MORE ON SEQUENTIALLY COMPACT IMPLYING PSEUDORADIAL

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Abstract. We improve some results of [JSz] and answer the main problem raised there. Thus we show that if a compact T_2 space X is not pseudoradial then there is a $Y \subset X$ with $|Y| < \mathfrak{s}$ so that $\overline{Y}$ is not pseudoradial. Also, we show that if any number of Cohen reals are added to a model of CH then CSC spaces are pseudoradial in the extension. As a byproduct, we also show that any ω_1-compact and sequentially compact separable T_3 space is compact in such an extension.

The aim of this note is to answer the main open problem of [JSz]: Does $2^\omega = \mathfrak{c} > \omega_2$ imply the existence of a compact T_2 and sequentially compact (in short: CSC) space that is not pseudoradial? As we shall show, the answer to this is negative: No such spaces exist in generic extensions obtained by adding Cohen reals to a ground model satisfying CH.

We follow the notation of [JSz]. A set S in a space X is said to be relatively sequentially compact (or RSC) if every ω-sequence of points of S has a subsequence that converges in X (of course a space X is sequentially compact if every subset is RSC). For any cardinal number κ we denote by $\ell_\kappa(S)$ the set of limit points of all the converging κ-sequences consisting of points of S. A subset S of a space X is radially closed in X if $\ell_\kappa(S) \subset S$ for all cardinals κ and a space X is said to be pseudoradial if every radially closed subset is actually closed. Note that, for example, $\beta\omega$ is not pseudoradial since the subset ω is radially closed and certainly not closed.

One of the main results of [JSz] states that if a compact T_2 space X is not pseudoradial then there is a set $S \in [X]^{<\mathfrak{c}}$ such that $\overline{S}$ is not pseudoradial either. Here we start with a theorem that easily yields that

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the cardinal $\mathfrak{c}$ in this result can actually be replaced by the cardinal $\mathfrak{s}$. It is pointed out in [Sh] (P. 583 Theorem 3.11) that this result was actually first noted by Šapirovskiĭ. Let us recall that a set A in a space X is said to be λ-closed if for every $B \in [A]^\lambda$ we have $\overline{B} \subset A$, moreoever A is $< \kappa$-closed if it is λ-closed for every $\lambda < \kappa$.

Theorem 1. *Let σ be an uncountable cardinal and X be a compact T_2 space that cannot be mapped (continuously) onto the Tychonov cube I^σ. Then every subset $A \subset X$ which is both $< \sigma$-closed and radially closed is actually closed.*

Proof. Assume, indirectly, that $A \neq \overline{A}$. We may assume also that $X = \overline{A}$. Let $K \subset \overline{A} \setminus A$ be a non-empty compact set of minimum possible character, i.e. $\chi(K, X) = \lambda(A, X)$ using the notation and terminology of [JSz]. Let $\kappa = cf(\chi(K, X))$, then clearly there is a κ-sequence of points of A that converges to K, hence as A is $< \sigma$-closed we must have $\sigma \leq \kappa$.

Applying Šapirovskiĭ's celebrated result (see [J1]) there is a point $p \in K$ with $\pi\chi(p, K) = \rho < \sigma \leq \kappa$. So let $\{K^\nu : \nu \in \rho\}$ be a family of closed G_δ sets in K that forms a local π-network at the point p in K. Then, by our choice of K, we have $\chi(K^\nu, X) = \chi(K, X)$ for each $\nu \in \rho$, hence every such K^ν can be written as

$$K^\nu = \cap\{K^\nu_\alpha : \alpha \in \kappa\}$$

where K^ν_α is closed, $\chi(K^\nu_\alpha, X) = |\alpha| + \omega$ and so $K^\nu_\alpha \cap A \neq \emptyset$ for all $\alpha \in \sigma$, hence lemma 3.7 of [J2] can be applied to get a κ-sequence of elements of A that converges to p. However this contradicts that A is radially closed. $\dashv$

Since no CSC space can be mapped onto $I^{\mathfrak{s}}$, moreover a compact space that is not SC has a countable subset which is radially closed but not closed, we immediately get the above promised strengthening of Theorem 6 from [JSz]:

Corollary 2. *If a compact T_2 space is not pseudoradial then there is a set $Y \subset X$ with $|Y| < \mathfrak{s}$ such that $\overline{Y}$ is not pseudoradial.* $\dashv$

The following observation is trivial but, as we are going to use it repeatedly, we state it separately.

Proposition 3. *If the countable set S is RSC in X and $\mathcal{F}$ is a filter base of cardinality $\aleph_1$ on S that converges to a point $p \in X$ then $p \in \ell_{\omega_1}(\ell_\omega(S))$.*

Combining Proposition 3 with Corollary 2 one immediately obtains a proof of Šapirovskiĭ's result (see [Š]) that under CH every CSC space X is pseudoradial, in fact

$$(1) \qquad \begin{cases} \text{for any } S \in [X]^\omega \text{ one has} \\ \overline{S} = \ell_{\omega_1}(\ell_\omega(S)) = \overline{S}^r \end{cases} .$$

The main result of [JSz] improves Šapirovskiĭ's since it says that even under $2^\omega = \mathfrak{c} \leq \omega_2$ every CSC space is pseudoradial. It was also shown there that this result is sharp in the sense that $\mathfrak{c} = \omega_3$ is consistent with the existence of a CSC space that is not pseudoradial. Our main result here implies that this is indeed the most one could hope for: If we add any number of Cohen reals to a ground model satisfying CH, then in the extension every CSC space X is pseudoradial. In fact we show that every $S \in [X]^\omega$ satisfies (1), hence as $s = \omega_1$ holds in such an extension (see [F]), Corollary 2 applies. What we will actually show is that the extension satisfies the following property $(*)$ which, in addition to (1), has other interesting consequences.

$(*)$ If X is an ω_1-compact space and $S \in [X]^\omega$ is RSC in X then for any ultrafilter u on S there is a point $x \in X$ and a subfamily $v \in [u]^{\omega_1}$ such that $v \to x$.

It is interesting to compare the property $(*)$ with Michael's bisequential property (see [M] or 6.12 of [N]): a space X is bisequential if for each $S \in [X]^\omega$ and ultrafilter u on S converging to a point $x \in X$ there is a countable subfamily $v \subset u$ such that $v \to x$.

Our main theorem now reads as follows. (We use $\mathcal{C}_\kappa$ to denote the standard notion of forcing that adds κ Cohen reals.)

Theorem 4. *If $V \models CH$ and κ is any cardinal then $V^{\mathcal{C}_\kappa} \models (*)$. In addition, in $V^{\mathcal{C}_\kappa}$, every CSC space is pseudoradial.*

Proof. Let $G \subset \mathcal{C}_\kappa$ be generic over V. For any regular cardinal $\lambda > \kappa$ consider H_λ and let $M \prec H_\lambda$ be an elementary submodel that is closed under countable sequences (i.e. $M^\omega \subset M$). It is well-known that then $N = M[G] \prec H_\lambda[G] = H_\lambda^{V[G]}$, moreover $|N| = |M|$ and N satisfies the following property (i):

(i) $N \cap [N]^\omega$ is cofinal in $\langle [N]^\omega, \subset \rangle$.

A key point of our proof is the following, actually quite easy, folklore lemma that establishes another property of N (see Lemma 5.1 in [D]).

Lemma 5. *If N is as above then it satisfies the following in $V[G]$:*

(ii) for every set $A \subset \omega$ we have a countable subfamily of $\mathcal{P}(A) \cap N$ that is cofinal in it w.r.t. inclusion, in short

$$cf(\mathcal{P}(A) \cap N, \subset) \leq \omega \ .$$

Proof. We first establish that

$$\mathcal{P}(A) \cap N = \mathcal{P}(A) \cap M[G] = \mathcal{P}(A) \cap V[G \restriction (\kappa \cap M)].$$

First of all, if $B \in N \cap \mathcal{P}(A) = M[G] \cap \mathcal{P}(A)$, it follows from the definition of $M[G]$ that B has a name, $\dot{B}$, in M. We may assume that $\dot{B}$ is a *nice* name (see 5.11 of Kunen's text [K]) and so $\dot{B}$ is actually a subset of M. Therefore B is actually a member of $V[G \restriction (\kappa \cap M)]$. For the other inclusion, since M is countably closed, it follows that every member of $\mathcal{P}(\omega) \cap V[G \restriction (\kappa \cap M)]$ has a name which is an element of M.

Since $A \subset \omega$, it has a nice name $\dot{A}$ whose support is some $I \in [\kappa]^{\omega}$. Now we work with $W = V[G \restriction (\kappa \cap M)]$ as our ground model using the standard forcing factoring results. Thus we see that a set B (in W) will be a member of $\mathcal{P}(A)$ if and only if there is a condition $p \in \mathcal{C}_I \cap G$ such that $p \Vdash B \subset \dot{A}$. But for such a p and ground model B, we clearly have $B \subset A_p = \{n \in \omega : p \Vdash n \in \dot{A}\} \subset A$. Consequently, $\{A_p : p \in \mathcal{C}_I \cap G\} \subset \mathcal{P}(A) \cap W$ is the required countable cofinal family in $\mathcal{P}(A) \cap N = \mathcal{P}(A) \cap W$. $\dashv$

Let us now get back to the proof of Theorem 4. We first show that $V^{\mathcal{C}_\kappa}$ is a model of $(*)$. By the above, for each regular $\lambda > \kappa$ the collection of those elementary submodels N of H_λ (in $V^{\mathcal{C}_\lambda}$) that satisfy both (i) and (ii) are cofinal (in fact even stationary) in $[H_\lambda]^{\omega_1}$. So given X, S and u as in $(*)$, first note that we may assume that $S = \omega$ and then choose λ such that $X, u \in H_\lambda$ and an elementary submodel $N \in [H_\lambda]^{\omega_1}$ satisfying (i) and (ii) such that $\{X, u\} \cup \omega_1 \subset N$.

Let us now put $v = u \cap N$. By (i) we may pick for each $\alpha \in \omega_1$ a family $v_\alpha \in [v]^\omega \cap N$ such that $\alpha < \beta < \omega_1$ imply $v_\alpha \subset v_\beta$, moreoever $v = \cup\{v_\alpha : \alpha \in \omega_1\}$. By elementarity and $v_\alpha \in N$, for each $\alpha \in \omega_1$ there is a point $x_\alpha \in X \cap N$ and a sequence $S_\alpha \in [S]^\omega \cap N$ such that $S_\alpha \to x_\alpha$, moreover $|S_\alpha \backslash V| < \omega$ for every $V \in v_\alpha$.

Now, applying that X is ω_1-compact, we can choose a point $x \in X$ that is a complete accumulation point of $\{x_\alpha : \alpha \in \omega_1\}$. We do not claim that $x \in N$, but we will show that $v \to x$.

Indeed, let U be an arbitrary open set containing x and set $A = S \cap U = \omega \cap U$. Since $|\{\alpha : x_\alpha \in U\}| = \omega_1$ there is a fixed $n \in \omega$ such that $|\{\alpha : S_\alpha \setminus n \subset A\}| = \omega_1$ as well. But for each such α we also have $S_\alpha \setminus n \in N$, hence by (ii) applied to A there is some set $B \in \mathcal{P}(A) \cap N$ such that $|\{\alpha : S_\alpha \setminus n \subset B \subset A\}| = \omega_1$. But then we clearly have $B \cap V \neq \emptyset$ for each $V \in v = u \cap N$, consequently we must have $B \in u$, hence $B \in v$, because $B, u \in N \prec H_\lambda$. However, this is exactly what we needed to show that $v \to x$. Noticing that $|v| = \omega_1$ we have completed the proof of $(*)$. It is useful to note that by applying Proposition 3 we may also conclude that $x \in \ell_{\omega_1}(\ell_\omega(S))$, hence any X as above will also satisfy (1).

Now let X be a CSC space. Note that any countable $S \subset X$ is certainly RSC in X and X is ω_1-compact, hence X satisfies (1). This means, of course, that every separable subspace of X is pseudoradial. Finally we recall that $V^{\mathcal{C}_\kappa} \models \mathfrak{s} = \omega_1$ and conclude, by Corollary 2, that X is pseudoradial. $\dashv$

In our next result we point out another interesting consequence of $(*)$.

Theorem 6. *If $(*)$ holds (hence in $V^{\mathcal{C}_\kappa}$ if $V \models CH$), the closure of a countable RSC set S in an ω_1-compact T_3 space is compact. In particular, every ω_1-compact, SC, separable T_3 space is compact.*

Proof. Since X is T_3, it is clearly sufficient to show that every ultrafilter u on S converges in X. This, however, is immediate from $(*)$. $\dashv$

Another result from [JSz] states that under $MA(\sigma\text{-centered})$ (i.e. $\mathfrak{p} = \mathfrak{c}$) every countably tight CSC space is pseudoradial. As one more application of Theorem 1, we present here the following strengthening of this result. Note that in this case $\mathfrak{p} = \mathfrak{s} = \mathfrak{c}$.

Theorem 7. *If $\mathfrak{p} = \mathfrak{c}$ then for every countable set S in a CSC space X we have*

$$\overline{S} = \ell_c(\ell_\omega(S)) = \overline{S}^r \; . \tag{1'}$$

In particular, if X does not map onto I^{ω_1} (e.g. $t(X) = \omega$ or X is T_5) then X is pseudoradial.

Proof. We may assume that $X = \overline{S}$. It is well known that if for some compact set $K \subset X \setminus S$ we have $\chi(K, X) < \mathfrak{c}$ then there is an ω-sequence of points of S that converges to K. In particular, for every $p \in X$ with $\chi(p, X) < \mathfrak{c}$ we have $p \in \ell_\omega(S)$.

So now assume that $p \in X \setminus S$ with $\chi(p, X) = \mathfrak{c}$. Then there is a decreasing sequence $\{K_\alpha : \alpha \in c\}$ of compact sets such that $\{p\} =$

$\cap\{K_\alpha : \alpha \in \mathfrak{c}\}$ and $\chi(K_\alpha, X) = |\alpha| + \omega$ for every $\alpha \in \mathfrak{c}$. According to our above remark, and using that X is SC, for every $\alpha \in \mathfrak{c}$ we have a point $x_\alpha \in K_\alpha \cap \ell_\omega(S)$. Since $x_\alpha \to p$ this clearly implies that $p \in \ell_{\mathfrak{c}}(\ell_\omega(S))$, and the proof of $(1')$ is completed.

The second part of the theorem now follows immediately from Theorem 1. $\dashv$

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