map like q using any sequence $\boldsymbol{a}$ that increases to infinity: Simply put $q_{\boldsymbol{a}}(n, x) = a_n + x(a_{n+1} - a_n)$; we shall call $q_{\boldsymbol{a}}$ the parameterization of $\mathbb{H}^*$ determined by $\boldsymbol{a}$. The map q from the previous paragraph will be called the *standard parameterization.*

1.4. Standard subcontinua. If $q_{\boldsymbol{a}} : \mathbb{M}^* \to \mathbb{H}^*$ is a parameterization then, as noted before, the map $q_{\boldsymbol{a}}$ is one-to-one on every continuum $\mathbb{I}_u$. Therefore $q_{\boldsymbol{a}}[\mathbb{I}_u]$ and $\mathbb{I}_u$ are homeomorphic; we call a subcontinuum of $\mathbb{H}^*$ of the form $q_{\boldsymbol{a}}[\mathbb{I}_u]$ a *standard subcontinuum* of $\mathbb{H}^*$.

The basic facts about standard subcontinua that we shall use here are that every proper subcontinuum K of $\mathbb{H}^*$ is the intersection of the family of all standard subcontinua that contain it and if K is nontrivial then it contains at least one standard subcontinuum. Theorem 2.6 substantially improves this last sentence.

Usually standard subcontinua are defined via embeddings of $\beta\mathbb{M}$ into $\beta\mathbb{H}$ but one readily checks that the two approaches are equivalent.

1.5. Decomposable versus indecomposable. A continuum is *decomposable* if it can be written as the union of two proper subcontinua and *indecomposable* otherwise. One can see whether a subcontinuum of $\mathbb{H}^*$ is decomposable or not by looking at its position inside standard subcontinua.

Lemma 1.1. *A subcontinuum of $\mathbb{H}^*$ is decomposable iff it is a nondegenerate interval of some standard subcontinuum.*

Here we take the intervals with respect to the quasi-order introduced before. It follows that decomposable subcontinua have cut points: intervals in standard subcontinua have (relative) interior and hence contain one of the cut points from Subsection 1.2. So, whenever an indecomposable continuum sits inside a standard subcontinuum it sits inside a layer of it. In fact layers of standard subcontinua are themselves indecomposable. Lemma 1.1 may also be used to prove the following one.

Lemma 1.2. *If of the two subcontinua K and L one is indecomposable and if their intersection is nonempty then either $K \subseteq L$ or $L \subseteq K$.*

We shall use Lemma 1.2 frequently in showing that one continuum must be contained in another one.

1.6. The known subcontinua. As noted in the introduction up to now nine nonhomeomorphic subcontinua of $\mathbb{H}^*$ were known in ZFC. Because we will want to show that the continuum that we will construct in Section 3 is new we give a brief description of the nine continua.

The first, K_1, consists of one point.

The next six will be constructed from a standard subcontinuum. Let us fix one such continuum $\mathbb{I}_u$. We take $K_2 = \mathbb{I}_u$. Next we take a layer L_1 such that the interval $[0_u, L_1)$ has countable cofinality and we let $K_3 = [0_u, L_1]$ and $K_4 = [L_1, 1_u]$. Next we take $L_2 <_u L_1$ such that $(L_2, 1_u]$ has countable coinitiality and we let $K_5 = [L_2, L_1]$. Finally we take $L_3 >_u L_1$ such that $(L_3, 1_u]$ has countable coinitiality as well and we let $K_6 = [L_2, L_3]$ and $K_7 = [L_1, L_3]$. These six continua are all decomposable and have two distinguished end sets; the number of one-point- or G_δ-end sets may be used to show that they are mutually nonhomeomorphic; for example K_3 and K_4 are nonhomeomorphic: K_3 has a one-point end set and a G_δ-end set, namely L_1, but in K_4 the end set L_1 is not a G_δ-set nor does it consist of one point.

The continuum K_8 will be L_1 and the ninth continuum K_9 is such that it has an increasing sequence $\langle C_n \rangle_n$ of proper indecomposable subcontinua such that $K_9 = \operatorname{cl} \bigcup_n C_n$. These two continua are indecomposable and distinguished by the presence of a proper dense F_σ-subset — K_9 has one and K_8 has none.

2. Making subcontinua from shift-invariant subsets of ω^*

In this section we describe a method by which all indecomposable subcontinua of $\mathbb{H}^*$ may be constructed. The idea is quite simple: For a subset F of ω^* we abbreviate $\bigcup_{u \in F} \mathbb{I}_u$ by $\mathbb{M}_F$ and we will denote the image of $\mathbb{M}_F$ under q by C_F. We shall identify F with the set $\{0_u : u \in F\}$ and also with the image of this set under q.

We shall see that C_F is an indecomposable continuum whenever F is closed, σ-invariant, dense-in-itself and not the union of two proper closed disjoint σ-invariant subsets. Conversely, if K is an indecomposable subcontinuum of $\mathbb{H}^*$ then there are a subset F of ω^* and a piecewise linear autohomeomorphism h of $\mathbb{H}$ such that $K = h[C_F]$.

The first lemma enables us to recognize σ-invariant subsets of ω^* by the behavior of $\mathbb{M}_F$ and C_F.

Lemma 2.1. *The set F is σ-invariant iff $\mathbb{M}_F = q^{\leftarrow}[C_F]$.*

Proof. Assume F is σ-invariant and let $x \in q^{\leftarrow}[C_F]$. Choose $y \in \mathbb{M}_F$ and $u \in F$ such that $q(y) = q(x)$ and $y \in \mathbb{I}_u$. If $y \neq x$ then either $y = 0_u$ and $x = 1_{u-1}$ or $y = 1_u$ and $x = 0_{u+1}$; in either case $x \in \mathbb{M}_F$ because $u-1, u+1 \in F$.

Conversely assume $\mathbb{M}_F = q^{\leftarrow}[C_F]$ and let $u \in F$. Because $q(1_u) = q(0_{u+1})$ we must have $0_{u+1} \in \mathbb{M}_F$ but this means that $u+1 \in F$. Likewise one concludes that $u - 1 \in F$. $\square$

Now we can show prove the following proposition.

Proposition 2.2. *If C_F is an indecomposable continuum then F is closed and σ-invariant.*

Proof. Let $u \in F$. Because $q[\mathbb{I}_{u+1}]$ meets $q[\mathbb{I}_u]$ it also meets C_F. It can not contain C_F because it does not contain $q[\mathbb{I}_u]$, therefore $q[\mathbb{I}_{u+1}] \subseteq C_F$ and so $u+1 \in F$. Likewise one shows that $u-1 \in F$.

Finally, F is closed because $F = \pi\bigl[q^{\leftarrow}[C_F]\bigr]$ and π is a closed map. $\square$

Next we need to know for which closed σ-invariant subsets F of ω^* the space C_F is connected.

Theorem 2.3. *The set C_F is connected iff F can not be written as the union of two disjoint proper closed σ-invariant subsets.*

Proof. Necessity: If $F = G \oplus H$ with G and H both σ-invariant then $\mathbb{M}_F = \mathbb{M}_G \oplus \mathbb{M}_H$ hence $C_F = C_G \oplus C_H$ by Lemma 2.1.

Sufficiency: If $C_F = G \oplus H$ the we can divide F into two sets $G' = \{u : q[\mathbb{I}_u] \subseteq G\}$ and $H' = \{u : q[\mathbb{I}_u] \subseteq H\}$. Because every $\mathbb{I}_u$ is connected this is a partition of F.

For every $u \in F$ the set $q[\mathbb{I}_{u-1}] \cup q[\mathbb{I}_u] \cup q[\mathbb{I}_{u+1}]$ is connected, so if u is in G' (or H') then so are $u-1$ and $u+1$. This shows that G' and H' are σ-invariant.

To see that G' and H' are closed just observe that $G = C_{G'}$ and so (again by the Lemma) $G' = \pi\bigl[q^{\leftarrow}[G]\bigr]$ and likewise for H. $\square$

We see that every 'indecomposable' σ-invariant closed set determines a subcontinuum of $\mathbb{H}^*$. Next we determine when C_F is an indecomposable continuum.

Lemma 2.4. *If $u \in F$ then u is an isolated point of F iff $q[\mathbb{I}_u]$ has nonempty interior in C_F.*

Proof. If u is isolated in F then $\mathbb{I}_u \setminus \{0_u, 1_u\}$ is open in $\mathbb{M}_F$ and saturated with respect to q so its image is open if C_F and contained in $q[\mathbb{I}_u]$.

Conversely, if there is an open set O such that $\varnothing \neq O \cap C_F \subseteq q[\mathbb{I}_u]$ then $q^{\leftarrow}[O]$ does the same thing for $\mathbb{I}_u$ with respect to $\mathbb{M}_F$. Pick an open rectangle $O(a,b,U)$ in $\mathbb{M}^*$ such that $O(a,b,U) \cap \mathbb{M}_F$ is nonempty and contained in $\mathbb{I}_u$. Then $U \cap F = \{u\}$. $\square$

Now we can characterize indecomposability of C_F.

Theorem 2.5. *If C_F is a continuum then it is indecomposable iff F is dense-in-itself.*

Proof. We just proved necessity: If $u \in F$ is isolated then $\mathbb{I}_u$ is a subcontinuum of C_F with nonempty interior (if $F = \{u\}$ then certainly $C_F = \mathbb{I}_u$ is decomposable).

To prove sufficiency assume that C_F is decomposable. By Lemma 1.1 we can find a standard subcontinuum $\mathbb{J}_u$ such that C_F has nonempty interior in $\mathbb{J}_u$ and therefore contains a cut point x of $\mathbb{J}_u$. Choose $v \in F$ such that $\mathbb{I}_v$ contains x. Then $\mathbb{I}_v$ is a subinterval of $\mathbb{J}_u$ and hence has nonempty interior in $\mathbb{J}_u$. So certainly $\mathbb{I}_v$ has nonempty interior in C_F; it follows that v is an isolated point of F. $\square$

We conclude that we get an indecomposable continuum in $\mathbb{H}^*$ each time we get a dense-in-itself indecomposable σ-invariant subset of ω^*. The converse holds as well.

Theorem 2.6. *Let K be a nontrivial indecomposable subcontinuum of $\mathbb{H}^*$. Then there are a σ-invariant subset F of ω^* and a (piecewise linear) autohomeomorphism h of $\mathbb{H}$ such that $K = \beta h[C_F]$.*

Proof. Take a standard subcontinuum that is contained in K, that is take a parameterization $q_{\boldsymbol{a}}$ of $\mathbb{H}$ and $u \in \omega^*$ such that $q_{\boldsymbol{a}}[\mathbb{I}_u] \subseteq K$.

We claim that for every point $v \in \omega^*$ either $q_{\boldsymbol{a}}[\mathbb{I}_v] \subseteq K$ or $q_{\boldsymbol{a}}[\mathbb{I}_v] \cap K = \varnothing$. For if $q_{\boldsymbol{a}}[\mathbb{I}_v]$ meets K then it can not contain K as it can not contain $q_{\boldsymbol{a}}[\mathbb{I}_u]$, so it must be contained in K.

Now we consider $F = \{v \in \omega^* : q_{\boldsymbol{a}}[\mathbb{I}_v] \subseteq K\}$. From what we have established above it follows that $K = q_{\boldsymbol{a}}[\mathbb{M}_F] = \beta h[C_F]$, where $h : \mathbb{H} \to \mathbb{H}$ is the unique piecewise linear map for which $h \circ q = q_{\boldsymbol{a}}$.

Because C_F apparently is an indecomposable continuum we see that F must be σ-invariant. $\square$

Remark 2.7. Let us note that we cannot distinguish nontrivial the subcontinua of $\mathbb{H}^*$ by means of cellularity, density or (π-)weight.

As $\mathbb{H}^*$ has weight $\mathfrak{c}$ it suffices to show that each nontrivial subcontinuum has cellularity $\mathfrak{c}$. Indeed, for every $x \in (0,1)$ we can take the open subset O_x of $\mathbb{M}^*$ determined by $\bigcup_n \{n\} \times (x - 2^{-n}, x + 2^{-n})$; the family $\bigl\{O_x : x \in (0,1)\bigr\}$ is pairwise disjoint and every O_x meets every $\mathbb{I}_u$.

Now if K is a subcontinuum of $\mathbb{H}^*$ then we take a parameterization $q_{\boldsymbol{a}} : \mathbb{M} \to \mathbb{H}$ and $u \in \omega^*$ such that $q_{\boldsymbol{a}}[\mathbb{I}_u] \subseteq K$. Then $\bigl\{q[O_x] \cap K : x \in (0,1)\bigr\}$ is a family of pairwise disjoint nonempty open subsets of K. (Note that $O_x = q_{\boldsymbol{a}}^{\leftarrow}\bigl[q_{\boldsymbol{a}}[O_x]\bigr]$ for every x, so that $q_{\boldsymbol{a}}[O_x]$ is open in $\mathbb{H}^*$.)

3. A New Continuum

In this section we apply the construction from Section 2, with very special input, to construct a new subcontinuum of $\mathbb{H}^*$. The input that we shall use is a minimal closed σ-invariant subset of ω^*. Such sets are easily constructed: Apply Zorn's Lemma to the family of all closed σ-invariant subsets of ω^*.

It is readily verified that a closed set F is minimally σ-invariant iff for every $u \in F$ the orbit $O_u = \{u + n : n \in \mathbb{Z}\}$ is dense in F. From this one easily deduces that for every $u \in F$ the set $S_u = \bigcup_n q[\mathbb{I}_{u+n}]$ is proper dense F_σ-subset of C_F.

Each minimal σ-invariant subset is clearly indecomposable and dense-in-itself so it follows that C_F is an indecomposable continuum whenever F is closed and minimally σ-invariant. The following theorem shows that C_F is indeed new.

Theorem 3.1. *If F is a minimal closed σ-invariant subset of ω^* then C_F is not homeomorphic to any one of the continua K_1 through K_9.*

Proof. Since K_1 consists of one point and since the continua $K_2, \ldots, K_7$ are decomposable we only have to deal with K_8 and K_9. Now K_8 has the property that every nonempty G_δ-subset of it has nonempty interior but K_9 and C_F do not have this property, so we are left with K_9.

Remember that K_9 has a proper, dense and meager F_σ-subset $H = \bigcup_n C_n$, where each C_n is indecomposable and nowhere dense in C_{n+1}. Assume $h : K_9 \to C_F$ is a homeomorphism and choose $x \in h[C_0]$; also pick $u \in F$ such that $x \in q[\mathbb{I}_u]$.

Because $h[H]$ is dense in C_F we know that there is an n such that $h[C_n] \nsubseteq q[\mathbb{I}_u]$ and consequently — because C_n is indecomposable — $q[\mathbb{I}_u] \subseteq h[C_n]$. But now one shows inductively that $q[\mathbb{I}_{u+k}] \subseteq h[C_n]$ for every k and hence that $S_u \subseteq h[C_n]$.

This leads to a contradiction since S_u is dense in C_F and C_n is nowhere dense in K_9. $\square$

Henceforth we shall denote the continuum of Theorem 3.1 by K_{10}.

The *composant* of a point x in a continuum X is the union of all proper subcontinua of X that contain the point. In an indecomposable continuum the composants are the equivalence classes under the relation "there is a proper subcontinuum containing both x and y".

The following proposition identifies the composants of K_{10}. In its proof we apply Theorem 5.3 from Hart [1992] (due to Van Douwen) which gives exact information on how two standard subcontinua intersect (if they intersect). Indeed if K and L are standard subcontinua, K with end points a and b, and L with end points c and d that intersect then $K\cap L$ equals L if $c,d\in K$, it equals K if $c,d\notin L$, and it equals one of the standard subcontinua $[a,d]$, $[c,b]$, $[a,c]$ and $[d,b]$ depending on which of the end points of L is in K and which of the end points of K is in L. Also, if K and L intersect then $K\cup L$ is a standard subcontinuum as well.

Proposition 3.2. *If $u\in F$ then S_u is the composant of u in K_{10}.*

Proof. Denote the composant of u by K_u. Clearly $S_u\subseteq K_u$, because S_u is the union of a family of proper subcontinua of K_{10} that contain the point u.

Conversely, let K be a nontrivial proper subcontinuum of K_{10} that contains u. If K is indecomposable then either $K\subseteq q[\mathbb{I}_u]$ or $q[\mathbb{I}_u]\subseteq K$. The latter case will give rise to a contradiction; for much as in the proof of Proposition 2.2 we can show that for each $n\in\mathbb{Z}$ if $q[\mathbb{I}_{u+n}]\subseteq K$ then also $q[\mathbb{I}_{u+n-1}]\subseteq K$ and $q[\mathbb{I}_{u+n+1}]\subseteq K$. This would imply that $K_{10}=\operatorname{cl}S_u\subseteq K$.

So assume K is decomposable and choose a standard subcontinuum $\mathbb{J}_v$ such that K is a nondegenerate interval of it. Now if for all $n>0$ we would have $u+n\in\mathbb{J}_v$ then, by the remarks preceding this proposition we would also have $q[\mathbb{I}_{u+n}]\subseteq\mathbb{J}_v$ for all $n\in\omega$. But then

$$K_{10}\subseteq\operatorname{cl}\bigcup_{n\geqslant 0}q[\mathbb{I}_{u+n}]\subseteq\mathbb{J}_v.$$

Now $K\subseteq K_{10}$ implies that K_{10} would be a nondegenerate interval of $\mathbb{J}_v$ and hence decomposable.

We conclude that there is $n>0$ such that $u+n\notin\mathbb{J}_v$ and likewise that $u+m\notin\mathbb{J}_v$ for some $m<0$. Applying the remarks preceding the proposition again we see that $\mathbb{J}_v$ is contained in the standard subcontinuum $\bigcup_{m\leqslant i<n}q[\mathbb{I}_{u+i}]$ and hence that $K\subseteq S_u$. $\square$

Thus composants of K_{10} correspond to orbits in F and we see that K_{10} has $2^{\mathfrak{c}}$ composants — the first continuum with $2^{\mathfrak{c}}$ composants was constructed by Smith in [1976]. This is of interest because the number of composants of $\mathbb{H}^*$ and of all countable cofinality layers is independent of ZFC: Under CH the number is $2^{\mathfrak{c}}$ and under NCF the number is 1 — we refer to Hart [1992] for references. We do not know the number of composants of K_9.

4. Four more subcontinua from CH

In this section we show how our construction may be employed to produce, using CH, four more subcontinua of $\mathbb{H}^*$. At the end of this section we use a result of Dow to show that under $\neg$CH there are always at least six continua different from K_1 through K_{10}.

We shall show that, under CH, the continuum K_{10} can be constructed as a layer in some standard subcontinuum $\mathbb{I}_u$. To this end we first investigate how we can make the parameterization of the layers as in Section 2 more concrete.

4.1. Concrete descriptions of layers. We fix a nontrivial layer L of some standard subcontinuum $\mathbb{I}_u$ and we find F such that $L = C_F$ (more precisely, we find a parameterization q of $\mathbb{H}$ and $F \subseteq \omega^*$ such that $L = q[\mathbb{M}_F]$).

Instead of ω we consider the set $K_f = \{\langle n, m\rangle : m < f(n)\}$, where f is as in Subsection 1.2, and order it lexicographically. The shift σ on K_f is defined by

$$\sigma(n, m) = \begin{cases} \langle n, m+1\rangle & \text{if } m+1 < f(n) \text{ and} \\ \langle n+1, 0\rangle & \text{otherwise.} \end{cases}$$

Furthermore put, for $m < f(n)$,

$$\mathbb{I}_{n,m} = \{n\} \times \left[\frac{m}{f(n)}, \frac{m+1}{f(n)}\right].$$

We define a filter $\mathcal{F}_L$ on K_f as follows:

$$\mathcal{F}_L = \{P : L \subseteq \mathrm{cl}_{\beta\mathbb{M}} \bigcup_{\langle n,m\rangle \in P} \mathbb{I}_{n,m}\}.$$

This filter has a base consisting of sets of the form

$$[g, h, U] = \{\langle n, m\rangle : g(n) \leqslant m \leqslant h(n), n \in U\},$$

where g and h are such that $g/f \in A_L$, $h/f \in B_L$ and $U \in u$. From this it follows that for every subset P of K_f

$$P \in \mathcal{F}_L \qquad \text{iff} \qquad \sigma[P] \in \mathcal{F}_L.$$

We can now describe F; it is the set $\bigcap_{P\in\mathcal{F}_L} P^*$ in K_f^*. Then $L = q^*[\mathbb{M}_F]$.

An alternative description of F is as follows. First we consider the map $\pi : K_f \to \omega$ defined by $\pi(n, m) = n$ and its Čech-Stone extension $\pi : \beta K_f \to \beta\omega$. The set $T_u = \pi^{\leftarrow}(u)$ contains a dense set of isolated points: Those determined by graphs of functions below f. For if $g \in {}^\omega\omega$ is such that $g(n) \leqslant f(n)$ for all n then the closure of its graph meets T_u in exactly one point and this point is therefore isolated. This set of isolated points is linearly ordered by $\leqslant_u$ and the gaps in this order are filled — in K_f^* — by sets like F. One might call F a layer in the space T_u.

4.2. A new layer. We shall now construct, assuming CH, an ultrafilter u, a function f and a layer F in T_u such that F is minimally σ-invariant. The corresponding continuum C_F is then a layer in $\mathbb{I}_u$

The following Lemma provides a way to recognize minimal σ-invariant sets.

Lemma 4.1. *A closed and σ-invariant set F is minimal iff for every open set that meets F there are finitely many translates that cover F.*

Proof. If O is open and meets the minimal set F then the family $\{\sigma^n[O] : n \in \mathbb{Z}\}$ covers F, because every orbit is dense and meets O. As F is compact this family has a finite subcover.

Conversely, if F is not minimal and G is a proper closed σ-invariant subset of F then $\omega^* \setminus G$ is an open set that meets F and is equal to all of its translates. $\square$

Using this Lemma we can describe what we need to construct. We must find an ultrafilter u and a gap $\langle \mathcal{F}, \mathcal{G}\rangle$ in ${}^\omega\omega/u$ below some function h, with the following property: if $X \subseteq K_h$ meets every set of the form $[f,g,U]$ with $f \in \mathcal{F}$, $g \in \mathcal{G}$, $U \in u$ then there are $f \in \mathcal{F}$, $g \in \mathcal{G}$, $U \in u$ and $k \in \omega$ such that $[f,g,U] \subseteq \bigcup_{|i|\leqslant k} \sigma^i[X]$. Here, as above

$$[f,g,U] = \{\langle n,m\rangle : f(n) \leqslant m \leqslant g(n), n \in U\}$$

for $f, g \in {}^\omega\omega$ and $U \subseteq \omega$.

The point is that the family of sets $[f,g,U]^*$ is a neighbourhood base for the set F, so that the previous paragraph expresses the condition of Lemma 4.1 in terms of the natural base for K_h^*.

The desired gap may be constructed, using CH, in an induction of length ω_1. One constructs $\mathcal{F} = \{f_\alpha : \alpha < \omega_1\}$, $\mathcal{G} = \{g_\alpha : \alpha < \omega_1\}$ and $\{U_\alpha : \alpha < \omega_1\}$ such that

(1) $\{U_\alpha : \alpha < \omega_1\}$ is decreasing mod finite and generates a P-point u.
(2) The pair $\langle \mathcal{F}, \mathcal{G}\rangle$ is a gap below the identity function Id in ${}^\omega\omega/u$.

To take care of our condition and to ensure that we get a gap we enumerate the family of subsets of K_h as $\{X_\alpha : \alpha < \omega_1\}$ and ${}^\omega\omega$ as $\{h_\alpha : \alpha < \omega_1\}$.

We start the induction by letting f_0 be identically 0, $g_0 = \mathrm{Id}$ and $U_0 = \omega$. We make sure that for every α one has

(1) $\sup\{g_\alpha(n) - f_\alpha(n) : n \in U_\alpha\} = \omega$ (to keep the induction alive).
(2) For all $n \in U_{\alpha+1}$ either $h_\alpha(n) < f_{\alpha+1}(n)$ or $h_\alpha(n) > g_{\alpha+1}(n)$ (to create a gap).
(3) If no finite number of translates of X_α covers $[f_\alpha, g_\alpha, U_\alpha]$ then X_α is disjoint from $[f_{\alpha+1}, g_{\alpha+1}, U_{\alpha+1}]$ (to meet our condition for minimality).

All steps are routine except when we have to take care of (3).

Assume no finite number of translates of X_α covers the set $[f_\alpha, g_\alpha, U_\alpha]$ and consider for each $n \in U_\alpha$ the longest interval I_n in $\{\langle n,m\rangle : m \leqslant h(n)\}$ that is contained in $[f_\alpha, g_\alpha, U_\alpha] \setminus X_\alpha$. The claim is that $\sup_{n\in U_\alpha} |I_n| = \omega$. For if the supremum equals $k < \omega$ then $\bigcup_{|i|\leqslant k} \sigma^i[X_\alpha]$ contains $[f_\alpha, g_\alpha, U_\alpha]$. This enables us to find functions p and q between f_α and g_α such that $[p, q, U_\alpha] \cap X_\alpha = \varnothing$.

Further modifications of p, q and U_α will be necessary so as to satisfy (2) and to ensure that the U_β will in the end generate an ultrafilter.

4.3. The four extra continua. It is clear that we can construct infinitely many K_{10}-like layers in $\mathbb{I}_u$ just by varying the gap. This implies that we can take four layers L_1, L_2, L_3 and L_4 in $\mathbb{I}_u$ such that $[0_u, L_1)$ and $[0_u, L_3)$ have countable cofinality, L_2 and L_4 are like K_{10} and $L_1 <_u L_2 <_u L_3 <_u L_4$.

We claim that the none of the four intervals $[0_u, L_2]$, $[L_1, L_2]$, $[L_2, L_3]$ and $[L_2, L_4]$ is homeomorphic to one of the continua K_1 through K_{10}.

As the intervals are nontrivial and decomposable they are different from K_1, K_8, K_9 and K_{10}. They are also different from K_2 through K_7: none of these have an end set with a proper dense F_σ-subset.

Finally, that the intervals are mutually nonhomeomorphic can be seen by looking at the 'other' end set: one point (0_u), not a G_δ-set (L_1), a G_δ-set (L_3) and not a G_δ-set but not homeomorphic to L_1 (L_4).

4.4. Six extra continua from $\neg$CH. If CH fails then Theorem ?? from Dow [1984] implies that there are u and v in ω^* such that

- if L is a layer in $\mathbb{I}_u$ such that $[0_u, L)$ has countable cofinality then $(L, 1_u]$ has coinitiality ω_1, and
- if L is a layer in $\mathbb{I}_v$ such that $[0_v, L)$ has countable cofinality then $(L, 1_v]$ has coinitiality larger than ω_1.

So, if we use $\mathbb{I}_u$ to make K_2 through K_7 and $\mathbb{I}_v$ to make the corresponding K'_2 through K'_7 then the K_i are mutually nonhomeomorphic, the K'_i are mutually nonhomeomorphic, and no K_i is homeomorphic to any K'_j because K_i has layers of type $\langle\omega, \omega_1\rangle$ whereas K'_j has none.

5. Concluding remarks

We have, in ZFC, ten 'honest' different subcontinua of $\mathbb{H}^*$. Under CH we can produce four more and under $\neg$CH even six more. Thus, ZFC proves that $\mathbb{H}^*$ has at least fourteen different subcontinua.

This should not be the best we can get. One would expect, given the way things usually go with the Čech-Stone compactification, to find $2^{\mathfrak{c}}$ different subcontinua in $\mathbb{H}^*$ but at present this number seems very hard to reach.

For instance, in Dow and Hart [1993] the authors have shown that under CH all standard subcontinua of $\mathbb{H}^*$ are homeomorphic, as are all layers that determine an interval with countable cofinality.

Although trying to vary minimal σ-invariant subsets of ω^* may look promising at first, one should realize that in [1990] Balcar and Błaszczyk showed that all such sets are all homeomorphic to the absolute of the Cantor cube of weight $\mathfrak{c}$

This leads to the (depressing) conjecture that under CH all versions of K_{10} are homeomorphic.

We finish with a question that, if answered positively, would give us five more intervals.

Question 5.1. Can the K_9-type continua be (homeomorphic to) layers of standard subcontinua?

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