ON MAD FAMILIES AND SEQUENTIAL ORDER

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1. Introduction

We investigate the question of the existence of a compact sequential space with sequential order more than 2. We establish a connection with an interesting problem about maximal families of almost disjoint subsets of ω. We also obtain a partial result by showing that it is consistent that every countably tight compact space of weight b is Frechet-Urysohn. The main task is to work with Shelah's forcing for obtaining a model of $b < s = a$.

The sequential order of a space is the supremum of the number of times that one must iterate the taking limits of convergent sequences in order to get the closure; that is, for $A \subset X$, $A^{(0)} = A$, $A^{(1)} = \{x \in X : (\exists \{x_n : n \in \omega\} \subset A)\ \{x_n : n \in \omega\}$ converges to $x\}$, and $A^{(\alpha)} = (\cup_{\beta<\alpha} A^{(\beta)})^{(1)}$. A space is sequential if and only if it has a sequential order ($\leq \omega_1$), and a space is Fréchet-Urysohn if its sequential order is one. A simple example of a compact space with sequential order two is the one point compactification of a Ψ-space, i.e. a space defined from a maximal almost disjoint family of a subsets of ω.

It was shown, by Bashkirov [Bas74], to follow from CH that there are compact spaces of any sequential order up to ω_1 and that MA implies the existence of a compact space of sequential order three.

2. Totally MAD families

Let us consider a simple construction of a compact space with sequential order three from the assumption $b = c$. We will then introduce an interesting property and question about mad families arising from this construction (what we will call totally mad families).

Proposition 2.1 ($b = c$). *There is a compact space with sequential order 3.*

Proof. We define two families of subsets of ω, $\{a_\alpha : \alpha < c\}$ and $\{b_\alpha : \alpha < c\}$, by induction on α. The inductive hypotheses are

(1) $\{a_\beta : \beta < \alpha\}$ are almost disjoint,
(2) for each $\beta, \gamma < \alpha$, a_β is either almost contained in b_β or is almost disjoint from it,
(3) for $\beta < \gamma < \alpha$, there is a finite subset (possibly empty) $F_{\beta,\gamma}$ of $\gamma + 1$ such that $b_\beta \cap b_\gamma$ is almost equal to $\cup\{a_\xi : \xi \in F_{\beta,\gamma}\}$.

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The goal is to have that $\{a_\alpha : \alpha < c\}$ is a maximal almost disjoint family and for each infinite $A \subset c$ there is an infinite $A' \subset A$ and a $\beta < c$ so that $a_\alpha \subset^* b_\beta$ for each $\alpha \in A'$. Therefore we simply fix enumerations, $\{x_\alpha : \alpha < c\}$ and $\{y_\alpha : \omega \leq \alpha < c\}$ of $\mathcal{P}(\omega)$ and $[c]^\omega$ respectively, where $y_\alpha \subset \alpha$ for all $\alpha \geq \omega$.

So we add two more conditions to our inductive hyptheses:

(4) for each $\beta < \alpha$, x_β meets some member of $\{a_\gamma : \gamma \leq \beta\}$ in an infinite set,

(5) for each $\gamma < \alpha$, there is $\beta \leq \gamma$ so that $a_\xi \subset^* b_\beta$ for infinitely many ξ in y_γ.

Let us show that we can find suitable a_α and b_α. Consider x_α; if $x_\alpha \cap a_\beta$ is infinite for some $\beta < \alpha$ then let $a_\alpha = \emptyset$. So assume that x_α does not meet a_β in an infinite set for any $\beta < \alpha$. If there is a $\beta < \alpha$ such that x_α meets b_β in an infinite set, then let $a_\alpha = x_\alpha \cap b_\beta$ (note that, by inductive hypothesis 3, this β is unique). If there is no such β, then we can let $a_\alpha = x_\alpha$.

Next consider the sequence $\{a_\xi : \xi \in y_\alpha\}$. If there is a $\beta < \alpha$ such that $a_\xi \cap b_\beta$ is infinite (hence cofinite) for infinitely many $\xi \in y_\alpha$, then let $b_\alpha = \emptyset$. Otherwise note that since $b = c$, we can find a function $h \in {}^{y_\alpha}\omega$ such that for each $\beta < \alpha$, there is a finite $F_{\alpha,\beta} \subset y_\alpha$ such that $a_\xi \subset b_\beta$ for each $\xi \in F_{\alpha,\beta}$, $a_\xi \cap b_\beta$ is finite for each $\xi \in y_\alpha - F_{\alpha,\beta}$ and $(a_\xi - h(\xi)) \cap b_\beta$ is empty for all but finitely many $\xi \in y_\alpha$. Furthermore, $a_\beta \cap \bigcup_{\xi \in y_\alpha} a_\xi - h(\xi)$ is finite for all $\beta \leq \alpha$. So we let $b_\alpha = \bigcup_{\xi \in y_\alpha} a_\xi - h(\xi)$. Note that it would suffice if we could do this for an infinite subset of y_α.

The Stone space of the Boolean subalgebra of $\mathcal{P}(\omega)$ generated by the union of the two families $\{a_\alpha : \alpha < c\}$ and $\{b_\alpha : \alpha < c\}$ together with the finite sets is our desired space X. The usual Ψ-space constructed from the maximal almost disjoint family $\{a_\alpha : \alpha < c\}$ is a subspace of X. The two scattering levels of Ψ constitute the first two scattering levels of X. The next scattering level of X consists of ultrafilters corresponding to each b_α, i.e. $\{b_\alpha - (\cup F \cup n) : n \in \omega$ and F is a finite subset of $\{a_\gamma : \gamma < c\}\}$ generates an ultrafilter of B. There is only one other point in the space, i.e. we have the one-point compactification of the points described so far. It is therefore trivial that X is compact and sequential. The inductive hypotheses 4 and 5 guarantee that the sequential order of X is at least 3, in fact it is equal to 3. $\square$

In the above construction, consider the step in which we chose b_α. We were given the family $\{a_\xi : \xi \in y_\alpha\}$ (an arbitrary countable subfamily of $\{a_\xi : \xi < c\}$). We can fix an enumeration $\{\xi_n : n \in \omega\}$ of y_α and, by removing a finite set from each, we may assume that the a_{ξ_n} are pairwise disjoint. Now, for each γ, there is a partial function f_γ from a subset of ω to ω such that $a_\gamma \cap a_{\xi_n}$ is not empty iff n is in the domain of f_γ and, for n in the domain of f_γ, $f_\gamma(n)$ is the largest element of $a_\gamma \cap a_{\xi_n}$. Similarly, we may assume that $b_\gamma \cap a_{\xi_n}$ is either empty or infinite for each n not in the domain of f'_γ and, for other n, $f'_\gamma(n)$ is the largest value of $a_{\xi_n} \cap b_\gamma$. In the above proof, we found a function h which bounded all of the f_γ and f'_γ for $\gamma < \alpha$. However it was not necessary to have a total function, it would have sufficed to pass to an infinite subset of $\{\xi_n : n \in \omega\}$. We would say that $\{a_\gamma : \gamma < \alpha\}$ is not totally unbounded with respect to $\{a_{\xi_n} : n \in \omega\}$ as per the following definition.

Definition 2.2. (1) Given $\mathcal{B}$ and $\mathcal{A}$, infinite families of subsets of ω, say that $\mathcal{A}$ is totally unbounded with respect to $\mathcal{B}$, if for each infinite $\mathcal{B}' \subset \mathcal{B}$ and each $h \in {}^{\mathcal{B}'}\omega$, there is an $a \in \mathcal{A}$ such that $a \cap \bigcup\{b - h(b) : b \in \mathcal{B}'\}$ is infinite.

(2) A maximal almost disjoint family $\mathcal{A}$ is totally mad if for each infinite $\mathcal{B} \subset \mathcal{A}$ no subset of $\mathcal{A}$ of cardinality less than c is totally unbounded with respect to $\mathcal{B}$.

Question 1. (1) Is there a totally mad family?
(2) Does $b = \omega_1$ imply there is a totally mad family?

Remark 2.1. A totally mad family has a refinement which is a completely separable mad family. A mad family is said to be completely separable (introduced by Hechler and Shelah [Hec71, ErSh72]) if each $X \subset \omega$ which meets infinitely many in an infinite set will actually contain one of them. The existence of a completely separable mad family has been established under various cardinal hypotheses ([BS89]) but it is not known to follow from $b = \omega_1 < s$. Our hope is that that there is a modification of Shelah's model of $b < s$ (discussed later in the paper) in which every compact separable sequential space will have sequential order at most 2 and that in all liklihood there will be no totally mad family.

The next result will be an integral tool in our construction of a totally mad family (Proposition 2.6).

Proposition 2.3. *Let $\mathcal{R} = \{R_n : n \in \omega\}$ be a partition of ω into infinite parts. Then there is a family $\mathcal{D}$ such that*

(1) *$\mathcal{R} \cup \mathcal{D}$ is a mad family on ω;*
(2) *every $D \in \mathcal{D}$ is a transversal of $\mathcal{R}$, i.e., $|D \cap R_n| \leq 1$ for every $n \in \omega$;*
(3) *whenever $\mathcal{B} \in [\mathcal{R}]^\omega$ and $\mathcal{D}_0 \in [\mathcal{D}]^{<\mathfrak{c}}$, then there is an infinite $\mathcal{B}' \subseteq \mathcal{B}$ and a function $g : \mathcal{B}' \to \omega$ such that the set $D \cap \bigcup\{B \setminus g(B) : B \in \mathcal{B}'\}$ is finite for every $D \in \mathcal{D}_0$.*

Therefore the family $\mathcal{R} \cup \mathcal{D}$ is a mad family in which no subset of $\mathcal{D}$ with cardinality less than $\mathfrak{c}$ is totally unbounded with respect to $\mathcal{R}$.

Proof. By [BPS80], we may fix $\mathcal{T}$, a tree π-base for $[\omega]^\omega$ of height at most $\mathfrak{b}$ (i.e. $\mathcal{T} \subset [\omega]^\omega$ forms a tree under the ordering of reverse inclusion mod finite and $\mathcal{T}$ is dense in $[\omega]^\omega$ with respect to this ordering). It is easily arranged that $\mathcal{T}$ is dense in $[\omega]^\omega$ with respect to the usual reverse inclusion ordering since we can just ensure that for each $T \in \mathcal{T}$ and each $n \in \omega$, there is a $T' \in \mathcal{T}$ with $T' \subset T - n$. For $X \subseteq \omega$, let $\operatorname{dom} X$ denote the set $\{n \in \omega : X \cap R_n \neq \emptyset\}$. For every $X \subseteq \omega$ with $\operatorname{dom} X$ infinite, choose some $T(X) \in \mathcal{T}$ so that $T(X) \subseteq \operatorname{dom} X$ and $T(X) \neq T(X')$, whenever $X \neq X'$. This is clearly possible since for every infinite $M \subseteq \omega$, the set of all $T \in \mathcal{T}$ with $T \subseteq M$ is of size $\mathfrak{c}$.

Enumerate $[\omega]^\omega$ as $\{X_\alpha : \alpha < \mathfrak{c}\}$ and proceed by transfinite induction. Suppose we have found $\{D_\beta : \beta < \alpha\}$ for some $\alpha < \mathfrak{c}$. If X_α is almost disjoint with all R_n, $n \in \omega$, and all D_β, $\beta < \alpha$, then let D_α be a subset of X_α, which is a transversal of $\mathcal{R}$ and such that $\operatorname{dom} D_\alpha = T(X_\alpha)$, otherwise let $D_\alpha = \emptyset$. Let $\mathcal{D} = \{D_\alpha : \alpha < \mathfrak{c}, D_\alpha \neq \emptyset\}$.

The conditions (1) and (2) immediately follow from the inductive construction. In order to verify (3), let $\mathcal{D}_0 \in [\mathcal{D}]^{<\mathfrak{c}}$ and $\mathcal{B} \in [\mathcal{R}]^\omega$. There is some $T_0 \in \mathcal{T}$ such that for every $n \in T_0$, $R_n \in \mathcal{B}$. There are $\mathfrak{c}$ almost disjoint members of $\mathcal{T}$, all contained in T_0 and $|\mathcal{D}_0| < \mathfrak{c}$, hence there is some $T \subseteq T_0$, $T \in \mathcal{T}$ such that the inclusion $\operatorname{dom} D \subseteq^* T$ holds for no $D \in \mathcal{D}_0$. Since $\mathcal{T}$ is a tree under ${}^*\supseteq$ and since $\operatorname{dom} D \in \mathcal{T}$ for every $D \in \mathcal{D}$, we have, for $D \in \mathcal{D}_0$, that either $T \cap \operatorname{dom} D$ is finite or $T \subseteq^* \operatorname{dom} D$. For $D \in \mathcal{D}_0$, let $f_D \in {}^T\omega$ be defined by $f_D(n) = 0$, if $n \notin \operatorname{dom} D$, and by $f_D(n) = k$, if $n \in T$ and $\{k\} = D \cap R_n$. Since the height of $\mathcal{T}$ is less or equal to $\mathfrak{b}$, there is some $f \in {}^T\omega$ with $f \ {}^*\!\!> f_D$ for all $D \in \mathcal{D}_0$ such that $\operatorname{dom} D \ {}^*\!\!\supseteq T$. Now, it remains to put $\mathcal{B}' = \{R_n : n \in T\}$, define $g(R_n) = f(n)$ and (3) holds with this choice of $\mathcal{B}'$ and g. □

Corollary 2.4. *Let $\alpha \in \omega_1$ be a limit ordinal, let $\mathcal{C} = \{C_\beta : \beta < \alpha\}$ be a decreasing mod finite chain of subsets of ω. Then there is a family $\mathcal{D}(\mathcal{C}) \subseteq [\omega]^\omega$ such that*

1. *for every $D \in \mathcal{D}(\mathcal{C})$ and every $\beta < \alpha$, $D \subseteq^* C_\beta$;*
2. *$\mathcal{D}(\mathcal{C})$ is an almost disjoint family;*
3. *$\mathcal{D}(\mathcal{C})$ is maximal with respect to (1) and (2);*
4. *if $\mathcal{B}$ is an almost disjoint family of sets such that for every $B \in \mathcal{B}$ there is some $\beta < \alpha$ with $B \subseteq^* (C_\beta \setminus C_{\beta+1})$ and if $\mathcal{D}_0 \in [\mathcal{D}(\mathcal{C})]^{<\mathfrak{c}}$, then there is some infinite $\mathcal{B}' \subseteq \mathcal{B}$ and a function $g : \mathcal{B}' \longrightarrow \omega$ such that the set*
$$D \cap \bigcup\{B \setminus g(B) : B \in \mathcal{B}'\}$$
is finite for every $D \in \mathcal{D}_0$.

Proof. Choose an increasing sequence $\langle \alpha_n : n \in \omega\rangle$ cofinal in α. Let $R_0 = \omega - C_{\alpha_0}$ and for $n > 0$ define $R_n = C_{\alpha_n} \setminus (C_{\alpha_{n+1}} \cup \bigcup_{k<n} R_k)$. Apply Proposition 2.3 to get $\mathcal{D} = \mathcal{D}(\mathcal{C})$. (1),(2) and (3) of the corollary follow from (1) and (2) of Lemma 2.3. If there is some $\beta < \alpha$ such that the set $\mathcal{B}' = \{B \in \mathcal{B} : B \cap C_\beta \text{ is finite}\}$ is infinite, put g to be the constant 0 and (4) follows. Otherwise there is some infinite set $\mathcal{B}'$ such that the set $\{\beta < \alpha : B \subseteq^* C_\beta, B \in \mathcal{B}'\}$ is cofinal in α and moreover, for every $n \in \omega$, there is at most one $B \in \mathcal{B}'$ with $B \subseteq^* R_n$. Now, Lemma 2.3 easily applies. □

Lemma 2.5. *Let $\mathcal{R} = \{R_n : n \in \omega\}$ be a partition of ω with each R_n infinite, let $Y \in [\omega]^\omega$ be almost disjoint with all R_n. Then there is a family $\mathcal{H} = \{H_\alpha : 0 < \alpha < \mathfrak{b}\}$ such that:*

1. *$H_{\alpha+1}$ (for each $\alpha < \mathfrak{b}$) is an infinite set, which is almost disjoint with R_n for each $n \in \omega$;*
2. *H_α (for each limit $\alpha < \mathfrak{b}$) is a decreasing mod finite chain, $H_\alpha = \{C_{\alpha,\beta} : \beta < \alpha\}$ with each $C_{\alpha,\beta}$ almost disjoint with R_n for each $n < \omega$, ;*
3. *if $\alpha < \beta < \mathfrak{b}$, then $H_{\alpha+1} \cap H_{\beta+1}$ is finite;*
4. *if $\alpha < \beta \leq \gamma < \delta < \mathfrak{b}$, then $C_{\beta,\alpha} \cap C_{\delta,\gamma}$ is finite;*
5. *if $\alpha < \beta < \gamma < \mathfrak{b}$ or $\beta < \gamma \leq \alpha < \mathfrak{b}$, then $H_{\alpha+1} \cap C_{\gamma,\beta}$ is finite;*
6. *whenever $X \in [\omega]^\omega$ is almost disjoint with each R_n ($n \in \omega$), then either $X \cap H_{\alpha+1}$ is infinite for some $\alpha < \mathfrak{b}$ or there is some limit $\alpha < \mathfrak{b}$ such that $X \cap C_{\alpha,\beta}$ is infinite for all $\beta < \alpha$;*
7. *$|Y \cap H_1| = \omega$.*

Proof. Enumerate $R_n = \{r(n,k) : k \in \omega\}$ and fix a family of strictly increasing functions $\{f_\alpha : \alpha < \mathfrak{b}\} \subset {}^\omega\omega$ which is unbounded with respect to $<^*$ and, moreover, $f_\alpha <^* f_\beta$ for $\alpha < \beta < \mathfrak{b}$. In addition, assume that $\{n : Y \cap \{r(n,k) : k < f_1(n)\} \neq \emptyset\}$ is infinite. Let $H_1 = \{r(n,k) : n \in \omega, k \in \omega, k < f_1(n)\}$; $H_{\alpha+1} = \{r(n,k) : n \in \omega, k \in \omega, f_\alpha(n) \leq k < f_{\alpha+1}(n)\}$ for $0 < \alpha < \mathfrak{b}$, and finally let $C_{\alpha,\beta} = \{r(n,k) : n \in \omega, k \in \omega, f_\beta(n) \leq k < f_\alpha(n)\}$ for $\beta < \alpha < \mathfrak{b}$, α limit.

It is straightforward to verify that, since for all $\alpha < \beta < \mathfrak{b}$, $f_\alpha(n) < f_\beta(n)$ holds for almost all $n \in \omega$, (1) up to (5) are satisfied. (6) holds because the family $\{f_\alpha : \alpha < \mathfrak{b}\}$ is unbounded and all the functions f_α are strictly increasing. By the choice of f_1, (7) holds. $\square$

Proposition 2.6 ($\mathfrak{b} = \mathfrak{s} = \omega_1$ or $\mathfrak{b} = \mathfrak{c}$). *There is a totally mad family on ω.*

Proof. Of course the statement follows trivially from $\mathfrak{b} = \mathfrak{c}$, so we assume $\mathfrak{b} = \mathfrak{s} = \omega_1$. Fix a splitting family $\{S_\xi : \xi < \omega_1\}$ and an unbounded family of functions $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq {}^\omega\omega$ satisfying: for $\alpha < \beta < \mathfrak{b}$, $f_\alpha <^* f_\beta$, moreover, $\lim f_\beta(n) - f_\alpha(n) = \infty$ and $f_0 \geq \mathrm{id}$.

We shall construct simultaneously, for $\xi < \omega_1$, a family $\mathfrak{M}_\xi \subset [\omega]^\omega$ consisting of decreasing mod finite chains of countable limit length, a mad family $\mathcal{A}_\xi$ on ω and an almost disjoint family $\mathcal{D}_\xi$, with the aim to get $\bigcup_{\xi<\omega_1} \mathcal{D}_\xi$ totally mad.

The starting point is quite simple: let $\mathcal{D}_0 = \{D_n : n \in \omega\}$ be an arbitrary partition of ω with each D_n infinite, let $\mathfrak{M}_0$ consist of just one chain $\mathcal{C}$, where $\mathcal{C} = \{\omega \setminus \bigcup_{k<n} D_k : n \in \omega\}$ and let $\mathcal{A}_0$ be the union of $\mathcal{D}_0 \cup \mathcal{D}(\mathcal{C})$, where $\mathcal{D}(\mathcal{C})$ is the family described in Corollary 2.4.

Let $\xi < \omega_1$ and suppose that all $\mathcal{D}_\eta$, $\mathfrak{M}_\eta$ and $\mathcal{A}_\eta$ are known for $\eta < \xi$. The induction assumptions are as follows:

(a) for every $\eta < \zeta < \xi$, $\mathcal{A}_\zeta$ refines $\mathcal{A}_\eta$ and $\bigcup_{\xi\leq\zeta} \mathcal{D}_\xi \subset \mathcal{A}_\zeta$;
(b) for every $\eta < \zeta < \xi$ and for every chain $\mathcal{C} \in \mathfrak{M}_\eta$ there is some $\mathcal{C}' \in \mathfrak{M}_\zeta$ with $\mathcal{C} \subseteq \mathcal{C}'$ and $\mathcal{C}'$ is an end-extension of $\mathcal{C}$;
(c) for every $\eta < \xi$, for every $D \in \bigcup_{\zeta\leq\eta} \mathcal{D}_\zeta$ and for every chain $\mathcal{C} \in \mathfrak{M}_\eta$ there is some $C \in \mathcal{C}$ with $D \cap C$ finite, similarly, for two distinct chains $\mathcal{C}, \mathcal{C}' \in \mathfrak{M}_\eta$ there is $C \in \mathcal{C}$ and $C' \in \mathcal{C}'$ with $C \cap C'$ finite;
(d) for every $\eta < \xi$ and for every chain $\mathcal{C} \in \mathfrak{M}_\eta$ there is a family $\mathcal{D}(\mathcal{C})$ chosen as in Corollary 2.4, such that for each $A \in \mathcal{A}_\eta$ either there is a $C \in \mathcal{C}$ such that $A \cap C$ is finite, or there is a $D \in \mathcal{D}(\mathcal{C})$ such that $A \subset D$;
(e) for every $\eta < \xi$ and for every infinite $X \subseteq \omega$, either $|X \cap D| = \omega$ for some $D \in \bigcup_{\zeta\leq\eta} \mathcal{D}_\zeta$ or there is a chain $\mathcal{C} \in \mathfrak{M}_\eta$ such that $|X \cap C| = \omega$ for all $C \in \mathcal{C}$;
(f) if $\eta + 1 < \xi$, then for every $\mathcal{C} \in \mathfrak{M}_{\eta+1}$ there is some $C \in \mathcal{C}$ with $C \subseteq S_\eta$ or $C \cap S_\eta = \emptyset$;
(g) if $\eta + 1 < \xi$ and $\mathcal{C} \in \mathfrak{M}_\eta$, then for every infinite $\mathcal{B} \subset \mathcal{D}_{\eta+1}$ such that, for each $D \in \mathcal{B}$, $D \subset^* C$ for all $C \in \mathcal{C}$, there is a $\mathcal{C}' \in \mathfrak{M}_{\eta+1}$ and a descending mod finite sequence $\{C_n : n \in \omega\} \subset \mathcal{C}'$ such that, for each n, there is a $B \in \mathcal{B}$ which is contained mod finite in $C_n - C_{n+1}$ while if $\eta < \xi$ is a limit ordinal, then $\mathcal{D}_\eta$ is empty.

If $\xi < \omega_1$ is limit, let $\mathfrak{M}_\xi$ consist of all chains $\mathcal{C}$ such that for every $\eta < \xi$ there is some $\mathcal{C}_\eta \in \mathfrak{M}_\eta$ with $\mathcal{C} = \bigcup_{\eta<\xi} \mathcal{C}_\xi$. For every $\mathcal{C} \in \mathfrak{M}_\xi$, let $\mathcal{D}(\mathcal{C})$ be the almost disjoint family resulting from the application of Corollary 2.4 to $\mathcal{C}$. Then $\mathcal{B}$ defined as $\bigcup_{\eta<\xi} \mathcal{D}_\eta \cup \bigcup\{\mathcal{D}(\mathcal{C}) : \mathcal{C} \in \mathfrak{M}_\xi\}$ is a maximal almost disjoint family on ω; let $\mathcal{A}_\xi$ be an arbitrary mad family on ω which refines $\mathcal{B}$ as well as all $\mathcal{A}_\eta$, $\eta < \xi$. Since $\xi < \omega_1$, such an $\mathcal{A}_\xi$ is easy to find. Finally, put $\mathcal{D}_\xi = \emptyset$.

For a successor ordinal $\xi = \eta + 1$, we proceed more carefully. First, denote by $\mathcal{Y}$ the family of all $Y \in [\omega]^\omega$ such that the set $\{\mathcal{C} \in \mathfrak{M}_\eta : \text{for every } C \in \mathcal{C}, |Y \cap C| = \omega\}$ is of size $\mathfrak{c}$. Then for every $Y \in \mathcal{Y}$ select one chain $\mathcal{C}(Y) \in \mathfrak{M}_\eta$ such that Y meets all $C \in \mathcal{C}(Y)$ in an infinite set and do it so that distinct Y's are paired to distinct $\mathcal{C}(Y)$'s.

Consider $\mathcal{C} \in \mathfrak{M}_\eta$. There is some $\alpha < \omega_1$ such that $\mathcal{C} = \{C_\beta : \beta < \alpha\}$ and in the previous induction step some cofinal sequence $\langle \alpha_n : n \in \omega\rangle$ was selected in order to find $\mathcal{D}(\mathcal{C})$ using Corollary 2.4. Let $R_n = C_{\alpha_n} \setminus (C_{\alpha_{n+1}} \cup \bigcup_{k<n} R_k)$ and apply Lemma 2.5 to the family $\mathcal{R} = \{R_n : n \in \omega\}$ to select the family $\{H_\alpha : \alpha < \mathfrak{b}\}$. Since all members of $\mathcal{D}(\mathcal{C})$ are transversals of $\mathcal{R}$ and since $\lim f_{\alpha+1}(n) - f_\alpha(n) = \infty$, every $H_{\alpha+1}$ meets uncountably many D's from $\mathcal{D}(\mathcal{C})$, hence also uncountably many A's from $\mathcal{A}_\eta$. For every $\alpha < \mathfrak{b}$ $(= \omega_1)$ choose a countable family $\{A_n(\mathcal{C},\alpha) : n \in \omega\} \subseteq \mathcal{A}_\eta$ such that $A_n(\mathcal{C},\alpha) \cap H_{\alpha+1}$ is infinite for every n, and put $D_n(\mathcal{C},\alpha) = \{A_n(\mathcal{C},\alpha) \cap H_{\alpha+1} : n \in \omega\}$. In the case that $\mathcal{C} = \mathcal{C}(Y)$, just one modification is necessary: we shall guarantee that $D_0(\mathcal{C},1)$ meets Y in an infinite set. This is always possible with the aid of Lemma 2.5 (7), and due to the fact that $\mathcal{A}_\eta$ is a mad family. Let $\mathcal{D}_\xi$ be the set of all infinite members of $\{D_n(\mathcal{C},\alpha) \cap S_\eta : n \in \omega, \alpha \in \omega_1, \mathcal{C} \in \mathfrak{M}_\eta\} \cup \{D_n(\mathcal{C},\alpha) - S_\eta : n \in \omega, \alpha \in \omega_1, \mathcal{C} \in \mathfrak{M}_\eta\}$.

If $\mathcal{C} \in \mathfrak{M}_\eta$, then $\mathfrak{M}_\xi(\mathcal{C})$ will denote the family, which we now define, of all chains to which $\mathcal{C}$ extends. We now take care of condition (f). For $\alpha + 1 < \omega_1$, if all sets $H_{\alpha+1} \cap S_\eta \setminus \bigcup_{k<n} D_k(\mathcal{C},\alpha)$ are infinite, then $\mathcal{C} \cup \{H_{\alpha+1} \cap S_\eta \setminus \bigcup_{k<n} D_k(\mathcal{C},\alpha) : n \in \omega\}$ belongs to $\mathfrak{M}_\xi(\mathcal{C})$. Similarly, if all sets $H_{\alpha+1} \cap (\omega \setminus S_\eta) \setminus \bigcup_{k<n} D_k(\mathcal{C},\alpha)$ are infinite, then $\mathcal{C} \cup \{H_{\alpha+1} \cap (\omega \setminus S_\eta) \setminus \bigcup_{k<n} D_k(\mathcal{C},\alpha) : n \in \omega\}$ belongs to $\mathfrak{M}_\xi(\mathcal{C})$ as well. If $\alpha < \omega_1$ is limit, then $\mathcal{C} \cup H_\alpha$ is a chain, however, we again take the member S_η from the splitting family into account: $\mathcal{C} \cup \{C_{\alpha,\beta} \cap S_\eta : \beta < \alpha\}$ and $\mathcal{C} \cup \{C_{\alpha,\beta} \setminus S_\eta : \beta < \alpha\}$ will be members of $\mathfrak{M}_\xi(\mathcal{C})$, provided they do not contain finite sets. Clearly, for each $0 < \alpha < \omega_1$, at least one of the chains defined above is put into $\mathfrak{M}(\mathcal{C})$. Set $\mathfrak{M}_\xi = \bigcup\{\mathfrak{M}_\xi(\mathcal{C}) : \mathcal{C} \in \mathfrak{M}_\eta\}$.

It remains to define $\mathcal{A}_\xi$. Similar to the limit step, let $\mathcal{D}(\mathcal{C})$ be the almost disjoint family resulting from the application of Corollary 2.4 to the chain $\mathcal{C}$. Then $\mathcal{B} = \bigcup_{\eta \le \xi} \mathcal{D}_\eta \cup \bigcup\{\mathcal{D}(\mathcal{C}) : \mathcal{C} \in \mathfrak{M}_\xi\}$ is a mad family and it remains to define $\mathcal{A}_\xi = \{A \cap B : A \in \mathcal{A}_\eta, B \in \mathcal{B}, |A \cap B| = \omega\}$.

This completes the inductive definitions. The verification of the induction assumptions is straightforward and will be left to the reader.

We have to verify that $\mathcal{D} = \bigcup_{\xi<\omega_1} \mathcal{D}_\xi$ is totally mad. The almost disjointness of $\mathcal{D}$ should be clear, so let us first prove its maximality. Let $Y \in [\omega]^\omega$ be arbitrary, assume that $\xi(0) < \omega_1$ is the first ordinal such that both sets $Y \cap S_{\xi(0)}$ and $Y \setminus S_{\xi(0)}$ are infinite. If Y does not meet any $D \in \bigcup_{\eta \le \xi(0)} \mathcal{D}_\eta$ in an infinite set, then by (e), there is some chain $\mathcal{C}$ in $\mathfrak{M}_\xi$ such that $Y \cap C$ is infinite for all $C \in \mathcal{C}$, however, by (f), there must be two chains $\mathcal{C}_0$ and $\mathcal{C}_1$ in $\mathfrak{M}_\xi$ with this property. Choose two infinite subsets Y_0 and Y_1 of Y such that $Y_0 \subseteq^* C$ for all $C \in \mathcal{C}_0$ and $Y_1 \subseteq^* C$ for all $C \in \mathcal{C}_1$. Let $\xi(1)$ be maximum of the first ζ with both $S_\zeta \cap Y_0$ and $Y_0 \setminus S_\zeta$ infinite and of the first η with both $S_\eta \cap Y_1$ and $Y_1 \setminus S_\eta$ infinite. If Y still meets no $D \in \bigcup_{\eta \le \xi(1)} \mathcal{D}_\eta$ in an infinite set, then Y_0 is compatible with two chains $\mathcal{C}_{00}$, $\mathcal{C}_{01} \in \mathfrak{M}_{\xi(1)}$, both extending $\mathcal{C}_0$, and similarly for $Y(1)$. Proceeding further, if the branching continues, we arrive finally to a limit $\xi = \sup\{\xi(n) : n \in \omega\}$. Since Y meets all members from $\mathfrak{c}$ chains from $\mathfrak{M}_\eta$, our construction ensures that $Y \in \mathcal{Y}$ at the $\xi + 1$ st step of the induction and so $Y \cap D$ is infinite for some $D \in \mathcal{D}_{\xi+1}$. Since Y was arbitrary, we conclude that $\mathcal{D}$ is a mad family.

Let $\mathcal{B} \in [\mathcal{D}]^\omega$, $\mathcal{E} \in [\mathcal{D}]^{<\mathfrak{c}}$ be two disjoint subsets of $\mathcal{D}$. Then there is some infinite

$\{B_n : n \in \omega\} \subseteq \mathcal{B}$, some $\xi < \omega_1$ and a chain $\mathcal{C} = \{C_\beta : \beta < \alpha\} \in \mathfrak{M}_\xi$ such that for some increasing sequence of ordinals $\langle \beta_n : n \in \omega \rangle$ cofinal in α, $B_n \subseteq^* C_{\beta_n} \setminus C_{\beta_n+1}$. To see this, let δ be minimal such that $\mathcal{B} \cap \bigcup_{\eta<\delta} \mathcal{D}_\eta$ is infinite; hence we may assume that $\mathcal{B} \subset \bigcup_{\eta<\delta} \mathcal{D}_\eta$. If $\delta = 0$ then it is clear that we may take $\mathcal{C}$ to be the unique member of $\mathfrak{M}_0$. Otherwise, recursively choose a sequence $\mathcal{C}_\eta \in \mathfrak{M}_\eta$ so that $\beta < \eta$ implies $\mathcal{C}_\beta \subset \mathcal{C}_\eta \in \mathfrak{M}_\eta$ and there are infinitely many $B \in \mathcal{B}$ which are contained mod finite in every member of $\mathcal{C}_\eta$. Clearly there is some $\xi \leq \delta$ for which we cannot choose $\mathcal{C}_\xi$. If $\xi = \beta + 1$ is a successor then we can apply inductive hypothesis (g) to find the required $\mathcal{C} \in \mathfrak{M}_\xi$ and sequence $\langle \beta_n : n \in \omega \rangle$. If on the other hand, ξ is a limit, then $\mathcal{C} = \bigcup_{\eta<\xi} \mathcal{C}_\eta \in \mathfrak{M}_\xi$ is the desired chain and it should be clear, by inductive hypotheses (a) and (d), that there is a sequence $\langle \beta_n : n \in \omega \rangle$ as required (i.e. there is an infinite sequence $\langle B_n : n \in \omega \rangle \subset \mathcal{B}$ and an increasing sequence $\langle \eta_n : n \in \omega \rangle \subset \xi$ such that B_n is almost contained in every member of $\mathcal{C}_{\eta_n}$ but is almost disjoint from some member of $\mathcal{C}_{\eta_n+1}$).

Having found $\mathcal{C}$, let α be the length of $\mathcal{C}$ and let $\langle \alpha_n : n \in \omega \rangle$ be the increasing sequence that was used in the application of Corollary 2.4 when defining $\mathcal{D}(\mathcal{C})$. Choose an infinite subset $\mathcal{B}_0$ of $\mathcal{B}$ so that for each $n \in \omega$, there is at most one $B \in \mathcal{B}_0$ such that $B \subset^* C_{\alpha_n} - C_{\alpha_{n+1}}$. Now, by construction, we have that for every $E \in \mathcal{E}$, either there is an n such that $E \cap C_{\alpha_n}$ is finite or there is a $D \in \mathcal{D}(\mathcal{C})$ such that $E \subset^* D$ (by hypotheses (a) and (d)). Therefore by Lemma 2.3, there is a suitable $\mathcal{B}' \subset \mathcal{B}$ and function $h \in {}^{\mathcal{B}'}\omega$ witnessing that $\mathcal{E}$ is not totally unbounded with respect to $\mathcal{B}$. The proposition is proved. $\square$

3. Shelah's $b < a$ poset

For the reader's convenience we will reproduce the results from [She84] that we will need in the next section.

Definition 3.1. 1) Let K_n be the family of pairs (s, h), s a hereditarily finite set, h a partial function from $\mathcal{P}(s)$ to $n + 1$ such that

(1) $h(s) = n$
(2) if $h(t) = \ell + 1$ $(t \subseteq s)$, $t = t_1 \cup t_2$ then $h(t_1) \geq \ell$ or $h(t_2) \geq \ell$.

2) Let $K_{\geq n} = \bigcup_{k \geq n} K_k$ and $K = \bigcup_n K_n$.

One example of a member of $(s, h) \in K_n$ is to take s to be the set of integers $[0, 2^n]$ and to define $h(t) = \log_2(|t|)$ for $t \subseteq s$.

Definition 3.2. An ordering on K is defined in two steps:

(1) Suppose (s, h) and (s', h') are members of K, we say that $(s, h) \leq^d (s', h')$ (we think of (s, h) as refining (s', h')) if:
    (a) $s = s'$;
    (b) $\mathrm{dom}(h) \subset \mathrm{dom}(h')$;
    (c) for some integer i, $h(u) = \max\{0, h'(u)\}$ for all $u \in \mathrm{dom}(h)$.
(2) $(s, h) \leq^e (s', h')$ if for some $\bar{s} \in \mathrm{dom}\, h'$, $(s, h) = (\bar{s}, h' \restriction \mathcal{P}(\bar{s}))$.
(3) $(s, h) < (s', h')$ if for some $(\bar{s}, \bar{h})$,

$$(s, h) \leq^d (\bar{s}, \bar{h}) \leq^e (s', h') \ .$$

We next define a set L consisting of finite trees of members of K whose maximal elements will be integers. These will form the component pieces of the conditions of our poset and they represent a certain measure of how likely it is that the finite sets of integers will be contained in the generic filter.

Definition 3.3. L is the union of all the L_n where for each n, L_n is the family of pairs (S, H) such that:

(1) S is a finite tree with a root;
(2) $int(S)$ is the set of maximal nodes of S and will be a subset of ω, the set of non-maximal nodes of S will be denoted $in(S)$ and we will assume that $in(S)$ is disjoint from ω;
(3) H is a function whose domain is $in(S)$ and let H_x denote $H(x)$ for $x \in in(S)$;
(4) For $x \in in(S)$, $(Suc_S(x), H_x) \in K_{\geq n}$ where $Suc_S(x)$ is the set of immediate successors of x in S.

If $(S, H) \in L$, then $lev(S, H) = \max\{n : (S, H) \in L_n\}$.

The ordering on K will lift naturally to an ordering on L as follows.

Definition 3.4. We say that $(S^0, H^0) \leq (S^1, H^1)$ if $S^0 \subseteq S^1$, they have the same root, $in(S^0) = S^0 \cap in(S^1)$ and, for every $x \in in(S^0)$, $(Suc_{S^0}(x), H^0_x) \leq (Suc_{S^1}(x), H^1_x)$.

If $t \in L$, we use (S^t, H^t) to denote its components. For $(S, H) \in L$ and $x \in in(S)$, let $(S, H)^{[x]} = (S^{[x]}, H \restriction S^{[x]})$ where $S^{[x]} = \{y \in S : x \leq y \text{ in S }\}$. An important tool in the proofs is to define half(S, H) for (S, H) in L_n, where half$(S, H) = (S', H')$ is defined by $S' = S$, $\mathrm{dom}(H'_x) = \{A \in \mathrm{dom}(H_x) : H_x(A) \geq n/2\}$ and $H'_x(A) = [H_x(A) - n/2]$ (the greatest integer function). The usefulness is in the following

Fact 1. If $t \in L_n$ and $t' \leq$ half(t), then $\bar{t}$ (canonically obtained from t' and t) is such that $\bar{t} \leq t$, $int(t') = int(\bar{t})$, and $\bar{t} \in L_{[n/2]}$. Define $\bar{t}$ as follows: $S^{\bar{t}} = S^{t'}$ and $H^{\bar{t}} = H^t \restriction S^{t'}$ in the sense that for each $x \in S^{\bar{t}}$, $H^{\bar{t}}_x = H^t_x \restriction \mathcal{P}(Suc_{S^{t'}}(x))$

An important fact for getting the splitting number to go up, as well as in some of the proofs of the properties of the poset we will define is the following.

Fact 2. If $(S, H) \in L_{n+1}$ and $int(S) = A_0 \cup A_1$, then there is $(S', H') \leq (S, H)$ such that $int(S') \subseteq A_0$ or $int(S') \subseteq A_1$ and $(S', H') \in L_n$.

Definition 3.5. The forcing-notion Q is defined as follows:

(1) $p \in Q$ if $p = (W, T)$ where W is a finite subset of ω and T is a countably infinite set of pairwise disjoint members of L such that $T - L_n$ is finite for each n. Let $int(p)$ and $int(T)$ denote the set $\bigcup\{int(t) : t \in T\}$.
(2) Given $t_1 = (S_1, H_1), \ldots, t_k = (S_k, H_k)$ all from L such that $S_i \cap S_j = \emptyset$ for $i \neq j$, t is <u>built</u> from $t_1, \ldots, t_k$ if: There are incomparable nodes $a_1, \ldots, a_k$ of S such that every node of S is comparable with one of the a_i and such that $(S, H)^{[a_i]} = (S_i, H_i)$ for $i = 1, \ldots, k$.

(3) $(W,T) \leq (W',T')$ if $W' \subset W \subset W' \cup int(T')$ and letting $T' = \{t'_0, t'_1, \dots\}$ and $T = \{t_0, t_1, \dots\}$, there are $\bar{t}_i \leq t'_i$ for each i and pairwise disjoint finite subsets, $B_0, B_1, \dots$, of $\{\bar{t}_i : i \in \omega\}$ such that, for each $m \in \omega$, t_m is built from B_m. It follows that for each n only finitely many of the B_m's will meet $L - L_n$.

(4) We also demand that for $(W,T) \in Q$ and $t, t' \in T$, $\max W < \min(int(t))$ and either $\max(int(t)) < \min(int(t'))$ or vice-versa.

It is natural to think of W as the root of the condition $(W,T) \in Q$ and root preserving extensions will be important. The reader familiar with Axiom A forcings will observe that Q is an Axiom A poset and there is a natural choice for $<_n$ similar to the $<_0$ defined next.

Definition 3.6. For $p \in Q$ we use (W^p, T^p) and $p <_0 q$ will mean that $p < q$ and $W^p = W^q$. In addition, we will use $\{t^p_n : n \in \omega\}$ to denote T^p listed so that $\max(int(t_n))$ is increasing. If $p \in Q$, $n \in \omega$, and $W \subseteq 1 + \max(int(t^p_n))$ then p^n_W denotes the condition $(W, \{t^p_\ell : \ell > n\})$.

Definition 3.7. For a proper ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ (which includes all finite sets) let $Q[\mathcal{I}]$ denote the set of $q \in Q$ such that for each $A \in \mathcal{I}$, there are infinitely many $t \in T^q$ such that $int(t) \cap A$ is empty.

Proposition 3.8. *(1) If $p \in Q$ and τ_n are Q-names of ordinals, then there is a $q <_0 p$, such that for each $k \leq n < \omega$ and each $W \subseteq \max[int(t^q_n)] + 1$, q^n_W forces a value on τ_k iff some $<_0$ extension of q^n_W forces a value on τ_k.*

(2) Q is proper.

(3) $\Vdash_Q$ "$\{n : (\exists p \in G_Q)[n \in W^p]\}$ is an infinite subset of ω which $\mathcal{P}(\omega)^V$ does not split".

Proof. Parts (1) and (2) are proven below for $Q[\mathcal{I}]$, the more difficult case, so we will not prove them here. Item (3) follows from Fact 2 above. $\square$

Proposition 3.9. *Let $\mathcal{C}_{\omega_1}$ denote the poset for adding ω_1 Cohen reals. In $V^{\mathcal{C}_{\omega_1}}$ the following hold for an ideal $\mathcal{I} \in V$ with $[\omega]^{<\omega} \subset \mathcal{I} \subset \mathcal{P}(\omega)$:*

(1) If $p \in Q[\mathcal{I}]$ and τ_n are $Q[\mathcal{I}]$-names of ordinals then there is a $<_0$ extension q of p such that: $q \in Q[\mathcal{I}]$, and for each $k \leq n < \omega$ and each $W \subset \max[int(t_n)] + 1$, q^n_W forces a value on τ_k iff some $<_0$ extension of q^n_W forces a value on τ_k.

(2) $Q[\mathcal{I}]$ is proper.

(3) $\Vdash_{Q[\mathcal{I}]}$ "$\{n : (\exists p \in G_{Q[\mathcal{I}]})\ \ n \in W^p\}$ is an infinite subset of ω which is almost disjoint from every $A \in I$".

Proof. (1) Let λ be a large enough regular cardinal and let M be a countable elementary submodel of $(H(\lambda), \in, V \cap H(\lambda))$ to which all of $\mathcal{I}$, the set of ω_1 many Cohen reals given by $\mathcal{C}_{\omega_1}$, $Q[\mathcal{I}]$, p, and τ_n, for each $n \in \omega$, all belong. Let $\delta = M \cap \omega_1$. Define by induction on $n < \omega$, $q^n \in Q[\mathcal{I}] \cap M$, t_n and $k_n < \omega$ such that:

(1) each q^n is a $<_0$ extension of p,

(2) $q^n \leq q^\ell$ for $\ell < n$ and if $W \subset k_n$, $m < n+1$ and some $<_0$ extension of (W, T^{q^n}) forces a value on τ_m, then (W, T^{q^n}) does so.

(3) $k_n > k_\ell$ and $k_n > \max(int(t_\ell))$ for $\ell < n$.

(4) every $\ell \in int(q^n)$ is greater than k_n.

(5) $t_n \in T^{q^n}$ and $\mathrm{lev}(t_n) > n$ and $\min(int(t_n))$ is greater than k_n.

To do so, first choose k_n, then q^n, and at last t_n. We want in the end to let $T^q = \{t_n : n < \omega\}$. One point is missing. Why does $q = (W^p, T^q)$ belong to $Q[\mathcal{I}]$ (not just to Q)? The answer is that it may not but we may choose a real $r \in {}^\omega\omega$ which is Cohen generic over $V[M]$ and choose t_n to be the $r(n)$-th possible one according to some fixed well-ordering from V of the hereditarily finite sets (i.e. all possible t's). Then for any $A \in \mathcal{I}$ there are infinitely many such t_n for which $int(t_n) \cap A = \emptyset$, hence there will be infinitely many n such that $int(t_n) \cap A$ is empty.

Now the proof of (2) is quite standard. Let M etc. be as above. Recall that from [She82], it suffices to show that there is a condition $q < p$ such that for each $\tau \in M$ which is a $Q[\mathcal{I}]$-name of an ordinal, q forces that τ takes on a value from M. So in condition (1) we may take $\{\tau_n : n \in \omega\}$ to be the list of all $Q[\mathcal{I}]$-names of ordinals which are members of M. Let r be any extension of q which forces a value on some τ_k. Let $n > k$ be chosen so that $W^r \subset \max[int(t_n)] + 1$. Clearly $(W^r, \{t_\ell : \ell > n\})$ has a $<_0$-extension (namely r) which forces a value on τ_k. Therefore $q^n_{W^r}$ forces a value on τ_k and, since $q^n_{W^r}$ is a member of M, this value is in M.

Statement (3) follows trivially from the fact that for each $A \in \mathcal{I}$, the set $\{q \in Q[\mathcal{I}] : int(T^q) \cap A = \emptyset\}$ is dense in $Q[\mathcal{I}]$. $\square$

We will also need the fact that Q is almost ${}^\omega\omega$-bounding – this is certainly the property of Q which is the most difficult to obtain (and which does not hold for $Q[\mathcal{I}]$ for many choices of $\mathcal{I}$). This proof we will need to reproduce carefully as we will need to generalize it in the next section. We refer the reader to [She82] for the basic properties of almost ${}^\omega\omega$-bounding forcings: e.g. that a countable support iteration of almost ${}^\omega\omega$-bounding proper posets is itself weakly bounding. We give the definitions below.

Lemma 3.10. *Let $(\emptyset, T) \in Q$ and let $\mathcal{W}$ be a family of finite subsets of $int(T)$ so that*

$$\text{for every } (\emptyset, T') < (\emptyset, T), \text{ there is a } W \subset int(T') \text{ with } W \in \mathcal{W}\ . \tag{3.1}$$

Then for each $k \in \omega$, there is a $t \in L_k$ appearing in some $(\emptyset, T') < (\emptyset, T)$ such that, for every $t' \le t$, there is a $W \in \mathcal{W}$ with $W \subseteq int(t')$.

Proof. Let T be enumerated as $\{t_n : n \in \omega\}$ and assume with no loss of generality that $\mathcal{W}$ is closed upwards. We first do the case $k = 1$. The desired tree t will be built from a set $\{t'_n : n \in u\}$ where $t'_n \le t_n$ for each n. For this we need a finite subset u of ω together with a partial function $H : \mathcal{P}(u) \to \omega$ such that $H(u) = 1$ and $H(u_1) \ge 0$ or $H(u_2) \ge 0$ whenever $u = u_1 \cup u_2$ and whenever $H(v) \ge 0$, then for every choice $\bar{t}_n \le t'_n \ \ (n \in v)$ the set $\bigcup_{\ell \in v} int(\bar{t}_n)$ belongs to $\mathcal{W}$.

This suggests defining a partial function $H : [\omega]^{<\omega} \to \omega$ by the following conditions:

1. $u \in \mathrm{dom}(H)$ iff for every $\ell \in u$ the tree $\mathrm{half}(t_\ell)$ belongs to L_1 and for all possible choices $t'_\ell \le \mathrm{half}(t_\ell) \ \ (\ell \in u)$ the set $\bigcup_{\ell \in u} int(t'_\ell)$ belongs to $\mathcal{W}$, and
2. $H(u) \ge n + 1$ iff whenever $u = u_1 \cup u_2$ one has $H(u_1) \ge m$ or $H(u_2) \ge m$.

It is assumed tacitly that $H(u)$ is always chosen as large as possible so that for example, $H(u) = 0$ implies there are $v, w \subset u$ such that $u = v \cup w$ but $v, w \notin \mathrm{dom}(H)$. The need to use $\mathrm{half}(t_\ell)$ will become apparent shortly.

Once we have u with $H(u) \ge 1$ we define $t = (S^t, H^t)$ by taking u as the root of the tree S^t and putting the trees $\mathrm{half}(t_\ell)$ as its immediate successors. The function

H^t_u is taken to be H on the successors of u and the appropriate $H^{\mathrm{half}(t_\ell)}$ higher up. This defines an element of L_1 that is built using trees that were below elements of T and as such it appears in a condition $(\emptyset, T')$ below $(\emptyset, T)$.

Now if $t' < t$ then the successors of u that are in t' form a set u' that is in $\mathrm{dom}(H^{t'})$ and hence in $\mathrm{dom}(H)$ so that we have $H(u') \geq 0$. But t' is built using $t'_\ell \leq \mathrm{half}(t_\ell)$ with $\ell \in u'$, so $int(t') = \bigcup_{\ell \in u'} int(t'_\ell)$ belongs to $\mathcal{W}$.

We show that there is such a u in three stages.

<u>Stage A:</u> There is n such that if $t'_\ell < \mathrm{half}(t_\ell)$ is chosen for each $\ell < n$, then $\bigcup_{\ell<n} int(t'_\ell) \in \mathcal{W}$. This is because, otherwise, the family of all $\langle t'_\ell : \ell < n\rangle$ such that $n < \omega$, $t'_\ell < \mathrm{half}(t_\ell)$ for $\ell < n$ and $\bigcup_{\ell<n} int(t'_\ell) \notin \mathcal{W}$ form an ω-tree with finite branching. By König's lemma there is an infinite branch, say $\langle t'_n : n \in \omega\rangle$. But now there is a T' such that $(\emptyset, T') < (\emptyset, T)$ and $int(T') = \bigcup_{n\in\omega} int(t'_n)$. The only problem with taking $T' \subset \{t'_n : n \in \omega\}$ is that $\{lev(t'_n) : n \in \omega\}$ may be bounded. However by Fact 1, we can replace each t'_n by some $\bar{t}_n$ where the level of $\bar{t}_n$ is at least one-half of that of t_n. It follows that $T' = \{\bar{t}_n : n \in \omega\}$ is a suitable choice. Of course, the hypothesis on $\mathcal{W}$ contradicts the assumption that $\langle t'_n : n \in \omega\rangle$ is a branch through the above tree – hence we finish. Note that if $A \subset \omega$ is such that $\{n : int(t_n) \cap A = \emptyset\}$ is infinite, then this is true of T' as well. Therefore this stage generalizes directly for $Q[\mathcal{I}]$.

<u>Stage B:</u> Repeated applications of Stage A will give $0 = n(0) < n(1) < \ldots$ such that for every i and choice of $t'_\ell < \mathrm{half}(t_\ell)$ $(n(i) \leq \ell < n(i+1))$, the set $\bigcup\{int(t'_\ell) : n(i) \leq \ell < n(i+1)\}$ is an element of $\mathcal{W}$. Note that if i is large enough then $\mathrm{half}(t_\ell)$ belongs to L_1 for all $\ell \geq n(i)$, so that $[n(i), n(i+1)) \in \mathrm{dom}(H)$ for all but finitely many i.

<u>Stage C:</u> Similarly we can produce a sequence $0 = m(0) < m(1) < \ldots$ such that for every $j \in \omega$ and function $h\colon [m(j), m(j+1)) \to \omega$ with $h(i) \in [n(i), n(i+1))$ for all $i \in \mathrm{dom}(h)$ and for every choice $t'_{h(i)} \leq \mathrm{half}(t_{h(i)})$, the set $\cup\{int(t'_{h(i)} : m(j) \leq i < m(j+1)\}$ belongs to $\mathcal{W}$. Let us remark however that if A is some subset of ω for which there are infinitely many n such that $int(t_n)$ is disjoint from A, then this may not be true for a subsequence of t_n's. Therefore this stage does not easily generalize to $Q[\mathcal{I}]$.

Now, take j large enough so that $[n(i), n(i+1)) \in \mathrm{dom}(H)$ for $i \geq m(j)$ and consider $u = [n(m(j)), n(m(j+1))\ )$; we claim that $H(u) \geq 1$. Indeed, let $v \subseteq u$. If there is i such that $[n(i), n(i+1)\ ) \subset v$, then $v \in \mathrm{dom}(H)$, otherwise we may define $h\colon [m(j), m(j+1)\ ) \to u$ with $h(i) \in [n(i), n(i+1)\ ) \setminus v$ for all i. It then follows, by Stage C, that $u \setminus v \in \mathrm{dom}(H)$.

Now observe that the above arguments apply to all $(\emptyset, T') \leq (\emptyset, T)$ and hence that the family $\mathcal{W}_1$ consisting of those W that contain sets of the form $\bigcup_{\ell\in u} int(t'_\ell)$, where $H(u) \geq 1$ and $t'_\ell \leq \mathrm{half}(t_\ell)$ for all ℓ, also satisfies (3.1). We may define its associated function H_1. We note that $H_1(u) \geq 0$ implies $H(u) \geq 1$ and so $H_1(u) \geq 1$ implies $H(u) \geq 2$. Using high enough t_ℓ we can then find our required $t \in L_2$. An obvious induction completes the proof. $\square$

We can now reformulate this lemma in terms of forcing.

Lemma 3.11. *Fix $p \in Q$, a sequence $\{\tau_n : n \in \omega\}$ of Q-names of ordinals, and let $q = (W^q, \{t_n : n \in \omega\})$ be as in Proposition 3.8, then there is an $r <_0 q$ such that if $T^r = \{\bar{t}_n : n \in \omega\}$ the following holds.*

$$\begin{array}{ll} & \textit{For every } n < \omega, W \subseteq [0, \max(int(\bar{t}_n)) + 1] \textit{ and } t''_{n+1} \leq \bar{t}_{n+1}, \\ (3.2) & \textit{there is } W' \subset int(t''_{n+1}) \textit{ such that } (W \cup W', \{\bar{t}_\ell : \ell > n+1\}) \\ & \textit{forces a value on } \tau_m\ (m \leq n) \end{array}$$

Proof. Choose, inductively $\bar{t}_n$ using Lemma 3.10 as follows. Having chosen $\bar{t}_{n-1}$, let m be large enough so that $\max[int(\bar{t}_{n-1})] < \min[int(t_m)]$. Let $\mathcal{W}$ be the set of all $W \subset [\min[int(t_m)], \omega)$ such that for each W' with $W^q \subset W' \subset \max[int(\bar{t}_{n-1})] + 1$ there is an $r \leq (W', \{t_\ell : m \leq \ell < \omega\})$ such that $W^r - W' \subseteq W$ and r forces a value on each of τ_k for $k \leq n$. Clearly $\mathcal{W}$ satisfies the conditions in Lemma 3.10. Apply Lemma 3.10 to $\{t_\ell : \ell \geq m\}$ to obtain a condition $\bar{t}_n \in L_n$. □

Recall that a poset Q is *almost ${}^\omega\omega$-bounding* if for every Q-name of a function from ω to ω and $p \in Q$, there is some $g \in {}^\omega\omega$ such that for every infinite $A \subset \omega$, there is a condition $p' < p$ such that

$$p' \Vdash_Q \{n \in A : f(n) < g(n)\} \text{ is infinite .}$$

A poset is *weakly bounding* if every unbounded family of functions from ω to ω remains unbounded in the extension.

Proposition 3.12. *The forcing Q is almost ${}^\omega\omega$-bounding.*

Proof. Let $\dot{f}$ be a Q-name of a function from ω to ω and let $p \in Q$. For each n, let $\tau_n = \dot{f}(n)$ and obtain $r = (W^r, T^r)$ as in Lemma 3.11. Let $g\colon \omega \to \omega$ be the function where $g(n)$ is the largest k such that for some $W \subset \max(int(t^r_{n+1})) + 1$, the condition $(W, \{t^r_\ell : \ell > n+1\}$ forces that $\dot{f}(n) = k$. Given any infinite $A \subset \omega$, let $p_A = (W^p, \{t^r_{n+1} : n \in A\})$. Let $\tilde{r} < p_A$ and let $n \in A$ be such that some $\tilde{t} \leq t^r_n$ is used in building some member of $T^{\tilde{r}}$. Of course $W^{\tilde{r}}$ is one of the subsets used in defining $g(n)$ and by Lemma 3.8, it follows that the value $\tilde{r}$ forces on τ_n is the same as that forced by $(W^{\tilde{r}}, \{t^r_\ell : \ell > n+1\})$. This completes the proof. □

4. The Modification of $Q[\mathcal{I}]$

One thing that was not proven in the previous sections is that $Q[\mathcal{I}]$ is almost ${}^\omega\omega$-bounding. Indeed, in general, we essentially only expect $\mathcal{C}_{\omega_1} * Q[\mathcal{I}]$ to be almost ${}^\omega\omega$-bounding and not even for all ideals $\mathcal{I}$. In [She84], Shelah shows that this is the case when $\mathcal{I}$ is the ideal generated by an almost disjoint family of subsets of ω. We can describe this another way. Consider the usual Ψ-space (so-called) generated by an almost disjoint family $\mathcal{A}$, i.e. the space is $\omega \cup \{x_A : A \in \mathcal{A}\}$ and a neighbourhood base for x_A is $\{\{x_A\} \cup A - n : n \in \omega\}$. Let X be the one point compactification of this space and let x denote the new point added. What we want to remark is that X is a compact sequential space and that $\mathcal{I}$ is equal to $\mathcal{I}_x = \{A \subset \omega : x \notin cl_X(A)\}$. In general, if X is a sequential space which is a compactification of ω and $x \in X$, let $\mathcal{I}_{X,x} = \{A \subset \omega : x \notin cl_X(A)\}$. We intend to show in this section that there is a forcing P which is a countable support iteration of proper almost ${}^\omega\omega$-bounding forcings so that $Q[\mathcal{I}_{X,x}]$ is almost ${}^\omega\omega$-bounding in V^P (and so can be used in an iteration which will be weakly bounding).

We begin with the following. As usual, if X is a compact space and P is any forcing notion, then X generates canonically a compact space in the extension V^P as follows: assume that X is a subset of $[0,1]^\kappa$ for $\kappa = w(X)$. In the extension, we let X refer to the closure of X^V in $[0,1]^\kappa$. In the case that X is zero-dimensional, we are simply refering to the Stone space of the algebra of clopen sets in both models.

Lemma 4.1 (CH). *Let X be a compactification of ω and let Q_{ω_1} be the countable support ω_1-length iteration in which Q is the iterand for each $\alpha < \omega_1$. Let $P = Q_{\omega_1} * \mathcal{C}_{\omega_1}$. In V^P, if $(\emptyset, \{t_n : n \in \omega\}) \in Q$ then, there is an infinite $J \subset \omega$ and a sequence $t'_n \leq t_n$ for $n \in J$ so that $(\emptyset, \{t'_n : n \in J\}) \in Q$ and there is an $x \in X$ so that $\cup\{int(t'_n) : n \in J\}$ converges to x in the space X.*

Proof. It should be clear that we may assume that $\delta < \omega_1$ is large enough so that $\{t_n : n \in \omega\}$ is a member of the model $V^{Q_\delta * \mathcal{C}_\delta}$ where, of course, Q_δ is the iteration of δ copies of Q and $\mathcal{C}_\delta$ is the obvious countable sub-poset of $\mathcal{C}_{\omega_1}$. Working in the model V^{Q_δ} we can inductively choose t'_n in such a way that for each forcing condition $p \in \mathcal{C}_\delta$ there are infinitely many n such that there is a $p' < p$ which forces t_n to equal t'_n and such that $(\emptyset, \{t'_n : n \in \omega\})$ is a member of Q (i.e. $t'_m \in L - L_n$ for at most finitely many m). Now find a coordinate $\alpha > \delta$ so that $(\emptyset, \{t'_n : n \in \omega\})$ is a member of $G(\alpha)$ (i.e. the generic subset of Q given by the α-th coordinate in Q_{ω_1}). It is easy to check that, by genericity, $J_1 = \{n : (\exists \bar{t}_n)\, \bar{t}_n \leq t'_n \text{ and } (\exists q \in G(\alpha))\, int(\bar{t}_n) \subset W^q\}$ is infinite. In addition we may assume (by passing to a subset) that $\mathrm{lev}(\bar{t}_n)$ diverges to infinity. For each $n \in J_1$, fix $\bar{t}_n \leq t'_n$ such that there is a $q \in G(\alpha)$ with $int(\bar{t}_n) \subset W^q$. Finally, by the inductive construction of the t'_n's, there is an infinite set $J \subset J_1$ in V^P such that $t_n = t'_n$ for each $n \in J$. This is our desired set J. Note that $(\emptyset, \{\bar{t}_n : n \in J\})$ is a member of Q. The only thing remaining is to see that $\cup\{int(\bar{t}_n) : n \in \omega\}$ converges to some member of X. This however follows from the fact (Lemma 3.8) that $\{m : (\exists q \in G(\alpha))\, m \in W^q\}$ is not split by any basic open subset of X. $\square$

The key property that seems to make this go through is the following.

Definition 4.2. A poset $P = Q[I_{X,x}]$ will be said to have the *madf property* if for each $p \in P$ with $T^p = \{t_n : n \in \omega\}$ there are pairwise disjoint subsets of ω, $\{J_n : n \in \omega\}$ and a sequence $\{t'_n : n \in \omega\}$ with $t'_n \leq t_n$ for each n, so that for each $B \subset \omega$ such that $B \cap J_n$ is infinite for each n, $(\emptyset, \{t'_n : n \in B\})$ is a member of P.

Lemma 4.3. *Suppose that X is a compactification of ω with countable tightness and suppose that for each $p \in Q$, there is a $y \in X$ and $p' \leq p$ such that $int(T^{p'})$ converges to y. Then $Q[\mathcal{I}_{X,x}]$ has the madf property for each $x \in X$.*

Proof. Let S be the set of $y \in X$ such that there is a $J_y \subset \omega$ such that there is a $T_y \leq T_{J_y}$ for which $int(T_y)$ converges to y (where $T_J = \{t_n : n \in J\}$). Note that by the hypothesis on X, every infinite subset of ω contains some such J. Since $p \in Q[\mathcal{I}_{X,x}]$, x is in the closure of S. Since X has countable tightness, there is a countable subset $\{s_n : n \in \omega\} \subset S$ which has x as a limit point. Note that if $x \in S$, then we may stop since it would follow that, below $(\emptyset, T_x)$, the posets $Q[\mathcal{I}_{X,x}]$ and Q are the same.

For each n, fix J_{s_n} and $\{t^n_m : m \in J_{s_n}\}$ so that $t^n_m \leq t_m$ for each m, the set $\cup\{int(t^n_m) : m \in J_{s_n}\}$ converges to s_n and $(\emptyset, \{t^n_m : m \in J_{s_n}\})$ is in Q. Since we may shrink J_{s_n}, we may assume that $\{J_{s_n} : n \in \omega\}$ are pairwise disjoint (countable families of infinite sets have disjoint refinements). With no loss of generality (and it isn't important) we may assume that $\cup_{n\in\omega} J_{s_n}$ is all of ω. For each $m \in \omega$, there is a unique n, such that $m \in J_{s_n}$, define t'_m to be t^n_m. Observe that $(\emptyset, \{t'_m : m \in \omega\})$ is a member of $Q[\mathcal{I}_{X,x}]$. Now suppose $B \subset \omega$ is such that $B \cap J_{s_n}$ is infinite for each n, then $(\emptyset, \{t'_m : m \in B\})$ is a member of $Q[\mathcal{I}_{X,x}]$. To see this, fix any $A \in \mathcal{I}_{X,x}$. Since $x \notin cl_X(A)$, there is an n such that $s_n \notin cl_X(A)$. It follows that the intersection of A with $\cup\{int(t^n_m) : m \in J_{s_n}\}$ is finite. Therefore there are infinitely many members, m, of $B \cap J_{s_n}$ such that $int(t^n_m) \cap A$ is empty. $\square$

Lemma 4.4. *Suppose that X is a compactification of ω and suppose that $x \in X$ is such that $Q[\mathcal{I}_{X,x}]$ has the madf property. Let $p = (\emptyset, T) \in Q[\mathcal{I}_{X,x}]$ and suppose that $\mathcal{W}$ is a set of finite subsets of ω such that for each $(\emptyset, T') < (\emptyset, T)$, there is a $W \in \mathcal{W}$ with $W \subset int(T')$.*
Then, for every $k \in \omega$, there is a $t \in L_k$ appearing in some $(\emptyset, T') < (\emptyset, T)$ such that, for every $t' \leq t$, there is a $W \in \mathcal{W}$ with $W \subset int(t')$.

Proof. Let $T = \{t_n : n \in \omega\}$ and proceed as in Lemma 3.10. As mentioned in the proof of Lemma 3.10, the proofs of Stage A and Stage B are unchanged. We generalize the idea from [She84], where this is shown for the case X is the one point compactification of a Ψ-space, to prove Stage C. The rest of the proof is as in Lemma 3.10.

Fix the sequences $\{t'_n : n \in \omega\}$ and $\{J_n : n \in \omega\}$ as in Definition 4.2. The rest of the proof is virtually the same as in [She84].

In Stage A, we may assume that $n(i+1)$ is chosen large enough so that $[n(i), n(i+1)) \cap J_k$ is not empty for each $k \leq n(i)$. Now fix a function ϕ from ω to ω so that $\phi(i) \leq n(i)$ for each i and so that $\phi^{-1}(n)$ is infinite for each n (ϕ can be defined by induction).

<u>Stage C:</u> We wish to define the sequence $m(\ell)$ so that for each function h with domain $[m(\ell), m(\ell+1))$ and with $h(i) \in [n(i), n(i+1)) \cap J_{\phi(i)}$ for each relevent i, and for each selection of $\bar{t}_{h(i)} \leq half(t'_{h(i)})$, the set $\bigcup int(\bar{t}_{h(i)})$ is in $\mathcal{W}$. As in the proof of Stage A in Lemma 3.10, the idea is to note that if $m(\ell+1)$ cannot be chosen, then we obtain an infinite sequence $\{\bar{t}_{h(i)} : m(\ell) \leq i < \omega\}$ such that $h(i) \in [n(i), n(i+1)) \cap J_{\phi(i)}$ for each i and so that $\bar{t}_{h(i)} \leq half(t'_{h(i)})$. The role of the $half(t'_n)$'s is the same in that they guarantee that we can find a suitable $(\emptyset, \bar{T}) \leq (\emptyset, T)$ in Q which does not contain a member of $\mathcal{W}$. However we must ensure that this $(\emptyset, \bar{T})$ is a member of $Q[\mathcal{I}_{X,x}]$ to obtain a contradiction. We do obtain such a contradiction, since B, the range of h, meets each J_n in an infinite set. Therefore, just as above, there are, for each $A \in \mathcal{I}_{X,x}$ infinitely many i so that $\bar{t}_{h(i)}$ misses A. $\square$

Corollary 4.5 (CH). *If X is a compactification of ω and is such that in $V^{\mathcal{C}_{\omega_1}}$, $Q[I_{X,x}]$ has the madf property, then, in $V^{\mathcal{C}_{\omega_1}}$, $Q[\mathcal{I}_{X,x}]$ is almost ${}^{\omega}\omega$-bounding.*

Proof. We work in $V^{\mathcal{C}_{\omega_1}}$ and let $\dot{f}$ be a $Q[\mathcal{I}_{X,x}]$-name of an increasing function from ω to ω. Fix $\delta < \omega_1$ large enough so that $\dot{f} \in V^{\mathcal{C}_\delta}$. We may invoke Lemmas 4.1 and 4.4, to find a sequence $\{\bar{t}_n : n \in \omega\}$ just as we did in Lemma 3.11 except that we may use a real which is Cohen generic over $V^{\mathcal{C}_\delta}$ to choose the next value for $\bar{t}_n$ from among all those satisfying the requirements. This genericity will ensure that $(\emptyset, \{\bar{t}_n : n \in \omega\}) \in Q[\mathcal{I}_{X,x}]$. We define a g similar to that defined in Lemma 3.12.

First of all, since there is no harm in refining the sequence of $\bar{t}_n$'s, we may assume that there are pairwise disjoint J_n's so that for each B which meets each J_n in an infinite set, $(\emptyset, \{\bar{t}_n : n \in B\})$ is in $Q[\mathcal{I}_{X,x}]$. Choose an increasing sequence $n(0,0) < n(0,1) < n(1,1) < n(0,2) < \ldots < n(0,\ell) < \ldots < n(\ell,\ell) \ldots$ so that $n(m,\ell) \in J_m$ for each $m \le \ell < \omega$. Clearly $(\emptyset, \{t(n(m,\ell)) : m \le \ell < \omega\})$ is in $Q[\mathcal{I}_{X,x}]$. For each ℓ, define $g(\ell)$ large enough so that for each $W \subset \min int(n(0,\ell))$ and each $t \le \bar{t}(m,\ell)$ $(m \le \ell)$, there is a $W' \subset int(t)$ such that $(W \cup W', \{\bar{t}_n : n(\ell,\ell) < n\})$ forces a value on $\dot{f}(n(m,\ell))$ which is less than $g(\ell)$. We may assume that g is increasing.

Fix any $A \in V^{\mathcal{C}_{\omega_1}}$, and use the condition $p_A = (W^p, \{\bar{t}_{n(m,\ell)} : m \le \ell \in A\})$. To finish the proof, suppose that r is any condition below p_A. There is an $m < \ell \in A$ such that $\bar{t}_{(m,\ell)}$ is used in building a condition t' of T^r. We know that $int(t')$ contains a set W' so that $(W^r \cup W', \{\bar{t}_n : n > n(\ell,\ell)\})$ forces a value on $\dot{f}(n(m,\ell))$ which is less than $g(\ell)$. Since $\dot{f}$ is increasing, it follows that $g(\ell)$ is forced to be greater than $\dot{f}(\ell)$. □

The payoff in this section is the following interesting observation.

Theorem 4.6. *There is a model of $b = \omega_1 < \omega_2 = s = a = c$ in which every compact space of countable tightness and weight ω_1 is actually Fréchet-Urysohn.*

Proof. Start with a model of CH, and force with the countable support iteration of posets of the form $Q_{\omega_1} * \mathcal{C}_{\omega_1} * Q[\mathcal{I}_{X,x}]$ where a suitable diamond sequence is used to choose which X and x to use. More precisely, diamond will predict a space X containing ω and a point $x \in X$. If X has countable tightness after forcing with $Q_{\omega_1} * \mathcal{C}_{\omega_1}$ then we will use $Q[\mathcal{I}_{X,x}]$ otherwise just use the trivial poset. By the Lemmas in this section it follows that this iteration is weakly bounding, hence b will be ω_1.

In the extension assume that X is a compact space of countable tightness and weight ω_1. Let x be a limit point of the countable set $\omega \subset X$. We may assume that ω is dense in X. The diamond sequence will have chosen a space X' with the same base as X and the point x at some stage in the iteration. If we did not use $Q[\mathcal{I}_{X,x}]$ it was because the reflected copy of X did not have countable tightness. However uncountable tightness in compact spaces is upwards absolute (if cardinals are preserved), hence we were able to force with $Q[\mathcal{I}_{X,x}]$. This forcing introduces a subset of ω which converges to x. The fact that $a = \omega_2$ follows from the fact that the one-point compactification of a Ψ-space is sequential but not Frechet-Urysohn. Finally, $s = \omega_2$ since we forced with Q cofinally. □

Question 2. Does Q_{ω_1} preserve countable tightness in compact spaces? It is easy to see that $\mathcal{C}_{\omega_1}$ does but it is certainly known that there are proper posets which do not [Fre88].

Question 3. Are there any totally mad families in this model? What if we do not use Cohen reals at ω_1 limits?

The only thing blocking a proof, for us, of an affirmative answer to the first part of the previous question is that we need a "no" answer to the following question. However it probably has a "yes" answer, but then we would ask about towers which are maximal chains in $\bigcup_{\xi<\omega_1} \mathfrak{M}_\xi$ from Proposition 2.6.

Question 4. Can Q fill maximal towers?

Recall that the proper forcing axiom, PFA, implies that all compact spaces of countable tightness are sequential. However at present the most complicated (in terms of sequential order) compact sequential space that we know under this axiom has sequential order only three!

Question 5. Does PFA imply that all compact spaces of countable tightness have sequential order at most three?

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