EXTENDING REAL-VALUED FUNCTIONS IN $\beta\kappa$

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ABSTRACT. An Open Coloring Axiom type principle is formulated for uncountable cardinals and is shown to be a consequence of the Proper Forcing Axiom. Several applications are found. We also study dense C^*-embedded subspaces of ω^*, showing that there can be such sets of cardinality $\mathfrak{c}$ and that it is consistent that $\omega^* \setminus \{p\}$ is C^*-embedded for some but not all $p \in \omega^*$.

1. Introduction

We establish a consequence of the proper forcing axiom (PFA) which is a combinatorial principle for uncountable cardinals which is similar to the Open Coloring Axiom of Todorčević [Tod89a] (see also [AS81]), see Definition 4.1. This principle is more general than another similar principle formulated by Todorčević in [Tod89b]. In the second section we show that this principle implies several reflection results concerning extending real-valued continuous functions on subspaces of $\beta\kappa$ (for discrete κ) and the reflection of first-countability in spaces of countable tightness. In the third section we show, in response to a question in [vDKvM89], that it is consistent that there is a "small" dense subset of $\beta\omega \setminus \omega$ which is C-embedded (i.e. every real-valued continuous function has a continuous extension to all of $\beta\omega$). It is shown in [vDKvM89] that $\omega^* \setminus \{p\}$ is C-embedded for every $p \in \omega^*$ and it is asked in [HvM90] if there is a model in which $\omega^* \setminus \{p\}$ is C-embedded for some but not all p. We show that in the Miller model $\omega^* \setminus \{p\}$ is not C-embedded if and only if p is a P-point of ω^*. A set X is said to be C^*-embedded in Y if every bounded real-valued continuous function on X has a continuous extension to Y. The distinction is not significant in this article because in each case the space under discussion will not possess any unbounded continous real-valued functions.

2. Uncountable Domains

There are many situations in which one has two families $\mathcal{I}$ and $\mathcal{J}$ which are subsets of a cardinal κ such that every member of $\mathcal{I}$ is almost disjoint from every member of $\mathcal{J}$ and one is interested in determining if the two families can be separated by a single set X, i.e. a set X such that every member of $\mathcal{I}$ is almost contained in X and every member of $\mathcal{J}$ is almost disjoint from X (where *almost disjoint* means the intersection is in some fixed ideal, e.g. the ideal of finite sets). Of course it is well-known that there are situations where there is no such separating set X. However it is perfectly reasonable to expect that if there is no such X then there should be small subsets (in terms of cardinality) of $\mathcal{I}$ and $\mathcal{J}$ for which there is also no such separating set X (a useful reflection principle results). It is easily seen that countable subfamilies can always be separated hence $\aleph_1$ would be the target. If $\mathcal{I} \cup \mathcal{J}$ forms a maximal almost disjoint family of sequences of rationals which converge to reals and $\mathcal{I}$ is the set of all those which converge to some member of a Bernstein set, then $\mathcal{I}$ and $\mathcal{J}$ cannot be separated. However it follows from $\mathrm{MA}(\omega_1)$

that any subsets of $\mathcal{I}$ and $\mathcal{J}$ of cardinality $\aleph_1$ can be separated. We will avoid this problem by assuming that our ideal $\mathcal{I}$ is a P-ideal, that is, if $\mathcal{I}' \subset \mathcal{I}$ is countable, then there is a member $I \in \mathcal{I}$ such that $I' \setminus I$ is finite for each $I' \in \mathcal{I}'$. One other situation must be avoided in order to expect $\aleph_1$-sized reflection (under PFA); this is formulated in the next definition.

Definition 2.1. We will say that two ideals $\mathcal{I}$ and $\mathcal{J}$ can be *countably separated* if there are countably many sets X_n such that for each pair $(I, J) \in \mathcal{I} \times \mathcal{J}$, there is an n such that I is almost disjoint from X_n and J is almost contained in X_n . We will say that the ideals are *strongly countably separated* if the X_n's have the additional property that $X_n \cap I$ is finite for each $I \in \mathcal{I}$ and each $n \in \omega$ (i.e. $[X_n]^\omega \cap \mathcal{I} = \emptyset$ for each n).

If we say that a set of pairs $\{(I_\alpha, J_\alpha) : \alpha \in \omega_1\}$ are (strongly) countably separated then we mean that the family $\{I_\alpha : \alpha \in \omega_1\}$ is (strongly) countably separated from $\{J_\alpha : \alpha \in \omega_1\}$.

Notice that the definition of countably separated is symmetric in $\mathcal{I}$ and $\mathcal{J}$ and is obviously more general than that of strongly countably separated. However we will actually be using the stronger notion in all cases because of the following lemma.

Lemma 2.2. *If $\mathcal{I}$ and $\mathcal{J}$ are countably separated and $\mathcal{I}$ is a P-ideal, then they are strongly countably separated.*

Proof. Assume that $\mathcal{I}$ is a P-ideal and that $\{X_n : n \in \omega\}$ countably separates $\mathcal{I}$ and $\mathcal{J}$. Simply take the family of only those X_n such that $[X_n]^\omega \cap \mathcal{I} = \emptyset$, this family will also separate the two ideals. Indeed, for each n such that there is an $I \in \mathcal{I}$ such that $X_n \cap I$ is infinite, let I_n be such a member of $\mathcal{I}$. Since $\mathcal{I}$ is a P-ideal, there is an $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for each n. Therefore I has the property that $I \cap X_n$ is infinite for each n such that there is some $I' \in \mathcal{I}$ such that $I' \cap X_n$ is infinite. Now $\{X_n : I \cap X_n \text{ is finite }\}$ separates $\mathcal{I}$ and $\mathcal{J}$ since this family will have to separate the family $\{I' \cup I : I' \in \mathcal{I}\}$ and $\mathcal{J}$. □

The remarkable thing about the following result is that there is no hypothesis on the ideal $\mathcal{J}$. This result is the combinatorial essence to many reflection results, some of which are discussed in the next section.

Theorem 2.3 (PFA). *If $\mathcal{I}$ and $\mathcal{J}$ are ideals of countable sets with $\mathcal{I}$ a P-ideal then either*

1. *$\mathcal{I}$ and $\mathcal{J}$ can be countably separated,*
   or
2. *there are $\aleph_1$-generated subideals $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{J}' \subset \mathcal{J}$ which cannot be countably separated.*

Stated as above this theorem most resembles the Open Colouring Axiom principle discussed above. However it is useful to explicitly state a more detailed structure theorem that yields Theorem 2.3 as a Corollary. The terms used in this statement are defined below.

Lemma 2.4. *If $\mathcal{I}$ and $\mathcal{J}$ are ideals, with $\mathcal{I}$ a P-ideal, which cannot be countably separated then there is a proper poset which introduces a subcollection $\{(I_\alpha, J_\alpha) : \alpha \in \omega_1\} \subset \mathcal{I} \times \mathcal{J}$ which forms either a Hausdorff-Luzin type family or a finite-to-one graph type family and such that $\{I_\alpha : \alpha \in \omega_1\}$ generates a P-ideal.*

Theorem 2.5 (PFA). *If $\mathcal{I}$ and $\mathcal{J}$ are ideals, with $\mathcal{I}$ a P-ideal, which cannot be countably separated then there is a subcollection $\{(I_\alpha, J_\alpha) : \alpha \in \omega_1\} \subset \mathcal{I} \times \mathcal{J}$ which forms either a Hausdorff-Luzin family or a finite-to-one graph family. In addition, $\{I_\alpha : \alpha \in \omega_1\}$ can be chosen to generate a P-ideal.*

Clearly what is happening is that the Hausdorff-Luzin families and the finite-to-one graph families are the only families of size $\aleph_1$ which are "absolutely" not strongly countably separated – i.e. passing to a larger (ω_1 preserving) model will not make them strongly countably separated.

The best known example of such absolutely unseparated ideals are the Hausdorff-Luzin type.

Definition 2.6. We say that $\{(I_\alpha, J_\alpha) : \alpha \in \omega_1\}$ is a Hausdorff-Luzin type family if for each $\alpha < \omega_1$ and $m \in \omega$, the set of $\beta < \alpha$ such that $I_\alpha \cap J_\beta$ has cardinality at most m is finite.

Definition 2.7. We say that $\{(I_\alpha, J_\alpha) : \alpha \in \omega_1\}$ is a finite-to-one graph type family if the J_β's are pairwise disjoint and there is a sequence of enumerations, $\{j(\beta, k) : k \in \omega\}$, of the J_β's so that for each α there is a finite-to-one function g_α from α to ω such that $j(\beta, g_\alpha(\beta)) \in I_\alpha$ for each $\beta < \alpha$ (roughly speaking, I_α contains the graph of a finite-to-one function)

Proposition 2.8. *Hausdorff-Luzin type families cannot be strongly countably separated.*

Proof. Let $\{(I_\alpha, J_\alpha) : \alpha \in \omega_1\}$ be a Hausdorff-Luzin type family. Assume that $\{X_n : n \in \omega\}$ strongly separates the family $\{I_\alpha : \alpha < \omega_1\}$ from $\{J_\alpha : \alpha < \omega_1\}$. There is an n and an m so that there is an uncountable set S such that for each $\beta \in S$, $|J_\beta \setminus X_n| \leq m$. In addition, for each $\beta \in S$, assume that $|I_\beta \cap X_n| \leq m$. Now fix $\alpha \in S$ such that $S \cap \alpha$ is infinite, hence there is a $\beta < \alpha$ such that $|I_\alpha \cap J_\beta|$ is bigger than $2m$. Consider how $I_\alpha \cap J_\beta$ is split by X_n. $|(I_\alpha \cap J_\beta) \cap X_n| \leq |X_n \cap I_\alpha| \leq m$ and $|(I_\alpha \cap J_\beta) \setminus X_n| \leq |J_\beta \setminus X_n| \leq m$. Therefore $|I_\alpha \cap J_\beta| \leq 2m$. ☐

Lemma 2.9. *A finite-to-one graph type family cannot be strongly countably separated.*

Proof. Fix a family as in Definition 2.7 and assume that $\{X_n : n \in \omega\}$ strongly separates the I_α's from the J_α's. Certainly there is some X_n which contains (mod finite) uncountably many of the J_β. Therefore, there is an m so that there is an uncountable set S such that for each $\beta \in S$, $J_\beta \setminus X_n \subset \{j(\beta, \ell) : \ell < m\}$. Choose any α such that $S \cap \alpha$ is infinite. Since g_α is finite-to-one $g_\alpha(\beta) < m$ for at most finitely many β, hence I_α meets X_n in an infinite set. ☐

In the remainder of this section we prove Lemma 2.4. We leave it to the reader to observe that if we fix names for a Hausdorff-Luzin type family or a finite-to-one graph type family then there is a family of ω_1 dense subsets of the poset such that if a generic filter meets each of these dense sets then the interpretations of the above names will be either a Hausdorff-Luzin type family or a finite-to-one graph type family.

The proof will use the following result of Todorčević [Tod89b]. The proof is available in [Dow92] but since we require a slightly stronger form of it in one part of the proof (and for completeness) we include a proof in the form of Claim 4 of the proof of Theorem 2.3.

Lemma 2.10 ([Tod89b]). *If $\mathcal{I}$ is an ideal of countable sets then either there are S_n $(n \in \omega)$ such that $\bigcup \mathcal{I} = \bigcup_n S_n$ and $[S_n]^\omega \subset \mathcal{I}$ for each n, or there is a proper poset P which forces an uncountable $S \subset \bigcup \mathcal{I}$ such that S is almost disjoint from every member of $\mathcal{I}$.*

Proof of Lemma 2.4. Assume that $\mathcal{I}$ and $\mathcal{J}$ are not countably separated. Let κ be a regular cardinal larger than $|\mathcal{P}(\bigcup \mathcal{I})|$ (without loss of generality $\bigcup \mathcal{J} = \bigcup \mathcal{I}$). Let P be the countable condition collapse of $|H(\kappa)|$ and let G be a P-generic filter. The following claim will be useful:
Claim 1: If $X \in V[G]$ is such that $[X]^\omega \cap \mathcal{I}$ is empty, then there is Y in $H(\kappa)^V$ such that $X \subset Y$ and $[Y]^\omega \cap \mathcal{I}$ is empty.

Proof of Claim 1: Let $\dot{X}$ be a name for X and let $p_0 \in P$ force that $[\dot{X}]^\omega \cap \mathcal{I}$ is empty. Fix a countable elementary submodel M of a large enough $H(\theta)$ with $p_0, P, \dot{X}, \mathcal{I}$ all in M. Fix $I \in \mathcal{I}$ such that $I' \setminus I$ is finite for each $I' \in M \cap \mathcal{I}$. Recursively choose, if possible, descending $p_n \in P \cap M$, and $x_n \in I \cap M$ $(n \in \omega)$ such that $p_{n+1} \Vdash x_n \in \dot{X}$ and, for $n > 0$, $x_n \notin \{x_0, \ldots, x_{n-1}\}$. This process must stop for some n, since P is countably closed and a lower bound would force that $\dot{X} \cap I$ is infinite. Hence there is an n such that $p_n \Vdash \dot{X} \cap I \cap M$ is finite. Let Y be the set of all x such that there is a $q < p_n$ which forces x to be in $\dot{X}$. Clearly $p_n \Vdash \dot{X} \subset Y$. Since $\dot{X}$ and p_n are both in M, it follows that Y is in M. Since I almost contains every member of $\mathcal{I} \cap M$ it follows that Y meets every member of $\mathcal{I} \cap M$ in a finite set. Therefore Y meets every member of $\mathcal{I}$ in a finite set.

It follows by the claim that $\mathcal{I}$ and $\mathcal{J}$ are not countably separated in $V[G]$ and that the cardinality of $\mathcal{I} \cup \mathcal{J}$ is ω_1, in fact, $H(\kappa)^V$ has cardinality ω_1. Fix an increasing continuous chain $\{M_\alpha : \alpha \in \omega_1\}$ of countable elementary submodels of $H(\kappa)^V$ whose union is all of $H(\kappa)^V$ and such that $\{M_\beta : \beta \leq \alpha\} \in M_{\alpha+1}$ for each $\alpha < \omega_1$.

For each $\alpha < \omega_1$, choose $J_\alpha \in \mathcal{J} \cap M_{\alpha+1}$ such that J_α is not contained (mod finite) in any member of M_α which is almost disjoint from every member of $\mathcal{I}$, and so that, if possible $J_\alpha \subset M_\alpha$. Note that such a set J_α will always exist because $\mathcal{I}$ and $\mathcal{J}$ are not countably separated. Note that if J_α cannot be taken to be a subset of M_α, then $J_\alpha \setminus M_\alpha$ is a suitable choice for J_α, hence we assume that either $J_\alpha \subset M_\alpha$ or J_α is disjoint from M_α. Similarly, we may choose $I_\alpha \in \mathcal{I} \cap M_{\alpha+1}$ so that I_α contains, mod finite, every member of $\mathcal{I} \cap M_\alpha$. We will prove that we can force an uncountable $Y \subset \omega_1$ so that $\{I_\alpha : \alpha \in Y\}$ and $\{J_\alpha : \alpha \in Y\}$ is either a Hausdorff-Luzin type family or a finite-to-one graph type family.

Claim 2: $\{I_\alpha : \alpha \in \omega_1\}$ and $\{J_\alpha : \alpha \in \omega_1\}$ are not countably separated in $V[G]$.
Proof of Claim 2: Since $\{I_\alpha : \alpha < \omega_1\}$ is cofinal (mod finite) in $\mathcal{I}$, it suffices to show that $\mathcal{I}$ and $\{J_\alpha : \alpha < \omega_1\}$ are not countably separated. By Claim 1, it suffices to show that there is no countable collection in $H(\kappa)^V$ which witnesses that they are countably separated. However any countable subset of $H(\kappa)^V$ will be contained in M_α for some $\alpha < \omega_1$ and J_α will witness that this countable collection does not separate the two collections.

Let $\mathcal{K}$ be defined by

$$K \in \mathcal{K} \leftrightarrow (\exists m \in \omega)(\forall \alpha \in \omega_1)\{\beta \in K : |I \cap J_\beta| > m\} \text{ is finite.}$$

The proof splits into two cases according to the application of Lemma 2.10 to the ideal $\mathcal{K}$.

Case 1: There is no cover $\{Y_n : n \in \omega\}$ of ω_1 such that $[Y_n]^\omega \subset \mathcal{K}$ for all $n \in \omega$. We will show that there is a proper poset $R * Q$ which introduces an uncountable set $Y' \subset \omega_1$ such that $\{(I_\alpha, J_\alpha) : \alpha \in Y'\}$ is a Hausdorff-Luzin type family.

Case 2: There is a cover $\{Y_n : n \in \omega\}$ of ω_1 such that $[Y_n]^\omega \subset \mathcal{K}$. In this case we fix any indexing $\{j(\beta, n) : n \in \omega\}$ of J_β for each $\beta \in \omega_1$ and we will show that there is an n and a proper poset Q which introduces a $Y' \in [Y_n]^{\omega_1}$ such that $\{(I_\alpha, J_\alpha) : \alpha \in Y'\}$ is a finite-to-one graph type family. The steps in this case are more involved. We first show that we may assume that the J_β are pairwise disjoint (as required in the definition of a finite-to-one graph type family). We define $g_\alpha : \alpha \to \omega$ by

$$g_\alpha(\beta) = \min\{m : I_\alpha \cap J_\beta \subset \{j(\beta, k) : k \leq m\}\} .$$

We will then show that there is a forcing similar to the one used for the proof of Lemma 2.10 which will introduce Y' so that $g_\alpha \restriction (\alpha \cap Y')$ is finite-to-one as required.

In the sequel, when we refer to one of our ideals in an extension of the model in which it was defined, then we will mean the ideal generated by the ground model ideal.

Proof of Case 1: It follows, by Lemma 2.10, that there is a proper poset R such that R introduces an uncountable set Y such that $[Y]^\omega \cap \mathcal{K}$ is empty. Define the poset $Q = [Y]^{<\omega}$ ordered by $a < b$ if $a \supset b$ and for each $\alpha \in b$ and each $\beta \in \alpha \cap (a \setminus b)$ $|I_\alpha \cap J_\beta| > |b|$. Forcing with Q will introduce a $Y' \subset Y$ such that for each $\alpha \in Y'$ and each m, $\{\beta \in Y' \cap \alpha : |I_\alpha \cap J_\beta| < m\}$ is finite. Showing that Q is ccc (and a simple density argument) establishes that Y' can be forced to be uncountable.

To see that Q is ccc, let $\{F_\alpha : \alpha \in \omega_1\}$ be a family of finite subsets of Y. We may assume that all the F_α have cardinality m and that they form a Δ-system with root F. Let L_0 be any infinite subset of $\bigcup_n F_n \setminus F$. Since $L_0 \subset Y$, $L_0 \notin \mathcal{K}$, hence there is an $I'_0 \in \mathcal{I}$ such that there is no bound on the cardinalities of $I'_0 \cap J_\beta$ for $\beta \in L_0$. Choose $L'_0 \subset L_0$ so that the cardinalities of $I'_0 \cap J_\beta$ diverge to infinity for $\beta \in L'_0$. Choose L_1 an infinite subset of $\bigcup\{F_n \setminus (F \cup L'_0) : L'_0 \cap F_n \neq \emptyset\}$. Again, find $I'_1 \in \mathcal{I}$ and $L'_1 \subset L_1$ so that the cardinalities of $I'_1 \cap J_\beta$ diverge to infinity for $\beta \in L'_1$. Repeat, obtaining I'_k and L'_k for $k < K \leq m$, until $T = \{n : F_n \subset \bigcup_{k<K} L'_k\}$ is infinite. Let $I \in \mathcal{I}$ be the union of the I'_k and choose α such that I is almost contained in I_ξ for each $\xi \in F_\alpha \setminus F$. It is easily checked that for each $\xi \in F_\alpha \setminus F$ there are at most finitely many $n \in T$ for which there is a $\beta \in F_n$ such that $|I_\xi \cap J_\beta| < m$. Therefore, there is an $n \in T$ such that F_α is compatible with F_n.

Proof of Case 2:

As mentioned above we first show that we can assume that the J_β are pairwise disjoint (by restricting to a co-countable set).

Claim 3: The set of β such that $J_\beta \subset M_\beta$ is countable.

Let S be the set of α such that $J_\alpha \subset M_\alpha$. We first show, by contradiction, that S is not stationary. Fix n such that $S_0 = Y_n \cap S$ is stationary and let $Z_0 = \omega_1$. For each $\gamma < \omega_1$, there is a minimal $m_\gamma < \omega$ such that, for each $\alpha \in \omega_1$, there are only finitely many $\beta \in Y_n \cap \gamma$ for which $|I_\alpha \cap J_\beta|$ is larger than m_γ. Since the I_α are increasing mod finite, it follows that the m_γ's are bounded by a single m.

Since $\{I_\gamma : \gamma \in Z_0\}$ is cofinal in $\mathcal{I}$ it follows that if X_0 is the set consisting of those x which are members of I_γ for uncountably many γ, then X_0 almost contains each $I \in \mathcal{I}$. Fix a countable elementary submodel N_0, of some $H(\theta)$ which contains everything mentioned, such that $N_0 \cap \omega_1 = \alpha_0 \in S_0$. By Claim 1 (dualized), there is $X_0' \in H(\kappa)^V \cap N_0$ such that $X_0' \subset X_0$ and $I \setminus X_0'$ is finite for each $I \in \mathcal{I}$. It follows that $X_0' \in M_{\alpha_0}$ (i.e. $N_0 \models (\exists \beta)\ X_0' \in M_\beta$, hence there is such a $\beta < \alpha$). Choose $x_0 \in J_{\alpha_0} \cap X_0$, which we may do since the complement of X_0' contains mod finite every member of $\mathcal{I}$ and so cannot contain J_{α_0}. Since $J_{\alpha_0} \subset M_{\alpha_0}$, and $\alpha_0 \in S_1 = \{\beta \in S_0 : \{x_0\} \subset J_\beta\} \in M_{\alpha_0}$, it follows that S_1 is stationary. Also, $Z_1 = \{\gamma \in Z_0 : \{x_0\} \subset I_\gamma\}$ is cofinal in Z_0, hence $\{I_\gamma : \gamma \in Z_1\}$ is cofinal in $\mathcal{I}$ and again $X_1 = \{x : x \in I_\gamma$ for uncountably many $\gamma \in Z_1\}$ almost contains every member of $\mathcal{I}$. Therefore, by induction we can find $\{x_0, \ldots, x_m\}$ so that $S_{m+1} = \{\beta \in Y : \{x_0, \ldots, x_m\} \subset J_\beta\}$ is stationary and $Z_{m+1} = \{\gamma : \{x_0, \ldots, x_m\} \subset I_\gamma\}$ is uncountable. However, if $\gamma \in Z_{m+1}$ is such that $S_{m+1} \cap \gamma$ is infinite, then $\{\beta \in S_{m+1} \cap \gamma : |I_\gamma \cap J_\beta| > m\}$ is infinite contradicting that m "works" for $Y_n \cap \gamma$. This completes the proof that S is not stationary.

Now we show that for all but countably many β, $J_\beta \cap M_\beta$ is empty. Indeed, there must be a $\mathcal{C} \subset [H(\kappa)^V]^\omega$ which is closed and unbounded and such that for each $M \in \mathcal{C}$ and $J \in \mathcal{J}$, there is a $Y \in M$ such that $[Y]^\omega \cap \mathcal{I} = \emptyset$ and $J \cap M \subset Y$. Now if $\gamma < \omega_1$ is large enough such that $\mathcal{C} \in M_\gamma$, then for all $\beta \geq \gamma$, $M_\beta \in \mathcal{C}$, hence $\beta \notin S$.

For each β, fix an enumeration $\{j(\beta, m) : m \in \omega\}$ of J_β. Recall that we defined $g_\alpha : \alpha \to \omega$ by

$$g_\alpha(\beta) = \min\{m : I_\alpha \cap J_\beta \subset \{j(\beta, k) : k \leq m\}\}\ .$$

Clearly if $\{I_\alpha : \alpha \in \omega_1\}$ and $\{J_\beta : \beta \in Y_n\}$ are countably separated for each n, then so are $\mathcal{I}$ and $\mathcal{J}$. Therefore we may fix an n so that $\{I_\alpha : \alpha \in \omega_1\}$ and $\{J_\beta : \beta \in Y_n\}$ are not countably separated. In fact, it follows immediately that we may assume that $\{I_\beta : \beta \in Y_n\}$ and $\{J_\beta : \beta \in Y_n\}$ are not countably separated.

To help us find our desired Y' we define another auxiliary ideal $\mathcal{L}$.

$$L \in \mathcal{L} \leftrightarrow (\exists m \in \omega)\ (\forall \alpha \in Y_n)\{\beta \in L : I_\alpha \cap J_\beta \not\subset \{j(\beta, \ell) : \ell \leq m\}\} \text{ is finite}$$

The idea is that members L of $\mathcal{L}$ are "bad" if $\{J_\beta : \beta \in L\}$ are to be part of a finite-to-one graph family. Clearly we need to show that we can force a set $Y' \in [Y_n]^{\omega_1}$ such that $Y' \cap L$ is finite for each $L \in \mathcal{L}$. Just as in Lemma 2.10, we need to show that Y_n cannot be written as a countable union, $\bigcup_n Z_n$, such that $[Z_n]^\omega \subset \mathcal{L}$ for each n. We show that such a sequence of Z_n will contradict that $\{I_\alpha : \alpha \in Y_n\}$ and $\{J_\beta : \beta \in Y_n\}$ are not countably separated.

Indeed, observe that if $Z \subset \omega_1$ is such that $[Z]^\omega \subset \mathcal{L}$, then there is an m such that

$$Z' = \bigcup \{J_\beta \setminus \{j(\beta, \ell) : \ell < m\} : \beta \in Z\}$$

is almost disjoint from every member of $\mathcal{I}$. Indeed, for each γ, there is an m_γ witnessing that $Z \cap \gamma \in \mathcal{L}$ and so there is an m so that $m_\gamma = m$ for uncountably many γ. So to see that $I \cap Z'$ is finite (for some $I \in \mathcal{I}$), choose γ so that $m_\gamma = m$ and $I \cap J_\beta = \emptyset$ for $\beta > \gamma$.

By Lemma 2.10, there is a proper poset R which adds an uncountable $Y \subset Y_n$ such that $[Y]^\omega \cap \mathcal{L}$ is empty. However as we need a bit more we prove it and Lemma 2.10 now. The proof of Lemma 2.10 follows from Claim 4 by ignoring $\mathcal{I}$ since we will not use any properties of $\mathcal{L}$ other than the fact that Y_n cannot be expressed as a countable union of "$\mathcal{L}$-homogeneous" sets. For each $I \in \mathcal{I}$ and $\beta \in \omega_1$, let $g_I(\beta)$ be the smallest integer such that $I \cap J_\beta \subset \{j(\beta,\ell) : \ell < g_I(\beta)\}$.

Claim 4: There is a proper poset R such that R adds an uncountable set $Y \subset Y_n$ such that $[Y]^\omega \cap \mathcal{L}$ is empty and, for each $\alpha \in Y$, there is an $I_\alpha \in \mathcal{I}$ such that $g_{I_\alpha} \restriction \alpha$ is finite-to-one.

Define $\tilde{\mathcal{L}}$ to be all $Z \subset \omega_1$ with the property that $[Z]^\omega \subset \mathcal{L}$. Recall that Y_n is not covered by a countable subcollection of $\tilde{\mathcal{L}}$. As is usual (see §6 [Dow92]) define $p \in R$ iff

$$p = A_p \cup \mathcal{L}_p \cup \mathcal{M}_p \text{ where } \begin{cases} A_p \subset Y_n \text{ is finite,} \\ \mathcal{L}_p \subset \mathcal{L} \text{ is finite,} \\ \mathcal{M}_p \text{ is a finite } \in\text{-chain,} \\ \text{for } \alpha < \beta \in A_p\ \exists M \in \mathcal{M}_p \text{ s.t. } \alpha \in M, \beta \notin M, \\ \alpha \notin \bigcup(M \cap \tilde{\mathcal{L}}) \text{ for any } \ M \in \mathcal{M}_p \text{ and } \alpha \in A_p \setminus M, \\ M \in \mathcal{M}_p \text{ implies } M \prec H(\theta) \text{ and } |M| = \omega\ . \end{cases}$$

and $p < q$ if $p \supset q$ and for $\alpha \in A_q$ and $\beta \in \alpha \cap (A_p \setminus A_q)$, β is not in $\bigcup \mathcal{L}_q$ and $I_\alpha \cap J_\beta \setminus \{j(\beta,\ell) : \ell < |A_q|\}$ is not empty. Note that elementarity will guarantee that I_α contains mod finite every member of $\mathcal{I} \cap M$ if $M \cap \omega_1 \leq \alpha$.

To show that R is proper, fix a countable $M \prec H(\rho)$ for a suitably large ρ and let $p \in M \cap R$. We show that $p \cup \{M \cap H(\theta)\}$ is (M,R)-generic. Let $D \in M$ be a dense open subset of R and let $r \in D$ be less than $p \cup \{M \cap H(\theta)\}$. Let $\{\alpha_0, \ldots, \alpha_{n-1}\}$ list $A_r \setminus M$ in increasing order. Also, let $r_0 = r \cap M$; note that $r_0 \in R$. Let S denote the tree of finite sets s of ordinals in $\omega_1 \setminus (\max(A_{r_0}) + 1)$ (ordered by end-extension) for which there is a $q \in D$ extending r_0 such that $|A_q| = |A_r|$ and $A_q \setminus A_{r_0}$ end-extends s. Prune S (starting from the top) to get T so that for each $t \in T$, $T_t = \{\xi : t^\frown\xi \in T\}$ is not covered by countably many members of $\tilde{\mathcal{L}}$. One uses the fact that $\mathcal{M}_r$ separates A_r and that $\alpha \in A_r \setminus M'$ (for $M' \in \mathcal{M}_r$) implies that α is not in any member of $\tilde{\mathcal{L}} \cap M'$ to show that $\{\alpha_0, \ldots, \alpha_i\}$ is in T for each $i < n$ (including $i = 0$ which yields that $\emptyset \in T$). The reason for introducing T is that T can be shown to be a member of M_{α_0} while D certainly is not. We must now work in M_{α_0} rather than M because we will want to know that the sets I_{α_i} are large enough with respect to certain other sets that arise.

Let $L = \bigcup \mathcal{L}_{r_0}$ and begin a finite recursion. Since $\emptyset \in T$, and $T \in M_{\alpha_0}$, $T_\emptyset \in M_{\alpha_0}$ and $M_{\alpha_0} \models T_\emptyset \notin \tilde{\mathcal{L}}$, hence there is a $T' \in M_{\alpha_0}$ such that $T' \in [T_\emptyset]^\omega \setminus \mathcal{L}$. Since $T' \notin \mathcal{L}$ and $\mathcal{I}$ is a P-ideal, there is an $I \in \mathcal{I} \cap M_{\alpha_0}$ such that, for each m, there is a $\beta_m \in T'$ such that $j(\beta_m, \ell) \in I$ for some $\ell > m$. Clearly $\{\beta_m : m \in \omega\}$ meets every member of $\mathcal{L}$ in a finite set, so we may assume that, in fact, $T' \cap L$ is empty. Since I_{α_i} almost contains I for each $i < n$, there is a $\xi_0 \in T'$ such that $I_{\alpha_i} \cap J_{\xi_0} \setminus \{j(\xi_0,\ell) : \ell < |A_{r_0}|\} \neq \emptyset$ for each $i < n$. Continue choosing $\xi_i \in M_{\alpha_0} \setminus L$ $(i < n)$ so that $t = \{\xi_0, \ldots, \xi_i\} \in T$ and so that $I_{\alpha_j} \cap J_{\xi_i} \setminus \{j(\xi_i,\ell) : \ell < |A_{r_0}|\}$ is not empty for each $j < n$. The argument is the same as above with T_t replacing $T_\emptyset$. When we have chosen $\{\xi_0, \ldots, \xi_{n-1}\} \in T \cap M_{\alpha_0}$, we have that there is a $q \in D \cap M$ such that $A_q = A_{r_0} \cup \{\xi_0, \ldots, \xi_{n-1}\}$. One easily checks that $q \cup r$ is a member of R and by the careful choice of the ξ (i.e. the condition on $I_{\alpha_j} \cap J_{\xi_i}$) we have that $q \cup r$ is less than both q and r. This completes the proof that R is proper.

If G is R-generic, we let $Y' = \bigcup\{A_p : p \in G\}$. The condition on extension guarantees that $g_\alpha \restriction (Y' \cap \alpha)$ is finite-to-one for each $\alpha \in Y$. In addition $[Y']^\omega \cap L$ is finite for each $L \in \mathcal{L}$ since it is easily seen that there is a $p \in G$ such that $L \in \mathcal{L}_p$ and then, by the definition of extension, it follows that $Y' \cap L \subset A_p$. Finally we must show that Y' is uncountable. This follows directly from the fact that Y_n is not covered by countably many members of $\tilde{\mathcal{L}}$. Indeed, if $p \in R$ is arbitrary, and M is any countable elementary submodel of $H(\theta)$ containing p, then $Y_n \setminus (\tilde{\mathcal{L}} \cap M)$ is uncountable. $\square$

3. Applications of Theorem 2.3

The results in this section were shown to hold in models in which a supercompact cardinal is collapsed to $\aleph_2$ by either the Levy or the Mitchell collapse. In particular Corollary 3.3 is from [DJW89], Corollaries 3.4 and 3.6 are from [DM90], and Corollary 3.5 is from [Dow88]. The notion of a coherent κ-matrix (of integer-valued functions) was introduced in [DJW89] (see Definition 3.1). It was shown that, for a regular uncountable cardinal κ, every space which can be written as an increasing κ-chain of first-countable subspaces would itself be first-countable if and only if every coherent κ-matrix had an extension to a coherent $\kappa + 1$-matrix.

Recall that for a cardinal κ, $\beta\kappa$ denotes the Stone-Čech compactification of the space κ with the discrete topology. As usual we identify $\beta\kappa$ with the Stone space of the Boolean algebra $\mathcal{P}(\kappa)$ in which the fixed ultrafilters are identified with the ordinals in κ. The subspaces κ^* and $U(\kappa)$ of $\beta\kappa$ consist of the free ultrafilters and the uniform ultrafilters, respectively; where an ultrafilter is *uniform* if each of its members has cardinality κ.

Definition 3.1. For any ordinal number κ, call $\{f_{\alpha,n} : \alpha \in \kappa, n \in \omega\}$ a κ-matrix if each $f_{\alpha,n} \in {}^\alpha\omega$ and $k < n$ implies $f_{\alpha,k} \leq f_{\alpha,n}$. We say that it is a *coherent* κ-matrix if for each $\beta < \alpha < \kappa$, $\sup\{f_{\alpha,n}(\beta) : n \in \omega\} = \omega$ and, for each n there is an m such that $f_{\beta,n} \leq f_{\alpha,m} \restriction \beta$ and $f_{\alpha,n} \restriction \beta \leq f_{\beta,m}$.

Remark 3.1. The condition that $f_{\alpha,k} \leq f_{\alpha,n}$ for $k < n$ was inadvertently left out of the definition of κ-matrix in [DJW89].

Definition 3.2. For a function g into ω, let $g^\downarrow$ denote the set of pairs (x, m) such that $m \leq g(x)$ (and x is in the domain of g).

Corollary 3.3 (PFA). *If $\{g_{\alpha,n} : \alpha < \kappa, n \in \omega\}$ is a coherent κ-matrix of functions for a regular cardinal $\kappa > \omega_1$, then the matrix extends to a $\kappa + 1$-matrix.*

Proof. We define a pair of ideals $\mathcal{I}$ and $\mathcal{J}$ and then apply Theorem 2.3. Let $I \in \mathcal{I}$ if and only if I is a countable subset of $\kappa \times \omega$ and such that $I \cap g^\downarrow_{\alpha,n}$ is finite for each $\alpha < \kappa$ and $n \in \omega$. It is easily checked that $\mathcal{I}$ is a P-ideal. Next, let $J \in [\kappa \times \omega]^\omega$ be a member of $\mathcal{J}$ if and only if there is an $\alpha < \kappa$ and $n \in \omega$ such that $J \subset g^\downarrow_{\alpha,n}$.

We first show that any $\mathcal{I}' \in [\mathcal{I}]^{\omega_1}$ and $\mathcal{J}' \in [\mathcal{J}]^{\omega_1}$ can be countably separated. Indeed, choose $\lambda < \kappa$ large enough so that $K \subset \lambda \times \omega$ for each $K \in \mathcal{I}' \cup \mathcal{J}'$. The fact that the matrix is coherent easily implies that $\{g^\downarrow_{\lambda,n} : n \in \omega\}$ countably separates $\mathcal{I}'$ and $\mathcal{J}'$.

Therefore by Theorem 2.3 and Lemma 2.2, there are $\{X_n : n \in \omega\}$ which strongly separate $\mathcal{I}$ and $\mathcal{J}$. Without loss of generality $X_n \subset X_{n+1}$ for each n. For each n, define $h_n \in {}^\kappa\omega$ by

$$h_n(\alpha) = \min\{k : X_n \cap (\{\alpha\} \times [k, \omega)) \text{ is empty}\} .$$

Let K be the set of γ such that the set $\{h_n(\gamma) : n \in \omega\}$ is bounded in ω. If we show that K is countable, then by adding only finitely points to each X_n we can actually assume that K is empty. Suppose that K is uncountable and fix any $\alpha \in \kappa$ so that there is an m and uncountably many $\gamma \in K \cap \alpha$ such that $h_n(\gamma)$ is bounded by m. Choose $k \in \omega$ so that $g_{\alpha,k}(\gamma) > m$ for uncountably many of these γ. But now this gives rise to a $J \in \mathcal{J}$ (i.e. a subset of $g^{\downarrow}_{\alpha,k}$) which is not contained, mod finite, in any X_n.

Now we show that we can use $\{h_n : n \in \omega\}$ to extend the matrix. Fix any $\alpha < \kappa$, $n \in \omega$ and $m \in M$. By the definition of h_m, there is a k such that $h_m \restriction \alpha$ is bounded by $g_{\alpha,k}$. We must also show there is a k so that $g_{\alpha,n} < h_k$. This is the same as showing that $g^{\downarrow}_{\alpha,n}$ is contained in some X_k. If there were no such k, we could easily define a countable $J \subset g^{\downarrow}_{\alpha,n}$ such that J is not contained in any X_k. But now $J \in \mathcal{J}$ hence there is a k such that J is almost contained in X_k . Finally we finish by noting that the h_n's are an increasing family and $\{h_n(\gamma) : n \in \omega\}$ is unbounded for each γ. □

Corollary 3.4 (PFA). *For each regular cardinal $\kappa > \omega_1$, the space $\kappa^* \setminus U(\kappa)$ is C^*-embedded in $\beta\kappa$.*

Proof. Suppose that Z_0 and Z_1 are disjoint zero-set subsets of the space $\kappa^* \setminus U(\kappa)$. Now define $\mathcal{I}$ (respectively $\mathcal{J}$) to be the ideal of all those countable subsets, I (respectively J) of κ such that $I^* \subset Z_0$ (respectively $J^* \subset Z_1$). We first note that $\aleph_1$-sized subideals $\mathcal{I}', \mathcal{J}'$ of $\mathcal{I}$ and $\mathcal{J}$ respectively are countably separated. Indeed, given such subideals there is a λ which contains $I \cup J$ for each $I, J \in \mathcal{I}' \cup \mathcal{J}'$. Furthermore, there is a $Y \subset \lambda$ such that $Y^* \cap Z_0$ is empty and $Y^* \supset Z_1 \cap \lambda^*$. This Y alone will countably separate $\mathcal{I}'$ and $\mathcal{J}'$.

Therefore there is a family of X_n which strongly separates the ideals. In this case $\mathcal{J}$ is also a P-ideal so we will show that some X_n actually contains, mod finite, every member of $\mathcal{J}$. Otherwise, choose, for each n, a $J_n \in \mathcal{J}$ such that $J_n \setminus X_n$ is infinite. Clearly then any $J \in \mathcal{J}$ which contains, mod finite, each J_n will not be contained, mod finite, in any X_n. It follows easily that $\overline{X_n}$, for some n, is a clopen subset of $\beta\kappa$ which contains Z_1 and is disjoint from Z_0. □

Corollary 3.5 (PFA). *A countably tight space is first countable if and only if every $\leq \aleph_1$-sized subspace is first countable. Equivalently, a sequential or Fréchet space is first countable if every $\leq \aleph_1$-sized subspace is first countable.*

Proof. Let Y be a countably tight space and assume that each $\leq \aleph_1$-sized subspace is first countable. Fix any $y \in Y$. Define $\mathcal{I}$ to be the ideal of countable subsets of $Y \setminus \{y\}$ which converge to y. Since each countable subspace is first countable, $\mathcal{I}$ is a P-ideal. Let $\mathcal{J}$ be the ideal of countable subsets of Y which do not have y as a limit point. It is easily seen that if $\{(I_\alpha, J_\alpha) : \alpha < \omega_1\}$ is a family as in (2), then the subspace $\{y\} \cup \bigcup_\alpha J_\alpha$ is not first countable. Therefore we assume that $\{X_n : n \in \omega\}$ is as in (1) and show that Y is first countable at y. Again, without loss of generality, each X_n is almost disjoint from every member of $\mathcal{I}$. Suppose that Z has cardinality $\aleph_1$ and is a subset of $Y \setminus \bigcup_n X_n$ (if this set is uncountable). If $Z \cup \{y\}$ is first countable, then there is an infinite $J \subset Z$ which does not have y as a limit point, i.e. $J \in \mathcal{J}$. Since there is no X_n which almost contains J, $Y \setminus \bigcup_n X_n$ must be countable. Therefore we may assume $Y \setminus \{y\} = \bigcup_n X_n$. Since X_n is almost disjoint from each member of $\mathcal{I}$ and since Y is sequential (this follows from countable tightness and first countable on countable subsets), it follows that

y is not a limit point of any X_n. Now suppose that y is not in the closure of some $Z \subset Y$. To prove that Y is first countable at y we show that Z is almost contained in some X_n. Indeed, if it were not, then Z would have a countable subset J which also was not contained (mod finite) in any of the X_n. However y is not a limit of $J \subset Z$, hence $J \in \mathcal{J}$. This contradicts that some X_n should almost contain J. $\square$

Corollary 3.6 (PFA). *If $\{Z_\alpha : \alpha < \kappa\}$ (with $cf(\kappa) > \omega_1$) is a sequence of zero-subsets of κ^* with the property that $Z_\alpha = Z_\beta \cap \alpha^*$ for each $\alpha < \beta < \kappa$, then there is a zero set Z of $\beta\kappa$ such that $Z \cap \alpha^* = Z_\alpha$ for each $\alpha < \kappa$.*

4. More on ω^*

In this section we answer questions that have been raised about the variety of dense C^*-embedded subsets of ω^*. Van Douwen, Kunen and van Mill [vDKvM89] have shown that it is consistent that there are proper dense (even co-dense) subsets of ω^* which are C^*-embedded and have asked if it is consistent that there are dense subsets of size $\mathfrak{c}$ which are C^*-embedded. We show that this follows from PFA (in fact MA plus Todorčević's strong form of the Open Coloring Axiom). We also show that it holds in the Cohen model. It is well-known that it follows from CH that no proper dense subset is C^*-embedded. We let $C(X)$ denote the ring of real-valued continuous functions on a space X. Let $[X]^2$ denote the set of two-element subsets of X (which we identify with the set of ordered pairs (x, y) such that $x < y$ in the case that X is ordered).

Definition 4.1. OCA denotes the statement that if $[X]^2 = K_0 \cup K_1$ where $X \subset \mathbf{R}$ and where K_0 is open in $[X]^2$, then either there is an uncountable 0-homogeneous set or X is the countable union of 1-homogeneous sets. A set $Y \subset X$ is 0-homogeneous (respectively 1-homogeneous) if $[Y]^2 \subset K_0$ (respectively K_1).

Todorčević shows that this form of OCA follows from PFA; hence we may assume MA plus OCA is consistent with ZFC. A much weaker form of OCA was first shown to be consistent in [AS81]. For a family $\mathcal{A} \subset [\omega]^\omega$, let $\partial^\infty\mathcal{A}$ denote the set of complete accumulation points of $\mathcal{A}$, i.e. $\partial^\infty\mathcal{A} = \{x \in \omega^* :$ for each u in the ultrafilter x, $|\mathcal{A}| = |\{a \in \mathcal{A} : u \cap a \neq^* \emptyset\}|\}$.

The main theorem of this section follows easily from Lemmas 4.3 and 4.4 by applying simple cardinal arithmetic.

Theorem 4.2. *If V is a model of Martin's Axiom plus OCA or if it is obtained by adding at least 2^λ Cohen reals to some ground model satisfying $\mathfrak{c} = \lambda$ then there are dense C-embedded subspaces of ω^* which have cardinality $\mathfrak{c}$.*

Proof. By Lemma 4.3 or 4.4, simply choose an $X \in [\omega^*]^{\mathfrak{c}}$ which satisfies the hypotheses of the lemma. That this can be done simply relies on the fact that there are only continuum many $\mathcal{A} \subset [\omega]^\omega$ which must be considered. In the case that Martin's Axiom holds we have that $2^{<\mathfrak{c}} = \mathfrak{c}$ while in the case that V is obtained by adding at least 2^λ Cohen reals, then it suffices to observe that $[\mathcal{P}(\omega)]^{\leq\lambda}$ has cardinality $\mathfrak{c}$. $\square$

Lemma 4.3 (MA+OCA). *Assume that $X \subset \omega^*$ is such that for any $\mathcal{A} \cup \mathcal{C} \subset [\omega]^\omega$ such that $|\mathcal{A} \cup \mathcal{C}| < \mathfrak{c}$ and if $\partial^\infty\mathcal{A} \cap \partial^\infty\mathcal{C} \neq \emptyset$ then X meets this intersection. Then X is C-embedded in ω^*.*

Proof. Let $X \subset \omega^*$ satisfy the hypotheses of the Lemma and assume that $f \in C(X)$. By Lavrentieff's theorem, f will extend continuously to a G_δ-set D with $X \subset D \subset \omega^*$. Since X meets $\partial^\infty \mathcal{A}$ for each countable $\mathcal{A} \subset [\omega]^\omega$ and non-empty G_δ subsets of ω^* have dense interior, it should be clear that f is a bounded function. Therefore if f does not extend continuously to ω^*, there are reals $r < s$ such that the closure of $A_1 = f^{-1}([s, \infty))$ meets the closure of $A_0 = f^{-1}((-\infty, r])$. Let $\mathcal{X}$ denote all disjoint pairs (a, b) of infinite subsets of ω such that $a^* \subset A_0$ and $b^* \subset A_1$. We may think of $\mathcal{X}$ as a subset of the square of the Cantor set by identifying a subset of ω with its characteristic function. Define a partition $K_0 \cup K_1$ of $[\mathcal{X}]^2$ by $\langle (a, b), (c, d)\rangle \in K_0$ if $(a \cup c) \cap (b \cup d)$ is not empty. It is easily seen that K_0 is open in $[\mathcal{X}]^2$. By OCA, there is either

1. an uncountable 0-homogeneous $\mathcal{Y} \subset \mathcal{X}$, or
2. a countable family of 1-homogeneous sets whose union is $\mathcal{X}$.

We show that both situations lead to a contradiction. In the first instance, assume that $\{(a_\alpha, b_\alpha) : \alpha < \omega_1\}$ is a 0-homogeneous set. We first show that $\bigcup\{a^*_\alpha : \alpha < \omega_1\}$ and $\bigcup\{b^*_\alpha : \alpha < \omega_1\}$ do not have disjoint closures. Indeed, if they did then there would be an $a \subset \omega$ such that $a_\alpha \setminus a$ and $b_\alpha \cap a$ are finite for all $\alpha \in \omega_1$. Then we find an uncountable J and an $n \in \omega$ so that for all $\alpha \in J$, $a_\alpha \setminus a \subset n$ and $b_\alpha \cap a \subset n$. In addition, we may arrange that $a_\alpha \cap n$ and $b_\alpha \cap n$ are the same for all $\alpha \in J$. But now, if $\alpha < \beta$ are both in J, it follows that $(a_\alpha \cup a_\beta) \cap (b_\alpha \cup b_\beta)$ is empty, which contradicts that this family was to be 0-homogeneous. Note that each a_α is almost disjoint from each b_β since $a^*_\alpha \subset A_0$ and $b^*_\beta \subset A_1$. By MA (i.e. $\mathfrak{b} > \omega_1$), it follows that $\partial^\infty\{a_\alpha : \alpha < \omega_1\}$ meets $\partial^\infty\{b_\alpha : \alpha < \omega_1\}$, and therefore this intersection should meet X. However this would clearly contradict that f is continuous on X.

Now suppose that $\mathcal{X}$ is a countable union of 1-homogeneous sets and fix p in $\overline{A_0} \cap \overline{A_1}$. By induction on cardinality it follows that if $\lambda < \mathfrak{c}$ and $\{y_\alpha : \alpha < \lambda\}$ is a subset of the ultrafilter p, then

$$D \cap A_0 \cap \bigcap\{y^*_\alpha : \alpha < \lambda\}$$

has non-empty interior (and similarly with A_1 replacing A_0). Indeed, for each $\alpha < \lambda$, choose, by the inductive hypothesis, a_α so that a^*_α is contained in

$$D \cap A_0 \cap \bigcap\{y^*_\beta : \beta < \alpha\} .$$

Clearly $\partial^\infty\{a_\alpha : \alpha < \lambda\}$ meets itself, hence there is an element, x, of X in $\partial^\infty\{a_\alpha : \alpha < \lambda\}$. Since D is a G_δ, there are $b_n \in x$ ($n \in \omega$) so that $\bigcap_n b^*_n \subset D$ and $b^*_n \cap D \subset f^{-1}((-\infty, r + 1/n))$. Observe that $x \in y^*_\beta$ for each $\beta < \lambda$, hence, by MA,

$$\bigcap\{b^*_n : n \in \omega\} \cap \bigcap\{y^*_\beta : \beta < \lambda\}$$

has interior contained in $D \cap A_0$, as was to be proved.

Now fix an enumeration, $\{y_\alpha : \alpha < c\}$, of p and choose, for each $\alpha < \mathfrak{c}$, disjoint infinite a_α and b_α so that

$$a^*_\alpha \subset A_0 \cap \bigcap\{y^*_\beta : \beta < \alpha\}$$

and

$$b^*_\alpha \subset A_1 \cap \bigcap\{y^*_\beta : \beta < \alpha\} .$$

Note that for each $J \subset \mathfrak{c}$ with $|J| = \mathfrak{c}$, p is a limit point of both $\bigcup\{a^*_\alpha : \alpha \in J\}$ and $\bigcup\{b^*_\alpha : \alpha \in J\}$, hence $\bigcup\{a_\alpha : \alpha \in J\}$ is not disjoint from $\bigcup\{b_\alpha : \alpha \in J\}$.

However since $\{(a_\alpha, b_\alpha) : \alpha \in c\}$ is a countable union of 1-homogeneous sets, there is a $J \subset \mathfrak{c}$ of cardinality $\mathfrak{c}$, so that $\{(a_\alpha, b_\alpha) : \alpha \in J\}$ is 1-homogeneous. This is our desired contradiction since 1-homogeneity guarantees that $\bigcup\{a_\alpha : \alpha \in J\}$ is, in fact, disjoint from $\bigcup\{b_\alpha : \alpha \in J\}$. $\square$

Lemma 4.4. *Let G be $\mathrm{Fn}(\kappa, 2)$-generic over a model V and let $\lambda = \mathfrak{c}^V$. In $V[G]$, suppose $X \subset \omega^*$ is such that for any $\mathcal{A} \cup \mathcal{C} \subset [\omega]^\omega$ such that $\partial^\infty \mathcal{A} \cap \partial^\infty \mathcal{C} \neq \emptyset$ and $|\mathcal{A} \cup \mathcal{C}| \leq \lambda$, then X meets this intersection. Then X is C-embedded in ω^*.*

Proof. Let $X \subset \omega^*$ satisfy the hypotheses of the Lemma and assume that $f \in C(X)$. By Lavrentieff's theorem, f will extend continuously to a G_δ-set D with $X \subset D \subset \omega^*$. Just as in the proof of Lemma 4.3, assume that f has been extended continuously to all of D and it does not extend continuously to ω^*. Therefore there are reals $r < s$ such that the closure of $f^{-1}([s, \infty))$ meets the closure of $f^{-1}((-\infty, r])$. Let $\mathcal{X}$ denote all infinite subsets a of ω such that $f[a^*] \subset (-\infty, r]$ and similarly $\mathcal{Y}$ is all $b \subset \omega$ such that $f[b^*] \subset [s, \infty)$. Since D is a dense G_δ and non-empty G_δ's of ω^* have dense interior, it follows that $\mathcal{X}$ cannot be separated from $\mathcal{Y}$, i.e. if $A \subset \omega$ is such that $a \setminus A$ is finite for each $a \in \mathcal{X}$, then there is a $b \in \mathcal{Y}$ such that $A \cap b$ is infinite.

Let θ be a sufficiently large regular cardinal in V and let $M \prec H(\theta)$ contain $\mathrm{Fn}(\kappa, 2)$-names for each of $\mathcal{X}$ and $\mathcal{Y}$ and suppose that $M^\omega \subset M$. It follows that $M[G]$ is an elementary submodel of $H(\theta)[G]$ and that $H(\theta)[G]$ is the $H(\theta)$ in the sense of $V[G]$. In addition, since $M^\omega \subset M$, it follows that $M[G] \cap [\omega]^\omega$ is contained in $V[G \cap M]$ and, countable subsets of $M[G] \cap [\omega]^\omega$ which are members of $V[G \cap M]$ are also members of $M[G]$. For proofs of these facts see §4 of [Dow92].

We will show that the interpretations of $\mathcal{A} = \mathcal{X} \cap M$ and $\mathcal{C} = \mathcal{Y} \cap M$ also cannot be separated in $V[G]$. Therefore if $|M| \leq \lambda$, it will follow that $X \cap \partial^\infty \mathcal{A} \cap \partial^\infty \mathcal{C} \neq \emptyset$ contradicting that f is continuous on X.

Working in $V[G \cap M]$, suppose that there is a name $\dot{A}$ which is forced to (mod finite) contain every member of $\mathcal{X} \cap M[G]$ and to be (mod finite) disjoint from every member of $\mathcal{Y} \cap M[G]$. For each condition p (in the countable name $\dot{A}$), let $\mathcal{X}_p$ denote all those members of $\mathcal{X} \cap M[G]$ which are forced by p to be contained in $\dot{A}$. Clearly, then $\mathcal{X}_p$ is separated from $\mathcal{Y} \cap M[G]$ by the set $\{n : p \Vdash n \in \dot{A}\}$ which is a member of $M[G]$ hence it follows by elementarity that $\mathcal{X}_p$ is separated from all of $\mathcal{Y}$. It follows then that it suffices to show that if $\{X_n : n \in \omega\} \subset \mathcal{P}(\omega)$ (in $V[G \cap M]$) is such that $X_n \cap b$ is finite for each $b \in \mathcal{Y}$ and $n \in \omega$, then there is an $a \in \mathcal{X} \cap M[G]$ such that $a \setminus X_n$ is infinite for each n. As explained above, it follows that $\{X_n : n \in \omega\} \in M[G]$, hence, by elementarity, it suffices to show that there is an $a \in \mathcal{X}$ such that $a \setminus X_n$ is infinite for each n. Observe that, since $X_n^* \cap f^{-1}([s, \infty))$ is empty for each n, it follows that $f^{-1}((\infty, r]) - \bigcup_{k<n} X_k^*$ is not empty for each n. For each n, choose $a_n \in \mathcal{X}$ such that $a_n \cap X_k = \emptyset$ for each $k < n$. Let $x \in X$ be a member of $\partial^\infty\{a_n : n \in \omega\}$. Clearly, $f(x) \leq r$. Since D is a G_δ, we may choose $c_n \in x$ $(n \in \omega)$ so that $\bigcap_{n \in \omega} c_n^*$ is contained in D and, for each n, $f(c_n^* \cap D) \subset (-\infty, r + \frac{1}{n})$. Let $a \in [\omega]^\omega$ be such that $a \setminus c_n$ is finite for each n. It is easily seen that $a \in \mathcal{X}$, and since $c_n \cap X_n$ is empty, it follows that $a \setminus X_n$ is infinite for each n. $\square$

5. ω^* minus a point in the Miller model

It is shown in [vDKvM89] that it is consistent to suppose that $\omega^* \setminus \{p\}$ is C^*-embedded for every $p \in \omega^*$ (e.g. from PFA) and Malyhin has shown that this also holds in the Cohen model. It is asked in [HvM90] if there is a model in which $\omega^* \setminus \{p\}$ is C^*-embedded for some but not all $p \in \omega^*$. We establish that such a model exists: the one obtained by iterating Miller forcing. We obtain that, in this model, $\omega^* \setminus \{p\}$ is C^*-embedded iff p is not a P-point.

Recall (see [Mil84]) that the Miller poset Q consists of rooted trees $T \subset {}^{<\omega}\omega$ with the property that for each $t \in T$, there is an extension $t' \in T$ of t which is *branching* in the sense that t' has infinitely many immediate successors in T. The ordering on Q is inclusion: $T < T'$ if $T \subset T'$. Blass and Shelah [BS87] have shown that the character of every P-point ultrafilter on ω is ω_1 in the Miller model (see Corollary 5.2 for a very brief sketch). For convenience, we will assume that if $T \in Q$ and $t \in T$, then t consists of an increasing function (we can work below any T with this property).

Lemma 5.1. *If $p \in \omega^*$ has a base of cardinality ω_1, then $\omega^* \setminus \{p\}$ is not C^*-embedded in ω^*.*

Proof. This is essentially well-known. If p is a P-point then $\omega^* \setminus \{p\}$ is homeomorphic to $\omega_1^* \setminus U(\omega_1)$ (the so-called sub-uniform ultrafilters on ω_1) which, obviously, has a non-trivial compactification, namely ω_1^*. Therefore the Stone-Čech compactification of $\omega^* \setminus \{p\}$ is not ω^*, i.e. the one-point compactification.

If p is not a P-point then fix a pairwise disjoint sequence $\{b_n : n \in \omega\} \subset [\omega]^\omega \setminus p$ so that every member of p meets infinitely many of the b_n in an infinite set. Let $\{i_\alpha : \alpha < \omega_1\}$ enumerate a generating set for the dual ideal of p such that $i_n \supset b_n$ for each n. We will choose a partition, $a_\alpha \cup b_\alpha$, of each i_α so that if $\beta < \alpha$, then $a_\alpha \cap i_\beta$ is almost equal to $a_\beta \cap i_\alpha$. In addition, we must ensure that for each β there is an α so that $a_\alpha \setminus i_\beta$ is infinite. We have already chosen b_n and let $a_n = i_n \setminus b_n$. Now consider stage α. If i_α is in the ideal generated by $\{i_\beta : \beta < \alpha\}$ then simply let a_α be a suitable finite union of previously chosen $a_\beta \cap i_\alpha$. Otherwise, fix some infinite $j_\alpha \subset i_\alpha$ such that $j_\alpha \cap i_\beta$ is finite for all $\beta < \alpha$. We wish to choose $a_\alpha \subset i_\alpha$ so that

1. $j_\alpha \subset a_\alpha$,
2. for each $\beta < \alpha$, $a_\beta \cap i_\alpha$ is almost contained in a_α,
3. for each $\beta < \alpha$, $b_\beta \cap i_\alpha$ is almost disjoint from a_α.

That is, we are simply asking for a set to separate the family $\{j_\alpha\} \cup \{a_\beta \cap i_\alpha : \beta < \alpha\}$ from the family $\{b_\beta \cap i_\alpha : \beta < \alpha\}$. Since every member of the first family is almost disjoint from every member of the second and both families are countable, this is easily done. We let $b_\alpha = i_\alpha \setminus a_\alpha$.

Clearly there are uncountably many α such that i_α is not in the ideal generated by $\{i_\beta : \beta < \omega_1\}$ and, for each such α, there was a j_α chosen to witness that a_α is not in the ideal generated by $\{i_\beta : \beta < \alpha\}$.

Now by the coherence property of the a_α's with respect to the i_β's, it follows that $U = \bigcup\{a_\alpha^* : \alpha < \omega_1\}$ is a clopen subset of $\omega^* \setminus \{p\}$. In addition, since no i_β contains mod finite all the a_α, U is not a compact set (it has p in its closure). Since $b_n^* \cap U$ is empty for each n, it follows that $\omega^* \setminus U$ also has p as a limit. Therefore there is a two-valued function on $\omega^* \setminus \{p\}$ which does not extend continuously to ω^*. □

Corollary 5.2. *If G is P_{ω_2}-generic over V, where $V \models CH$ and P_{ω_2} is the ω_2-length countable support iteration of Miller forcing, then $\omega^* \setminus \{p\}$ is not C^*-embedded in ω^* for any P-point ultrafilter $p \in \omega^*$.*

Proof. Suppose that p is a P-point ultrafilter on ω in $V[G]$. Fix a name for p and choose an elementary submodel M of $H(\theta)$ (in V) for a large enough θ so that M contains the name, $M^\omega \subset M$ and $|M| = \omega_1$. Again, standard arguments yield that $p \cap M[G]$ is a member of $V[G \cap M]$ and, in that model, is a P-point ultrafilter on ω (see, for example, §4 of [Dow92]). Also, $V[G]$ is obtained from $V[G \cap M]$ by forcing with a countable support iteration of Miller posets. Miller [Mil84] has shown that single stage Miller forcing preserves P-points and it is shown in Blass and Shelah [BS87], that any countable support iteration of proper forcings that preserve P-points also preserves them. Therefore, $p \cap M[G]$ generates p in the final model. Since $V[G \cap M] \models$ CH, it follows that p has a base of cardinality ω_1. Now we apply Lemma 5.1. □

Theorem 5.3. *Let, for $\lambda \leq \omega_2$, P_λ be the λ-length countable support iteration of Miller forcing Q. If G is P_{ω_2}-generic over V, a model of CH, and if, in $V[G]$, $p \in \omega^*$ is not a P-point, then $\omega^* \setminus \{p\}$ is C^*-embedded in ω^*.*

Proof. In $V[G]$, fix a partition $\{C_n : n \in \omega\}$ of ω so that for each $U \in p$, there are infinitely many n so that $U \cap C_n$ is infinite. Also let $f \in C^*(\omega^* \setminus \{p\})$ and assume that p is in the closure of $f^{-1}(0)$ and $f^{-1}(1)$. Let C denote the set $\bigcup_n C_n^*$ and define $\mathcal{I}_0$ to be the ideal of subsets, a, of ω such that $a^* \subset f^{-1}(0) \setminus \overline{C}$ and similarly define $\mathcal{I}_1$ to be those a such that $a^* \subset f^{-1}(1) \setminus \overline{C}$.
Claim: Each of $\mathcal{I}_0$ and $\mathcal{I}_1$ are P-ideals and every member of p contains a member of $\mathcal{I}_0$ and a member of $\mathcal{I}_1$.

By symmetry we just prove the claim for $\mathcal{I}_0$. Let $u \in p$. First we find an $x \in u^* \setminus C$ such that $f(x) = 0$. Since $(u \setminus \bigcup_{n<k} C_n)^* \cap f^{-1}(0)$ is not empty for each k, choose x_k in this intersection for each k. The sequence $\{x_k : k \in \omega\}$ has $2^{\mathfrak{c}}$ limit points, none of which are in C; let x be such a limit point. Fix a $v \subset u$ such that $v \in x \setminus p$. Now f restricted to v^* is continuous and v^* is homeomorphic to ω^* hence the zero set $v^* \cap f^{-1}(0) \setminus C$ has non-empty interior. Now suppose that I is a countable subset of $\mathcal{I}_0$. Since each member of $\mathcal{I}_0$ is almost disjoint from each C_n, it follows that there is a set u which is almost disjoint from C_n for each n and which contains mod finite each member of I. Again, restricting f to u^* allows us to utilize the fact that the ideal of $a \subset u$ such that $a^* \subset f^{-1}(0) \cap u^*$ is a P-ideal. Since I is contained in this ideal this completes the proof of the Claim.

Fix P_{ω_2} names for p, the C_n's, and the ideals $\mathcal{I}_0$ and $\mathcal{I}_1$. Since P_{ω_2} is proper and has the $\aleph_2$-c.c., there is a $\lambda < \omega_2$ of uncountable cofinality so that

1. $p' = p \cap V[G_\lambda]$ is an ultrafilter on ω in $V[G_\lambda]$;
2. $\langle C_n : n \in \omega\rangle \in V[G_\lambda]$;
3. $\mathcal{I}_0' = \mathcal{I}_0 \cap V[G_\lambda]$ and $\mathcal{I}_1' = \mathcal{I}_1 \cap V[G_\lambda]$ are members of $V[G_\lambda]$ and are P-ideals; and
4. every member of p' contains an infinite member of each of $\mathcal{I}_0'$ and $\mathcal{I}_1'$.

Since $V[G_\lambda]$ is a model of CH, there are $\{a_\alpha : \alpha < \omega_1\}$ increasing mod finite and cofinal in $\mathcal{I}_0'$ and $\{b_\alpha : \alpha < \omega_1\}$ increasing mod finite and cofinal in $\mathcal{I}_1'$. Let g_λ be the generic real added by Q_λ (recall that g_λ is an increasing function). Let $C^\downarrow = \bigcup_n (C_n \cap g_\lambda(n))$ and for each α, let $a_\alpha^\downarrow = a_\alpha \cap C^\downarrow$ and $b_\alpha^\downarrow = b_\alpha \cap C^\downarrow$. We finish the proof by showing that, in $V[G]$, the family $\{a_\alpha^\downarrow : \alpha < \omega_1\}$ is not separated from

$\{b^{\downarrow}_{\alpha} : \alpha < \omega_1\}$. This will complete the proof since $(C^{\downarrow})^* \cap f^{-1}(0)$ is contained in a clopen subset of ω^* which is disjoint from $(C^{\downarrow})^* \cap f^{-1}(1)$ because $C^{\downarrow} \notin p$. We proceed in two steps, the first is to show that they are not separated in $V[G_{\lambda+1}]$ and the second is an easy appeal to results of Shelah [She84] for the preservation by the rest of the iteration.

Claim: for every $a \in [\omega]^\omega \cap V[G_{\lambda+1}]$ such that $a^{\downarrow}_{\alpha} \subset^* a$ for all $\alpha < \omega_1$, there is an α such that $a \cap b^{\downarrow}_{\alpha}$ is infinite.

To prove this we work in $V[G_\lambda]$ and force with Q. Recall that $Q \subset \mathcal{P}({}^{<\omega}\omega)$ and that g_λ is the union of roots of conditions T which are in the generic filter G such that $G_{\lambda+1} = G_\lambda * G$. Let $\dot{a}$ be the Q-name of a subset of $C^{\downarrow}$ and assume that $T \in Q$ is such that $T \Vdash |a^{\downarrow}_{\alpha} \setminus \dot{a}| < \omega$ for each $\alpha < \omega_1$. Let M be a countable elementary submodel of some $H(\theta)$ which includes $\langle C_n : n \in \omega\rangle, T, \dot{a}$ and the sequence $\{a_\alpha, b_\alpha : \alpha < \omega_1\}$. Since $a^{\downarrow}_{\alpha}$ is simply equal to $a_\alpha \cap C^{\downarrow}$ and $C^{\downarrow}$ is defined in terms of $\langle C_n : n \in \omega\rangle$ and the canonical generic real it shouldn't be necessary to specify names for $a^{\downarrow}_{\alpha}$ (and similarly $b^{\downarrow}_{\alpha}$).

It suffices to show that there is a $T' < T$ such that $T' \Vdash \dot{a} \cap b^{\downarrow}_{\alpha}$ is infinite where $\alpha = M \cap \omega_1$. This is done by a standard fusion argument in which the following is the main step.

Claim: If $\tilde{T} < T$ is a member of M and $m \in \omega$, there is a $T' < \tilde{T}$ (again in M) such that $T' \Vdash (\exists k > m)\ k \in \dot{a} \cap b^{\downarrow}_{\alpha}$.

Let $A = \{\ell : (\exists T' < \tilde{T})\ T' \Vdash \ell \in \dot{a}\}$. Towards a contradiction, assume that there is a β such that $a_\beta \setminus A$ is infinite. Since $a_\beta \cap C_n$ is finite for each n, there is a $J \in [\omega]^\omega$ and a function $h \in {}^J\omega$ such that $h(n) \in a_\beta \setminus A$ for each $n \in J$. Let $S \in Q$ and $m \in \omega$ be arbitrary, let $s \in S$ be the root and let n be the smallest member of J which is larger than $m + |s|$. Since s has infinitely many immediate successors in S, there is an $s^\frown j$ in S with $j > h(n)$. It follows that S has an extension which forces $g_\lambda(n)$ to be larger than $h(n)$. Since S and m were arbitrary and since $h \in V[G_\lambda]$ it follows that, in $V[G_{\lambda+1}]$, the set $\{n \in J : g_\lambda(n) > h(n)\}$ is infinite. But this is a contradiction since, $\tilde{T}$ forces that $a^{\downarrow}_{\beta} \subset^* \dot{a} \subset A$, while $\{h(n) : n \in J \text{ and } h(n) < g_\lambda(n)\}$ is an infinite subset of $a^{\downarrow}_{\beta} \setminus A$. So we have established that A is a member of $V[G_\lambda]$ which contains mod finite each a_β, hence it follows that there is a β such that $A \cap b_\beta$ is infinite. Since A is in M and b_α contains mod finite each $b_\beta \in M$, it follows that $A \cap b_\alpha$ is infinite. Fix any $k > m$ in this intersection and since $k \in A$ there is $T' < T$ so that $T' \Vdash k \in \dot{a}$. Since $T \Vdash \dot{a} \subset C^{\downarrow}$, it follows that $T' \Vdash k \in b^{\downarrow}_{\alpha} = b_\alpha \cap C^{\downarrow}$.

Now for the fusion argument. For a condition $T \in Q$ and a $t \in T$, the set $T_t = \{s \in T : s \text{ is comparable with } t\}$ is again a member of Q. If A is a maximal set of pairwise incomparable elements of $T \in Q$ and for each $t \in A$ we are given a $T'_t \leq T_t$ in Q, then $T(A) = \bigcup\{T'_t : t \in A\} < T$ is again a member of Q. Note that the set of branching nodes of T which have precisely n branching predecessors is a maximal set of pairwise incomparable elements. Furthermore, for each branching $t \in T$ which is contained in a member of A, t has exactly the same set of immediate successors in $T(A)$ defined as above as it had in T. If, $T_n = T_{n-1}(A_n)$ is defined recursively, where $T_0 = T$ and A_n is the set of branching nodes of T_{n-1} which have precisely n branching predecessors, then $T_\omega = \bigcap T_n$ is a member of Q. Note that if, for each n, there is a statement φ_n of the forcing language such that, at stage n, each T'_t, for $t \in A_n$, which is used in the construction of T_n, forces that φ_n holds, then T_ω forces that φ_n holds for each n. We use this exact process where φ_n is the assertion that there is a $k > n$ in $\dot{a} \cap b^{\downarrow}_{\alpha}$. The only inductive assumption necessary is that each T'_t is chosen to be a member of M (it is not necessary that T_n itself be a member of M). The inductive step is simply as follows: suppose we have chosen T_n and $t \in A_{n+1}$. There is a unique $t' \leq t$ in A_n and, by assumption $T'_{t'}$ is a member of M. By the construction of T_n, it follows that $(T_n)_t = (T'_{t'})_t$, is again a member of M. By the above claim, $(T_n)_t$ has an extension T'_t as required.

The proof of the theorem is completed by the next lemma. □

The following result is also given in [Laf93].

Lemma 5.4. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are P-ideals in $\mathcal{P}(\omega)$ which cannot be separated. Then $\mathcal{A}$ and $\mathcal{B}$ remain unseparated after forcing with a countable support iteration of Miller forcings.*

Proof. Obviously we may assume that every member of $\mathcal{A}$ is almost disjoint from every member of $\mathcal{B}$. Following the notation of Shelah [She84], unseparated gaps can be coded as a *nice* pair. For each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, let $f_{a,b}(n) = 1$ if $n \in a$ and $f_{a,b}(n) = 2$ if $n \in b \setminus a$ and let $f_{a,b} = 0$ for other values of n. Let F be the set of all $f_{a,b}$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Let R be the following (absolute) two-place relation on ω^ω: $g\ R\ f$ iff $\{n : f(n) = 1 \neq g(n)\}$ is infinite or $\{n : g(n) = 1 \text{ and } f(n) = 2\}$ is infinite (i.e. the idea is that $f = f_{a,b}$ and $g^{-1}(1)$ either meets b infinitely or it does not contain a). Recall that F is said to be R-bounding if for every g there is an $f \in F$ such that $g\ R\ f$ (that is to say that $\mathcal{A}$ and $\mathcal{B}$ are not separated). To apply the results of [She84] we must show the following: given a countable elementary submodel M there is an $f \in F$ so that for every m_0, Player II has an absolute winning strategy for the following game. On the k-th move, player I chooses $g_k \in {}^\omega\omega$, $f_k \in F \cap M$, such that $g_k \restriction m_{\ell+1} = g_\ell \restriction m_{\ell+1}$ for $0 < \ell < k$ and $g_k R f_k$, then player II chooses $m_{k+1} > m_k$. In the end, player II wins if $\bigcup_k g_k \restriction m_k\ R\ f$.

Given M, choose $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that a contains mod finite every member of $M \cap \mathcal{A}$ and b contains mod finite every member of $M \cap \mathcal{B}$. Let $f = f_{a,b}$. In a play of the game, we are given $g_k R f_k$. Since $f_k \in F \cap M$, there are $a' \in M \cap \mathcal{A}$ and $b' \in \mathcal{B}$ such that $f_k = f_{a',b'}$. Since $\{n : f_{a',b'}(n) = 1 \neq f_{a,b}(n)\} \cup \{n : f_{a',b'}(n) = 2 \neq f_{a,b}(n)\}$ is finite, there is an m_{k+1} such that $\{n < m_{k+1} : f_{a,b}(n) = 1 \neq g_k(n)\} \cup \{n : g_k(n) = 1 \text{ and } f_{a,b}(n) = 2\}$ has size at least k. It follows then that $\bigcup_k g_k \restriction m_k\ R\ f$.

By the results of [She84] then, it is sufficient to show that Q itself preserves that $\mathcal{A}$ cannot be separated from $\mathcal{B}$. This is somewhat easier than the argument in Theorem 5.3 since we do not have to worry about intersection with $C^{\downarrow}$ so we will omit the details. □

References

[AS81] U. Avraham and S. Shelah. Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic. *Israel J. Math*, 38:161–176, 1981.

[BS87] A. Blass and S. Shelah. There may be simple $P_{\aleph_1}$ and $P_{\aleph_2}$-points and the Rudin-Keisler order may be downward directed. *Ann. Pure Appl. Logic*, 83:213–243, 1987.

[DJW89] Alan Dow, Istvan Juhász, and William Weiss. Integer-Valued Functions and Increasing Unions of First Countable Spaces. *Israel Journal of Mathematics*, 67:181–192, 1989.

[DM90] A. Dow and J. Merrill. $\omega_2^* - U(\omega_2)$ can be C^*-embedded in $\beta\omega_2$. *Top. and its Appl.*, 35:163–175, 1990.

[Dow88] A. Dow. An introduction to applications of elementary submodels to topology. *Topology Proceedings*, 13:17–72, 1988.

[Dow92] A. Dow. Set theory in topology. In M. Hušek and J. van Mill, editors, *Recent Progress in General Topology*, chapter 4, pages 169–197. Elsevier Science Publishers B.V., 1992.

[HvM90] K.P. Hart and J. van Mill. Open problems on $\beta\omega$. In J. van Mill and G.M. Reed, editors, *Open problems in Topology*, pages 97–125. North-Holland, 1990.

[Laf93] C. Laflamme. Bounding and dominating number of families of functions on N. Logic Eprints University of Florida, 1993

[Mil84] A. Miller. Rational perfect set forcing. In *Axiomatic Set Theory*, pages 143–159. Contemporary Mathematics, AMS, 1984.

[She84] S. Shelah. On cardinal invariants of the continuum. In *Axiomatic Set Theory*. Contemporary Mathematics, AMS, 1984.

[Tod89a] S. Todorcevic. *Partition relations in topology*. Contemporary Mathematics, 1989.

[Tod89b] S. Todorčević. Tightness in products. *Interim Reports of the Prague Topological Symposium*, 4:7–8, 1989.

[vDKvM89] E.K. van Douwen, K. Kunen, and J. van Mill. There can be proper dense C^*-embedded subspaces in $\beta\omega - \omega$. *Proc. AMS*, 105:462–470, 1989.