Set theory in Topology

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1. Introduction

This article is intended to give a survey of the primarily set-theoretic techniques which, in the author's view, have played a major role in the development of set-theoretic topology over the past several years. The focus is on the techniques rather than the results themselves. We have made a selection of results which we feel are both interesting and which illustrate the techniques. However we have not tried to make this a survey of the many recent results which one might normally associate with the contents of this article. Of course, the interested reader will have found many of these results in the other articles contained in this volume.

Much of set-theoretic topology is focussed on the study of reflection – when we have it and when we do not. When we construct interesting examples it is often of a space which witnesses some kind of non-reflection. When we prove theorems we prove that some kind of reflection holds. It is even true that in many cases in which we produce examples of spaces to witness non-reflection we have to utilize reflection in the construction – as we will see in the second section in a beautiful construction of a non-collectionwise Hausdorff space by M.E.Rudin. In the third section, we use countable elementary submodels to investigate stationary set reflection and relate it to reflecting collectionwise Hausdorffness in locally small spaces. The fourth section explores the technique of factoring large iterations of forcing posets, again using reflection. The techniques developed in §4 are then applied in §5 by discussing the wonderful result of Balogh that there may be no locally compact normal non-collectionwise normal spaces. In §6 – **PFA and OCA** – we have tried to emphasize the Ramsey theoretic approach to applying these axioms as outlined in Todorčević's book ([Tod89]). We also include a proof of the underlying combinatorial result to Balogh's solution of the Moore-Mrowka problem. The final section – Kurepa families – was kindly supplied by Stevo Todorčević. These structures should find many applications in the inductive constructions of interesting topological spaces.

2. A technique of Rudin

In this section we give a proof that there is a space which is normal but not collectionwise Hausdorff (hereafter CWH). The purpose is to discuss a prediction type principle formulated in [Rud84]. Many inductions are based on the fact that all the countable structures on c can be listed in length c. The point of this technique is that we actually use this list as a kind of oracle in the induction.

Let θ be a regular cardinal which is much larger than c.

Definition 2.1. Given a model (set) M, the Mostowskii collapse is a well-known isomorphism from $\{M, \in\}$ to its transitive collapse. Simply

$$\widehat{M} = \{\widehat{m} : m \in M\}$$

where, for $m \in M$,

$$\widehat{m} = \{\widehat{x} : x \in M \cap m\} \ ,$$

It is easily checked that $m \mapsto \widehat{m}$ is an isomorphism from $(M, \in)$ to $(\widehat{M}, \in)$.

The idea of the Mostowski collapse can be applied quite generally. In case we are building an object on c it is useful to define a similar kind of isomorphism (collapse), say $F_M : M \to F_M(M)$, which would be the identity on c. For simplicity we will send all ordinals to ordinals.

By induction on rank: if $F_M(Y)$ has been defined for all $Y \in M \cap X$, then

$$F_M(X) = \begin{cases} X & \text{if } X \in M \cap c, \\ c \cup \{F_M(Y) : Y \in M \cap X\} & \text{if } X \in M \cap [c, \theta), \\ F_M(X) = \{F_M(Y) : Y \in M \cap X\} & \text{otherwise.} \end{cases}$$

Hence for ordinals $\alpha \in M - c$, $F_M(\alpha) \in [c, c + \omega_1)$.

Note that if X is any set (not necessarily in M), then $F_M(X)$ is defined. Hence $F_M(M) = \{F_M(X) : X \in M\}$. If the set M is clear from the context, we will use F rather than F_M. Perhaps we should point out that although F is an isomorphism from the model $(M, \in)$ to $(F_M(M), \in)$ (provable by induction on rank), it does not preserve even the subset relation. For example, if $X = c + 2 - \{c\}$, then $F_M(X)$ is simply the countable set $\{F_M(\alpha) : \alpha \in M \cap X\}$ and so does not contain the set $c = F_M(c)$. Therefore we have that $F_M(c) \not\subset F_M(X)$, but

$$(F_M(M), \in) \models (\forall x)(x \in F_M(c) \rightarrow x \in F_M(X)) \equiv (F_M(c) \subset F_M(X)) .$$

Since $F(M)$ is uniquely determined by the Mostowskii collapse of M, $\widehat{M}$, which is in H_{ω_1}, and the countable set $F(M) \cap [0, c+\omega_1)$, there are only c many such sets (or even ω-sequences of such sets). For this construction we want to predict both a topology τ on a set X containing c and possible separations, f, of c as a subspace. Let $\{\langle M_\alpha, X_\alpha, \tau_\alpha, f_\alpha\rangle : \alpha < c\}$ enumerate the set

$$\{\langle F_M(M), F_M(X), F_M(\tau), F_M(f)\rangle : \\ \{X, \tau, f\} \subset H(\theta) \text{ and } M \prec H(\theta) \text{ is countable}\} .$$

We may assume that $M_\alpha \cap c \subset \alpha$. We could also restrict the list to those $\langle X, \tau, f\rangle$ which have some special properties (e.g. being a topological space) and this would be reflected in M. The idea now is that we define the β^{th} object by looking at M_β and we can also look ahead to all those M_α which contain β.

We intend to build a space which is not CWH. The ordinals in c will form the closed discrete set which cannot be separated. Any potential separation will be "trapped" by some M_α – i.e. after we are done we take a countable elementary submodel M which contains the separation (in fact, as in our list above, we add the separation as a distinguished predicate in the language). Some M_α will equal $F(M)$ and since we added the predicate to the language we know what corresponds to the separation. We ensure that the point α will be a limit point of this separation by virtue of the points which are in M_α. Naturally this will do nothing to ensure normality. This is where the trick of making the list in advance comes in. The points in M_α will know that α is going to be used in this way.

Similar to the space in [Rud84], the structure of the space itself is quite simple. The space X will be $c \cup [c]^2$. Each of the pairs $\{\alpha, \beta\}$ will be isolated in X. For an $\alpha \in c$, we will specify the neighbourhood base at α, $U(\alpha, A)$ for $A \subset c$ such that $\alpha \in A$. We will ensure that for $\alpha \in A$ and $\beta \notin A$, $U(\alpha, A)$ and $U(\beta, c - A)$ are disjoint. If we do so, it is clear that X is normal and that c will be a closed discrete subset of X.

The simplest way to ensure that α will "ruin" the potential separation predicted by f_α will be to pick a sequence $\{\beta_n^\alpha\} \subset M_\alpha$ such that α will be a limit point of the assigned neighbourhoods of $\{\beta_n^\alpha\}$. The only way this is possible is to ensure that $\{\beta_n^\alpha, \alpha\}$ is in each of the respective assigned neighbourhoods and to ensure that at least one of these pairs is in every neighbourhood of α – i.e. that the sequence $\{\beta_n^\alpha, \alpha\}$ has α as a limit point.

The difficulty is with normality. Let $A \subset c$ be such that $\alpha \in A$; for each n such that $\beta_n^\alpha \notin A$ we must choose between (Case 1) $\alpha \notin U(\beta_n^\alpha, c - A)$ and (Case 2)

$\beta_n^\alpha \notin U(\alpha, A)$. We will use an ultrafilter to decide. Fix any free ultrafilter p_α on c such that $\{\beta_n^\alpha : n \in \omega\} \in p_\alpha$. Now Case 1 will correspond to $A \in p_\alpha$ and Case 2 with $A \notin p_\alpha$. Hence the neighbourhood trace of α on $\{\{\beta_n^\alpha, \alpha\} : n \in \omega\}$ will be like p_α.

More formally, once we have chosen the sequences $\{\beta_n^\alpha : n \in \omega\}$, we define, for $A \subset c$ and $\alpha \in c$,

$$\begin{aligned} U(\alpha, A) \quad = \quad & \{\alpha\} \cup \{\{\beta_n^\alpha, \alpha\} : n \in I_\alpha(A)\} \cup \{\{\alpha, \gamma\} : \\ & \alpha < \gamma \text{ and either } \gamma \in A \text{ or } \ \alpha \notin \{\beta_n^\gamma\} \text{ or } A \cap \{\beta_n^\gamma\} \notin p_\gamma\} \end{aligned}$$

where $I_\alpha(A)$ is the set $\{n : \beta_n^\alpha \in A\}$ if it is in p_α, otherwise it is ω. It can be shown that these sets form a base at α.

These sets are essentially the largest possible sets to accomplish the above normality-type separations. All that remains is to show that we can choose the sequences so as to ensure that X is not CWH. The tricky part now is that we must use our knowledge of what the restrictions to M_α of the neighbourhood assignment given by a potential separation f look like – $U(\beta, A_\beta)$ for some A_β – in order to try to ensure that $\{\beta, \alpha\}$ will be in $U(\beta, A_\beta)$ for "p_α-many" β. Notice that if A_β is the restriction of a set, say $\tilde{A}_\beta$, which contains α then $\{\beta, \alpha\} \in U(\beta, \tilde{A}_\beta)$. So in the situation when we actually have the restriction of a separation, it will be the case that α is not in $\tilde{A}_\beta$ for all β in some member of the ultrafilter p_α. Therefore, choose, if possible an increasing sequence $\{\beta_n^\alpha : n \in \omega\}$ cofinal in $M_\alpha \cap \alpha$ such that $\beta_n^\alpha \notin \cup_{k<n} A_k$ (reflecting the case that $\alpha \notin \cup_{k<n} A_k$). If this is not possible, then as discussed above, M_α is not predicting a separation so the sequence is chosen to be any increasing cofinal sequence.

The proof that the space is not CWH is to simply note that for any potential separation f, we can choose a countable elementary submodel, M, of $\langle H_\theta, f\rangle$. Then there is an α such that $F(M) = M_\alpha$ and $F(f) = f_\alpha$. In this case, the choice of $\{\beta_n^\alpha : n \in \omega\}$ was successfully carried out. That is, let I be the infinite set $\{n \in \omega : \{\beta_n^\alpha, \alpha\} \in f(\alpha)\}$. Fix any $n \in I$ and note that since $\{\beta_n^\alpha, \alpha\}$ is in $f(\alpha)$, it follows that $\{\beta_n^\alpha, \alpha\}$ is not in $f(\beta_n^\alpha)$. Let $f(\beta) = U(\beta, \tilde{A}_\beta)$ for all β. In keeping with the notation of the previous paragraph, $F(f(\beta)) = U(\beta, A_\beta)$ where $A_\beta = \tilde{A}_\beta \cap M_\alpha$. Now, for each integer $n \in I$, $\alpha \notin U(\beta_n^\alpha, \tilde{A}_{\beta_n^\alpha})$ (hence $\alpha \notin \tilde{A}_{\beta_n^\alpha}$). By the elementarity of M, $c - \bigcup\{\tilde{A}_{\beta_k^\alpha} : k \in I \cap n\}$ is cofinal in c for each integer n.

3. More reflection

Recall that if X is any set then there are notions of *closed and unbounded* (*cub*) and *stationary* for the structure $[X]^\omega$ ordered by $\subset$. In this section we show how we can take advantage of the fact that the set of countable elementary submodels of H_θ is *cub* in $[H(\theta)]^\omega$. It was shown by Shelah that if one Levy collapses a strongly compact cardinal then every first countable locally separable space is CWH if it is $\aleph_1$-CWH. It was later noticed that any proper collapse will yield the same result. The idea behind these proofs is Fleissner's lemma about the closure "going up" stationarily often.

Fix a space X which is locally separable and has countable tightness. Suppose that D is a closed discrete subset of X and that τ is the topology on X. Let

$$S = \{U \in [X]^\omega : U \subset \mathrm{int}_X \overline{U} \text{ and } D \cap \overline{U} \not\subset U\} \ .$$

Claim 1. D can be separated in X if and only if S is not stationary.

Proof. Suppose $f\colon DMApsto\tau$ is a separation of D. Let $\tilde{S} = \{a \in [H(\theta)]^\omega : A \cap X \in S\}$. By Claim 1.12 of [She82], $\tilde{S}$ is stationary in $[H(\theta)]^\omega$ if and only if S is stationary in $[X]^\omega$. Since the set, C, of countable elementary submodels M of $H(\theta)$ which contain the function f is a *cub* it suffices to show that $C \cap \tilde{S}$ is empty. Fix any $M \in C$ and $d \in D - M$. We claim that $f(d)$ is a neighbourhood of d which misses $M \cap X$. Indeed, if $x \in M \cap f(d)$, then by elementarity $M \models (\exists y \in D) x \in f(y)$. Hence there is such a $y \in D \cap M$ which contradicts that f is a separation.

For the other direction, suppose that S is not stationary. We prove this (as in [Fle74]) by induction on $\kappa = |D|$. Let C be a *cub* in $[X]^\omega$ which misses S. Fix an elementary chain of elementary submodels $\{M_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ such that S, D, C etc. are all in M_0, $|M_\alpha| < \kappa$ for all α and $D \subset \cup M_\alpha$. Since X is locally separable,

$$D \cap M_\alpha \subset \mathrm{int}_X \overline{M_\alpha \cap X} \quad \text{for all} \quad \alpha .$$

Therefore, by the induction hypothesis, it suffices to show that if $d \in D - M_\alpha$, then $d \notin \overline{M_\alpha \cap X}$. To do so, let $\{x_n : n \in \omega\}$ enumerate any countable subset of $M_\alpha \cap X$. Recursively choose $a_n \in C \cap M_\alpha$, such that $a_n \cup \{x_{n+1}\} \subset a_{n+1}$. Clearly, $U = \cup a_n \in C$ and $d \notin U$. Since $U \notin S$, it follows that $d \notin \overline{U}$. Since X has countable tightness, this completes the proof. $\square$

Definition 3.1. A subset S of $[H_\theta]^\omega$ will be said to $\in$-reflect to a set $Y \subset H_\theta$ if $(S \cap Y) \cap [Y]^\omega)$ is stationary in $[Y]^\omega$.

The idea here is that many members of $S \cap [Y]^\omega$ are also members of Y. This is probably stronger than demanding only that $S \cap [Y]^\omega$ is stationary and Y is unbounded in $[Y]^\omega$. The statement that stationary subsets of $[H_\theta]^\omega$ reflect will mean that every stationary subset of $[H_\theta]^\omega$ $\in$-reflects to some size $\aleph_1$ subset of H_θ.

Corollary 3.2. *If stationary subsets of $[H_\theta]^\omega$ reflect, then every $\aleph_1$-CWH locally separable countably tight space is θ-CWH.*

Proof. Let X be a locally separable countably tight space and suppose that $D \in [X]^{\leq\theta}$ is discrete. Since X is locally separable, there is a subspace X' of X of cardinality no more than θ with the property that $D \subset \mathrm{int}_X \overline{X'}$. We can assume that the base set of X' is just the ordinal θ. As above, let

$$S = \{U \in [X']^\omega : U \subset \mathrm{int}_X \overline{U} \text{ and } D \cap \overline{U} \not\subset U\} .$$

Now let κ be a large enough so that $\{X, D, S\}$ is in H_κ. Let $\tilde{S}$ be all countable subsets of H_θ which are of the form $M \cap H_\theta$ for some $M \prec H_\kappa$ such that $\{X, D, S\} \in M$ and such that $M \cap X'$ is a member of S. Then $\tilde{S}$ is stationary if and only if S is.

Let $A \subset H_\theta$ be of size ω_1 and witness that $\tilde{S}$ $\in$-reflects. Consider $s \in \tilde{S} \cap A \cap [A]^\omega$. We claim that $\overline{s \cap X} \cap D - s$ meets A. Indeed, since $s \in A$, there is an $a \in \tilde{S} \cap [A]^\omega$ such that $s \in a$. It will actually suffice to know that there is a $y \in a \cap [A]^\omega$ such that $s \subset y$. Fix $M \prec H_\kappa$ with $X, D \in M$ such that $M \cap H_\theta = a$. Since $s \in \tilde{S}$, it follows that

$$M \models \overline{s \cap X'} \cap D - s \neq \emptyset,$$

hence this intersection meets A. Note that we can assume that a contains $\overline{y \cap X} \cap D$ since this set must be countable in order for X to be $\aleph_1$-CWH. $\square$

Proposition 3.3. *The statement that every stationary subset reflects is equivalent to Fleissner's axiom R.*

Proof. Recall that axiom R is the statement that if S is a stationary subset of $[\kappa]^\omega$ and C is a t.u.b. subset of $[\kappa]^{\omega_1}$ (i.e. C is unbounded and closed under unions of strictly increasing ω_1 chains), then there is a $Y \in C$ such that $S \cap [Y]^\omega$ is stationary. Axiom R implies that every stationary set reflects since H_θ may be identified with some κ and C may be taken to be all those elementary submodels of H_θ which are equal to a union of an $\in$-chain of countable elementary submodels.

For the other direction, assume that every stationary subset reflects. Let S and C be as in axiom R. Choose a large enough θ and let $\tilde{S}$ be the pull back of S to $[H_\theta]^\omega$. Let A be chosen to witness that $\tilde{S}$ will $\in$-reflect (we may assume that we have shrunk $\tilde{S}$ to only those members which are elementary submodels of H_θ which also contain C as an element). Inductively choose members c_α of $C \cap A$ so as to form a cofinal increasing chain of length ω_1. Suppose that $\{c_\beta : \beta < \alpha\}$ has been chosen. There is a member, say a, of $\tilde{S} \cap A$ containing this countable subset of A. By elementarity, there is a member c_α of C which contains the union of $a \cap C$. We may also assume that c_α includes the α^{th} member of $A \cap \kappa$ under some ordering in order type ω_1. Therefore the union $Y = \cup\{c_\alpha : \alpha \in \omega_1\}$ will be a member of C and it will equal $A \cap \kappa$. Now we must show that $S \cap [Y]^\omega$ is stationary. This is direct. If D is cub in $[Y]^\omega$, then $\tilde{D} = \{c \subset [A]^\omega : c \cap Y \in D\}$ is cub in $[A]^\omega$. Fix an $\tilde{s} \in \tilde{S} \cap \tilde{D}$. Now simply check that $\tilde{s} \cap \kappa = \tilde{s} \cap Y$ hence $\tilde{s} \cap \kappa$ is a member of $S \cap D$. $\square$

4. A standard iteration lemma

Let us try to formulate some useful (cross between general and understandable) reflection lemmas for iterations.

Definition 4.1 ([Kun80]). For a set M (usually a model of most of ZF) and a generic filter G of a poset P (often $P \in M$),

$$M[G] = \{\ \mathrm{val}_G(\dot{X}) : \dot{X} \in M \text{ is a } P\text{-name}\,\}\,.$$

Recall that a filter G is P-generic over M if $(P \cap M) \cap D$ is not empty for each $D \in M$ which is a dense open subset of P. A condition $p \in P$ is said to be (M,P)-generic ([She82]) if every generic filter which contain p is P-generic over M. We use "generic" to mean P-generic over V.

Theorem 4.2 ([She82], III.211). *Suppose that P is a poset in an elementary submodel $M \prec H_\theta$ and suppose that G is P-generic over both V and M. Then $M[G] \prec H_\theta[G]$.*

Proof. Note that since P is in H_θ, it follows that $|P| < \theta$. Therefore, for every $\dot{X}$ and $p \in P$ such that $p \Vdash \dot{X} \in H_\theta$, there is a $\dot{Y} \in H_\theta$ with $|\dot{Y}| < \theta$ such that $p \Vdash \dot{X} = \dot{Y}$. Hence the set of hereditarily $< \theta$ sets in $V[G]$ is equal to $(\check{H}_\theta)[G]$.

We use the Tarski-Vaught criteria to establish that $M[G] \prec H_\theta[G]$. To prove this, one simply uses elementarity of M in H_θ to assert that the necessary names reside in M (and the fact that the forcing relation is absolute between M and H_θ). $\square$

The important thing about $M[G]$ in the previous result is that it is isomorphic to a model in $V[G \cap M]$ - namely its transitive collapse. It is also useful to consider a pseudo-collapse which is the identity on the ordinals similar to the one we used in Section 2. Let j denote the inverse of the Mostowskii collapsing map. Hence j is an elementary embedding of $\widehat{M[G]}$ into $H_\theta[G]$ which takes $\widehat{M[G]}$ to $M[G]$. In such a situation, it is customary to let j" be defined by

$$j\text{"}X = \{j(x) : x \in X\}\ .$$

Also define e recursively, as follows,

$$\mathrm{e}(\alpha) = j(\alpha) \quad \text{for ordinals } \alpha \in \widehat{M}$$

and

$$\mathrm{e}(X) = \mathrm{e}"X = \{\mathrm{e}(x) : x \in X\} \quad \text{for} \quad X \in \widehat{M} - \text{ ON} \ .$$

The pseudo-collapse, F, we are speaking of is just the composition of the Mostowskii collapse with e. That is, for $\alpha \in M$, $F(\alpha) = \mathrm{e}(\widehat{\alpha}) = \alpha$ and for other $X \in M$, $F(X) = \mathrm{e}(\widehat{X}) = \{\mathrm{e}(\widehat{x}) : x \in X \cap M\}$.

Now, if we consider $\widehat{M[G]}$ (or $F(M[G])$), we have that for each member $\widehat{X}$ of $\widehat{M[G]}$ there is a $\widehat{P}$-name in $\widehat{M}$ whose valuation by $\widehat{G}$ is equal to $\widehat{X}$. Furthermore $pMApsto\widehat{p}$ is an isomorphism from the poset $P \cap M$ to $\widehat{P}$ which takes $G \cap (P \cap M)$ to $\widehat{G}$. Since the Mostowski collapse is definable, (hence $\widehat{G} \in V[G \cap M]$) and $\widehat{M[G]} = \widehat{M}[\widehat{G}]$ it follows that $\widehat{M[G]}$ is in $V[G \cap M]$. An alternative way to view all this is that $F = \mathrm{e} \circ \widehat{}$ is obtained by recursively re-interpreting the P-names of M as $P \cap M$-names of their restrictions to $M[G]$ (using the fact that G is P-generic for M). Hence if we are interested in the properties in $V[G]$ of a structure $\mathcal{X} \in M[G]$ then $F(\mathcal{X})$ will be in $V[G \cap M]$ and should be an interesting substructure of $\mathcal{X}$. Since $F(M[G])$ is isomorphic to $\widehat{M[G]}$ we can restrict our discussion to the transitive collapse.

So, $\widehat{M[G]}$ will be a submodel of $H_\theta[G \cap M]$ (not elementary) which is elementarily embedded into $H_\theta[G]$ by the inverse of the Mostowskii collapse, j. So to show that the structure $\mathcal{X}$ with a certain property has a small substructure with that property we show that $\widehat{\mathcal{X}}$ has the property. The proof splits into two steps. What happens when we pass from $\widehat{M[G]}$ to $H_\theta[G \cap M]$ – i.e. is M "closed" enough to give enough absoluteness? Next, what happens when we pass from $H_\theta[G \cap M]$ to $H_\theta[G]$ – i.e. is the forcing "mild" enough to preserve the property?

In summary, we think of $M[G]$ as being equal to $\widehat{M[G]}$, hence it is a member of $V[G \cap M]$ but is an elementary submodel of $V[G]$.

Theorem 4.3 (CH). *Suppose that $P = \cup\{P_\alpha : \alpha < \omega_2\}$ is a proper poset such that $|P_\alpha| \leq \aleph_1$ for each $\alpha < \omega_2$ and such that $\{\lambda < \omega_2 : P_\lambda$ is completely embedded in $P\}$ is stationary in ω_2. Let G be P-generic over V and suppose that $\mathcal{X}$ is a structure in $H_\theta[G]$. Then there is a $\lambda < \omega_2$ and an $M \in H_\theta[G]$ of cardinality $\aleph_1$ such that*

$$\widehat{M} \in V[G_\lambda] \textit{ and } V[G_\lambda] \models \widehat{M}^\omega \subset \widehat{M}$$

and

$$V[G] \models \langle M, \mathcal{X} \cap M, [\mathcal{X}]^\omega \cap M, \epsilon\rangle \prec \langle H_\theta, \mathcal{X}, [\mathcal{X}]^\omega, \epsilon\rangle \ .$$

The usefulness of the previous result is that since $\widehat{M}$ is closed under ω-sequences in $V[G_\lambda]$ there will be a lot of absoluteness between $\widehat{M}$ and $V[G_\lambda]$. The final model is obtained by forcing with the tail P/P_λ, hence the standard next step in an argument is to use the reflected properties of M (usually some maximality type property) to show that the tail will preserve this and contradict some property of $\mathcal{X}$ in $V[G]$.

In many instances it is not necessary to worry about $\widehat{M}$ since the collapsing function may be the identity on a large part of M. That is the idea of the next two corollaries.

Corollary 4.4. *Suppose that P is a poset in an elementary submodel $M \prec H_\theta$ and that G is P-generic over both V and M. Let κ be the least cardinal such that $M \cap \kappa \notin \kappa$ and assume (with no loss of generality) that the base set for P is an ordinal. Then*

$$(M \cap \check{H}_\kappa)[G \cap M] = (M \cap \check{H}_\lambda)[G \cap M] \prec H_\lambda[G]$$

where λ is the minimum member of $(M \cup \{\theta\}) \cap [\kappa, \theta]$.

Proof. The Mostowskii collapse is just the identity function on $M \cap H_\kappa$. If $\lambda = \theta$, then we are just restating Proposition 4.2. If $\lambda < \theta$, then $\lambda \in M$, hence $H_\lambda \in M$ and $M \cap H_\lambda \prec H_\lambda$. Then again we have that $(M \cap H_\lambda)[G] \prec H_\lambda[G]$. Finally we observe that, for any $\dot{X} \in M \cap H_\lambda$, $\mathrm{val}_{G \cap M}(\dot{X}) = \mathrm{val}_G(\dot{X})$. □

Corollary 4.5. *Suppose that $P = \cup\{P_\alpha : \alpha < \omega_2\}$ is a proper poset such that $P_\alpha \in H_{\omega_2}$ for each $\alpha < \omega_2$ and such that $S = \{\lambda < \omega_2 : P_\lambda$ is completely embedded in $P\}$ is stationary in ω_2. Let $\mathcal{X}$ be any P-name and let M be any $\aleph_1$-sized elementary submodel of H_θ (in V) such that $\{\mathcal{X}, P\} \subset M$, $\lambda = M \cap \omega_2 \in S$ and such that $M^\omega \subset M$. Then, $(M \cap H_{\omega_2})[G_\lambda] \prec H_{\omega_2}[G]$ and $(M \cap H_{\omega_2})[G_\lambda]$ is closed under ω-sequences.*

The method outlined above is almost always used when establishing consistency results about βN. Typically $\mathcal{X}$ is some family of subsets of N which may be one of: an ultrafilter, a tower, a tree π-base, or a splitting family. As an example, let us prove that there are no towers of lenth ω_2 in the Mathias model – the model obtained by the ω_2-lenth countable support iteration of the Mathias poset. This was established for an slightly modified iteration of Mathias forcings by Dordal [BD85]. The only reason that they modified the support is that they did not have at hand the very useful Lemma 1.13 of [She84]. Recall that *Mathias forcing* is the poset $\{(a, A) : a \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega\}$ where $(a, A) \leq (b, B)$ if $a \supset b$ and $(a - b \cup A) \subset B$. Given a filterbase $\mathcal{S} \subset [\omega]^\omega$, we say that $Y \in [\omega]^\omega$ **diagonalizes** $\mathcal{S}$ if $Y - S$ is finite for all $S \in \mathcal{S}$. It is interesting that Mathias forcing adds a filter and then diagonalizes it ([Mat77]) but that it preserves *undiagonalizability* for most existing filters (Dordal [BD85]). It cannot preserve this for all filters because of the following easily established fact.

Proposition 4.6. *There is an undiagonalizable filter $\mathcal{F}$ on $\mathcal{P}(N)$ such that any forcing which adds a real also diagonalizes $\mathcal{F}$.*

We can see (e.g. from [She84]) that we have to strengthen the notions that we are trying to preserve to countably complete versions. So let us say that a filter base $\mathcal{S}$ is *ω-diagonalizable* if there is a countable family $\mathcal{Y}$ such that each member of $\mathcal{S}$ contains a member of $\mathcal{Y}$.

Proposition 4.7. *In the Mathias model, if a filter $\mathcal{S} \subset [\omega]^\omega$ cannot be ω-diagonalized, then there is a subset $\mathcal{S}' \in [\mathcal{S}]^{\omega_1}$ which cannot be ω-diagonalized.*

Proof. Let P_λ (for $\lambda \leq \omega_2$) be the countable support iteration of λ-many copies of Mathias forcing. In $V[G]$, suppose that $\mathcal{S} = \{S_\alpha : \alpha < \omega_2\}$ is a filter base of infinite subsets of ω which cannot be ω-diagonalized (e.g. a tower). Let λ, M and $\widehat{M}$ be as in Theorem 4.3. By elementarity, $\widehat{M} \models \widehat{\mathcal{S}}$ cannot be ω-diagonalized. For each $\alpha \in \lambda$, $\widehat{S_\alpha}$ is equal to S_α – hence $\widehat{\mathcal{S}} = \{S_\alpha : \alpha \in \lambda\}$. Since $V[G_\lambda] \models \widehat{M}^\omega \subset \widehat{M}$, it follows that $\widehat{\mathcal{S}}$ cannot be ω-diagonalized in $V[G_\lambda]$. We have shown that this property of $\mathcal{S}$ has reflected to $V[G \cap M]$ which completes the first step of a proof of this type. The next two lemmas handle the second step. □

To complete the proof we reproduce the result of Baumgartner and Dordal and sketch the proof of this special case of Shelah's powerful iteration technique developed in [She84].

Proposition 4.8. *If $\mathcal{S} \subset [\omega]^\omega$ is not ω-diagonalizable then this remains true of $\mathcal{S}$ after Mathias forcing.*

Proof. Suppose that $\{\dot{Y}_n : n \in \omega\}$ are Mathias names of a infinite subsets of ω. Let M be a countable elementary submodel containing $\{\dot{Y}_n : n \in \omega\}$ and A. Let B be an infinite subset of ω which will be chosen as a generic object for M over some forcing. Let S be any member of $\mathcal{S}$ which does not contain (mod finite) any infinite member of M. We want for all $n \in \omega$, $(a, B) \Vdash |\dot{Y}_n - S| = \omega$. It will simplify the proof if we just show this for $\dot{Y}$ (any one of the $\dot{Y}_n$'s) and leave it to the reader to observe that since a hasn't changed this can be done not only for all of the $\dot{Y}_n$'s but even for every name in M.

Given any $(b, B) \in M$, there is an $B' \in [B]^\omega \cap M$ and $(b', B') \leq (b, B)$ such that $(b', B') \Vdash (\exists k > \max b)\ k \in \dot{Y} - S$. This is simply because $\{k : (b, B) \not\Vdash k \notin \dot{Y}\}$ is an infinite member of M – hence S does not contain this set (mod finite). Next, in a usual Ramsey type argument, choose a descending sequence A_k for $k \in \omega$ and let $j_k = \min A_k$. Choose A_{k+1} so that for every $b \subset \{j_i : i \leq k\}$, if there is an $A'' \subset A_{k+1}$ such that $(a \cup b, A'')$ forces some integer bigger than $k+1$ to be in $\dot{Y} - S$, then already $(a \cup b, A_{k+1})$ does this. It follows that if (b, B) is any extension of $(a, \{j_k : k \in \omega\})$, where $b \subset \{j_i : i < k\}$, then $(a \cup b, B) \Vdash \dot{Y} - S \not\subset k$. □

There seems to be an additional difficulty in proving the iteration preserves the property. The approach in [She84] is to strengthen further the notion of non-ω-diagonalizability. We refer the interested reader there for the notion of a bounding set for a nice relation. The nearly equivalent alternative we take here is to demand a stronger preservation property of each poset of the iteration. For the sake of brevity we will be somewhat informal, the interested reader is referred to Steprāns [DoSt91] for a more complete treatment of a similar type of problem. Let us notice that in the previous proof, we showed that if M is a countable elementary submodel (containing the poset) and if $S \in \mathcal{S}$ does not contain any member of M, then for any condition $p \in M$, there is a $q < p$ (which is (M, P)-generic) and which forces that S does not contain any member of $M[G]$. Let us say then that P strongly preserves that $\mathcal{S}$ is not ω-diagonalizable.

Corollary 4.9. *If $\mathcal{S} \subset [\omega]^\omega$ is not ω-diagonalizable then this is strongly preserved by Mathias forcing.*

Lemma 4.10. *If P is the countable support iteration of proper posets each of which strongly preserves that $\mathcal{S} \subset [\omega]^\omega$ cannot be ω-diagonalized, then P strongly preserves that $\mathcal{S}$ cannot be ω-diagonalized.*

Proof. Since P is proper and we are only interested in the existence of countably many subsets of N, we may assume that P is the iteration of countably many proper posets Q_n. Let $P \in M$ be a countable elementary submodel of a suitably large H_θ and choose $S \in \mathcal{S}$ which does not contain any infinite member of M. Since we are assuming that P_n strongly preserves that $\mathcal{S}$ cannot be diagonalized, then for any condition $p \in P_n \cap M$, there is an (M, P)-generic condition $q < p$ such that if $G_n \ni q$ is P_n-generic over V then S does not contain any infinite member of $M[G_n]$.

Let $\dot{Y}$ be a P-name in M and suppose that $p \in P \cap M$. We first show that we may suppose that p is such that $n \in \dot{Y}$ is decided by conditions in P_n – that is, if $q < p$ and $q \Vdash n \in \dot{Y}$ then $(q \restriction n \wedge p) \Vdash n \in \dot{Y}$ ([She82]). For $n \in \omega$, let P^n denote the P_n-name of the poset such that $P = P_n * P^n$ and D_n be the P_n-name of the dense open subset of P^n of conditions which decide the statement $n \in \dot{Y}$. Now inductively define $p_n \in P$ so that

$$p_{n+1} \leq p_n \quad \text{and} \quad p_{n+1} \restriction n = p_n \restriction n$$
$$p_n \restriction n \Vdash_{P_n} p_{n+1} \restriction [n, \omega] \in D_n$$

Now let $p(n) = p_{n+1}(n)$ for each n, i.e. $p = \bigwedge p_n$. It follows that p has the desired property.

Let $\{D_n : n \in \omega\}$ be an enumeration of the dense subsets of P which are in M. Fix n and now suppose that q_n has been chosen so that $q_n \restriction n < p \restriction n$ is (M, P_n)-generic and if $q_n \restriction n \in G_n$ a P_n-generic filter, then $\mathrm{val}_{G_n}(q_n) \in M[G_n] \cap D_n$– i.e. $q_n \restriction n \Vdash q_n^- = q_n/P_n = q'/P_n$ for some q' in $M \cap D_n$. So working in $M[G_n]$ we can find a $q_{n+1}^- < q_n^-$ which is in $M[G_n]$ and which forces that some integer larger than n is in $\dot{Y} - S$ (here we are using the hypothesis that P_n preserves that S does not contain any infinite subset of $M[G_n]$). Furthermore, since q_n is (M, P_n)-generic, $M[G_n] \models \mathrm{val}_{G_n}(D_n) = \{\mathrm{val}_{G_n}(q) : q \in D_n\}$ is a dense subset of P/P_n. Therefore we suppose that $q_{n+1}^- \in \mathrm{val}_{G_n}(D_n)$. Next we can "strengthen" q_{n+1}^- on just the n^{th}-coordinate so that it is P_{n+1}/P_n-generic over $M[G_n]$. Since $q_n \restriction n$ forces all this to happen, there is a P_n-name of q_{n+1}^- such that $q_n \restriction n$ forces that it has the above properties. The condition $q_{n+1} = q_n \restriction n * q_{n+1}^-$. It turns out that the fusion q of the q_n, i.e. $q = \cup\{q_n \restriction n : n \in \omega\}$, is a condition which is (M, P)-generic and which forces that $\dot{Y} - S$ is infinite. The main point is that $q \restriction n \Vdash q/P_n \leq q_{n+1}^-$. $\square$

Now we turn to applications of the method to more topological questions. For example we show that if ω_2 Cohen subsets of ω_1 are added then we get CWN wrt ω_1 copies of ω_1 in first countable spaces – hence the hypothesis $\diamondsuit^+$ in Rudin's construction of a normal non-CWN manifold cannot be weakened to just $\diamondsuit$ ([Kun80, VIII,J5]).

Lemma 4.11. *If G is $P = Fn(\omega_2, 2, \omega_1)$-generic over a model V of CH, then, in $V[G]$, normal first countable spaces are collectionwise normal with respect to ω_1 copies of ω_1.*

Proof. In $V[G]$, let X be a normal first countable space and, for each $\alpha \in \omega_1$, let $\{x_{\alpha,\beta} : \beta \in \omega_1\}$ be a topological copy of ω_1 in X such that the collection of all ω_1 many copies of ω_1 is discrete. Let $\mathcal{X}$ be the structure consisting of X, the ω_1 copies of ω_1, and a countable neighbourhood assignment for each point. Let M and λ be given by the Lemma 4.3. By the following Proposition, the forcing does not introduce any new neighbourhoods of any of the copies of ω_1 in the subspace $X \cap M$. Therefore, if we collapse each of the copies of ω_1 to a point, the quotient space of $X \cap M$ is a subspace of the quotient space of X (in $V[G]$ under the same collapse). Furthermore, since X is normal the closed discrete set in the quotient space corresponding to the discrete family of copies of ω_1 is normalized. Then Proposition 4.13 finishes the proof. $\square$

Proposition 4.12 (Fleissner)**.** *If ω_1 is embedded in a first countable space, then the collection of ground model neighbourhoods of ω_1 remains a local base for ω_1 after any countably closed forcing.*

Proof. Let $U(\alpha, n)$ for $n \in \omega$ be a neighbourhood base for x_α in X. Let P be a countably closed forcing and let $\dot{f}$ be a P-name of a function from ω_1 into ω. Assume that for each $g \in {}^{\omega_1}\omega$ from V, there is a p and an α such that $p \Vdash U(\alpha, g(\alpha)) - \bigcup_\beta U(\beta, \dot{f}(\beta)) \neq \emptyset$. Let M be a countable elementary submodel and let $\delta = M \cap \omega_1$. Suppose that $p \in P \cap M$ and $n \in \omega$. Claim: there is a $x \in M$ and a $q < p$ in M such that $q \Vdash x_n \in U(\delta, n) - \bigcup_{\beta<\alpha} U(\beta, \dot{f}(\beta))$. Suppose otherwise, then there is a $\gamma \in M$ (i.e. determined by $[\gamma, \delta] \subset U(\delta, n)$) such that for all $\alpha > \gamma$ there is an m such that $p \Vdash U(\alpha, m) \subset \bigcup_\beta U(\beta, \dot{f}(\beta))$; i.e. for each such α in M we can take m to be the minimum one such that $U(\alpha, m) \subset U(\delta, n)$. This would determine a ground model g.

Therefore by a countable induction we can choose a descending sequence of conditions p_n and a sequence x_n converging to δ such that $p_n \Vdash x_n \notin \bigcup_\beta U(\beta, \dot{f}(\beta))$ – an obvious contradiction. $\square$

Proposition 4.13 (Tall). *Let X is a space and let D be a closed discrete set of cardinality ω_1. If D is not separated, then D is not normalized in $V[G]$, where G is $Fn(\omega_1, 2, \omega_1)$-generic.*

The Lemma 4.3 also gives us the following factoring lemma.

Lemma 4.14 (CH). *Suppose that $P = \{P_\alpha, Q_\alpha : \alpha \in I\}$ (I a set of ordinals) is a finite support iteration of ccc posets of size $\aleph_1$, and $M^\omega \subset M$ is an elementary submodel of $\langle H_\theta, P, I, \in\rangle$. Then P factors as $P_I * Q_I$, where P_I is the iteration of the Q_α for $\alpha \in I \cap M$ and Q_I (in V^{P_I}) is the iteration of those Q_α for $\alpha \in I - M$.*

Proof. We may suppose that each Q_α is simply an ordering on ω_1. Therefore we can replace P by its dense set of elements which are functions into ω_1 and whose domain is a member of $[I]^{<\omega}$. This is also true for $P' = P_I * Q_I$ but for the sake of clarity let us represent P' differently. Let $I_0 = I \cap M$ and let $I_1 = I - I_0$. Hence members of P_I are finite functions from I_0 and, in V^{P_I}, members of Q_I are finite functions from I_1 into ω_1. So P' has a dense set of conditions of the form (p, q) where $p \in P_I$ and $q \in V$ is a finite function from I_1 into ω_1. Again, replace P' by this dense set of conditions. To finish the proof, we just check that $(p,q) MApstop \cup q$ is an isomorphism between P' and P. An easy induction on $\alpha \in I$ shows that

$$(p \restriction \alpha, q \restriction \alpha) <_{P'} (p' \restriction \alpha, q' \restriction \alpha) \text{ iff } (p \cup q) \restriction \alpha <_P (p' \cup q') \restriction \alpha$$

which completes the proof. $\square$

Since many properties (those obtained by P_I) are preserved by finite support iterations of $\aleph_1$-sized posets the above result has many applications.

Corollary 4.15. *If we have an iteration P as in the previous theorem which adds an ω_2-tower or an unbounded family of size ω_2 in ${}^\omega\omega$ or an (ω_2, λ)-gap in $\mathcal{P}(\omega)/fin$ etc., then these objects keep these properties in the final model.*

5. Supercompact

In this section we will discuss Balogh's important result that if supercompact Cohen or Random reals are added then in the resulting model locally compact normal spaces are collectionwise normal.

Recall that an elementary embedding $j\colon V \mapsto W$ is called a λ-supercompact embedding with critical point κ if $\kappa \le \lambda < j(\kappa)$, j is the identity on V_κ and W is closed under λ-sequences. Normally the inner model W is denoted by M but we intend to use M in the role it played in §4, also the j we introduced there was modelled after the j given here.

Another way to view this is that the model W has a very special submodel. That is, choose some large θ (and then choose λ much larger). Then $M = \{j(x) : x \in H_\theta\}$ is an elementary submodel of the interpretation of $H_{j(\theta)}$ in W. There are two very important properties of M. In the model W, $M^{<\theta} \subset M$ and the Mostowski collapse of M, $\widehat{M}$, is just equal to the H_θ of V. This is exactly the situation we described at the beginning of section 4. That is, we have a model $M \prec H_{j(\theta)}$ (of W) whose Mostowskii collapse, H_θ (that lives in a different "universe" – V) and the elementary embedding j from the Mostowskii collapse back to M. The function e from Section 4 corresponds to recursively applying j" (except to ordinals). The additional thing we have now is that $\theta < j(\kappa)$ where κ is fixed (the supercompact) and λ can be chosen, in advance, to be as large as we like.

The first thing to realize is that, since V is elementarily equivalent to W (via j), anything we prove about $j(X)$ in W, is also true of X in V. Then we use the above setup and results from § 4 to prove things like $j(X)$ has a small substructures sharing its properties – e.g. by all the elementarity and closure available $j(X) \cap M = j\text{"}X$ – therefore X has such substructures.

The new idea of Balogh is to be make use of more general kinds of substructures than simply subspace or subset with a weaker topology. That is, when we have a space $(X, \mathcal{T})$ which is a member of an elementary submodel M we frequently take $X_M = M \cap X$ as a set of points and consider the possibly different topologies on X – that inherited as a subspace of X and that given by $M \cap \mathcal{T}$. Perhaps in hindsight, it is not so surprising that one should consider a quotient space.

We do not expect to be able to find a small substructure on a subspace since the character can be large. However, there is no real reason to insist on keeping the same set of points. Instead we view a space as a collection of neighbourhood filters hence a substructure would be a smaller collection of smaller neighbourhood filters.

In case the space is zero-dimensional a natural object to take is a reflected subalgebra of the Boolean algebra of clopen sets. In case the space is Tychonoff a similar object is to take an embedding of the space into $[0,1]^\Lambda$ as in Tychonoff's theorem and to then consider the projection of X (as a subspace) into the space $[0,1]^{M\cap\Lambda}$. This turns out to be very fruitful. This idea was also discussed by E. Bandlow during the Symposium.

Let us explore the approach a little. If X is a zero-dimensional space then $CO(X)$ is an algebraic object. Stone's duality makes it easy to understand the idea of a substructure — a subset of the topology (i.e. a subalgebra) corresponds canonically to a continuous image of X. More generally we could look at $C^*(X)$ – the ring of continuous functions (or even any embedding of X into $[0,1]^\Lambda$ for some index set Λ). With this approach, we can regard a substructure as a subring of continuous functions which will yield a projection function, π, from $[0,1]^\Lambda$ onto $[0,1]^{M\cap\Lambda}$. There is also a well-developed duality between compact spaces and rings. A compact space X is equal to the space of maximal ideals of $C^*(X)$ (in general, closed subsets of X are determined by ideals, namely all functions which are constantly zero on the set). Note that $\pi(X)$ is almost always disjoint from M since the members of $\pi(X)$ are functions with domain $M \cap \Lambda$ which is not usually in M. However we can think of $x \in X \cap M$ as a representative of $\pi(x)$ – but it is even still likely that $\pi(X) - \{\pi(x) : x \in X \cap M\}$ is non-empty (even large). For this reason we are better off keeping our focus on $C^*(X) \cap M$.

Definition 5.1. Given a Tychonoff space X, in an elementary submodel M, let $\pi_{X,M}$ (or just π if X, M are clear from the context) denote the projection map from $[0,1]^{C^*(X)}$ onto $[0,1]^{M\cap C^*(X)}$. In this context, we will identify X with its homeomorphic image in $[0,1]^{C^*(X)}$ via the evaluation map. If X is locally compact, let X_M denote the space $\pi(\overline{X}) - \pi(\overline{X} - X)$. That is, since $\overline{X} = \beta X$, we are letting $X_M = \pi(X) - \pi(\beta X - X)$.

Now we consider the problem of separating normalized collections of closed sets in locally compact spaces. We need some simple topological properties of quotient maps on locally compact spaces. An idea first used by Tall is that one can recognize when a family is discrete in a locally compact space just by looking at countable subcollections.

Lemma 5.2. *Let X be a locally compact space and let $\mathcal{C}$ be a discrete collection of subsets of X. Suppose that $X, \mathcal{C}, C^*(X)$ are in an elementary submodel M such that every countable subset of M is a member of some countable element of M. Then the collection $\mathcal{C}_M = \{\pi(C) : C \in M \cap \mathcal{C}\}$ is discrete in X_M.*

Proof. Suppose that the collection is not discrete. Let C_M denote $\pi(C)$ for each $C \in M \cap \mathcal{C}$. If it is not discrete then there is a compact set $F \subset X_M$ which meets C_M for infinitely many $C \in \mathcal{C} \cap M$. Since every such countable set is contained in some countable element of M there is a countable $\mathcal{C}' \subset \mathcal{C}$ which is an element of M such that F meets infinitely many members of $\{C_M : C \in \mathcal{C}'\}$. Since $\mathcal{C}$ in discrete in X, there is a function $f \in C^*(X)$ such that $\{f(C) : C \in \mathcal{C}'\}$ is a closed discrete subset of $(0,1]$ and $f(x) > 0$ for all $x \in X$. By elementarity, there is such a function in M. Obviously, $f_M = f \circ \pi^{-1} \in C^*(X_M)$ for all $f \in C^*(X) \cap M$ (indeed, f and $f \circ \pi^{-1}$ are just restrictions of the projection function at the f-th coordinate in $[0,1]^{C^*(X)}$ and $[0,1]^{M \cap C^*(X)}$). Since $f_M(\pi(C)) = f(C)$ for $C \in \mathcal{C}'$ and $f_M(X_M) \subset f(X) \subset (0,1]$, it follows that $\mathcal{C}'$ is discrete in X_M. $\square$

Proposition 5.3. *Let X be a normal locally compact space. Let π be a continuous function of βX onto a space Y. Let $X_M = Y - \pi(\beta X - X)$.*

1. *If $A \subset X_M$ and U is a neighbourhood of $\pi^{-1}(A)$ in X, then $\pi(U)$ is a neighbourhood of A in X_M.*
2. *If $A, B \subset X_M$ are closed and are such that $\pi^{-1}(A)$ and $\pi^{-1}(B)$ have disjoint closures in X then A and B are separated in X_M.*

Proof.
1. Let F be the compact set $\beta X - U$. Therefore $\pi(F)$ is a compact subset of Y which is disjoint from A – hence A is contained in the open set $Y - \pi(F) \subset \pi(U)$.
2. Since X is normal, choose disjoint neighbourhoods U and V of the closure (in X) of $\pi^{-1}(A)$ and $\pi^{-1}(B)$. By part (i), $\pi(U)$ is a neighbourhood of A. Furthermore, the closure of $\pi(V)$ is contained in the image of the closure of V in βX (which is disjoint from A). Therefore $\pi(U) - \overline{\pi(V)}$ and $\pi(V)$ form the required separation of A and B.

$\square$

Corollary 5.4. *Let X be a normal locally compact space and let $\mathcal{C}$ be a discrete collection of subsets of X. Suppose that $X, \mathcal{C}, C^*(X)$ are in an elementary submodel M such that every countable subset of M is a member of some countable element of M. Then the family $\{\pi(C) : C \in \mathcal{C} \cap M\}$ is normalized in X_M.*

Proof. By Lemma 5.2, the family $\{\pi(C) : C \in \mathcal{C} \cap M\}$ is discrete. By similar reasoning as in 5.2, we can check that the family $\{\pi^{-1}(\pi(C)) : C \in \mathcal{C} \cap M\}$ is also discrete in X. Indeed, if $\mathcal{C}'$ is a countable subset of $\mathcal{C} \cap M$ then we may assume that $\mathcal{C}' \in M$ and choose a function $f \in C^*(X) \cap M$ which witnesses the discreteness of $\mathcal{C}$. Since $f \in M$, it follows that $f(\pi^{-1}(\pi(x))) = f(x)$ for all $x \in X$. Therefore, f witnesses the discreteness of $\{\pi^{-1}(\pi(C)) : C \in \mathcal{C}'\}$. Therefore if $\mathcal{C}' \subset \mathcal{C} \cap M$ and $A = \bigcup\{\pi(C) : C \in \mathcal{C}'\}$, it follows that $\pi^{-1}(A)$ is closed in X. The result now follows from 5.3. $\square$

Let κ be supercompact and let P be the forcing for adding either κ many Cohen reals or κ many random reals. Fix P-names of X, $\mathcal{C}$ and $C^*(X)$ such that p forces that X is a locally compact space and that $\mathcal{C}$ is a discrete normalized collection. We will show that p does not force that $\mathcal{C}$ is unseparated (and since p was arbitrary, p forces that $\mathcal{C}$ is separated). Let θ be a cardinal much larger than the cardinality of the transitive collapse of (the names of) X and $C^*(X)$. Since θ is so large, it suffices to show that for some generic filter G with $p \in G$, $H_\theta[G] \models \mathcal{C}$ is separated.

Assume, for convenience of notation, that $[H_\theta]^{<\theta} \subset H_\theta$ and let j be a λ-supercompact embedding for κ into a model W with $\lambda > |H_\theta|$. As above let M be the image of H_θ under j. Now in W, let G be a $j(P)$-generic filter such that $j(p) \in G$. Naturally $j(P)$ is just the forcing for adding $j(\kappa)$ many Cohen reals. Therefore $j(P) \cap M$ is completely embedded in $j(P)$ (hence G is $j(P)$-generic over M) and $j(P)/(j(P) \cap M)$ is isomorphic to $j(P)$. Furthermore, $j(P) \cap M$ is taken to P by the Mostowskii collapsing function and since $M^\omega \subset M$ (it is easily checked that) $\widehat{G} = \widehat{G \cap M}$ is P-generic over V (so is $G \cap H_\kappa$ but these are different generic filters).

¿From our results of § 4 we have.

Proposition 5.5. *A summary of observations:*

1. *In $W[G]$, $M[G]$ is an elementary submodel whose Mostowskii collapse is $\breve{H}_\theta[\widehat{G}]$.*
2. *$\breve{H}_\theta[\widehat{G}]$ is a submodel of $W[G \cap M]$.*
3. *In $W[G \cap M]$, $\breve{H}_\theta[\widehat{G}]$ is closed under $<\theta$-sequences.*
4. *In $W[G \cap M]$, $C^*(j(X)) \cap M$ (also taking into account the intersection of M with the ideal giving $\beta X - X$) corresponds to the subspace of $[0,1]^{M \cap C^*(j(X))}$ which is equal to $j(X)_M \cap W[G \cap M]$.*
5. *Both X and $\mathcal{C}$ are in the model $\breve{H}_\theta[\widehat{G}]$.*
6. *The image of $C^*(j(X)) \in M$ under the Mostowskii collapse is $C^*(X)$.*
7. *The image of $j(\mathcal{C})$ by the Mostowskii collapse is $\mathcal{C}$ in the sense that, for each $C \in \mathcal{C}$, the ideal $\{f \in C^*(j(X)) : f(j(C)) = 0\}$ is sent to $\{f \in C^*(X) : f(C) = 0\}$.*

Condition (iv) is not as complicated as it looks. For example, notice that even if $j(X)$ is just $[0,1]^\omega$ then $j(X)_M = j(X)$. Then $j(X)_M \cap W[G \cap M]$ will indeed be $[0,1]^\omega$ but as determined in the inner model, more reals will be added.

Corollary 5.6. *In $W[G \cap M]$, the space $j(X)_M \cap W[G \cap M]$ has a discrete collection $\{\pi(j(C)) \cap W[G \cap M] : C \in \mathcal{C}\}$ which is separated in $j(X)_M \cap W[G \cap M]$ if and only if $\mathcal{C}$ is separated in X.*

As discussed above, the final ingredient is taken from the Normal Moore Space investigations (see Fleissner [DTW90, Tal84, Fle84]).

Proposition 5.7. *The family $\{\pi(j(C)) \cap W[G \cap M] : C \in \mathcal{C}\}$ is separated in $j(X)_M \cap W[G \cap M]$ if and only if it is normalized after adding (enough) Cohen reals.*

Therefore, to finish the proof of Balogh's theorem, it now suffices to prove that $\{\pi(j(C)) : C \in \mathcal{C}\}$ is normalized in $W[G]$. But this follows immediately from Proposition 5.2 and Corollary 5.4.

6. PFA and OCA

Inspired by Todorčević's book [Tod89], we take Ramsey-theoretic approach to the proper forcing axiom, PFA. Consider the following Ramsey-type axiom: Suppose that $[X]^2 = K_0 \cup K_1$, then either X is the union of countably many 1-homogeneous sets (i.e. $X = \cup_n X_n$ such that $[X_n]^2 \subset K_1$) or there is an uncountable 0-homogeneous set. We know that this *axiom* is false as it stands but it is a highly informative prototype. That is, if one sees an application of it then there is a natural poset (described below) which forces the axiom to hold for the specific case (where "uncountable" means size $\breve{\omega}_1$). If the partition is reasonably nice then the poset (or some modification of condition 4) is proper and we have a consequence of PFA. Of course, OCA is the (consistent) statement(Corollary 6.3) obtained by restricting to the case when X is a set of reals and K_0 is an open set in the upper half plane.

Definition 6.1. Given a partition $[X]^2 = K_0 \cup K_1$, we define $p \in P$ if there are X_p and $\mathcal{M}_p$ such that

1. X_p is a finite 0-homogeneous subset of X,
2. $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of some H_θ (for a suitably large θ),
3. X_p is separated (ordered) by $\mathcal{M}_p$, i.e. for $x \neq y \in X_p$, either x comes before y in the sense that there is an $M \in \mathcal{M}_p$ such that either $x \in M$ and $y \notin M$ or vice versa,
4. for each $M \in \mathcal{M}_p$ and $x \in X_p - M$, x is not in any 1-homogenous set which is a member of M.

Proposition 6.2. *If X is a set of reals and K_0 is an open subset of X^2, then either X is a countable union of 1-homogeneous sets or the poset of Definition 6.1 is proper.*

Proof. Let P be a member of M (a countable elementary submodel of some large H_λ) and fix any $p \in P$. We show that $p' = (X_p, \{M \cap H_\theta\} \cup \mathcal{M}_p)$ is the desired M-generic condition. Fix any $D \in M$ which is a dense subset of P and choose $q \in D$ which is below p'. Since $X_q = \{z_1, \ldots, z_m, x_1, \ldots, x_n\}$ is ordered by $\mathcal{M}_q$, $\{M_0, \ldots, M_n\} \supseteq \mathcal{M}_q$ so that $M_0 = M \cap H_\theta$, $\{z_1, \ldots, z_m\} \subset M_0$, and for $i < j \leq n$, $x_j \in M_j - M_i$. Since K_0 is open there are rational intervals B_i for $i = 1, \ldots, n$ such that $\{z_1, \ldots, z_m\} \times \cup_{i \leq n} B_i \subset K_0$ and for $0 < i < j \leq n$ $B_i \times B_j \subset K_0$. We will inductively choose, for $i = 1, \ldots, n$, $y_i \in M_0 \cap B_i$ so that $\{y_n, \ldots, y_m\} \cup \{x_1 \ldots, x_m\}$ is 0-homogeneous and so that there is a $q' \in D \cap M_n$ such that $X_{q'} = \{z_1, \ldots, z_m\} \cup \{y_1, \ldots, y_m\}$. Let us first recall that simply by choosing $y_i \in B_i \cap M_n$ we obtain that $\{y_i, x_j\} \in K_0$ for all $i \neq j \leq m$. This is unusually simple but we encounter the more typical difficulties with the $i = j$ case – here is where we use the fact that x_i is not in any 1-homogeneous set which is a member of M_{i-1}. Notice, for example, that for each open rational interval B the set

$$\{x \in B : \{x\} \times B \subset K_1\}$$

is a 1-homogeneous set which is a member of $M_0 = M \cap H_\theta$ – hence does not contain x_i. Therefore, there are plenty of y_i in B_i such that $\{y_i, x_i\} \in K_0$, but of course we need enough so that we can get q' to be in D.

Here is a standard technique for dealing with this. Although, D is not in H_θ the set

$$\{X_r - \{z_1, \ldots, z_m\} : r \in D \text{ and } \{z_1, \ldots, z_m\} \subset X_r\} \text{ is.}$$

We first consider only

$$t \in \{X_r - \{z_1, \ldots, z_m\} : r \in D \text{ and } \{z_1, \ldots, z_m\} \subset X_r\}$$

which have size n and we consider them to be n-tuples in $\Pi_{i=1}^n B_i$. Then we let T' be the tree of all $\leq n$-tuples generated from this set. Then we prune T' to a tree T such that the non-maximal (in T') nodes of T have a "large" number of immediate successors. The notion of large we use should for an i-tuple t, should guarantee that there is a y such that $\{y, x_{i+1}\} \in K_0$ and $t \cup \{y\} \in T'$ and (of course) if $t \cup \{y\}$ is not maximal it also should have a large number of immediate successors. It will also be important that the tree T is a member of M_0 (note that D itself is not).

For $i = 1, \ldots, n$, let $t_i = \{x_j : j < i\}$. Now $t_n \in M_n$ and with $t = t_n$, $T'_t = \{y : t \cup \{y\} \in T'\}$ is a member of M_n. Furthermore,

$$\{x \in T_t : \{x\} \times T_t - \{x\} \subset K_1\}$$

is a 1-homogeneous set and so does not contain x_n. Therefore, there is a $w \in T_t - \{x_n\} \subset B_n$ such that $\{x_n, w\} \in K_0$. Choose basic open $B'_n, C_n \subset B_n$ such that $x_n \in B'_n$, $w \in C_n$ and $B'_n \times C_n \subset K_0$. Let T^n be the subtree of T' obtained by taking the downward closure of those nodes whose n^{th} point is in C_n. Note that $t = t_{n-1} \in T^n$. Also

$$\{x \in T^n_t : \{x\} \times T^n_t - \{x\} \subset K_1\}$$

is a 1-homogeneous set which is a member of M_{n-1}. Therefore we can choose basic open $B'_{n-1}, C_{n-1} \subset B_{n-1}$ such that

$$x_{n-1} \in B'_{n-1} \quad B'_{n-1} \times C_{n-1} \subset K_0 \quad \text{and } T^n_{t_{n-1}} \cap C_{n-1} \neq \emptyset\ .$$

Then let $T^{n-1} = \{t \in T^n : (\exists s \in T^n_t)\ s(n-1) \in C_{n-1}\}$. In this way we define $C_n, \ldots, C_1$ and $T^n, \ldots, T^1$. The desired tree T is T^1. It turns out that finding the tree T has already done all the work – there is no reason to inductively choose $y_1, \ldots, y_n$ since any member of T will work. First of all, T is not empty since, by induction, $t_i \in T^i$ – hence $\emptyset = t_1 \in T$.

To check that this works, let $r \in D$ be such that $\{z_1, \ldots, z_m\} \subset X_r$ and $X_r - \{z_1, \ldots, z_m\} = \{y_1, \ldots, y_n\} \in T$. By definition of T^i, it follows that $y_i \in C_i \subset B_i$. It is easily checked that $X_r \cup X_q$ is 0-homogeneous. □

Corollary 6.3 ((PFA) implies (OCA)). *Assume PFA. Suppose that X is a separable metric space and $[X]^2 = K_0 \cup K_1$ with K_0 open in X^2. Then either there is an uncountable K_0-homogeneous set or X is a countable union of K_1-homogenous sets.*

Here is a well-known application of OCA. Recall that Kunen proved that there are no (c,c)-gaps in $[\omega]^\omega$ if PFA holds ([Bau84]). A (λ,μ)-gap is a pair of chains of subsets of $[\omega]^\omega$, $(\{a_\alpha : \alpha < \lambda\}, \{b_\beta : \beta < \mu\}$, such that each a_α is almost disjoint from each b_β and with the property that if $a \,{}^*\!\supset a_\alpha$ for all $\alpha < \lambda$, then there are $\beta < \mu$ such that $a \cap b_\beta$ is infinite.

Proposition 6.4 (OCA). *If $\omega_1 \leq \lambda < \mu$, then there are no (λ,μ)-gaps in $[\omega]^\omega$.*

Proof. Suppose that $(\{a_\alpha : \alpha < \lambda\}, \{b_\beta : \beta < \mu\})$, is a gap. Let S be the set of $(a,b) \in \left([\omega]^\omega\right)^2$ such that $a \cap b = \emptyset$ and there are $\alpha < \lambda$ and $\beta < \mu$ such that $a - a_\alpha$ and $b - b_\beta$ are both finite. Define a partition $K_0 \cup K_1$ of S by

$$\langle (a,b),(c,d)\rangle \in K_0 \text{ iff } (a \cup c) \cap (b \cup d) \text{ is not empty}\ .$$

It is easily checked that if S is identified (canonically) with a subspace of ω^ω, then K_0 is relatively open. Furthermore an uncountable 0-homogeneous set, $\{(c_\alpha, d_\alpha) : \alpha < \omega_1\}$, results in a Hausdorff-Lusin style gap. That is, suppose that $A \subset \omega$ is such that $c_\alpha - A$ and $d_\alpha \cap A$ are finite for all $\alpha < \omega_1$. Fix $n \in \omega$ and uncountable $I \subset \omega_1$ such that $c_\alpha - A$ and $d_\alpha \cap A$ are contained in n for all $\alpha \in I$. By further refining I, we may suppose that $d_\alpha \cap n$ is the same for all $\alpha \in I$. But now it follows that $c_\alpha \cap d_\beta$ is empty for any α and β in I – contradicting 0-homogeneity. But this contradicts the existence of $A = \omega - b_\xi$ where $\xi < \mu$ is chosen such that $d_\alpha < b_\xi$ for all $\alpha < \omega_1$.

Therefore it suffices to show that S cannot be expressed as a countable union of 1-homogeneous sets. Here is a nice argument for this. Suppose that $S = \cup_n J_n$ and $[J_n]^2 \subset K_1$ for each n. Let M be a countable elementary submodel, with

$\{J_n : n \in \omega\}, S$ in M, of some large H_θ. Let α_0 and β_0 be the supremum of $M \cap \lambda$ and $M \cap \mu$ respectively. Let $a = a_{\alpha_0}$ and $b = b_{\beta_0} - a$. Fix n such that $(a, b) \in J_n$. Then, by virtue of the existence of (a, b),

$$M \models (\forall \alpha \in \lambda)\ (\forall \beta \in \mu)\ (\exists (a, b) \in J_n) \text{ such that } a_\alpha \subset^* a \text{ and } b_\beta \subset^* b\ .$$

Therefore,

$$A = \bigcup \{a' : (\exists b')\ (a', b') \in J_n\}$$

contains, mod finite, every a_α, and

$$B = \bigcup \{b' : (\exists a')\ (a', b') \in J_n\}$$

contains, mod finite, every b_β. Furthermore, A and B are disjoint since $[J_n]^2 \subset K_1$. $\square$

A recent very important result using this general technique (in hindsight(?)) is Balogh's solution of the Moore-Mrowka problem (see [Juh92, 3.1]). It depends on showing that countably compact regular spaces of countable tightness contain uncountable free sequences. Let us check if the Ramsey axiom is relevant.

Suppose that X is a regular space and that $\mathcal{F}$ is a countably complete maximal filter of closed subsets of X. Fix, for each point $x \in X$, a pair, W, U of neighbourhoods of x such that $\overline{W} \subset U$ and $X - U \in \mathcal{F}$.

Define a partition of $[X]^2$ as follows:

$$\{x, y\} \in K_0 \text{ iff } W_x \cap W_y \text{ contains exactly one of } \{x, y\}\ .$$

Claim: If some member of $\mathcal{F}$ is a countable union of 1-homogeneous sets, then X contains an uncountable closed discrete set.

Proof. Suppose that $\{I_n : n \in \omega\}$ are 1-homogeneous sets such that $\cup I_n \in \mathcal{F}$. For each n choose, by the maximality of $\mathcal{F}$, $F_n \in \mathcal{F}$ such that either $F_n \cap I_n' = \emptyset$ or $F_n \subset I_n'$. Let $F = \cap F_n \in \mathcal{F}$. Pick n such that $I_n \in \mathcal{F}^+$ and inductively choose $F_\alpha \in \mathcal{F}$, $x_\alpha \in I_n \in F_\alpha$, and and an increasing chain of countable elementary submodels M_α for $\alpha < \omega_1$ as follows. Having defined $\{F_\beta, x_\beta, M_\beta : \beta < \alpha\}$, let M_α be a countable elementary submodel containing this sequence together with $X, \mathcal{F}$. Let $F_\alpha = \bigcap \{F : F \in \mathcal{F} \cap M_\alpha\}$. Pick any $x_\alpha \in F_\alpha \cap I_n$.

By countable tightness, it suffices to show that $\{x_\beta : \beta < \alpha\}$ is closed. Suppose that $x \in F$ is a limit point and fix the minimum $\beta < \alpha$ such that $x_\beta \in W_x$. Choose m such that $x \in I_m$. Since $x \in F_1$ and $\mathcal{F}$ is countably complete it follows that $I_m \cap F$ is uncountable. If $x \notin I_m'$, then $I_m' \cap F = \emptyset$ and $I_m \cap F$ would be an uncountable closed discrete set. Therefore, $F \subset I_m'$, in particular, $x_\beta \in I_m'$. By countable tightness, there is a $y \in M_\beta \cap I_m$ such that $y \in W_x \cap W_{x_\beta}$. By the minimality of β, x is in $\overline{\{x_\gamma : \beta < \gamma < \alpha\}} \subset F_{\beta+1}$. Since $X - W_y \in \mathcal{F} \cap M_{\beta+1}$, $F_{\beta+1} \cap W_y = \emptyset$, and it follows that $y \in W_x \cap W_y$, but $x \notin W_y$. But this means that $\{x, y\} \in K_0$ contradicting that $x, y \in I_m$. $\square$

Similarly, suppose that X contains a 0-homogeneous set, I, which meets every member of $\mathcal{F}$. Inductively choose, $\{M_\alpha, F_\alpha\}$ and $x_\alpha \in F_\alpha \cap I$ for $\alpha < \omega_1$ just as above. By the construction $x_\alpha \notin W_{x_\beta}$ for $\beta < \alpha$. Since I is 0-homogenous, it follows that $\{x_\beta : \beta < \alpha\} \subset W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$. Furthermore, $F_{\alpha+1} \cap U_\alpha = \emptyset$ and $\{x_\gamma : \gamma > \alpha\} \subset F_{\alpha+1}$. Therefore $\{x_{\alpha+1} : \alpha < \omega_1\}$ is a free sequence.

If X is not a countable union of 1-homogeneous sets, then we can try the prototype poset P as given in Definition 6.1 for adding an uncountable 0-homogeneous set. Since this uncountable set will be added generically, we could even ensure that, if X had character at most $\aleph_1$, then the sequence $\{x_\alpha : \alpha < \omega_1\}$ would be a topological copy of ω_1. To do so we would simply have to include side conditions consisting of a finite sequence of neighborhoods of the members of X_p and restrict the definition of $<_P$ in a rather obvious manner. There is no essential change to the following proof in order to obtain the stronger result.

Theorem 6.5 (PFA). *Let X be a regular space. Suppose that Y is a subspace which has countable tightness and that $\mathcal{F}$ is a free countably complete maximal filter of closed subsets of Y. Furthermore, let f be any function with domain Y such that $\overline{f^{-1}(f(y))} \notin \mathcal{F}$ for all $y \in Y$. Suppose that Y has no infinite closed discrete subspace on which f is one-to-one. Then X contains, $\{x_\alpha : \alpha < \omega_1\}$, an uncountable free sequence which is a subset of Y (and on which, f is one-to-one). In addition, if $\{W_y : y \in Y\}$ is any neighbourhood assignment, then $\{x_\alpha : \alpha \in \omega_1\}$ can be chosen so that for each $\alpha < \omega_1$, there is a $\beta < \alpha$ such that $\{x_\gamma : \beta < \gamma \leq \alpha\} \subset W_{x_\alpha}$.*

Proof. For each $y \in Y$, fix an X-neighbourhood W_y and $F_y \in \mathcal{F}$ so that $\overline{W_y} \cap Y - U_y = \emptyset$ (i.e. $U_y = X - \overline{F_y}$ in the discussion above). We make two changes to the definition of P from definition 6.1. Replace conditions 4 and 1 by

1' X_p is a finite subset of Y,

4' If $M \in \mathcal{M}_p$ and x is the minimal point of $X_p - M$ (i.e. $M \in M' \in \mathcal{M}_p$ implies $x \in M'$), then $x \in \overline{F \cap M}$ for all $F \in \mathcal{F} \cap M$.

Also the elementary submodels in $\mathcal{M}_p$ should include X with its topology, Y, $\mathcal{F}$ and the functions f, $\{\langle y, (W_y, F_y)\rangle : y \in Y\}$.

We define $p < q$ if $X_p \supset X_q$, $\mathcal{M}_p \supset \mathcal{M}_q$, and for each $M \in \mathcal{M}_q$ if y is the minimal point of $X_q - M$, then for each $y' \in X_q$ such that $y \in W'_y$

$$X_p \cap M \subset W_{y'} \cup \bigcup \{W_z : z \in X_q \cap M\} \ .$$

This condition is slightly weaker than demanding that X_p is a 0-homogeneous set where K_0 is defined as in the discussion above. It is easier to view it as an approximation to a copy of ω_1. Indeed, given y as above, let z be the largest element of $M \cap X_q$. Then $W' = \cap \{W_{y'} : y \in W_{y'} \text{ and } y' \in X_q\}$ is a neighbourhood of y and q will force that W' will contain all members of X_p which are between z and y (with $p < q$). Furthermore, since the generic uncountable set will have a natural well-ordering in order type ω_1, this will guarantee that the existence of a stationary 0-homogenous set can be deduced from the pressing down lemma.

We now show that P is proper and that if Y has no infinite closed discrete subspace on which f is one-to-one then $\cup\{X_p : p \in G\}$ is uncountable for any P-generic filter G.

Suppose that $p \in P \cap M$ for some countable elementary submodel $M \ni P$ of a sufficiently large H_λ. Define $p_M = \langle X_p, \mathcal{M}_p \cup (M \cap H_\theta)\rangle$.

Fix any dense open subset $D \subset P$ with $D \in M$ and choose $q \leq p_M$ with $q \in D$. Let $n = |X_q - M|$ and $X'_q = X_q \cap M$. Let T' be the set of n element sets, t, for which there is an $r \leq p_M$ in D with $t = X_r - X'_q$. Just as in the proof of Proposition 6.2, T' is in $M \cap H_\theta$ although D is not.

First suppose that $n = 0$. Let $\mathcal{M}'_q = \mathcal{M}_q \cap M \in M$. Choose, by elementarity, any $r \in D \cap M$ such tha$X_r = X'_q$ and $\mathcal{M}'_q \subset M'$ for each $M' \in \mathcal{M}_r - \mathcal{M}'_q$. Such an r will be a member of $D \cap M$ which is compatible with q - as required.

Now suppose that $n > 0$. Choose an increasing sequence $\{M_1, \ldots, M_{n-1}\} \subset \mathcal{M}_q$ (with $M_1 = M \cap H_\theta$) to witness that $X_q - M$ is separated. And let $X_q = \{x_1, \ldots, x_n\}$ be enumerated such that $x_i \notin M_i$. The main difficulty in finding an $r \in D \cap M$ which is compatible with q is in ensuring that $X_r - X'_q \subset W'$, where $W' = \cap \{W_{y'} : y \in W_{y'} \text{ and } y' \in X_q\}$. Here is where we use condition 4'.

As in proposition 6.2, we prune T' to a subset T so that, in this case, for each non-maximal $t \in T$,

$$\overline{\{y \in Y : t \cup \{y\} \in T\}} \in \mathcal{F}$$

This is done by a reverse induction on the cardinality of $t \in T'$. Again, by induction, we show that $\{x_1, \dots, x_k\}$ remains in the $T^k \in M$ which is obtained after $n-k$ steps. Hence, after n steps we have our set T with $\emptyset \in T = T^0$ (in this case $\{x_1, \dots, x_n\}$ is also in T).

Suppose that we have T^{k+1} (where $T^n = T'$) and that $\{x_1, \dots, x_k\} \in T^{k+1}$. Let

$$\begin{aligned} T^k = \{t \in T^{k+1} :& \\ &(\exists s)\ (s \cup t \in T^{k+1},\ |s| = k \text{ and } \overline{\{y \in Y : s \cup \{y\} \in T^{k+1}\}} \in \mathcal{F}\}. \end{aligned}$$

To see that $\{x_1, \dots, x_{k-1}\}$ is in T^k, notice that

$$\overline{\{y \in Y : \{x_1, \dots, x_{k-1}\} \cup \{y\} \in T^{k+1}\}}$$

is in M_k and contains x_k. If it were not in $\mathcal{F}$ then, by the maximality of $\mathcal{F}$, there would be a member of $\mathcal{F} \cap M_k$ (containing x_k) which missed it.

Next, we work in M and choose inductively, $\{y_1, \dots, y_n\} \subset W' \cap M$ so that $\{y_1, \dots, y_k\}$ is in T for each k. To see that y_{k+1} can be chosen recall that

$$x_1 \in \overline{\overline{\{y \in Y : \{y_1, \dots, y_k\} \cup \{y\} \in T\}} \cap M}$$

and that, by countable tightness and elementarity,

$$\overline{\overline{\{y \in Y : \{y_1, \dots, y_k\} \cup \{y\} \in T\}} \cap M} = \overline{\{y \in Y : \{y_1, \dots, y_k\} \cup \{y\} \in T\} \cap M}\ .$$

Now suppose that Y does not contain an infinite closed set on which f is one-to-one. Fix any $\alpha \in \omega_1$ and define

$$D_\alpha = \{p \in P : (\exists M \in \mathcal{M}_p)\ (\alpha \in M \text{ and } \ X_p - M \neq \emptyset\}\ .$$

It suffices to show that D_α is dense. Fix any $p \in P$ and choose a countable elementary submodel $M \prec H_\lambda$ containing p and α. It is immediate that p_M (as defined above) is in P. Fix a descending base $\{F_n : n \in \omega\}$ for $\mathcal{F} \cap M$. Inductively choose $y_n \in F_n \cap M$, so that $y_n \notin \overline{f^{-1}(f[\{y_i : i < n\}])}$. Clearly f is one-to-one on $\{y_n : n \in \omega\}$, so pick a limit point x. Note that $x \in \overline{F_n \cap M}$ for each $n \in \omega$. Furthermore, $x \notin M$ since $M \cap \bigcap\{F_n : n \in \omega\}$ is empty. It follows, then that $(X_p \cup \{x\}, \mathcal{M} \cup \{M \cap H_\theta\})$ is in D_α and is below p. □

See [Juh92, 3.2,3.3] and [BDFN88], for other applications of the above theorem.

Corollary 6.6 (PFA). *If a locally compact space X maps onto ω_1 by a closed map, then X contains a perfect preimage of ω_1. If, in addition, the character of X is at most ω_1, then X contains a topological copy of ω_1.*

There are many applications of this method to the structure of $\mathcal{P}(\omega_1)$ and $\beta\omega_1$ (see [Tod89, Ch. 8] and [Dow88]). Here is a very strong failure of the principle ♣ which was announced in [Tod89a].

Theorem 6.7 (PFA). *If $\mathcal{I} \subset [\omega_1]^\omega$ is an $\aleph_1$-generated ideal then either*

(0) *there is an uncountable $S \subset \omega_1$ such that which is almost disjoint from each $a \in \mathcal{I}$, or*

(1) *there are S_n such that $\omega_1 = \bigcup_n S_n$ and $[S_n]^\omega \subset \mathcal{I}$ for each n.*

Proof. Assume that there is no family $\{S_n : n \in \omega\}$ as in (1); we will produce an S as in (0). Let $\mathcal{I}_0$ be a generating set for $\mathcal{I}$ of cardinality $\aleph_1$. Also, let $\widetilde{\mathcal{I}}$ denote the ideal $\{S \subset \omega_1 : [S]^\omega \subset \mathcal{I}\}$. Hence we are assuming that $\omega_1 - \cup\mathcal{S}$ is uncountable for any countable $\mathcal{S} \subset \widetilde{\mathcal{I}}$. The rough idea is to get an uncountable 0-homogeneous set where $(\alpha, \beta) \in K_0$ might mean something like α is not in the β^{th} member of $\mathcal{I}_0$.

Fix an appropriately large θ and define $p \in P$ iff

$$p = A_p \cup \mathcal{I}_p \cup \mathcal{M}_p \text{ where } \begin{cases} A_p \subset \omega_1 & \text{is finite,} \\ \mathcal{I}_p \subset \mathcal{I}_0 & \text{is finite} \\ \mathcal{M}_p & \text{is a finite } \epsilon\text{-chain separating } A_p, \\ \alpha \notin \bigcup(M \cap \widetilde{\mathcal{I}}) & \text{for any } M \in \mathcal{M}_p \text{ and } \alpha \in A_p - M, \\ \text{each } M \in \mathcal{M}_p & \text{is countable and } M \prec H_\theta \end{cases},$$

and $p < q$ if $p \supset q$ and $A_p - A_q$ is disjoint from $\cup\mathcal{I}_q$.

As usual, let $M \prec H_\kappa$ be countable, for a suitable κ and let $p \in M \cap P$. We show that $p \cup \{M \cap H_\theta\}$ is (M, P)-generic. Let $D \in M$ be a dense open subset of P and let $r \in D$ be less than $p \cup \{M \cap H_\theta\}$. Let $\{\alpha_0, \ldots, \alpha_{n-1}\}$ list $A_r - M$ in increasing order. Also, let $r_0 = r \cap M$ – note that $r_0 \in P$. As usual, we consider

$$\{A : (\exists q \in D)\ q < r_0\,, |A_q - A_{r_0}| = n \text{ and } A \subset A_q - A_{r_0}\}\ .$$

Next, we can show, by considering $\{\alpha_0, \ldots, \alpha_{n-1}\}$, that this has a definable subset T in $M \cap H_\theta$ such that for each $t \in T$ with $|t| < n$, the set $T_t = \{\xi : t \cup \{\xi\} \in T\}$ is not covered by countably many members of $\widetilde{\mathcal{I}}$.

Therefore, if we let $I = \cup\mathcal{I}_r$, we have, by elementarity, that $(M \cap T_t) - I$ is not empty for each $t \in T \cap M$. Since $\emptyset \in T \cap M$, we can recursively choose $\xi_i \in a$ so that $\{\xi_0, \ldots, \xi_{n-1}\}$ is in T. Finally by elementarity, there is a $q \in M \cap D$ so that $q < r_0$ and $A_q - A_{r_0} = \{\xi_0, \ldots, x_{n-1}\}$. It follows that q is compatible with r, completing the proof that P is proper. □

Recall that $U(\omega_1) \subset \beta\omega_1$ is the set of uniform ultrafilters on ω_1. Here is one interesting topological consequence of the previous result. It is also just a reformulation of a result in [Tod89] (see Proposition 6.9).

Corollary 6.8 (PFA). *If H and K are disjoint closed subsets of $\omega_1^* - U(\omega_1)$, then there is an uncountable set $I \subset \omega_1$ such that one of $I^* \cap H$ or $I^* \cap K$ is empty.*

Proof. For each $\alpha < \omega_1$, fix $a_\alpha \subset \alpha$ such that

$$a_\alpha^* \cap K = \emptyset \text{ and } H \cap \alpha^* \subset a_\alpha^*\ .$$

Let $\mathcal{I}$ be the ideal generated by $\{a_\alpha : \alpha < \omega_1\}$ and apply Theorem 6.7. □

Proposition 6.9. *The following are equivalent.*

1. *If X and Y are Lindelöf countably tight spaces each with a unique non-isolated point, then $X \times Y$ is countably tight.*
2. *If H and K are disjoint closed subsets of $\omega_1^* - U(\omega_1)$, then there is an uncountable set $I \subset \omega_1$ such that one of $I^* \cap H$ or $I^* \cap K$ is empty.*

Proof. Let X and Y be spaces as in a) with non-isolated points x and y respectively. Suppose that $A \subset X \times Y$ is uncountable and no countable subset of A has a limit point. It follows then that $A \cap (X \times \{y'\})$ and $A \cap (\{x'\} \times Y$ is countable for each $x' \in X$ and $y' \in Y$. Choose $\{(x_\alpha, y_\alpha) : \alpha < \omega_1\} \subset A$ such that $\alpha < \beta$ implies $x_\alpha \neq x_\beta$ and $y_\alpha \neq y_\beta$. Then b) will fail for H, K defined according to

$$H \cap \alpha^* = \bigcap\{S^* : S \subset \alpha \text{ and } x \notin \overline{\{x_\beta : \beta \in S\}}\}$$

and

$$K \cap \alpha^* = \bigcap \{S^* : S \subset \alpha \text{ and } y \notin \overline{\{y_\beta : \beta \in S\}}\}$$

for all $\alpha < \omega_1$.

Similarly, if H and K witness a failure for b), then $X = \omega_1 \cup \{x\}$ is obtained by taking the neighbourhood traces on ω_1 of $\overline{H}$ as the neighbourhood traces of x. Similarly for Y. □

Remark 6.1. The following consequence of PFA is given in [Dow90] (via the above translation). If H is a closed subset of $\omega_1^* - U(\omega_1)$ with the property that the closure of H (in ω_1^*) has character at most ω_1 and contains $U(\omega_1)$ then there is an uncountable $I \subset \omega_1$ such that $I^* - U(\omega_1) \subset H$. If the character restriction could be removed this would be a significant strengthening of Theorem 6.8. However, it also shown in [Dow90], that the character restriction cannot be dropped.

We close this section with a method of getting other models of MA from a model of PFA. The first example of this in the literature is [Vlc90]. To demonstrate the technique we will again use gaps in $[\omega]^\omega$. Recall that a (λ, μ)-gap, $(\{a_\alpha : \alpha < \lambda\}, \{b_\beta : \beta < \mu\}$, is said to be *strong* ([vDKvM89]) if there is a unique (limit) ultrafilter p such that $\omega - (a_\alpha \cup b_\beta) \in p$ for all $\alpha < \lambda$ and $\beta < \mu$.

Theorem 6.10. *It is consistent with MA + $c = \omega_2$ that there is a strong (c, c)-gap.*

Proof. Let V be a model of PFA. We define a poset $\mathcal{P}$ by

$$p = (a_p, b_p) \in \mathcal{P} \text{ iff } \begin{cases} a_p \cup b_p \subset \omega, & \\ a_p \text{ and } b_p & \text{are disjoint,} \\ \omega - (a_p \cup b_p) & \text{is infinite.} \end{cases}$$

Define $p \le q$ iff $a_p \ {}^*{\supset}\ a_q$ and $b_p \ {}^*{\supset}\ b_q$.

Now $\mathcal{P}$ is σ-closed and so does not add any new subsets of ω.

Let $\mathcal{A} = \{a_p : p \in G\}$ and $\mathcal{B} = \{b_p : p \in G\}$ for some generic filter G. A standard density argument shows that,

$$\{\omega - (a \cup b) : a \in \mathcal{A} \text{ and } b \in \mathcal{B}\} \in \omega^* \ .$$

And if P is ω_1-Baire, (i.e. the intersection of $\aleph_1$ dense open sets will again be dense open - hence no new subsets of ω_1 will be added) then $\mathcal{A}'$ and $\mathcal{B}'$ are separated for any $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$ and $\mathcal{B}' \in [\mathcal{B}]^{\omega_1}$. Furthermore, MA + $c = \omega_2$ will hold in $V[G]$ since all posets of size $\aleph_1$ and $\aleph_1$-sized families of dense subsets will be from the ground model V.

Lemma. $\mathcal{P}$ is ω_1-Baire.

To prove this we define an *Amoeba* forcing – i.e. a forcing whose generic object will result in a new condition in $\mathcal{P}$.

Let $\mathbf{A}(\mathcal{P}) = \{(p, n) : p \in \mathcal{P} \text{ and } n \in \omega\}$ and define

$$(p, n) < (q, m) \text{ if} \begin{cases} n \ge m, & \\ a_p \supset a_q & \text{and } a_p \cap m = a_q \cap m, \\ b_p \supset b_q & \text{and } b_p \cap m = b_q \cap m. \end{cases}$$

We show that $\mathbf{A}(\mathcal{P})$ is proper. Let M be a countable elementary submodel of some large H_θ and let $(p_0, m_0) \in \mathcal{P} \cap H_\theta$. Let $\{D_n : n \in \omega\}$ enumerate the dense open subsets of $\mathbf{A}(\mathcal{P})$ which are members of M.

Fact: For any D_n and $p \in P$ and $m \in \omega$, there is a $q \in P$ and an m' such that $(q, m) < (p, m)$ and

(6.10-1) if there is any $(r, k) \in D_n$ with $(r, 0) < (q, 0)$ then

($(a_q \cup (a_r \cap m), b_q \cup (b_r \cap m)), m)$ is in D_n.

To prove this fact, simply list all pairs of disjoint subsets of m as $\{(c_\ell, d_\ell)\}$. Define a descending sequence (q_ℓ, m) with $q_0 = p$ together with a sequence m'_ℓ of integers, such that, if possible, $(\ (a_{q_{\ell+1}} \cup c_\ell, b_{q_{\ell+1}} \cup d_\ell), m'_\ell)$ is in D_n. Then q is the last q_ℓ and m' is the maximum of the m'_ℓ.

Therefore, by applying the above fact for each n, there is a strictly decreasing sequence $\{(q_n, m_n) : n \in \omega\}$ with $(q_0, m_0) = (q_0, n_0) < (p, n_0)$ and $\{m_n : n \in \omega\}$ strictly increasing, such that (6.10-1) holds (for each n) with $q = q_n$, $m = m_n$ and m_{n+1} is larger than the m' given by the Fact.
Now let $a = \cup\{a_{q_n} : n \in \omega\}$ and $b = \cup\{b_{q_n} : n \in \omega\}$. It is immediate from the definition of $(q_{n+1}, m_{n+1}) < (q_n, m_n)$ that a and b are disjoint. However, it is not immediate that $\omega - (a \cup b)$ is infinite. To obtain this, we simply ensure that m_n is chosen large enough so that $|m_n - (a_n \cup b_n)| > n$. Since $(\ (a, b), m_n) < (q_n, m_n)$ this ensures that $|m_n - (a \cup b)| > n$ for all n.

Claim: $(q, n_0) = (\ (a, b), n_0) < (p_0, n_0)$ is (M, P)-generic.
Indeed, suppose that $(r, k) \in D_n$ and $(r, k) < (q, n_0)$. Clearly, $(r, k) < (q_n, 0)$, hence by (6.10-1), it follows that $(\ (a_{q_n} \cup (a_r \cap m_{n+1}), b_{q_n} \cup (b_r \cap m_{n+1})), m_{n+1})$ is in $D_n \cap M$. It is also the case that this condition is below (q, n_0).

To finish the proof of the Theorem, we let $\mathcal{D}$ be a collection of $\aleph_1$ dense open subsets of $\mathcal{P}$. We shall show that $\bigcap \mathcal{D}$ is dense by showing that if G is suitably generic over $\mathbf{A}(\mathcal{P})$, then

$$(a_G, b_G) \in \bigcap \mathcal{D} \text{ where } a_G = \bigcup_{p \in G} a_p \quad \text{and } b_G = \bigcup_{p \in G} b_b \ .$$

First of all, it is a simple matter to check that if $D \subset P$ is dense open, then the set $\{(p, n) : p \in D \text{ and } n \in \omega\}$ is a dense open subset of $\mathbf{A}(\mathcal{P})$. Furthermore, there is a countable family $\mathcal{E}$ of dense open subsets of $\mathbf{A}(\mathcal{P})$ such that if $G \subset \mathbf{A}(\mathcal{P})$ is $\mathcal{E}$-generic, then $(a_G, b_G) \in P$. $\square$

7. Kurepa families

A *Kurepa family* is a family $\mathcal{K}$ of countable sets such that for every countable set A, the family $\mathcal{K} \restriction A = \{K \cap A : K \in \mathcal{K}\}$ is countable ([Dev84, p.320]). $\mathcal{K}$ is *cofinal* iff for every countable $A \subset \bigcup \mathcal{K}$, there is a $K \in \mathcal{K}$ such that $A \subset K$. Note that the cardinality of a cofinal Kurepa family is determined by the cardinality of its union, i.e., for $|\bigcup \mathcal{K}| > \omega$,

$$|\mathcal{K}| = \mathrm{cf}\, [\bigcup \mathcal{K}]^{\aleph_0}$$

Note also that if there is a cofinal Kurepa family with union θ, then there is such a family $\mathcal{K}$ which is well-founded with respect to $\subset$. This fact is useful in inductive constructions that involve the existence of a cofinal Kurepa family. Note also the difference between cofinal Kurepa families and "global" ones, i.e., if A is a subset of $\bigcup \mathcal{K}$ of cardinality θ such that $\theta > \mathrm{cf}\, \theta = \omega$, then $\mathcal{K} \restriction A$ has size <u>bigger</u> than $|A|$. This means that any construction of a cofinal Kurepa family on the cardinal θ must assume the existence of such a family on any cardinal less than θ.

To give a typical application of cofinal Kurepa families, let us say that $X = X_0 \cup X_1$ is a *Bernstein decomposition* of a space X if for every $C \subseteq X$ homeomorphic to the Cantor set, $C \cap X_i \neq \emptyset$ for each $i < 2$.

Proposition 7.1. *Let X be the Σ-product of the unit interval $[0, 1]$ over an index set I and suppose there is a cofinal Kurepa family on I. Then X admits a Bernstein decomposition.*

Proof. For K in $\mathcal{K}$ let $X_K = X_K^0 \dot\cup X_K^1$ be a fixed Bernstein decomposition of

$$X_K = \{f \in X : \{i \in I : f(i) \neq 0\} \subseteq K\} .$$

For $f \in X$, let K be the minimal element of $\mathcal{K}$ that contains $\{i \in I : f(i) \neq 0\}$, and put f in X_i if f is an element of X_K^i. The fact that $\mathcal{K}$ is a Kurepa family easily gives that any Cantor set must intersect both X_0 and X_1. $\square$

The existence of a cofinal Kurepa family on θ for any cardinal θ follows from an appropriate form of Jensen's gap-one two-cardinal theorem in L ([Jen72]). In the case of an ordinary Kurepa family an analysis of Jensen-Silver proof showed (see [Tod84, Theorem 8.7]) that we actually have a sharper result in the sense that only one $\square$-sequence is needed and that, moreover, no form of GCH is necessary. As we have already indicated above, so sharp a result cannot be expected in the case of cofinal Kurepa families.

We say that a Kurepa family $\mathcal{L}$ *extends* a Kurepa family $\mathcal{K}$ if $\mathcal{K} \subset \mathcal{L}$ and for all $K \in \mathcal{K}$ and $L \in \mathcal{L}$, $K \cap L \in \mathcal{K}$. We use $\mathcal{K} \lhd \mathcal{L}$ to denote this relation. Note that the union of any $\lhd$-chain of cofinal Kurepa families is a Kurepa family which is, moreover, cofinal if the cofinality of the chain is uncountable.

Proposition 7.2. *If every Kurepa family of size at most θ extends to a cofinal Kurepa family, then the same is true about θ^+.*

Proof. Recursively construct a $\lhd$-chain of Kurepa families. $\square$

This shows that cofinal Kurepa families exist on any ω_n for $n \in \omega$. Unfortunately, this is all one can prove in ZFC. To show that such a family exists, in the constructible universe, on any θ, we need only show (by copying the Jensen-Silver proof) the following.

Proposition 7.3 (V=L)**.** *Suppose $\omega = cf\theta < \theta$ and that any Kurepa family of size $<\theta$ can be extended to a cofinal Kurepa family. Then the same is true about Kurepa families of size θ.*

Proof. We shall concentrate on proving the existence of a cofinal Kurepa family in $[\theta]^{\aleph_0}$ assuming $\square_\theta$ and $\mathrm{cf}\,[\theta]^{\aleph_0} = \theta^+$. The part of extending an arbitrary Kurepa family of size θ is obtained by a trivial modification of this proof.

Let $\langle C_\alpha : \alpha < \theta^+\rangle$ be a fixed $\square_\theta$-sequence and for $\alpha < \theta^+$, let S_α be the set of all limit points of C_α. Let $\{A_\alpha : \alpha < \theta^+\}$ be a cofinal subset of $[\theta]^{\aleph_0}$ such that $A_\alpha \subset \alpha$ for all α. The family $\mathcal{K}$ will be obtained as the union of an increasing $\lhd$-chain $\{\mathcal{K}_\alpha : \alpha < \theta^+\}$ of Kurepa families of size θ such that $\alpha \subset \bigcup\mathcal{K}_\alpha$ for all α. We shall also have that $\mathcal{K}_\alpha$ is closed under finite unions, intersections and differences. Moreover, we shall have an $r_\alpha\colon \mathcal{K}_\alpha \mapsto \omega$ respecting $\subseteq$ (i.e., $A \subseteq B$ implies $r_\alpha(A) \leq r_\alpha(B)$) such that if $\mathcal{K}_{\alpha,n} = \{K \in \mathcal{K}_\alpha : r_\alpha(K) \leq n\}$, then:

1. $\mathcal{K}_{\alpha,n} \lhd \mathcal{K}_{\alpha,n+1}$ for $n < \omega$,
2. $\mathcal{K}_{\alpha,n}$ is a cofinal Kurepa family of size $<\theta$ for all $n < \omega$,
3. $K - \bigcup\mathcal{K}_{\alpha,n} \in \mathcal{K}_{\alpha,n+1}$ for all $K \in \mathcal{K}_{\alpha,n+1}$ and $n < \omega$,
4. $\alpha \in S_\beta$ implies $r_\alpha = r_\beta \restriction \mathcal{K}_\alpha$.

To handle the successor step, find $K_\alpha \supseteq A_\alpha \cup \{\alpha\}$ such that $K_\alpha \cap K \in \mathcal{K}_\alpha$ for all K in $\mathcal{K}_\alpha$ and let $\mathcal{K}_{\alpha+1}$ be generated by $\mathcal{K}_\alpha \cup \{K_\alpha\}$. Let $r_{\alpha+1}(K_\alpha) = 0$ and extend $r_{\alpha+1}$ on the rest of $\mathcal{K}_{\alpha+1}$ using r_α and respecting $\subseteq$. Note that $r_{\alpha+1}$ does not extend r_α since we must have that $r_\alpha(K) = 0$ for all $K \subset K_\alpha$.

As for limit steps, the only non-trivial case is when $\mathrm{cf}\,\alpha = \omega$. For each $\xi \in S_\alpha$, since $\mathcal{K}_{\xi,n}$ is a cofinal Kurepa family and r_ξ preserves $\subset$, it follows that $\bigcup\mathcal{K}_{\xi,n}$ does not contain any member of $\mathcal{K}_{\xi,n+1} - \mathcal{K}_{\xi,n+1}$. In fact, since the r_ξ for $\xi \in S_\alpha$

cohere, it follows that $\bigcup \mathcal{K}_{\xi,n}$ does not contain any K such that $\bar{r}_\alpha(K) > n$ where $\bar{r}_\alpha = \cup_{\xi \in S_\alpha} r_\xi$. For each n, let $\bar{\mathcal{K}}_{\alpha,n} = \bar{r}_\alpha^{-1}([0,n]) = \bigcup_{\xi \in S_\alpha} \mathcal{K}_{\xi,n}$. In case S_α is cofinal in α we may now simply extend the $\bar{\mathcal{K}}_{\alpha,n}$'s to cofinal Kurepa families $\mathcal{K}_{\alpha,n}$'s (using the inductive hypothesis of Proposition 7.3).

In case $\gamma = \sup S_\alpha$ is not cofinal in α, we instead define $\bar{\mathcal{K}}$ to be $\mathcal{K}_{\gamma,n} \cup \bigcup_{\xi \in S_\alpha} \mathcal{K}_{\xi,n} - \mathcal{K}_\gamma$. Again extend making sure conditions 1-4 remain true. $\square$

Remark 7.1. Note that the above proof shows that the only assumption needed to construct a cofinal Kurepa family in any $[\kappa]^{\aleph_0}$ is the following:

$$(\forall\theta)\ (\omega = \mathrm{cf}\,\theta < \theta \rightarrow \square_\theta \ \wedge \mathrm{cf}\,[\theta]^{\aleph_0} = \theta^+\ ).$$

There are many other places in the literature where the idea of the Jensen-Silver proof has been applied. We mention the closely related construction of Juhász-Nagy-Weiss [JNW79] of a regular locally countable countably compact monolithic ($\equiv$ closure of every countable set is second countable) space of any size of cofinality $\neq \omega$ assuming the above hypothesis. The following fact explains this connection.

Proposition 7.4. *There is a cofinal Kurepa family of size θ which is well-founded and closed under finite intersections iff there is a regular locally countable countably compact monolithic space of size θ.*

Proof. Let $\mathcal{K}$ be a cofinal Kurepa family of size θ which is well-founded and closed under intersections. We may assume that $\mathcal{K}$ contains $\{x\}$ for every x in $\bigcup\mathcal{K}$. For K in $\mathcal{K}$, let $(.,K]$ denote the set $\{L \in \mathit{Kurepa}\ : L \subseteq K\}$. Let σ be the topology on $\mathcal{K}$ determined by $\{(.,K] \setminus (.,L] : K, L \in \mathcal{K}\ , L \subset K\}$ as a subbasis. Then $(\mathcal{K},\sigma)$ is a locally countable Hausdorff (because $\mathcal{K}$ is closed under intersections) space which is countably compact (because $\mathcal{K}$ is cofinal and well-founded) and first countable (because $\mathcal{K}$ is Kurepa). It follows that $(\mathcal{K},\sigma)$ is also regular and monolithic.

Conversely, let X be a locally countable, countably compact monolithic space of size θ. Let $\mathcal{K}_X$ be the family of all compact clopen subsets of X. Then it is easily checked that $\mathcal{K}_X$ is a cofinal Kurepa family of size θ. (The fact that $\mathcal{K}_X$ is a Kurepa family was first observed by W. Weiss in a slightly different context). $\square$

Another closely related object from the literature is the simplified morass of [Vlm84]. (Clearly every simplified $(\omega_1, 1)$-morass is a cofinal Kurepa family.) This suggests some natural strengthenings of the notion of a cofinal Kurepa family. Namely, we may require the family $\mathcal{K}$ to be, moreover, *stationary* in $[\bigcup\mathcal{K}]^{\aleph_0}$ or even *stationary-coding* ([Zwi84]) in the sense that $K \subsetneq L \in \mathcal{K}$ implies that $\sup K \in L$. It is, therefore, natural to expect that stationary Kurepa families (or stationary-coding Kurepa families) might have applications where morasses cannot be used because of the cardinality restrictions.

Remark 7.2. This entire section was simply copied from a preprint of [Tod90] except this sentence which was inserted into the following remark.

The sole purpose of this note is to unify several ideas scattered throughout the literature. All non-attributed facts mentioned in this note should be considered as parts of the folklore of this subject.

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