# ON THE CHARACTER OF NORMAL NON-CWH SPACES

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Abstract. It shown that it is consistent (and independent of the Continuum Hypothesis) that every normal space of character at most $\mathfrak{c}^+$ is collectionwise Hausdorff

It is known to be consistent that every normal space of character at most $\aleph_1$ is collectionwise Hausdorff (for example it follows from $V = L$). Bing's space is a well-known example of a normal space which is not collectionwise Hausdorff. The character of Bing's space is $2^{\aleph_1}$ and Watson shows [3] that it is consistent that there is natural modification of Bing's space with lower character. However following Nyikos' solution of the normal Moore space conjecture, we know that it is consistent that normal spaces of character less than $\mathfrak{c} = 2^{\aleph_1}$ are collectionwise Hausdorff. Watson asks in [4], if there is always a normal space of character $\max\{\aleph_2, \mathfrak{c}\}$ which is not collectionwise Hausdorff. We answer Watson's questions in the negative.

## 1. Preliminaries

A collection $\mathcal{A}$ of subsets of a space $X$ is *separated* if there is a neighbourhood assignment for the sets in $\mathcal{A}$ consisting of pairwise disjoint sets. A space is $\kappa$-*collectionwise Hausdorff* ($<\kappa$-*collectionwise Hausdorff*) if each discrete collection of points of cardinality $\leq \kappa$ ($< \kappa$) is separated. Collectionwise Hausdorff is usually abbreviated as CWH. The character of a space $X$, denoted $\chi(X)$, is the least cardinal $\kappa$ such that every point has a local base of that cardinality.

**Lemma 1.1** ([1],[2]). *Let $\kappa < \lambda$ be regular cardinals in $V$, such that $2^{<\kappa} = \kappa$. Let $\mathcal{P} = Fn(\lambda, 2, \kappa)$ be the partial order of partial functions from $\lambda$ into 2 with cardinality less than $\kappa$, ordered by reverse inclusion. Let $G$ be $\mathcal{P}$-generic over $V$. In $V[G]$ every normal space of character less than $\lambda$ is $\kappa$-collectionwise Hausdorff if it is $<\kappa$-collectionwise Hausdorff.*

The above result is Tall's and we shall also need the following results of Fleissner.

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**Lemma 1.2 (V = L).** *Let $\kappa$ be a regular cardinal and let $X$ be a normal space with $\chi(X) \leq \kappa$. If $X$ is $<\kappa$-collectionwise Hausdorff then $X$ is $\kappa$-collectionwise Hausdorff.*

**Lemma 1.3.** *Suppose $cf(\kappa) < \kappa$ and $X$ is normal and $<\kappa$-collectionwise Hausdorff. If $\chi(X) < \kappa$ and, for all $\lambda$ with $\chi(X) \leq \lambda < \kappa$, $2^\lambda = \lambda^+$, then $X$ is $\kappa$-collectionwise Hausdorff.*

**Corollary 1.4.** *The following statement holds in L:*

$$(\forall\kappa)(\forall\lambda)\ \lambda \text{ regular and } \lambda < 2^\kappa \text{ , implies} \\ \text{normal } <\kappa\text{-CWH spaces of character at most } \lambda \text{ are } \kappa\text{-CWH .} \tag{1.1}$$

## 2. A topology from names

In this section the following result is proven by constructing a natural topology from the $\mathcal{P}$-name of a topology. Recall that for a poset $\mathcal{P}$, $c(\mathcal{P})$ denotes the least cardinal such that every antichain of $\mathcal{P}$ has smaller cardinality, e.g. $c(\mathcal{P}) = \omega_1$ if $\mathcal{P}$ is ccc.

**Lemma 2.1.** *Suppose $\kappa, \lambda$ are cardinals such that, each normal $<\kappa$-CWH space of character at most $\lambda$ is $\kappa$-CWH. Let $\mathcal{P}$ be a poset such that $|\mathcal{P} \cdot \lambda|^{<c(\mathcal{P})} \leq \lambda$. Then, if $\dot\tau$ is a $\mathcal{P}$-name of a topology on a set $X$ such that*

$$1 \Vdash \chi(\langle X, \dot\tau\rangle) \leq \lambda \text{ and } \kappa \subset X \text{ is unseparated, discrete, and } <\kappa\text{-separated in } X \ ,$$

*there is a set $A \subset \kappa$ and a $p \in \mathcal{P}$ such that*

$$p \Vdash A \text{ is not separated from } \kappa \setminus A \text{ in } X \ .$$

*Proof.* Let $\dot\tau$ and $X$ be as in the statement of the lemma. Since we are only interested in when neighbourhoods of points of $\kappa$ have non-empty intersection, we may assume that $S = X \setminus \kappa$ is open and discrete and that $|X| \leq \kappa \cdot \lambda$. Fix an $\alpha < \kappa$. Since $1 \Vdash \chi(\alpha, X) \leq \lambda$, there is a name $\dot f_\alpha$ such that 1 forces that $\dot f_\alpha$ is a function from $\lambda$ into $\dot\tau$ and that $\{\dot f_\alpha(\xi) : \xi \in \lambda\}$ is a local base at $\alpha$. By the maximum principle we can fix a function $f_\alpha$ (in $V$) from $\lambda$ into a set of $\mathcal{P}$-names such that, for each $\xi \in \lambda$, $1 \Vdash f_\alpha(\xi) = \dot f_\alpha(\xi)$ (note that $1 \Vdash \alpha \in f_\alpha(\xi)$ for each $\xi \in \lambda$). However we would like to find a set $\mathcal{U}_\alpha$ consisting of $\lambda$ many $\mathcal{P}$-names so that if $1 \Vdash \alpha \in \dot O \in \dot\tau$, then there is a $\dot U \in \mathcal{U}_\alpha$ such that $1 \Vdash \alpha \in \dot U \subset \dot O$. While we know that $1 \Vdash$ "$(\exists\xi \in \lambda)\ f_\alpha(\xi) \subset \dot O$", we do not necessarily have that $1 \Vdash$ "$f_\alpha(\xi) \subset \dot O$" for some $\xi \in \lambda$. What we do have then, is that for each $\dot O$ such that $1 \Vdash \alpha \in \dot O$, there is a maximal antichain $A \subset \mathcal{P}$ such that for each $p \in A$, there is an $\xi_p \in \lambda$ such that $p \Vdash f_\alpha(\xi_p) \subset \dot O$. As in the proof of the maximum principle, there is a single name $\sigma$ such that for each $p \in A$, $p \Vdash \sigma = f_\alpha(\xi_p)$. It follows that $1 \Vdash \alpha \in \sigma \subset \dot O$ for such a $\sigma$. If we start with a maximal antichain $A$ and an assignment $\{\xi_p : p \in A\}$ and construct $\sigma$ as above, then we obtain a $\sigma$ such that $1 \Vdash \alpha \in \sigma \in \tau$. Since $|A| < c(\mathcal{P})$ and the function $p \mapsto \xi_p; (p \in A)$ is a subset of $P \times \lambda$, it follows that there are only $(|P| \cdot |\lambda|)^{<c(\mathcal{P})} = \lambda$

many (up to forcing equivalence) such names $\sigma$. This proves that we can find our set of names $\mathcal{U}_\alpha$ with the properties desired:

(1) the cardinality of $\mathcal{U}_\alpha$ is at most $\lambda$,
(2) $1 \Vdash \alpha \in \dot{U}$ for each $\dot{U} \in \mathcal{U}_\alpha$, and
(3) for each $\dot{O}$ such that $1 \Vdash \alpha \in \dot{O} \in \dot{\tau}$, there is a $\dot{U} \in \mathcal{U}_\alpha$ such that $1 \Vdash \dot{U} \subset \dot{O}$.

Let $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$. Now, given a $\mathcal{P}$-name, $\dot{U}$, such that $1 \Vdash \dot{U} \in \tau$, define $W(\dot{U}) \subset \kappa \cup (\mathcal{P} \times S)$ as follows

$$W(\dot{U}) = \{\alpha \in \kappa : 1 \Vdash \alpha \in \dot{U}\} \cup \{(p,s) \in \mathcal{P} \times S : p \Vdash s \in \dot{U}\} .$$

The collection $\mathcal{W} = \{\{(p,s)\} : (p,s) \in \mathcal{P} \times S\} \cup \{W(\dot{U}) : 1 \Vdash \dot{U} \in \dot{\tau}\}$ forms a base for a topology on $Y(\mathcal{P},1) = \kappa \cup (\mathcal{P} \times S)$ with character at most $\lambda$. To see that it is a base, note that if $1 \Vdash \dot{U}_0 \subset \dot{U}_1 \cap \dot{U}_2$, then $W(\dot{U}_0) \subset W(\dot{U}_1) \cap W(\dot{U}_2)$. To see that the character is at most $\lambda$ simply note that $\{W(\dot{U}) : \dot{U} \in \mathcal{U}_\alpha\}$ is a local base at $\kappa$. Furthermore, since $1 \Vdash \kappa$ is discrete in $(X, \dot{\tau})$, there is, for any fixed $\alpha < \kappa$, a $\dot{O}$ such that $1 \Vdash \dot{O} \cap \kappa = \{\alpha\}$. It follows that $W(\dot{O}) \cap \kappa = \{\alpha\}$; hence $\kappa$ is also discrete.

More generally, for $p \in \mathcal{P}$, let

$$Y(\mathcal{P},p) = \{(q,y) \in Y(\mathcal{P},1) : q \leq p\}$$

with the subspace topology. Now, if $\dot{U}, \dot{U}' \in \mathcal{U}$ and $p \in \mathcal{P}$, then

$$p \Vdash \dot{U} \cap \dot{U}' = \emptyset \quad \text{iff} \quad Y(\mathcal{P},p) \cap \left(W(\dot{U}) \cap W(\dot{U}')\right) = \emptyset .$$

The following facts are immediate consequences of this relationship.

*Fact* 1. Let $p \in \mathcal{P}$ and $\mathcal{A} \subset \mathcal{P}(\kappa)$, then

$$\mathcal{A} \text{ is separated in } Y(\mathcal{P},p) \quad \text{iff} \quad p \Vdash_{\mathcal{P}} \mathcal{A} \text{ is separated in } X$$

*Fact* 2. $Y(\mathcal{P},1)$ is $<\kappa$-collectionwise Hausdorff.

*Fact* 3. $\kappa$ cannot be separated in $Y(\mathcal{P},1)$.

*Fact* 4. If $\mathcal{A}$ cannot be separated in $Y(\mathcal{P},1)$, then there is a $p \in \mathcal{P}$ such that

$$p \Vdash_{\mathcal{P}} \mathcal{A} \text{ cannot be separated} \quad \text{in } X .$$

*Proof.* By fact 1, $1 \nVdash A$ is separated in $X$.

This completes the proof since by our hypothesis on $V$, we know that $Y(\mathcal{P},1)$ is not normal. Therefore there is an $A \subset \kappa$ which cannot be separated from $\kappa \setminus A$ in $Y(\mathcal{P},1)$. Let $\mathcal{A} = \{A, \kappa \setminus A\}$ and apply the facts to find a $p \in \mathcal{P}$ such that $p \Vdash A$ is not separated from $\kappa \setminus A$ in $X$. $\square$

## 3. A preservation Lemma

The case $\kappa = \aleph_1$ in the following result is due to Tall and is part of the proof of Lemma 1.1.

**Theorem 3.1 (CH).** *Let $G$ be $Fn(\lambda, 2, \omega_1)$-generic over $V$, for some $\lambda$, and suppose that $X$ is a space in $V$ with a closed discrete $\aleph_0$-separated subspace, $\{x_\alpha : \alpha < \kappa \leq \lambda\}$, such that $V[G] \models X$ is normal, then $V[G] \models \{x_\alpha : \alpha < \kappa\}$ is separated .*

*Proof.* We may assume that $\kappa$ is the minimal such cardinal. Clearly, $\aleph_0 < \kappa \leq \lambda$. The generic function added by $Fn(\kappa, 2, \omega_1)$ induces a canonical partition of $\{x_\alpha : \alpha < \kappa\}$ into two disjoint closed sets, call them $G(0)$ and $G(1)$ respectively. Assume that $U(0)$ and $U(1)$ are names of disjoint open subsets of $X$ so that, for some $p \in Fn(\lambda, 2, \omega_1)$, $p \Vdash G(\ell) \subset U(\ell)$ for each $\ell \in 2$. Since the poset is homogeneous, we simplify the notation by assuming that $p = 1$. For an arbitrary $p \in Fn(\lambda, 2, \omega_1)$ and $\gamma \in \mathrm{dom}\,(p)$, let $p^\frown\gamma$ be the condition obtained from $p$ by "flipping" the value at $\gamma$, i.e. $p^\frown\gamma(\gamma) = 1 - p(\gamma)$. For each $\alpha < \kappa$, fix a maximal antichain $A_\alpha \subset Fn(\lambda, 2, \omega_1)$ so that for each $\alpha$ and $p \in A_\alpha$, there is a neighbourhood of $x_\alpha$, $W(\alpha, p)$ so that $p \Vdash W(\alpha, p) \subset U(\ell)$ and $p^\frown\alpha \Vdash W(\alpha, p) \subset U(1 - \ell)$, where $p \Vdash x_\alpha \in G(\ell)$.

Let $\{M_\xi : \xi < cf(\kappa)\}$ be a continuous elementary chain of elementary submodels of $H(\theta)$ for a sufficiently large $\theta$, so that $\{\langle A_\alpha, \{W(\alpha, p) : p \in A_\alpha\}\rangle : \alpha \in \kappa\} \in M_0$, $(\omega_1 \cup \xi) \subset M_\xi$ and $|M_\xi| < \kappa$. Now, let $G \subset Fn(\lambda, 2, \omega_1)$ be generic over $V$, and for each $\alpha \in \kappa$, let $p_\alpha \in G \cap A_\alpha$.

Suppose $\alpha, \beta \in \kappa$ and $\xi < cf(\kappa)$ are such that $\{\alpha, \beta\} \cap M_\xi = \{\alpha\}$. Note that since $A_\alpha \in M_\xi$ and $|A_\alpha| \leq \omega_1 \subset M_\xi$, it follows that $p_\alpha \in M_\xi$. Furthermore, $p_\alpha \subset M_\xi$, hence $p_\alpha$ is compatible with both $p_\beta$ and $p_\beta^\frown\beta$. Now $W(\alpha, p_\alpha) \cap W(\beta, p_\beta) = \emptyset$ since either $p_\alpha \cup p_\beta$ or $p_\alpha \cup p_\beta^\frown\beta$ forces that $W(\alpha, p_\alpha) \subset U(\ell)$ and $W(\beta, p_\beta) \subset U(1 - \ell)$ for some $\ell \in 2$. But now, since $\kappa$ was minimal, it follows that, for each $\xi < cf(\kappa)$, $1 \Vdash \{x_\alpha : \alpha < \kappa\} \cap (M_{\xi+1} \setminus M_\xi)$ can be separated. If these separations are chosen to refine the cover given by $\{W(\alpha, p_\alpha) : \alpha < \kappa\}$, then we obtain a separation of $\{x_\alpha : \alpha < \kappa\}$. $\square$

Therefore, by replacing $\omega_1$ in the above result by an arbitrary regular cardinal, Lemma 1.1 can be improved.

**Corollary 3.2.** *Let $\kappa < \mu$ be regular cardinals in $V$, such that $2^{<\kappa} = \kappa$. Let $G$ be $Fn(\mu, 2, \kappa)$-generic over $V$. In $V[G]$ every normal space of character less than $\mu$ is $<\mu$-CWH if it is $<\kappa$-CWH.*

*Proof.* Let $X$ be a name for a potential counterexample with character $< \mu$. Working in $V[G]$ fix an arbitrary closed discrete subspace of size $\kappa' < \mu$, $\{x_\alpha : \alpha < \kappa'\}$. As usual, we may assume that $|X| \leq \kappa_1 = \sup\{\chi(x_\alpha) : \alpha < \kappa\}$—where $\chi(x_\alpha)$ is the character of $x_\alpha$. Choose names for $X$ and $\{x_\alpha : \alpha < \kappa'\}$ of cardinality at most $\kappa_1$ and choose a $\gamma < \mu$ so that $X$ (including the neighbourhood bases for each point of $X$)

and $\{x_\alpha : \alpha \in \kappa'\}$ are in $V[G \cap Fn(\gamma, 2, \kappa)]$. In the model $V[G \cap Fn(\gamma, 2, \kappa)]$ apply Theorem 3.1, to deduce that since $X$ is normal in $V[G]$, $\{x_\alpha : \alpha < \kappa'\}$ is separated in $V[G \cap Fn(\gamma, 2, \kappa)]$, hence also in $V[G]$. □

## 4. The Main Theorem

**Theorem 4.1.** *There is a model of CH in which normal spaces of character at most $\aleph_2$ are collectionwise Hausdorff.*

*Proof.* (assuming Lemma 4.3): Suppose we are working in the model $L[G]$ constructed in Lemma 4.3 and that $X$ is a normal space of character at most $\aleph_2$. Let $\kappa$ be such that $X$ is $<\kappa$-CWH and we wish to show that $X$ is $\kappa$-CWH. The case $\kappa \leq \omega$ is trivial since every regular space is $\omega$-CWH. If $\kappa \geq \aleph_1$, then $\lambda = \aleph_2$ is less than $2^\kappa$, and $\kappa$-CWH follows directly from Lemma 4.3. □

**Theorem 4.2.** *There is a model of $\neg CH$ in which normal spaces of character at most $\mathfrak{c}$ are collectionwise Hausdorff.*

*Proof.* Let $G$ be $\mathrm{Fn}(\aleph_3, 2, \aleph_1) * \mathrm{Fn}(\omega_2, 2)$-generic over $L$ and apply Lemma 4.4 in which $\kappa_1 = \aleph_3$ and $\dot{Q}$ is $\mathrm{Fn}(\omega_2, 2)$. Again assume that $X$ is a normal space of character at most $\lambda = \mathfrak{c} = \aleph_2$ which is $<\kappa$-CWH for some $\kappa$. We show that it is $\kappa$-CWH. Of course we assume that $\kappa \geq \omega_1$. Now $\lambda < 2^{\aleph_1} \leq 2^\kappa$, hence $X$ is $\kappa$-CWH. □

Note that by replacing $\aleph_3$ by a larger $\kappa$ in the above proof one can find a model in which normal spaces of character even larger than $\mathfrak{c}$ are CWH.

Before proving the following Lemmas, let us make a historical remark which puts the paper in context. Let $\psi(\mu)$ be the assertion that Lemma 4.3 holds for $\kappa_1 = \mu$. Tall was the first in proving $\psi(\omega_2)$ (i.e. Lemma 1.1). Fleisner then proved Lemma 1.2 which can be considered to be $\psi(0)$. If $0 < \mu < \omega_2$, then $\psi(\mu)$ follows by the relative constructibility form of Lemma 1.2 (i.e. $\diamondsuit$-type axioms). This paper completes the picture by proving $\psi(\mu)$ for $\mu > \omega_2$.

**Lemma 4.3.** *If $\kappa_1 > \aleph_2$ is a regular cardinal in $L$, and $G$ is $Fn(\kappa_1, 2, \aleph_1)$-generic over $L$, then*

$$L[G] \models (\forall\kappa)(\forall\lambda)\ \lambda \ \textit{ regular and } \ \lambda < 2^\kappa \ \textit{ implies}$$
$$\textit{normal } <\kappa\textit{-CWH spaces of character at most } \lambda \textit{ are } \kappa\textit{-CWH}. \tag{4.1}$$

*Proof.* Recall that the conclusion of the theorem holds in $L$. Let $\kappa$ be a cardinal in $L$ and let $\varphi(\kappa)$ be the assertion that the statement of the theorem holds for $\kappa$. If $\kappa \geq \kappa_1$, then $\varphi(\kappa)$ follows directly from Theorem 2.1. For $\kappa = \aleph_1$, $\varphi(\kappa)$ is just statement 1.1. For arbitrary $\kappa < \kappa_1$, we apply Corollary 3.2. □

Now there are quite a variety of models as in Theorem 4.2 since, as the proof shows, one simply needs to take any $\kappa_1 > \omega_2$ and a ccc poset $\dot{Q}$ such that $|\dot{Q}| < \kappa_1$ in the following Lemma.

**Lemma 4.4.** *Let $\kappa_1$ be a regular uncountable cardinal in $L$ let $\dot{Q}$ be such that*

$$1 \Vdash_{Fn(\kappa_1,2,\aleph_1)} \dot{Q} \text{ is ccc and } |\dot{Q}| < \kappa_1 \ .$$

*If $G$ is $Fn(\kappa_1, 2, \aleph_1) * \dot{Q}$-generic over $L$, then*

$$\begin{aligned}(4.2)\quad L[G] \models (\forall\kappa)(\forall\lambda)\ \lambda &\text{ regular and } \lambda < 2^\kappa \text{ implies} \\ &\text{normal } {<}\kappa\text{-CWH spaces of character at most } \lambda \text{ are } \kappa\text{-CWH}\ .\end{aligned}$$

*Proof.* Let $G_1$ be the $Fn(\kappa_1, 2, \aleph_1)$-generic set given by $G$. Let $V = L[G_1]$; hence $L[G]$ is given by forcing with the poset $\mathcal{P} = \mathrm{val}_{G_1}(\dot{Q})$ over $V$. Furthermore $V$ is a model of CH, $|\mathcal{P}| < \kappa_1 = 2^{\aleph_1}$, and, of course,

$$\begin{aligned}(4.3)\quad (\forall\kappa)(\forall\lambda)\ \lambda &\text{ regular and } \lambda < 2^\kappa \text{ implies} \\ &\text{normal } {<}\kappa\text{-CWH spaces of character at most } \lambda \text{ are } \kappa\text{-CWH}\ .\end{aligned}$$

Since $\mathcal{P}$ is ccc, $c(\mathcal{P}) = \aleph_1$ and the only $\kappa$ for which Theorem 2.1 does not apply is $\kappa = \aleph_0$. □

*Remark* 4.1. It may be worthwhile to point out an aspect of Theorem 2.1 which was not needed for the results of this article. In Theorem 3.1, the non-normality is established by introducing a new partition of $\kappa$ for which there is no separation. However, in 2.1, the set $A$ which cannot be separated from $\kappa \setminus A$ is from the ground model. This means, for example, that if $X$ is in $V$, then the forcing is not adding a separation of $A$ and $\kappa \setminus A$. It has been shown that $Fn(I, 2, \aleph_0)$ preserves unseparatedness but no such result is available for the higher cardinal analogues (except for very restrictive assumptions on the topology). The result is that there is an interesting alternate approach to proving Theorems 4.1 and 4.2 which does not require Theorem 3.1. Let us reconsider the proof of Lemma 4.3. Of course, $\varphi(\kappa)$ still follows from 2.1 and 1.1 for $\kappa \geq \kappa_1$ and $\kappa = \aleph_1$ respectively. Interestingly, we have no proof for $\aleph_2 < \kappa < \kappa_1$, so assume $\kappa_1 = \omega_3$. Now suppose that $\kappa = \omega_2$. We may assume that the name for $X$ is chosen so that $1 \Vdash X$ is normal. Choose $\gamma < \omega_3$ as in 4.3 and then choose $\gamma' = \gamma + \omega_2$. By 2.1, there is an $A \subset \omega_2$ and a $p \in Fn(\gamma', 2, \aleph_1)$ such that $p \Vdash A$ is not separated from $\omega_2 \setminus A$ in $X$. From the point of view of the model $L[G \cap Fn(\gamma, 2, \aleph_1)]$, $X$ is a space with a pair of disjoint closed sets, $A, \omega_2 \setminus A$, such that forcing with $Fn(\omega_2, 2, \aleph_1)$ does not introduce a separation. Therefore there is no separation in $L[G_1]$.

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