ON COMPACT SEPARABLE RADIAL SPACES

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Abstract. If $\mathcal{A}$ and $\mathcal{B}$ are disjoint ideals on ω, there is a *tower preserving* σ-centered forcing which introduces a subset of ω which meets every infinite member of $\mathcal{A}$ in an infinite set and is almost disjoint from every member of $\mathcal{B}$. We can then produce a model in which all compact separable radial spaces are Fréchet, thus answering a question of P. Nyikos. The question of the existence of compact ccc radial spaces which are not Fréchet was first asked by Chertanov (see [Arh78]).

1. Introduction

Nyikos has asked if there exists a compact separable radial space which is not Fréchet. A space is *radial* if, for each x and A with $x \in \overline{A}$, there is a well-ordered sequence of points from A which converge to x which means that every neighbourhood of x includes a final segment of the sequence. A space is *Fréchet* if it is radial in which the well-ordered sequence can always be chosen to be countable.

We produce a model in which every compact separable radial space is actually Fréchet. It is easily seen that if $\mathfrak{t} = \mathfrak{c}$ or if $\mathfrak{d} = \omega_1$, then there is a compact separable space which is radial but not Fréchet (see 2.1).

Our technique will be to modify the following combinatorial principle of Solovay.

> If $\mathcal{A}, \mathcal{B} \subset [\omega]^\omega$ are such that $|\mathcal{A} \cup \mathcal{B}| < \mathfrak{c}$, any finite union from $\mathcal{B}$ has infinite complement, and $A \cap B$ is finite for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then there is a $C \in [\omega]^\omega$ such that $C \cap B$ is finite for all $B \in \mathcal{B}$ and $C \cap A$ is infinite for all $A \in \mathcal{A}$.

By Bell's theorem [Bel81], it follows that "Solovay's Lemma" is equivalent to $\mathfrak{p} = \mathfrak{c}$. However, we weaken the above property by simply dropping the requirement that C be infinite (hence if $\mathcal{A}$ is empty there is nothing to do). We find some σ-centered forcings which will add sets C to meet every member of an ideal $\mathcal{A}$ while missing (mod finite) every member of a disjoint ideal $\mathcal{B}$ which have the additional property that they do not fill towers. This is a generalization of Baumgartner's result that the usual σ-centered forcing which adds an increasing dominating real also does not fill towers. Recall that a *tower* is a maximal descending (mod finite) chain of infinite subsets of ω.

If we assume a more restrictive condition on the family $\mathcal{A}$ (which we will call *weakly σ-bounded*) then we will in some sense introduce a new tower (NT). Say that $\mathcal{A}$ is dense in C if every infinite subset of C meets some member of $\mathcal{A}$ in an infinite set. The family of complements from a tower are dense in every infinite set. To formulate (NT) we will need one more simple notion. For each family $\mathcal{A} \subset \mathcal{P}(\omega)$,

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we will let $\mathcal{A}^{\downarrow}$ denote the downward closure of $\mathcal{A}$ in $[\omega]^{\omega}$, i.e. $B \in \mathcal{A}^{\downarrow}$ if B is an infinite subset of some member of $\mathcal{A}$.

Definition 1.1. Let (NT) be the statement: "for each weakly σ-bounded family $\mathcal{A} \subset \mathcal{P}(\omega)$ and each family $\mathcal{B} \subset \mathcal{A}$ of cardinality at most ω_1, there is a $C \subset \omega$ such that $\mathcal{A}$ is dense in C and $C \cap B$ is infinite for each infinite $B \in \mathcal{B}$". We say that a family $\mathcal{A}$ is *weakly σ-bounded*, if for each countable subset $\mathcal{A}'$ of $\mathcal{A}^{\downarrow}$, there is an $A \in \mathcal{A}$ such that $A \cap A'$ is infinite for all $A' \in \mathcal{A}'$

2. Topology

To motivate our approach we begin by recalling a classical (consistent) example of a space whose existence we must refute. In our new terminology, the hypothesis on the chain $\{a_\alpha : \alpha \in \omega_1\}$ is that it is dense only in those I which are actually contained in some a_α.

Proposition 2.1. *Suppose there is an increasing mod finite chain $\{a_\alpha : \alpha < \omega_1\}$ such that for each $I \in [\omega]^{\omega}$ either $I - a_\alpha$ is finite for some α or there is a $J \in [I]^{\omega}$ such that $J \cap a_\alpha$ is finite for all $\alpha < \omega_1$. Then there is a compact separable radial space which is not Fréchet.*

Proof. Let B be the subalgebra of $\mathcal{P}(\omega)$ which is generated by the finite sets together with the family $\{a_\alpha : \alpha < \omega_1\}$ and let X be the Stone space of B. It is clear that X is a compact separable space. The ultrafilters on B can be naturally partitioned into three sets. The first consists of the fixed ultrafilters which are isolated points of X. The second are the filters x_α such that there is an $\alpha < \omega_1$ such that x_α is generated by $\{a_\alpha - (a_\beta \cup n) : \beta < \alpha \text{ and } n \in \omega\}$. The third is the unique ultrafilter x_{ω_1} which is generated by $\{\omega - (a_\beta \cup n) : \beta < \omega_1 \text{ and } n \in \omega\}$. Since all points of X have a countable neighbourhood base except x_{ω_1}, we need only check the radial property at x_{ω_1}. It is easily seen that $\{x_\gamma : \gamma \leq \omega_1\}$ is homeomorphic to the ordinal space $\omega_1 + 1$, which is radial, so we need only consider the case when x_{ω_1} is a limit of some $A \subset \omega$. In this case $A - a_\alpha$ is infinite for all $\alpha < \omega_1$, hence, by hypothesis, there is a $J \subset A$ such that $a_\alpha \cap J$ is finite for all $\alpha < \omega_1$. Clearly J converges to x_{ω_1} which completes the proof that X is radial. □

Notice that X (as in 2.1) will remain compact separable and non-Fréchet in any extension which preserves ω_1. Therefore to destroy that X is a compact separable radial non-Fréchet space we must make it non radial. This requires that we introduce a set $I \subset \omega$ such that $\{a_\alpha \cap I : \alpha < \omega_1\}$ forms a non-extendible chain of coinfinite subsets of I. It is nearly immediate that (NT) implies there is such an I.

We now show that (NT) implies every compact separable radial space is indeed Fréchet and leave it for the final section to prove that (NT) is consistent. One key step is supplied by the following result of Juhász and Szentmiklossy. Recall that a space has *uncountable tightness* if there is a point x in the closure of some set $A \subset X$ such that x is not the limit point of any countable subset of A . Also, an ω_1-sequence $\{x_\alpha : \alpha \in \omega_1\}$ is said to be *free* if for each $\alpha < \omega_1$, the closures of the initial segment, $\{x_\beta : \beta < \alpha\}$, and of the final segment, $\{x_\beta : \alpha \leq \beta\}$, are disjoint. Finally, such a sequence is said to *converge* to x if every neighbourhood of x contains a final segment of the sequence.

Proposition 2.2. [JS92] *Each compact space of uncountable tightness contains a converging free ω_1-sequence.*

Of course a space is said to have countable tightness if it does not have uncountable tightness (perhaps the converse is more accurate). The following result is one of the first about radial spaces.

Proposition 2.3. *A compact radial space is Fréchet if and only if it has countable tightness.*

For the sake of completeness, in this context we can give a simple self-contained proof of the following corollary.

Corollary 2.4. *If a compact radial space is not Fréchet, then it contains a converging free ω_1-sequence.*

Proof. Assume that a compact radial space X is not Fréchet. Since X is not Fréchet we can find a point z and a set A such that z is a limit point of a set A but for which there is no ω-sequence from A converging to z. We have just established uncountable tightness: if z was a limit point of a countable subset of A, then that countable set, by the radial property, would have to have a well-ordered (hence ω) sequence which converged to z. By passing to a subset of A, we may assume that z is not a limit point of any subset of A which has strictly smaller cardinality.

But now, since X is radial, there is a minimum ordinal κ such that there is a κ-sequence of points from A which converges to z. Since any cofinal subsequence of this sequence will also converge to z, it immediately follows that κ does not have a countable cofinal sequence. In addition, by the minimality, κ is a cardinal and is equal to $|A|$. The fact that the sequence converges to z gives us one final key property: for any initial segement, S, of the sequence, there is a final segment, F, of the sequence so that S and F have disjoint closures. Indeed, since z is not a limit of S, we can simply choose F to be any final segment contained in a closed neighbourhood of z which avoids S.

Now by following along this sequence we can inductively choose points x_α for $\alpha < \omega_1$ which will form a free sequence. At each stage α, also choose a final segment F_α of the sequence which has the property that its closure is disjoint from the closure of $\{x_\beta : \beta < \alpha\}$. When choosing x_α be sure to choose it in F_γ for each $\gamma \leq \alpha$.

We are almost done. Since X is compact, there is certainly a point x which is in the closure of $\{x_\beta : \alpha < \beta < \omega_1\}$ for every $\alpha < \omega_1$ since this family has the finite intersection property. Finally, since x is a limit point of the sequence $\{x_\alpha : \alpha < \omega_1\}$ it must have a subsequence converging to x. We have ensured that this sequence will not be countable because $\{x_\alpha : \alpha \in \omega_1\}$ is a free sequence. □

Now we are ready to apply (NT).

Theorem 2.5. *The principle (NT) implies that every compact separable radial space is Fréchet.*

Proof. Assume that X is a compact separable radial space which is not Fréchet. By the previous corollary we may fix a point x together with a free ω_1-sequence $\{x_\alpha : \alpha \in \omega_1\}$ which converges to x. Since D has a countable dense set we may assume (by a trivial renaming) that this set is actually the set ω. For each $\alpha \in \omega_1$, choose a closed neighbourhood W_α of x which is disjoint from $\{x_\beta : \beta < \alpha\}$ (this uses the fact that we have a free sequence). Therefore $a_\alpha = \omega \setminus W_\alpha$ will not have x in its closure and, for each $\beta < \alpha$, x_β will not only be a limit point of a_α but will also not be a limit point of $\omega \setminus a_\alpha$.

Since X is radial we may also choose, for each $\beta < \alpha < \omega_1$, a sequence $y(\beta,\alpha) \subset a_\alpha$ such that $y(\beta,\alpha)$ converges to x_β. Observe that for $\beta < \alpha < \delta < \omega_1$, we have that $y(\beta,\alpha)$ meets a_δ in an infinite set (in fact $y(\beta,\alpha) \setminus a_\delta$ is finite). It follows easily now, that $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\} \cup \{y(\beta,\alpha) : \beta < \alpha < \omega_1\}$ is a weakly σ-bounded family. Finally we set $\mathcal{B} = \mathcal{A}$ and we show that the existence of the set C given to us by the principle (NT) contradicts that X is radial. First of all, since $C \cap y(\beta,\alpha)$ is not empty for each $\beta < \alpha$, it follows that each x_β, and therefore x, is a limit point of C. So to reach our contradiction we show that no subsequence from C will converge to x. This, however, follows immediately from assertion that $\mathcal{A}$ is *dense* in C. Indeed, if I is an infinite subset of C we find some $a \in \mathcal{A}$ such that $a \cap I$ is infinite. However, since $a \in \mathcal{A}$, there is an α such that a is disjoint from W_α. Clearly this implies that I does not converge to x. □

The following consequence of (NT) may also prove useful (and justifies the term "new tower").

Proposition 2.6. *Assume (NT) and that $\mathfrak{b}$ is greater than ω_1. For any family $\mathcal{A}$ which has a subfamily $\{b_\alpha : \alpha \in \omega_1\}$ such that for each $\alpha \in \omega_1$, there is an $A \in \mathcal{A}$ such that $b_\beta \setminus A$ is finite for all $\beta \in \alpha$, then there is a set C meeting each b_α in an infinite set such that the family $\{C \setminus A : A \in \mathcal{A}^{\downarrow}\}$ contains a tower of cofinality ω_1.*

Proof. Fix any subsequence $\{A_\alpha : \alpha \in \omega_1\} \subset \mathcal{A}$ such that $b_\beta \setminus A_\alpha$ is finite for each $\beta < \alpha$. It is well-known that we can choose a sequence, $\{c_\alpha : \alpha \in \omega_1\}$, of pairwise disjoint mod finite infinite sets so that each $c_\alpha \subset b_\alpha$. Choose, inductively, subsets $a_\alpha \subset A_\alpha$ so that, for $\beta < \alpha \leq \gamma$ $a_\beta \cup c_\beta$ is almost contained in a_α, and a_α is almost contained in $A_\gamma \cup c_\gamma$. One simply uses the assumption that $\mathfrak{b} > \omega_1$ to choose a finite subset, F_β, of $a_\beta \cup c_\beta$ for each $\beta < \alpha$, so that $\bigcup_{\beta<\alpha}[a_\beta \cup c_\beta] \setminus F_\beta$ is strictly contained mod finite in every $A_\gamma \cup c_\gamma$ with $\gamma \geq \alpha$. Set a_α equal to this union intersected with A_α. Since the family $\{a_\alpha : \alpha < \omega_1\}$ is (strictly) increasing mod finite, it is weakly σ-bounded. Apply (NT) to the collection $\{a_\alpha : \alpha < \omega_1\} \cup \{c_\alpha : \alpha < \omega_1\}$ with $\mathcal{B} = \{c_\alpha : \alpha \in \omega_1\}$. Since $\{a_\alpha : \alpha \in \omega_1\}$ is dense in C and $\{C \setminus a_\alpha : \alpha \in \omega_1\}$ is strictly descending mod finite, it is a tower. □

3. Set theory

Recall that the usual Solovay forcing for adding a set which meets every member of $\mathcal{A}$ and is almost disjoint from every member of $\mathcal{B}$ is the set consisting of (a, B) where $a \in [\omega]^{<\omega}$ and B is in the ideal generated by $\mathcal{B}$. A condition is stronger than another if each coordinate is larger but the new elements of the first coordinate are not from the weaker condition's second coordinate. We would like to first note that we will need a new forcing notion if we are to preserve towers.

Proposition 3.1. *For each tower there are ideals $\mathcal{A}$ and $\mathcal{B}$ so that $A \cap B$ is finite for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$ so that the usual Solovay forcing with respect to $\mathcal{A}$ and $\mathcal{B}$ will fill the tower.*

Proof. Let $\{t_\alpha : \alpha < \kappa\}$ be a tower and fix a family $\{T_n : n \in \omega\}$ of pairwise disjoint two element subsets of ω. We can simply define a_α and b_α, pairwise disjoint, so that $t_\alpha = \{n : T_n \cap (a_\alpha \cup b_\alpha)$ is empty $\}$. For $n \notin t_\alpha$, let $\min(T_n) \in a_\alpha$ and $\max(T_n) \in b_\alpha$. An easy density argument shows that if A is the new set added, then $\{n : T_n \subset A\}$ will be infinite and contained in t_α for each α. This example has the disadvantage that there is a set in the ground model which meets every a_α (in fact contains) and is disjoint from all the b_α (namely the set $\{\min(T_n) : n \in \omega\}$). If Martin's Axiom is assumed the example can be modified to ensure that there is no such set (be more random in whether $\min(T_n)$ goes in a_α or b_α). □

Definition 3.2. Fix a given an ideal $\mathcal{A} \subset \mathcal{P}(\omega)$. Define a forcing notion $Q_\mathcal{A}$ where $p \in A_\mathcal{A}$ if $p = (a_p, S_p)$ for some $a_p \in [\omega]^{<\omega}$ and $a_p \notin S_p \subset [\omega]^{<\omega}$ where S_p is such that for each $a \notin S_p$ and for each $A \in \mathcal{A}$, there is an $n \in \omega$ such that $(a \cup a') \notin S_p$ for any $a' \in [A \setminus n]^{<\omega}$.

Define $p < q$, if $a_p \supset a_q$, $S_p \supset S_q$, and $a_p \notin S_q$. It is clear that this is a transitive ordering.

For each ideal $\mathcal{A}$, $Q_\mathcal{A}$ is σ-centered. To see this suppose that $p, q \in Q_\mathcal{A}$ are such that $a_p = a_q$. We claim that $(a_p, S_p \cup S_q)$ is a member of $Q_\mathcal{A}$ which is below each of p and q. It is easily seen to be below each of p and q so we check that it is a member of $Q_\mathcal{A}$. Fix any $A \in \mathcal{A}$ and $b \in [\omega]^{<\omega} \setminus (S_p \cup S_q)$. Let n_p be large enough so that $b \cup a' \notin S_p$ for any $a' \in [A - n_p]^{<\omega}$ and similarly choose n_q. Let n be large than n_p and n_q and note, then, that $b \cup a' \notin S_p \cup S_q$ for any $a' \in [A - n]^{<\omega}$.

Note however, that if each member of $\mathcal{A}$ is finite, then $Q_\mathcal{A}$ is atomic. The purpose of $Q_\mathcal{A}$ is to add a subset of ω, $A_G = \bigcup\{a_p : p \in G\}$, which meets every member of $\mathcal{A}$ in an infinite set (a simple density argument), while if B is almost disjoint from every member of $\mathcal{A}$, and $p \in Q_\mathcal{A}$, then $(a_p, S_p \cup (\ [a_p \cup B]^{<\omega} - \{a_p\}\ ))$ is a member of $Q_\mathcal{A}$ which is below p and which forces that $A_G \cap B$ is contained in a_p. If $\mathcal{A}$ is $\mathcal{P}(\omega)$, then $Q_\mathcal{A}$ is just the usual Cohen forcing, while if $\mathcal{A}$ is a countable family of pairwise disjoint infinite sets, then $Q_\mathcal{A}$ is essentially just Hechler's forcing, Q_H, for adding a strictly increasing dominating real.

We will show that the forcings $Q_\mathcal{A}$ preserve towers which generalizes the following result (and proof) of Baumgartner and Dordal.

Proposition 3.3 ([BD85]). *The forcing Q_H preserves towers.*

Just as in [BD85] we proceed with a ranking on $[\omega]^{<\omega}$ associated with dense sets.

Lemma 3.4. *If $D \subset Q_\mathcal{A}$ is dense, then $[\omega]^{<\omega}$ can be written as an increasing union $\bigcup\{D_\alpha : \alpha < \omega_1\}$ where $D_0 = \{a_d : d \in D\}$ and, for each $\alpha > 0$, $a \in D_\alpha$ if there is an $A \in \mathcal{A}$ such that for each n, there is an $a' \subset A \setminus n$ such that $(a \cup a') \in \bigcup_{\beta<\alpha} D_\beta$.*

Proof. Let D and the D_α be as in the statement of the lemma. Our task is to show that $S = \bigcup\{D_\alpha : \alpha < \omega_1\}$ is all of $[\omega]^{<\omega}$. If $b \notin S$ and if $A \in \mathcal{A}$, then it is clear that there is an n such that $b \cup a \notin S (= \bigcup\{D_\alpha : \alpha < \omega_1\})$ for any $a \subset A \setminus n$. That is, (b, S) is a member of $Q_\mathcal{A}$. However the contradiction is now immediate since (b, S) can not have an extension in D, i.e. $d < (b, S)$ would imply that $a_d \in D_0 \subset S$ and $a_d \notin S$. □

Lemma 3.5. *The forcing $Q_\mathcal{A}$ preserves towers for each ideal $\mathcal{A} \subset \mathcal{P}(\omega)$. In fact, more generally, if a weakly σ-bounded family $\mathcal{T}$ is dense in ω, then it remains both weakly σ-bounded and dense in ω after forcing with $Q_\mathcal{A}$.*

Proof. As mentioned earlier, the family of complements of any tower is dense in ω and it is easily seen that the family is weakly σ-bounded, hence we fix a weakly σ-bounded family $\mathcal{T}$. Fix any ideal $\mathcal{A}$ (which we may assume has at least one infinite set in it).

Let $\{\dot{n}_i : i \in \omega\}$ be a sequence of $Q_\mathcal{A}$-names of integers (listed in increasing order). Fix any countable elementary submodel M (containing both $\mathcal{T}$ and $\mathcal{A}$ as elements) and let t_δ denote any member of $\mathcal{T}$ which meets every member of $M\cap\mathcal{T}^\downarrow$. We will show that $\{\dot{n}_i : i\in\omega\}$ is forced to meet t_δ in an infinite set. Since we use nothing more than that $\{\dot{n}_i : i\in\omega\}$ is a member of M, it also follows from this that $\mathcal{T}$ remains weakly σ-bounded.

The important thing about M and t_δ here is that if we have an infinite subset Z of ω which is a member of M, then we know that $Z\cap t_\delta$ is infinite since by elementarity there will be a $t\in M\cap\mathcal{T}^\downarrow$ such that $t\subset Z$.

We proceed by contradiction. Assume that p_0 is a member of $Q=Q_{\mathcal{A}}$ which forces that $\{\dot{n}_i : i\in\omega\}\cap t_\delta$ is finite. Then fix any p_1 below p_0 and an integer m_1 so that p_1 forces that $\dot{n}_i\in t_\delta$ for each $i>m_1$.

For each $a\in[\omega]^{<\omega}$, define $Z_\ell(a)=\{j : (\forall p\in Q) \text{ if } a_p=a \text{ then } p\not\Vdash \dot{n}_\ell\neq j\}$. That is $Z_\ell(a)$ is the set of all j such that any condition with first coordinate a has an extension which forces $\dot{n}_\ell$ to take on value j. Note that if $p\Vdash \dot{n}_\ell\notin t_\delta$, then $Z_\ell(a_p)$ is disjoint from t_δ and, being a member of M, must be finite.

Now we prove that $Z_\ell(a)$ is non-empty for each a such that there is a p such that $a_p=a$ and $p\Vdash \dot{n}_\ell\notin t_\delta$. Note that $Z_\ell(a')$ is finite for any $a'\supset a$ such that $a'\notin S_p$ since (a',S_p) would be less than p and so also forces that $\dot{n}_\ell\notin t_\delta$.

Fix such a p and $a=a_p$. Let D be the dense set of conditions which decide the value of $\dot{n}_\ell$. By Lemma 3.4 there is a minimal $\alpha<\omega_1$ such that $a\in D_\alpha$. We proceed by induction on α, i.e. for each q such that $q\Vdash \dot{n}_\ell\notin t_\delta$ and $a_q\in\bigcup_{\beta<\alpha}D_\beta$, we assume that $Z_\ell(a_q)$ is non-empty. By elementarity, there is an $A\in M\cap\mathcal{A}$ such that for each n there is an $a_n\subset[A\setminus n]^{<\omega}$ such that $a\cup a_n\in\bigcup_{\beta<\alpha}D_\beta$. Fix such a sequence $\{a_n : n\in\omega\}\subset\bigcup_{\beta<\alpha}D_\beta$, which is an element of M. Since this sequence is in M, so is the set $\bigcup_n Z_\ell(a\cup a_n)$. This set is disjoint from t_δ, hence it is finite. Therefore there is some z which is a member of $Z_\ell(a\cup a_n)$ for infinitely many n. Now $z\in Z_\ell(a)$. Indeed, suppose that q is such that $a_q=a$. By the definition of $Q_{\mathcal{A}}$, there is an n' so that $a_q\cup a'\notin S_q$ for each $a'\subset A\setminus n'$. Now choose $n>n'$ so that $z\in Z_\ell(a\cup a_n)$. Therefore $(a_q\cup a',S)\le q$, but since $z\in Z_\ell(a\cup a')$ there is an $r\le(a_q\cup a',S)$ such that $r\Vdash z=\dot{n}_\ell$. Since $r\le q$, it follows that $z\in Z_\ell(a)$.

We are ready for the final contradiction. Since $Z_\ell(a)\cap\ell$ is empty for each ℓ, it follows that $\bigcup_{\ell>m_1}Z_\ell(a_{p_1})$ is an infinite subset of ω. Since it is also a member of M, there is an $\ell>m_1$ such that $Z_\ell(a_{p_1})\cap t_\delta$ is not empty. However this is the contradiction we seek, since p_1 was assumed to force that $\dot{n}_\ell$ is not a member of t_δ and so $Z_\ell(a_{p_1})$ must disjoint from t_δ. $\square$

We will need a result about preserving dense families through an iteration. For towers this was proven in [BD85]. The argument for dense weakly σ-bounded families seems to be a little harder (unless we unnecessarily strengthen the hypothesis to reflect what we can really prove for the $Q_{\mathcal{A}}$'s). In fact we will need a preliminary Lemma; the technique seems interesting on its own.

Lemma 3.6. *For a σ-centered poset P the following are equivalent:*

1. *P preserves dense weakly σ-bounded families;*
2. *for each P-name τ of an infinite subset of ω, there is a countable family $\{C_n : n\in\omega\}$ of infinite subsets of ω such that P forces that $\tau\cap a$ is not empty (hence infinite) for every set a which meets each C_n in an infinite set;*
3. *P preserves weakly σ-bounded families.*

Proof. We start with the implication (1) implies (2) by proving the contrapositive; i.e. we assume that (2) fails and prove that (1) also fails. Let $P = \bigcup_{n\in\omega} P_n$ where each P_n is a centered subfamily of P. Fix a P-name τ and assume that it is forced by 1 to be an infinite subset of ω which is a witness to the failure of (2). Since P is ccc, a simple density argument establishes that there is a condition $p \in P$ such that for each $q < p$ and each family $\{C_n : n \in \omega\}$, there is an $r < q$ and an $a \subset \omega$ such that $|a \cap C_n| = |C_n|$ for each n, and $r \Vdash \tau \cap a$ is empty.

Now, to establish the failure of (1), set $\mathcal{I}$ to be the set of all infinite I such that $p \Vdash I \cap \tau$ is finite. We will be done once we have established that $\mathcal{I}$ is a dense weakly σ-bounded family. That is, if $\{C_n : n \in \omega\}$ is a family of infinite subsets of ω, there is some $I \in \mathcal{I}$ such that $I \cap C_n$ is infinite for each n. To construct I we define, inductively, an antichain $\{p_i : i \in \omega\}$ of conditions $p_i < p$ together with sets $\{b_i : i \in \omega\}$ so that, for each n and each i,

1. $C_n \setminus \bigcup_{j\leq i} b_j$ is infinite; and
2. $p_i \Vdash \tau \subset b_i$; and
3. if p_i can be chosen from $\bigcup_{j\leq i} P_i$, then it is.

We construct p_i and b_i. If every $q < p$ is comparable with some member of $\{p_j : j < i\}$, we simply stop and set $I = \bigcup\{C_n \setminus \bigcup_{j<i} b_j\}$. We leave it to the reader to check that $p \Vdash I \cap \tau$ is empty. Otherwise we find a minimal k such that some member q of P_k is incompatible with each p_j $(j < i)$. Next, for each n, set D_n equal to the infinite set $C_n \setminus \bigcup_{j<i} b_j$ and apply the hypothesis to choose $p_i < q$ (again minimizing k such that $p_i \in P_k$) such that there is an a meeting each D_n in an infinite set and yet $p_i \Vdash \tau \cap a$ is empty. Finally, set $b_i = \omega \setminus a$ and observe that all the conditions are met.

Finally, define I by choosing a sequence of pairwise disjoint finite sets, $\{I_n : n \in \omega\}$, so that, for each n, I_n is disjoint from $\bigcup_{i\leq n} b_i$ and $I_n \cap C_i$ is not empty for each $i \leq n$. This is easily done. Set I to be the union of the I_n . Clearly I meets each C_n in an infinite set, so it suffices to show that $p \Vdash \tau \cap I$ is finite. Notice that it follows immediately that, for each i, $p_i \Vdash \tau \cap I \subset \bigcup_{j<i} I_j$, and so is finite. So to finish, just check that, by the third inductive hypothesis, $\{p_i : i < \omega\}$ is dense below p since this implies that if G is P-generic and $p \in G$, then there is an i such that $p_i \in G$.

Now assume that (2) holds and let $\mathcal{A}$ be a weakly σ-bounded family. For each n, let τ_n be a P-name which is forced by 1 to be an infinite subset of some member of $\mathcal{A}$. Since P is σ-centered, there is a countable subfamily $\mathcal{A}' \subset \mathcal{A}$ such that for each n, τ_n is forced to be contained in some member of $\mathcal{A}'$.

Fix a single family $\{C_n : n \in \omega\}$ as given by (2) which works for each τ_k. By choosing infinite subsets of the C_n, we may arrange that for each n, either C_n is almost disjoint with every member of $\mathcal{A}$ or it a member of $\mathcal{A}^{\downarrow}$. We can also add $\mathcal{A}'$ to the list and then arrange that if $C_m \notin \mathcal{A}^{\downarrow}$, then C_m is disjoint from or contained in C_n for each $n < m$. Fix a set $a \in \mathcal{A}$ such that $a \cap C_n$ is infinite for each n such that $C_n \in \mathcal{A}^{\downarrow}$. Let $b = a \cup \bigcup\{C_k : C_k \notin \mathcal{A}^{\downarrow}\}$. Clearly b meets every C_n in an infinite set, hence in the extension, $b \cap \tau_n$ is infinite for each n. However, $(b \setminus a)$ is almost disjoint with each member of $\mathcal{A}'$, hence we actually have that $a \cap \tau_n$ is infinite for each n. This implies that $\mathcal{A}$ remains weakly σ-bounded.

Finally we show that (3) implies (1). Assume that $\mathcal{A}$ is a dense weakly σ-bounded family. We know that $\mathcal{A}$ will remain weakly σ-bounded but not that it will remain dense. Assume otherwise and fix a P-name τ which is forced by some $p \in P$ to be almost disjoint from every member of $\mathcal{A}$. It suffices to construct a new weakly σ-bounded family on $\omega \times \omega$ which P does not preserve. We put $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$ where

$$\mathcal{B}_0 = \{a \times \omega \cup \omega \times a : a \in \mathcal{A}\}$$

and

$$\mathcal{B}_1 = \{\{n\} \times \omega : n \in \omega\} \ .$$

For any sequence $\{b_n : n \in \omega\} \subset \mathcal{B}^{\downarrow}$, let $a_n \in \mathcal{A}$ be chosen so that $b_n = a_n \times \omega \cup \omega \times a_n$ in case $b_n \in \mathcal{B}_0$ and $b_n \supset \{k_n\} \times a_n$ in case $b_n \in \mathcal{B}_1$ (recall that $\mathcal{A}$ is dense). Choose $a \in \mathcal{A}$ such that $a \cap a_n$ is infinite for all n. It is easily checked that $b = a \times \omega \cup \omega \times a$ meets each b_n which shows that $\mathcal{B}$ is weakly σ-bounded.

Now in the extension (by a generic filter containing p), consider the sequence $\{\{n\} \times \tau \setminus n : n \in \tau\}$. Each member of this sequence is contained in a member of $\mathcal{B}_1$. However no member b of $\mathcal{B}$ meets each member. Clearly b could not come from $\mathcal{B}_1$, so fix any $a \in \mathcal{A}$. Pick $n \in \tau$ so that $a \cap \tau \setminus n$ is empty. Then $a \times \omega \cup \omega \times a$ is disjoint from $\{n\} \times \tau \setminus n$. $\square$

Lemma 3.7. *A finite support iteration of σ-centered forcings each of which preserves dense weakly σ-bounded families in $\mathcal{P}(\omega)$, will itself preserve such families.*

Proof. Suppose that for each $\mu < \lambda$, the poset P_μ from the iteration sequence of σ-centered posets $\{P_\alpha; Q_\alpha : \alpha < \lambda\}$ preserves dense weakly σ-bounded families and that λ is a limit. We leave it as an exercise for the more advanced reader that we may assume that λ is no larger than the successor of $\mathfrak{c}$ (the less ambitious reader would be satisfied with knowing that this is the only case for which we apply the Lemma). The advantage of this assumption is that we then know that P_μ is σ-centered and therefore P_μ satisfies condition (2) of the previous Lemma. We show that P_λ also satisfies this condition and then simply note that the implication (2) implies (1) does not require the assumption that P be σ-centered.

Let τ be any P_λ-name of an infinite subset of ω and choose a suitable countable elementary submodel M with $\tau \in M$. Let δ denote the supremum of $M \cap \lambda$. We claim that the family of infinite subsets of ω which are in M witness that (2) holds for τ. Assume otherwise and fix a set $a \subset \omega$ which meets every infinite $b \subset \omega$ which is in M and is such that there is a $p \in P_\lambda$ which forces that $\tau \cap a$ is empty.

Let F be the support of p in the iteration and, since F is finite, fix $\gamma \in M$ such that $F \cap \delta \subset \gamma$. By inductive assumption, for each $\sigma \in M$ which is a P_γ-name of an infinite subset of ω, $p \restriction \gamma$ forces, with respect to P_γ, that a meets σ. There is such a P_γ-name $\tilde{\tau}$, where for each $q \in P_\lambda$ and integer k

$$q \restriction \gamma \Vdash k \in \tilde{\tau} \quad \text{iff} \quad q \Vdash k \in \tau \ .$$

Since $p \restriction \gamma$ forces that a meets $\tilde{\tau}$, fix any $k \in a$ and $q < p \restriction \gamma$ in P_γ such that $q \Vdash k \in \tilde{\tau}$. Now, by the elementarity of M, there is a dense set of conditions, each in M, which decide the truth value of $k \in \tilde{\tau}$. Since q is incompatible with each of those that force $k \notin \tilde{\tau}$, it follows that $p \restriction \gamma$ is compatible with $q \restriction \gamma$ for some $q \in M$ such that $q \Vdash k \in \tau$. Since $q \in M$, its support is a subset of M and therefore p and q are compatible on their common domain. It follows that p and q are compatible which is the contradiction we seek, since any extension of p and q will force that $k \in a \cap \tau$. $\square$

The following combinatorial statement is an easy consequence.

Theorem 3.8. *It is consistent with $\mathfrak{t} < \mathfrak{c}$ that the following weakening of Solovay's principle holds: Given $\mathcal{A}, \mathcal{B} \subset [\omega]^\omega$ such that $|\mathcal{A} \cup \mathcal{B}| < \mathfrak{c}$ and $A \cap B$ is finite for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there is a $C \subset \omega$ such that $C \cap A$ is infinite for all $A \in \mathcal{A}$ and $C \cap B$ is finite for all $B \in \mathcal{B}$.*

However we must still work harder to prove that (NT) is consistent (it is easy to see that (NT) implies the statement in the previous theorem). The following is really the final key.

Lemma 3.9. *If $\mathcal{A} \subset \mathcal{P}(\omega)$ is weakly σ-bounded and G is a generic filter for $Q_{\mathcal{A}}$, then $\mathcal{A}$ is dense in C where $C = \bigcup\{a_p : p \in G\}$.*

Proof. Suppose that $\{\dot{n}_j : j \in \omega\}$ is a sequence of $Q_{\mathcal{A}}$-names and that some q forces that $\{\dot{n}_j : j \in \omega\}$ (indexed in increasing order) is contained in C and is almost disjoint from every member of $\mathcal{A}$. Fix a countable elementary submodel M which includes $\mathcal{A}$, $\{\dot{n}_j : j \in \omega\}$ and q. Since M is countable and $\mathcal{A}$ is weakly σ-bounded, there is an A in $\mathcal{A}$ such that $A \cap A'$ is infinite for each $A' \in M \cap \mathcal{A}^{\downarrow}$.

Fix $p < q$ and m such that p forces that $\dot{n}_j$ is not in A for all $j > m$. Notice that if some $p' \in Q_{\mathcal{A}}$ forces a value on some $\dot{n}_j$, then this value is a member of $a_{p'}$ (since otherwise we can easily force C to avoid the value). Choose any $j > m$ such that $a_p \subset j$ (hence $\dot{n}_j \notin a_p$). Let D be the dense set of conditions which force a value on $\dot{n}_j$. Since D is in M, there is an $A' \in \mathcal{A} \cap M$, by Lemma 3.4, such that for each n, there is a some d in D such that $a_d \setminus a_p \subset A' - n$. Again notice that the value this d forces on $\dot{n}_j$ is in $a_d \setminus j$, hence in $A' - n$. Now we look at the second coordinate of p (which need not be in M) and find an n such that $a_p \cup a'$ is not in S_p for any $a' \subset A' - n$. Therefore, as before, we have that any $(a_p \cup a', S) \in Q_{\mathcal{A}}$ is compatible with p if $a' \subset A' - n$.

By all the above then, it follows that

$$B = \{k : (\exists d \in D)\ (a_d \setminus a_p) \subset A' - n \text{ and } d \Vdash k = \dot{n}_j\}$$

is a member of $M \cap \mathcal{A}^{\downarrow}$ and for each $k \in B$ there is a $d \in D \cap M$ which is compatible with p and which forces the value k on $\dot{n}_j$. Since A meets B (in an infinite set) we have the contradiction to the fact that $p \Vdash \dot{n}_j \notin A$. $\square$

Theorem 3.10. *It is consistent (with $\mathfrak{c} = \mathfrak{b} = \omega_2$) to have the statement (NT) holding, that is, for each weakly σ-bounded family $\mathcal{A}$ and each $\mathcal{B} \subset \mathcal{A}$ of cardinality at most ω_1, there is a $C \subset \omega$ such that $\mathcal{A}$ is dense in C and $C \cap B$ is infinite for each infinite $B \in \mathcal{B}$.*

Proof. We start with a model V of GCH. We construct a finite support iteration $\{P_\alpha; Q_\alpha : \alpha < \omega_2\}$ in which each Q_α is the P_α-name of a σ-centered poset of cardinality $\aleph_1$ which preserves dense weakly σ-bounded subsets of $\mathcal{P}(\omega)$. Clearly then, if G is P_{ω_2}-generic over V, the $\aleph_2$ of V will be the continuum in $V[G]$.

We will need a standard enumeration technique (from Martin and Solovay's original proof of the consistency of Martin's Axiom). Since the exact details of this technique are standard but nonetheless somewhat technical we have to chosen to essentially just remind the reader of the key consequence of this technique. There is a list, in the ground model, $\{Y_\alpha : \alpha \in \omega_2\}$ so that for each $\lambda \in \omega_2$ and each family $\mathcal{A} \subset \mathcal{P}(\omega)$ of size ω_1 in the model $V[G_\lambda]$, there is an α greater than or equal to λ such that Y_α is a P_λ name which is forced by 1 to equal $\mathcal{A}$. Of course we can also assume that each Y_λ is a P_λ-name of a family of subsets of ω. The iteration sequence is simply that Q_λ is chosen to be the P_λ-name of the poset Q_{Y_λ}.

Having defined the iteration we check that (NT) holds in the resulting model. Since there will be cofinally many λ such that Q_λ is forced to be $Q_{\mathcal{A}}$ for some countable family of pairwise disjoint infinite sets (which adds a dominating real), we will have that $\mathfrak{b} = \omega_2$. Let $\{a_\alpha : \alpha < \omega_2\}$ be a set of P_{ω_2}-names for which it is forced, by 1, that the collection forms a weakly σ-bounded family. We may assume that the subfamily $\mathcal{B}$ is simply $\{a_\alpha : \alpha < \omega_1\}$. Fix a generic filter G and for each $\lambda < \omega_2$, let G_λ denote the P_λ-generic filter $G \cap P_\lambda$. By a standard closing off argument, there is a $\lambda < \omega_2$ (bigger than ω_1) such that, for each $\alpha < \lambda$, a_α is a member of $V[G_\lambda]$ and $\{a_\alpha : \alpha < \lambda\}$ is a weakly σ-bounded family in $V[G_\lambda]$. Choose any $\beta \geq \lambda$ such that Y_β is the family $\{a_\alpha : \alpha < \lambda\}$; hence Q_β was chosen to be $Q_{\mathcal{A}}$.

By Corollary 3.6, $\mathcal{A} = \{a_\alpha : \alpha < \lambda\}$ is still weakly σ-bounded in $V[G_\beta]$. Therefore, by Lemma 3.9, there is, in $V[G_{\beta+1}]$, an infinite set C which meets every member of $\mathcal{A}$ and such that $\mathcal{A}$ is dense in C. Clearly the family $\{\omega \setminus C\} \cup \mathcal{A}$ is a dense weakly σ-bounded family in $V[G_{\beta+1}]$. Finally, by Lemma 3.7, this family is still dense in $V[G]$, which implies that $\mathcal{A}$ is dense in C as required. □

It may be useful to note that we also proved that each (dense) weakly σ-bounded family contains a (dense) weakly σ-bounded family of cardinality ω_1.

Remark 3.1. Nyikos also asks about the existence of separable radial non-Fréchet spaces (i.e. no compactness assumption). If there is not one in ZFC it appears that it may be significantly more difficult to produce a model in which there are none. Indeed, by Example 2.1 we know there is a compact example if $\mathfrak{b} = \omega_1$ and let us note here that if $\mathfrak{b} = \mathfrak{c}$, then there is a non-compact example. Suppose that $\{f_\alpha : \alpha < \kappa\}$ is a scale in $(\omega^\omega, <^*)$, i.e. $f_\alpha <^* f_\beta$ for $\alpha < \beta < \kappa$ and for each $g \in \omega^\omega$, there is an $\alpha < \kappa$ so that $g <^* f_\alpha$. We define a space $X = (\omega \times \omega) \cup \{x_\alpha : \alpha \leq \kappa\}$. For each $\alpha < \kappa$, the neighbourhood base for x_α are sets of the form $\{x_\alpha\}$ union a cofinite subset of the graph of f_α. The points of $\omega \times \omega$ are isolated and a set is a neighourhood of x_κ if there is an $\alpha < \kappa$ such that it contains x_β for each $\beta > \alpha$ and also a cofinite subset of $\{(n, m) : f_\alpha(n) < m\}$.

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