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A UNIVERSAL CONTINUUM OF WEIGHT $\aleph$

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Abstract. We prove that every continuum of weight $\aleph_1$ is a continuous image of the Čech-Stone-remainder $\mathbb{R}^*$ of the real line. It follows that under CH the remainder of the half line $[0,\infty)$ is universal among the continua of weight $\mathfrak{c}$.

We complement this result by showing that 1) under MA every continuum of weight less than $\mathfrak{c}$ is a continuous image of $\mathbb{R}^*$, 2) in the Cohen model the long segment of length ω_2+1 is not a continuous image of $\mathbb{R}^*$, and 3) PFA implies that $\mathbb{I}_u$ is not a continuous image of $\mathbb{R}^*$, whenever u is a $\mathfrak{c}$-saturated ultrafilter.

Introduction

In [13] Parovičenko proved that every compact Hausdorff space of weight $\aleph_1$ is a continuous image of the space ω^* (the Čech-Stone remainder of the discrete space ω). It follows that under CH the space ω^* is a universal compact space of weight $\mathfrak{c}$, universal in the sense of onto mappings rather than embeddings.

The purpose of this paper is to prove a similar result for $\mathbb{H}^*$, the Čech-Stone remainder of the half line $\mathbb{H}=[0,\infty)$. As $\mathbb{H}^*$ is compact and connected (a *continuum*) the following theorem is the proper analogue of Parovičenko's result.

Theorem 1. *Every continuum of weight $\aleph_1$ is a continuous image of $\mathbb{H}^*$.*

We present proofs of our theorem using each of three quite different approaches to Parovičenko's theorem from the literature that have proven valuable.

It is clear that new ideas are needed to prove Theorem 1 since the fact that ω^* is zero-dimensional appears to be so essential in each of the proofs of Parovičenko's theorem.

We now give brief outlines of the proofs of Parovičenko's theorem and we indicate how our proofs parallel these arguments; more details, including definitions, will appear later in the paper. We shall finish this introduction with some remarks on universal spaces.

Two algebraic proofs. The first two proofs of Parovičenko's theorem use Stone's duality, see [15], between compact zero-dimensional spaces and Boolean algebras. One first uses a theorem of Alexandroff, from [3], to find, given the compact space X of weight $\aleph_1$, a compact zero-dimensional space Y of weight $\aleph_1$ and a continuous map from Y onto X, thus reducing the problem to the case of zero-dimensional spaces. Stone Duality then tells us that finding a continuous map from ω^* onto Y is equivalent to finding an embedding of the Boolean algebra $\mathbb{C}$ of clopen subsets of Y into the Boolean algebra $\mathbb{B}$ of clopen subsets of ω^*.

This gives rise to two proofs of Parovičenko's theorem. In the first one constructs the desired embedding by transfinite induction; this is Parovičenko's original proof.

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The second proof recognizes that Parivičenko's theorem is in fact a special case of a well-known result about $\aleph_1$-saturated models.

Our parallels of these proofs are naturally based on Wallman's generalization of Stone's duality to the class of distributive lattices and amount to embedding, for the given continuum, a suitable sublattice of its lattice of closed sets into the lattice of closed subsets of $\mathbb{H}^*$. The key new idea is to extract the hidden model-theoretic aspects of the abovementioned proofs.

A topological proof. The third proof is of a more topological nature, it appears in [5]. It starts by writing the compact space X as an inverse limit $\varprojlim X_\alpha$ of an ω_1-sequence of compact metrizable spaces. The continuous map from ω^* onto X is obtained as the limit of a family of continuous onto maps $f_\alpha : \omega^* \twoheadrightarrow X_\alpha$. These maps are constructed by transfinite induction and using the following lemma.

Lemma 1. *Let M and N be compact metrizable spaces and let $f : \omega^* \twoheadrightarrow N$ and $g : M \twoheadrightarrow N$ be continuous surjections. Then there is a continuous surjection $h : \omega^* \twoheadrightarrow M$ such that $g \circ h = f$.*

Our third proof will be along these lines and the main problem is that the above lemma is false if we simply replace ω^* by $\mathbb{H}^*$ and assume that M and N are continua — see Example 3.2. Intriguingly, if g is induced by an appropriate elementary embedding then the analogue of Lemma 1 will hold.

Universal spaces. As with Parovičenko's theorem, Theorem 1 becomes a result about universal spaces if one assumes the Continuum Hypothesis (CH).

Theorem 2 (CH). *The space $\mathbb{H}^*$ is a universal continuum of weight $\mathfrak{c}$.*

It is of interest to note that in the class of compact metrizable spaces the situation is somewhat different from what we just found for spaces of weight $\mathfrak{c}$. On the one hand the Cantor set *is* a universal compact metrizable space; this is a theorem of Alexandroff [2] and Hausdorff [8]. On the other hand, in [20] Waraszkiewicz constructed a family of plane continua such that no metric continuum maps onto all of them, hence there is no universal metric continuum.

Part of the balance is restored by the following theorem, due to Aarts and Van Emde Boas [1]; as we shall need this theorem and its proof later on we provide a short argument.

Theorem 3. *The space $\mathbb{H}^*$ maps onto every metric continuum.*

Proof. Consider a metric continuum M and assume it is embedded into the Hilbert cube $Q = [0,1]^\infty$. Choose a countable dense subset A of M and enumerate it as $\{a_n : n \in \omega\}$. Next choose, for every n, a finite sequence of points $a_n = a_{n,0}$, $a_{n,1}$, …, $a_{n,k_n} = a_{n+1}$ such that $d(a_{n,i}, a_{n,i+1}) < 2^{-n}$ for all i — this uses the connectivity of M. Finally then let e be map from $\mathbb{H}$ to $(0,1] \times Q$ with first coordinate $e_1(t) = 2^{-t}$ and whose second coordinate satisfies $e_2(n + \frac{i}{n}) = d_{n,i}$ for all n and i and is (piecewise) linear otherwise.

It is clear that e is an embedding and one readily checks that $\operatorname{cl} e[\mathbb{H}] = e[\mathbb{H}] \cup (\{0\} \times M)$; the Čech-Stone extension βe of e maps $\mathbb{H}^*$ onto M. $\square$

1. Algebraic Preliminaries

We already indicated that two of our proofs will involve the lattice of closed sets of our continua and that the base for this is Wallman's generalization, to the class of distributive lattices, of Stone's representation theorem for Boolean algebras. Wallman's representation theorem is as follows.

Theorem 1.1 (Wallman [19]). *If L is a distributive lattice then there is a compact T_1-space X with a base for its closed sets that is a homomorphic image of L. The homomorphism is an isomorphism iff L is* disjunctive, *that is if $a \neq b$ then there is $c \in L$ such that $c \wedge a = 0$ or $c \wedge b = 0$.*

The representing space X is Hausdorff iff the lattice L is *normal*; this means that given $a, b \in L$ with $a \wedge b = 0$ there are $c, d \in L$ such that $c \vee d = 1$, $a \wedge d = 0$ and $b \wedge c = 0$. Note that this mimics the formulation of normality of topological spaces in terms of closed sets only.

Also, if one starts out with a compact Hausdorff space X and a base $\mathcal{C}$ for its closed sets that is closed under finite unions and intersections then this base $\mathcal{C}$ is a normal lattice and its representing space is X.

The following theorem shows how one can create onto mappings from maps between lattices. In it we use 2^X to denote the family of closed subsets of the space X.

Theorem 1.2. *Let X and Y be compact Hausdorff spaces and let $\mathcal{C}$ be a base for the closed subsets of Y that is closed under finite unions and intersections. Then Y is a continuous image of X if and only if there is a map $\varphi : \mathcal{C} \to 2^X$ such that*

1. *$\varphi(\varnothing) = \varnothing$ and if $F \neq \varnothing$ then $\varphi(F) \neq \varnothing$;*
2. *if $F \cup G = Y$ then $\varphi(F) \cup \varphi(G) = X$; and*
3. *if $F_1 \cap \cdots \cap F_n = \varnothing$ then $\varphi(F_1) \cap \cdots \cap \varphi(F_n) = \varnothing$.*

Proof. Necessity is easy: given a continuous onto map $f : X \to Y$ let $\varphi(F) = f^{\leftarrow}[F]$. Note that φ is in fact a lattice-embedding.

To prove sufficiency let $\varphi : \mathcal{C} \to 2^X$ be given and consider for each $x \in X$ the family $\mathcal{F}_x = \bigl\{F \in \mathcal{C} : x \in \varphi(F)\bigr\}$. We claim that $\bigcap \mathcal{F}_x$ consists of exactly one point. Indeed, by Condition 3 the family $\mathcal{F}_x$ has the finite intersection property, so that $\bigcap \mathcal{F}_x$ is nonempty. Next assume that $y_1 \neq y_2$ in Y and take $F, G \in \mathcal{C}$ such that $F \cup G = Y$, $y_1 \notin F$ and $y_2 \notin G$. Then, by Condition 2, either $x \in \varphi(F)$ and so $y_1 \notin \bigcap \mathcal{F}_x$ or $x \in \varphi(G)$ and so $y_2 \notin \bigcap \mathcal{F}_x$.

We define $f(x)$ to be the unique point in $\bigcap \mathcal{F}_x$.

To demonstrate that f is continuous and onto we show that for every closed subset F of Y we have

$$f^{\leftarrow}[F] = \bigcap \bigl\{\varphi(G) : G \in \mathcal{C}, F \subseteq \operatorname{int} G\bigr\}. \tag{$*$}$$

This will show that preimages of closed sets are closed and that, by Condition 1, every fiber $f^{\leftarrow}(y)$ is nonempty.

To verify $(*)$, first let $x \in X \setminus f^{\leftarrow}[F]$. Then one can find $G, H \in \mathcal{C}$ such that $f(x) \notin G$, hence $x \notin \varphi(G)$, and also $H \cap F = \varnothing$ and $G \cup H = Y$ so that $F \subseteq \operatorname{int} G$.

Second, if $F \subseteq \operatorname{int} G$ then one can find $H \in \mathcal{C}$ such that $H \cup G = X$ and $H \cap F = \varnothing$. It follows that if $x \notin \varphi(G)$ we have $x \in \varphi(H)$, hence $f(x) \in H$ and so $f(x) \notin F$. □

An obvious corollary of this theorem is that Y is a continuous image of X iff there is an embedding of the lattice $\mathcal{C}$ into the lattice 2^X.

The reason that Theorem 1.2 is formulated the way it is is that we shall apply it in the following way: given the continuum K we shall take a base $\mathcal{C}$ for its closed sets and define a map as in the theorem into the canonical base $\mathcal{L}$ for the closed sets of $\mathbb{H}^*$, rather than into the family $2^{\mathbb{H}^*}$. We do not know whether it is actually possible to get a lattice-embedding of $\mathcal{C}$ into $\mathcal{L}$.

The canonical base in question is just

$$\mathcal{L} = \{A^* : A \text{ is closed in } \mathbb{H}\}.$$

Here, as is common, A^* abbreviates $\operatorname{cl} A \cap \mathbb{H}^*$.

The following lemma will be the first step in the constructions of both embeddings.

Lemma 1.3. *Let K be a metric continuum and let $x \in K$. There is a map φ from 2^K to $2^{\mathbb{H}}$ such that*

1. $\varphi(\varnothing) = \varnothing$ *and* $\varphi(K) = \mathbb{H}$;
2. $\varphi(F \cup G) = \varphi(F) \cup \varphi(G)$;
3. *if* $F_1 \cap \cdots \cap F_n = \varnothing$ *then* $\varphi(F_1) \cap \cdots \cap \varphi(F_n)$ *is compact; and*
4. $\mathbb{N} \subseteq \varphi(\{x\})$.

In addition, if some countable family $\mathcal{C}$ of nonempty closed subsets of K is given in advance then we can arrange that for every F in $\mathcal{C}$ the set $\varphi(F)$ is not compact.

Proof. As proved in Theorem 3 there is a map from $\mathbb{H}^*$ onto K.

The proof given is flexible enough to allow us to ensure that the embedding e of $\mathbb{H}$ into $(0, 1] \times Q$ is such that $e(n) = \langle 2^{-n}, x\rangle$ for every $n \in \mathbb{N}$ and that for every element y of some countable set C the set $\{t : e_1(t) = y\}$ is cofinal in $\mathbb{H}$. In our case we let C be a countable subset of K that meets every element of the family $\mathcal{C}$.

We now identify K and $\{0\} \times K$ and define a map $\psi : 2^K \to 2^{\mathbb{I}\times Q}$ by

$$\psi(F) = \bigl\{y \in: \mathbb{I} \times Q : d(y, F) \leqslant d(y, K \setminus F)\bigr\}.$$

In [12, § 21 XI] it is shown that for all F and G we have

- $\psi(F \cup G) = \psi(F) \cup \psi(G)$;
- $\psi(K) = \mathbb{I} \times Q$ and $\psi(\varnothing) = \varnothing$ — by the fact that $d(y, \varnothing) = \infty$ for all y; and
- $\psi(F) \cap K = F$.

Note that for every $y \in K$ we have $d\bigl(\langle t, y\rangle, \{y\}\bigr) = d\bigl(\langle t, y\rangle, K \setminus \{y\}\bigr) = t$ and hence $\mathbb{I} \times \{y\} \subseteq \psi\bigl(\{y\}\bigr)$.

Now define $\varphi(F) = e^{\leftarrow}\bigl[\psi(F)\bigr]$ or rather, after identifying $\mathbb{H}$ and $e[\mathbb{H}]$, set $\varphi(F) = \psi(F) \cap e[\mathbb{H}]$. All desired properties are easily verified: 1 and 2 are immediate; to see that 3 holds note that if $F_1 \cap \cdots \cap F_n = \varnothing$ then $\operatorname{cl} \varphi(F_1) \cap \cdots \cap \operatorname{cl} \varphi(F_n) \cap K = \varnothing$, so that $\operatorname{cl} \varphi(F_1) \cap \cdots \cap \operatorname{cl} \varphi(F_n)$ is a compact subset of $\mathbb{H}$. That 4 holds follows from the way we chose the values $e(n)$ for $n \in \mathbb{N}$.

Finally, if $F \in \mathcal{C}$ and $y \in C \cap F$ then the cofinal set $\{t : \pi\bigl(e(t)\bigr) = y\}$ is a subset of $\varphi(F)$ so that $\varphi(F)$ is not compact. $\square$

Remark 1.4. If φ is as Lemma 1.3 then the map $\psi : \mathcal{C} \to \mathcal{L}$ defined by

$$\psi : C \mapsto \varphi(C) \mapsto \varphi(C)^*$$

is as in Theorem 1.2. Observe that ψ is even a $\cup$-homomorphism; to get a lattice homomorphism the map φ would have to satisfy $3'$ instead of 3, where $3'$ reads:

$3'$. the symmetric difference of $\varphi(F \cap G)$ and $\varphi(F) \cap \varphi(G)$ is bounded in $\mathbb{H}$.

We do not know how to achieve $3'$.

2. Two algebraic proofs

We shall now show how to construct, given a continuum K, a map φ from a base for the closed sets of K into the base $\mathcal{L}$ as in Theorem 1.2.

A proof using Model Theory. Our plan is to find the map φ promised above by an appeal to some machinery from Model Theory. The first step would be to show that $\mathcal{L}$ is an $\aleph_1$-saturated lattice and hence an $\aleph_2$-universal structure; this latter notion says that every structure of size $\aleph_1$ that is *elementary equivalent* to $\mathcal{L}$ is embeddable into $\mathcal{L}$. The second step would then be to show that every lattice of size $\aleph_1$ is embeddable into a lattice of size $\aleph_1$ that itself is elementary equivalent to $\mathcal{L}$.

There are two problems with this approach: 1) we were not able to show that $\mathcal{L}$ is $\aleph_1$-saturated, and 2) Lemma 1.3 does not give a lattice-embedding. We shall deal with these problems in turn.

An $\aleph_1$-saturated structure. As mentioned above we do not know whether $\mathcal{L}$ is $\aleph_1$-saturated. We can however find an $\aleph_1$-saturated sublattice:

$$\mathcal{L}' = \{A^* : A \text{ is closed in } \mathbb{H}, \text{ and } \mathbb{N} \subseteq A \text{ or } \mathbb{N} \cap A = \varnothing\}.$$

This lattice is a base for the closed sets of the space $\boldsymbol{H}$, obtained from $\mathbb{H}^*$ by identifying ω^* to a point.

To see that $\mathcal{L}'$ is $\aleph_1$-saturated we introduce another space, namely $\mathbb{M} = \omega \times \mathbb{I}$, where $\mathbb{I}$ denotes the unit interval. The canonical base $\mathcal{M}$ for the closed sets of $\mathbb{M}^*$ is naturally isomorphic to the reduced power $(2^{\mathbb{I}})^\omega$ modulo the cofinite filter. It is well-known that this structure is $\aleph_1$-saturated — see [9]. The following substructure $\mathcal{M}'$ is $\aleph_1$-saturated as well:

$$\mathcal{M}' = \{A^* : A \text{ is closed in } \mathbb{M}, \text{ and } \boldsymbol{N} \subseteq A \text{ or } \boldsymbol{N} \cap A = \varnothing\},$$

where $\boldsymbol{N} = \{0,1\} \times \omega$. Indeed, if T is a type over a countable subset of $\mathcal{M}'$ then, because of maximality, either $\boldsymbol{N} \subseteq x$ or $\boldsymbol{N} \cap x = \varnothing$ belongs to T. Any element of $\mathcal{M}$ that satisfies T will automatically belong to $\mathcal{M}'$.

We claim that $\mathcal{L}'$ and $\mathcal{M}'$ are isomorphic. To see this consider the map $q : \mathbb{M} \to \mathbb{H}$ defined by $q(n,x) = n + x$. The Čech-Stone extension of q maps $\mathbb{M}^*$ onto $\mathbb{H}^*$ and it is readily verified that $L \mapsto q^{-1}[L]$ is an isomorphism between $\mathcal{L}'$ and $\mathcal{M}'$. (In topological language: the space $\boldsymbol{H}$ is also obtained from $\mathbb{M}^*$ by identifying $\boldsymbol{N}^*$ to a point.)

A new language. To remedy the problem that Lemma 1.3 does not provide a lattice embedding but rather a map that only partially preserves unions and intersections we introduce a first-order language $\mathfrak{L}$ with equality, with a binary predicate J and for every $n \geqslant 2$ an n-ary predicate M_n. Every lattice is a structure for this language under the following interpretations:

$$\begin{aligned} J(x,y) &\equiv x \vee y = 1 \\ M_n(x_1,\ldots,x_n) &\equiv x_1 \wedge \cdots \wedge x_n = 0. \end{aligned}$$

A direct consequence of Theorem 1.2 is that compact spaces are homeomorphic iff they have bases for the closed sets that are isomorphic as $\mathfrak{L}$-structures.

The proof. Now let $\mathcal{C}$ be a base of size $\aleph_1$ for the closed sets of the continuum K. We want to find an $\mathfrak{L}$-structure $\mathcal{D}$ of size $\aleph_1$ that contains $\mathcal{C}$ and that is elementary equivalent to $\mathcal{L}'$. To this end we consider the diagram of $\mathcal{C}$; that is, we add the elements of $\mathcal{C}$ to our language $\mathfrak{L}$ and we consider the set $T_{\mathcal{C}}$ of all atomic sentences from this expanded language that hold in $\mathcal{C}$. For example if $a \cap b = \varnothing$ and $c \cup d = K$ then $M_2(a,b) \wedge J(c,d)$ belongs to $T_{\mathcal{C}}$.

To $T_{\mathcal{C}}$ we add the theory $T_{\mathcal{L}}$ of $\mathcal{L}$, to get a theory T. Now Lemma 1.3 implies that for every countable subset $\mathcal{C}'$ of $\mathcal{C}$ there is an $\mathfrak{L}$-embedding into $\mathcal{L}'$; indeed, Condition 4 says that ω^* may be mapped onto a fixed point x. This implies, by the compactness theorem, that the theory T is consistent and hence has a model $\mathcal{D}$ of size $\aleph_1$. The model $\mathcal{D}$ is as required: it satisfies the same sentences as $\mathcal{L}'$ and it contains a copy of $\mathcal{C}$, to wit the set of interpretations of the constants from $\mathcal{C}$.

An extension. It is clear from the above reasoning and Lemma 2.2 below that the following theorem holds.

Theorem 2.1. *If X is a continuum with an $\aleph_1$-saturated base for its closed sets that maps onto every metric continuum then it maps onto every continuum of weight $\aleph_1$.*

Lemma 2.2. *Suppose that X is a compact space which has a base $\mathcal{C}$ for the closed sets that is $\aleph_1$-saturated as a lattice. Then, if X maps onto a metric space M, there is an embedding of a base $\mathcal{D}$ for the closed subsets of M into $\mathcal{C}$ such as in Theorem 1.2, that is an $\mathfrak{L}$-embedding.*

Proof. Simply fix a map f from X onto M and let $\mathcal{D} = \{D_n : n \in \omega\}$ be a base for the closed subsets of M such that each D_n is a regular closed subset of M (that such a base, which in addition is closed under finite unions and intersections, exists is established in [16]). We inductively choose $C_n \in \mathcal{C}$ so that $f(C_n) = D_n$. To begin we may assume that $D_0 = \varnothing$ and $D_1 = M$, so, of course, we let $C_0 = \varnothing$ and $C_1 = X$ (note that since $\mathcal{C}$ is a lattice we may assume that $\varnothing \in \mathcal{C}$). Suppose that we have chosen C_{n-1} and we must choose C_n. Since we plan to guarantee that $f(C_m) = D_m$ for each m we will have that $\bigcap\{C_k : k \in F\}$ is empty for each $F \in [\omega]^{<\omega}$ such that $\bigcap\{D_k : k \in F\}$ is empty and that C_m will be empty whenever D_m is empty. Therefore our only concern is that $C_k \cup C_n$ should be X whenever $D_k \cup D_n$ is equal to M. To accomplish we first prove that, for each closed $K \subset M$, there is a $C \in \mathcal{C}$ such that $f^{-1}(\operatorname{int}(K)) \subset C$ and $f(C) = K$. Indeed, let $\{F_i : i \in \omega\}$ be a family of closed subsets of M whose union is the interior of K. Also let $\{U_i : i \in \omega\}$ be a neighbourhood base for K in M. For each i, there are J_i and Y_i in $\mathcal{C}$ such that $f^{-1}(F_i) \subset J_i$, $J_i \subset f^{-1}(\operatorname{int}(K))$, $f^{-1}(K) \subset Y_i$, and $Y_i \subset f^{-1}(U_i)$. Use the $\aleph_1$-saturatedness of $\mathcal{C}$ to pick L which contains each J_i and is contained in each Y_i. Clearly L has the desired properties.

Now for each $k < n$ such that $D_k \cup D_n = M$, let $J_k \in \mathcal{C}$ be such that $f^{-1}(\operatorname{int}(M \setminus D_k)) \subset J_k$ and $f(J_k) = M \setminus \operatorname{int}(D_k)$ (recall that D_k is regular closed). If $D_k \cup D_n$ (for $k < n$) is not equal to M, then let J_k be the empty set. Also let $J_n \in \mathcal{C}$ be any set such that $f(J_n) = D_n$. Now define C_n to be the union of J_n with all J_k for $k < n$. Clearly $C_n \cup C_k = X$ for all $k < n$ such that $D_n \cup D_k = M$, so it suffices to show that $f(C_n) = D_n$. Since $J_n \subset C_n$, it follows that $D_n \subset f(C_n)$. Let $k < n$ and assume that $D_k \cup D_n = M$. Then $M \setminus \operatorname{int}(D_k) \subset D_n$ (again using that D_k is regular closed). Since $f(J_k) = M \setminus \operatorname{int}(D_k)$, it follows that $f(L) \subset D_n$. $\square$

A direct construction. Rather than relying on general model-theoretic results we can also give a direct construction of an $\mathfrak{L}$-embedding of a base for the closed sets of a continuum K into the lattice $\mathcal{L}$. We do this in order to point out what may have seemed to make our main result harder to prove than Parovičenko's theorem.

We found it necessary to use elementary substructures to guide the induction; as a consequence we shall build the embedding countably many steps at a time, rather than one.

Let K be a continuum of weight $\aleph_1$ and fix a base $\mathcal{C}$ for the closed sets of K of size $\aleph_1$ and closed under finite unions and intersections. We also assume $\{x\} \in \mathcal{C}$ for some fixed $x \in K$.

To construct an $\mathfrak{L}$-embedding φ of $\mathcal{C}$ into the base $\mathcal{L}$ for the closed sets of $\mathbb{H}^*$ we construct a map $\psi : \mathcal{C} \to 2^{\mathbb{H}}$ that satisfies

1. if $F \neq \varnothing$ then $\psi(F)$ is not compact;
2. if $F \cup G = K$ then there is a t in $\mathbb{H}$ such that $[t, \infty) \subseteq \psi(F) \cup \psi(G)$; and
3. if $F_1 \cap \cdots \cap F_n = \varnothing$ then $\psi(F_1) \cap \cdots \cap \psi(F_n)$ is compact.

As in Remark 1.4 we then define $\varphi(F) = \psi(F)^*$.

We take a large enough regular cardinal κ such that x, K, $\mathbb{H}$, 2^K, $2^{\mathbb{H}}$ and $\mathcal{C}$ all belong to $H(\kappa)$ — we can take $\kappa = (2^{\mathfrak{c}})^+$. We let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be a continuous chain of countable elementary submodels of $H(\kappa)$ such that

- x, K, $\mathbb{H}$, 2^K, $2^{\mathbb{H}}$ and $\mathcal{C}$ all belong to M_0;
- $\langle M_\beta : \beta \leqslant \alpha \rangle \in M_{\alpha+1}$ for all α; and
- $\mathcal{C} = \bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha$, where $\mathcal{C}_\alpha = \mathcal{C} \cap M_\alpha$.

By elementarity $\mathcal{C}_0$ is a normal distributive lattice and its Wallman representation K_0 is a metric continuum. We may therefore apply Lemma 1.3 and obtain a map $\psi_0 : \mathcal{C}_0 \to 2^{\mathbb{H}}$ such that

- $\psi_0(\varnothing) = \varnothing$, $\psi_0(K) = \mathbb{H}$ and if $F \neq \varnothing$ then $\psi_0(F)$ is not compact;
- $\psi_0(F \cup G) = \psi_0(F) \cup \psi_0(G)$;
- if $F_1 \cap \cdots \cap F_n = \varnothing$ then $\psi_0(F_1) \cap \cdots \cap \psi_0(F_n)$ is compact; and
- $\mathbb{N} \subseteq \psi_0(\{x\})$, where x is the point fixed above.

Note that we can have $\psi_0 \in M_1$ by elementarity.

We shall construct for each α a map $\psi_\alpha : \mathcal{C}_\alpha \to 2^{\mathbb{H}}$ with the desired properties and such that $\psi_\beta = \psi_\alpha \restriction \mathcal{C}_\beta$ if $\beta < \alpha$, and $\psi_\alpha \in M_{\alpha+1}$ for all α. We already have ψ_0 and at limit stages we simply take unions so we only have to show how to construct $\psi_{\alpha+1}$ from ψ_α.

To this end enumerate $\mathcal{C}_\alpha$ as $\{c_n : n \in \omega\}$ and $\mathcal{C}_{\alpha+1} \setminus \mathcal{C}_\alpha$ as $\{d_n : n \in \omega\}$. Now for each n we can find, by elementarity, an n-tuple $\langle c_i^n : i < n \rangle$ of nonempty elements of $\mathcal{C}_\alpha$ such that

- for all $i, j < n$: $c_i^n \cup c_j = K$ iff $d_i \cup c_j = K$;
- for all $i, j < n$: $c_i^n \cup c_j^n = K$ iff $d_i \cup d_j = K$;
- for all $i < n$: $x \in c_i^n$ iff $x \in d_i$; and
- for all $f, g \subseteq n$: $\bigcap_{i \in f} c_i^n \cap \bigcap_{j \in g} c_j = \varnothing$ iff $\bigcap_{i \in f} d_i \cap \bigcap_{j \in g} c_j = \varnothing$.

Given this n-tuple find $m_n \in \mathbb{N}$ such that

- if $c_i^n \cup c_j = K$ then $[m_n, \infty) \subseteq \psi_\alpha(c_i^n) \cup \psi_\alpha(c_j)$;
- if $c_i^n \cup c_j^n = K$ then $[m_n, \infty) \subseteq \psi_\alpha(c_i^n) \cup \psi_\alpha(c_j^n)$;
- if $\bigcap_{i \in f} c_i^n \cap \bigcap_{j \in g} c_j = \varnothing$ then $\bigcap_{i \in f} \psi_\alpha(c_i^n) \cap \bigcap_{j \in g} \psi_\alpha(c_j) \subseteq [0, m_n)$;
- if $x \in c_i^n$ then $\mathbb{N} \setminus \psi_\alpha(c_i^n) \subseteq [0, m_n)$; and
- if $x \notin c_i^n$ then $\mathbb{N} \cap \psi_\alpha(c_i^n) \subseteq [0, m_n)$.

We can and will assume in addition that $m_{n+1} > m_n$ for all n and also that $\psi_\alpha(c_i^n) \cap [m_n, m_{n+1}) \neq \varnothing$ for all i.

Now we define, for all i,

$$\psi_{\alpha+1}(d_i) = \bigcup_{n>i} \psi_\alpha(c_i^n) \cap [m_n, m_{n+1}).$$

We show that $\psi_{\alpha+1}$ has all the required properties.

The first thing to check is that $\psi_{\alpha+1}(d_i)$ is indeed closed. Note that only the points m_{n+1} with $n > i$ can possibly be in $\operatorname{cl} \psi_{\alpha+1}(d_i)$ but not in $\psi_{\alpha+1}(d_i)$. But if $m_{n+1} \in \operatorname{cl}\bigl(\psi_{\alpha+1}(d_i)\bigr)$ then $m_{n+1} \in \operatorname{cl}\bigl(\psi_\alpha(c_i^n) \cap [m_n, m_{n+1})\bigr)$ or $m_{n+1} \in \psi_\alpha(c_i^{n+1})$; in the latter case we are done and in the former case we have $m_{n+1} \in \psi_\alpha(c_i^n)$ and hence $x \in c_i^n$, by the choice of m_n, and so $x \in d_i$. It then follows that $x \in c_i^{n+1}$ and so again $m_{n+1} \in \psi_\alpha(c_i^{n+1}) \subseteq \psi_{\alpha+1}(d_i)$.

It is now a routine matter to check that $\psi_{\alpha+1}$ is as required. For example, if $d_i \cup d_j = X$ them $[m_n, \infty) \subseteq \psi_{\alpha+1}(d_i) \cup \psi_{\alpha+1}(d_j)$, where $n = \max(i, j) + 1$ and if $\bigcap_{i \in f} d_i \cap \bigcap_{j \in g} c_j = \varnothing$ then $\bigcap_{i \in f} \psi_{\alpha+1}(d_i) \cap \bigcap_{j \in g} \psi_\alpha \subseteq [0, m_n)$, where $n = \max(f \cup g) + 1$.

In the end $\psi = \bigcup_{\alpha < \omega_1} \psi_\alpha$ is the map that we want.

3. A topological proof

In this section we show how to modify the argument of Błaszczyk and Szymański from [5] to obtain a topological proof of Theorem 1.

As mentioned in the introduction the following statement at first sight seems a reasonable property that $\mathbb{H}^*$ should share with ω^*.

False lemma 3.1. *If f maps $\mathbb{H}^*$ onto the metric continuum X and if g maps the metric continuum Y onto X then there is a map h from $\mathbb{H}^*$ onto Y such that $f = g \circ h$.*

That this lemma is false is witnessed by the following example.

Example 3.2. Define $f : \mathbb{H} \to S^1$ and $g : \mathbb{I} \to S^1$ by $f(t) = g(t) = e^{2\pi i t}$. Now any map from $\mathbb{H}^*$ onto $\mathbb{I}$ is induced by a map $h : \mathbb{H} \to \mathbb{I}$ and if we want to have $f^* = g^* \circ h^*$ then we must have $\lim_{t\to\infty} \bigl|f(t) - g\bigl(h(t)\bigr)\bigr| = 0$. Choose $N \in \mathbb{N}$ such that $\bigl|f(t) - g\bigl(h(t)\bigr)\bigr| < \frac{1}{4}$ for $t \geqslant N$ and let $n > N$. The image, under f, of $[n - \frac{1}{4}, n + \frac{1}{4}]$ is the arc from $-i$, via 1, to i. Therefore the image under $g \circ h$ of this interval does not contain -1 and so its image under h does not contain $\frac{1}{2}$; however this last image does contain points above and below $\frac{1}{2}$. This contradicts the continuity of h.

The interested reader can certainly formulate the appropriate elementary submodel version of Lemma 3.1 by examining the inductive step in our direct algebraic construction and the relation between $\mathcal{C}_\alpha$ and $\mathcal{C}_{\alpha+1}$ exploited therein. We choose a more direct approach.

Let K be the continuum of weight $\aleph_1$ and assume that K is a subspace of the Tychonoff cube $\mathbb{I}^{\omega_1}$. We shall construct by induction maps $f_\alpha : \mathbb{H} \to \mathbb{I}$ such that the Čech-Stone extension of the diagonal map $f = \triangle_\alpha f_\alpha$ maps $\mathbb{H}^*$ onto K.

For this it suffices to make sure that the following conditions are met:

1. for every open set U around K there is $n \in \mathbb{N}$ such that $f\bigl[[n,\infty)\bigr] \subseteq U$, and
2. for every open set U that meets K the set of $t \in \mathbb{H}$ for which $f(t) \in U$ is cofinal in $\mathbb{H}$.

To this end we let κ be a reasonably large regular cardinal such that K, $\mathbb{H}$ and $\mathbb{I}^{\omega_1}$ belong to $H(\kappa)$. We also fix a continuous chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of $H(\kappa)$ such that

- K, $\mathbb{H}$ and $\mathbb{I}^{\omega_1}$ belong to M_0, and
- for every α the chain $\langle M_\beta : \beta < \alpha\rangle$ belongs to M_α.

Observe that automatically $\omega_1 \subseteq \bigcup_\alpha M_\alpha$. Denote $M_\alpha \cap \omega_1$ by δ_α and denote the projection of K onto the first δ_α coordinates by K_α.

The model $H(\kappa)$ also contains the base $\mathcal{B}$ for $\mathbb{I}^{\omega_1}$ that is built using only open intervals in $\mathbb{I}$ with rational end points and because this base is uniquely defined it also belongs to M_0.

We shall construct the maps f_β for $\beta < \delta_\alpha$, by induction on α. To facilitate this we assume that the point $\mathbf{0}$ with all coordinates zero belongs to K; this entails no loss of generality because $\mathbb{I}^{\omega_1}$ is homogenous.

We use the proof of Theorem 3 to find a map $h : \mathbb{H} \to \mathbb{I}^{\delta_0}$ such that conditions 1 and 2 are met with K replaced by K_0. We can and will assume that $h(n) = \mathbf{0}$ for all $n \in \mathbb{N}$ and that $h \in M_1$. We let f_β be the βth coordinate of h of course.

Now let $\alpha \in \omega_1$ and assume that the maps f_β have been found for $\beta < \delta_\alpha$ such that $e_\alpha = \triangle_{\beta<\delta_\alpha} f_\beta$ belongs to $M_{\alpha+1}$ and meets conditions 1 and 2 with K replaced by K_α, and such that $f_\beta(n) = 0$ for all β and n.

Denote the family of elements of $\mathcal{B}$ that are supported in $\delta_{\alpha+1}$ by $\mathcal{C}$ and observe that $\mathcal{C} \subseteq M_{\alpha+1}$ — every element is defined from finitely many elements of $M_{\alpha+1}$: finitely many ordinals and rational numbers.

Let $\{U_n : n \in \omega\}$ enumerate a decreasing neighbourhood base for $K_{\alpha+1}$, where each U_n is a finite union of elements of $\mathcal{C}$ and arcwise connected. Furthermore we enumerate the elements of $\mathcal{C}$ that meet $K_{\alpha+1}$ as $\{V_n : n \in \omega\}$.

Let $n \in \omega$ and consider U_n and the sets V_i for $i \leqslant n$. The set of ordinals determined by these open sets is finite and may be split into two parts: $F_{n,0}$, those below δ_α, and $F_{n,1}$, the rest. One can express the facts that $K_{\alpha+1} \subseteq U_n$ and that each V_i meets $K_{\alpha+1}$ in one formula that involves as constants the elements of $F_{n,1}$ and some elements of M_α, notably K, the elements of $F_{n,0}$ and some rational numbers. For example if $V_0 = \pi_{\eta_1}^{\leftarrow}\bigl[(q_1,q_2)\bigr] \cap \pi_{\eta_2}^{\leftarrow}\bigl[(q_3,q_4)\bigr]$ then $V_0 \cap K \neq \varnothing$ abbreviates

$$(\exists x \in K)(q_1 < x_{\eta_1} < q_2 \wedge q_3 < x_{\eta_2} < q_4)$$

By elementarity one can find, in the interval $(\max F_{n,0}, \delta_\alpha)$, a finite set of ordinals G_n that can replace $F_{n,1}$ without changing the validity of the formula. Using G_n we can build open sets U_n' and $V_{n,i}$ for $i \leqslant n$ that are reflections of U_n and the V_i. In the above example this means that if, say $\eta_1 \in M_\alpha$ but $\eta_2 \notin M_\alpha$, there must be an ϵ in M_α (larger than η_1) such that

$$(\exists x \in K)(q_1 < x_{\eta_1} < q_2 \wedge q_3 < x_\epsilon < q_4)$$

If this were the case $n=0$ then we would take $V_{0,0} = \pi_{\eta_1}^{\leftarrow}\bigl[(q_1,q_2)\bigr] \cap \pi_{\epsilon}^{\leftarrow}\bigl[(q_3,q_4)\bigr]$.

We can therefore find, given a natural number N, two natural numbers k and l bigger than N and such that $e_\alpha\bigl[[k,\infty)\bigr] \subseteq U_n'$ and every $V_{n,i}$ meets $e_\alpha\bigl[[k,l]\bigr]$.

Use this to find a strictly increasing sequence $\langle k_n\rangle_n$ in $\mathbb{N}$ such that for every n we have $e_\alpha\bigl[[k_n,\infty)\bigr] \subseteq U_n'$ and $V_{n,i} \cap e_\alpha\bigl[[k_n,k_{n+1}]\bigr] \neq \varnothing$ for $i \leqslant n$.

Observe that the sets $F_{n,1}$ are increasing and that their union is the interval $[\delta_\alpha, \delta_{\alpha+1})$. So given β in this interval the set of n with $\beta \in F_{n,1}$ is a final segment of ω. For each such n let β_n be the element of G_n that corresponds to β. Finally define

$$f_\beta(t) = \begin{cases} 0 & \text{if } t \in [k_n,k_{n+1}] \text{ and } \beta \notin F_{n,1}, \text{ and} \\ f_{\beta_n}(t) & \text{if } t \in [k_n,k_{n+1}] \text{ and } \beta \in F_{n,1}. \end{cases}$$

Elementarity considerations will tell us that these f_β are as required. We have $e_{\alpha+1}\bigl[[k_n,k_{n+1}]\bigr] \subseteq U_n$ *because* $e_\alpha\bigl[[k_n,k_{n+1}]\bigr] \subseteq U_n'$ and $V_i \cap e_{\alpha+1}\bigl[[k_n,k_{n+1}]\bigr] \neq \varnothing$ *because* $V_{n,i} \cap e_\alpha\bigl[[k_n,k_{n+1}]\bigr] \neq \varnothing$ whenever $i \leqslant n$.

Finally note that that the whole construction can be considered to have taken place in $M_{\alpha+2}$ so that the result is in $M_{\alpha+2}$ as well.

In the end the full diagonal mapping $\triangle_{\beta<\omega_1} f_\beta$ is as required.

Remark 3.3. Let us note that in all three proofs it was necessary to map the set ω^* to a single point. In the first proof to get an $\aleph_1$-saturated structure and in the other proofs to make certain inductions work.

4. Other universal continua

We can exhibit several other natural universal continua: for every $n \in \mathbb{N}$ the remainder $(\mathbb{R}^n)^*$ maps onto $\mathbb{H}^*$ and hence onto every continuum of weight $\aleph_1$. Thus we find, under CH, for every n a universal continuum of weight $\mathfrak{c}$ and of dimension n.

There is one more natural type of continuum that we want to deal with. These we find in the remainder of the space $\mathbb{M} = \omega \times \mathbb{I}$ considered above. For every $u \in \omega^*$ the set

$$\mathbb{I}_u = \bigcap_{A \in u} \operatorname{cl}(A \times \mathbb{I})$$

is a component of $\mathbb{M}^*$. Each of our three proof can be modified to show that every continuum $\mathbb{I}_u$ also maps onto every continuum of weight $\aleph_1$.

This is easiest for the model-theoretic approach. Indeed, it is very easy to show that $\mathbb{I}_u$ has an $\aleph_1$-saturated base for its closed sets: let $\mathcal{I}$ be the family of finite unions of closed intervals in $\mathbb{I}$ with rational end points. The product $\mathcal{I}^\omega$ is naturally isomorphic to a base for the closed sets of $\mathbb{M}$ and the ultrapower $\mathcal{I}^\omega/u$ is naturally isomorphic to a base for the closed subsets of $\mathbb{I}_u$. It is well-known that such ultrapowers are $\aleph_1$-saturated.

To show how to adapt the other two proofs we use the map q from $\mathbb{M}^*$ onto $\mathbb{H}^*$ defined by $q\bigl(\langle n,x\rangle\bigr) = n+x$. The modifications will be such that the final map will map each of the continua $q[\mathbb{I}_u]$ onto the target continuum as well.

The first step is to adapt the proof of Theorem 3. This is quite straightforward: if we make sure that, for every n, the finite set $\{a_{n,1},\ldots,a_{n,k_n}\}$ contains the points a_0, …, a_{n-1} then it follows easily that $A \subseteq \beta e[\mathbb{I}_u]$ and hence $\beta e[\mathbb{I}_u] = M$ for every u.

The direct algebraic construction. As a result of the adaptation of Theorem 3 we know that we can ensure that for every $F \in \mathcal{C}_0$ there is a natural number n such that $\psi_0(F)$ meets $[m,m+1]$ for every $m \geqslant n$.

All that is needed to do then is to choose the numbers m_n so large that for every $i<n$ and every $m \geqslant m_n$ the set $\psi_\alpha(c_i^n)$ meets the interval $[m,m+1]$.

The topological proof. In this case the fact that h maps every $q[\mathbb{I}_u]$ onto K_0 translates to: if V is an open set that meets K_0 then there is a natural number n such that V meets $h\bigl[[m,m+1]\bigr]$ for every $m \geqslant n$.

The modification needed now is to demand each time that $V_{n,i}$ meet $e_\alpha\bigl[[k,k+1]\bigr]$ for all $k>k_n$.

In [6] it is shown that, under CH, all continua $\mathbb{I}_u$ are mutually homeomorphic. This gives us a second one-dimensional universal continuum of weight $\mathfrak{c}$. It is different from $\mathbb{H}^*$ because $\mathbb{H}^*$ is indecomposable and $\mathbb{I}_u$ is not.

5. Extensions and limitations

In this section we find analogues of some standard results about continuous images of ω^*.

First we extend our main result by showing that under Martin's Axiom (MA) every continuum of weight less than $\mathfrak{c}$ is a continuous image of $\mathbb{H}^*$.

Next we give some limiting results. We show that in the Cohen model the long segment of length ω_2 is not a continuous image of $\mathbb{H}^*$. This shows that an extra assumption like MA is necessary to push Theorem 1 from $\aleph_1$ to $\aleph_2$. The Cohen model result also shows that in Theorem 1 we cannot replace $\aleph_1$ by $\mathfrak{c}$. We improve this by showing that the Proper Forcing Axiom (PFA) implies that there is a continuum of weight $\mathfrak{c}$ that is not a continuous image of $\mathbb{H}^*$, so not even MA is strong enough to allow us to prove Theorem 1 for $\mathfrak{c}$ instead of $\aleph_1$.

Martin's Axiom. We show that Martin's Axiom (MA) implies that all continua of weight less than $\mathfrak{c}$ are continuous images of $\mathbb{H}^*$. We need the following result from [17].

Theorem 5.1 (MA). *If X is a compact space of weight less than $\mathfrak{c}$ then there is a compactification of ω with X as its remainder.*

Using this theorem and the ideas from the proof of Theorem 3 it is now quite straightforward to prove the following theorem by an extra application of MA.

Theorem 5.2 (MA). *Let K be a continuum of weight less than $\mathfrak{c}$. Then K is a continuous image of $\mathbb{H}^*$.*

Proof. It suffices to construct a compactification of $\mathbb{H}$ with K as its remainder. Let κ be the weight of K and let $\gamma\omega$ be a compactification of ω with K as its remainder. We embed $\gamma\omega$ into $\mathbb{I}^\kappa$ and we note the following two facts.

1. If U is a neighbourhood of K in $\mathbb{I}^\kappa$ then $\omega \setminus U$ is finite; and
2. K has arbitrarily small arcwise connected neighbourhoods.

The first fact is immediate; the second follows by noting that every basic open set in $\mathbb{I}^\kappa$ is arcwise connected and that every neighbourhood of K contains a neighbourhood that is both a finite union of basic open sets and connected.

There is a map $h : \mathbb{H} \to \mathbb{I}^\kappa$ that almost gives the desired compactification of $\mathbb{H}$: the piecewise linear map that connects the natural numbers in their natural order. This map is defined coordinatewise by $h_\alpha(n+t) = (1-t)n_\alpha + t(n+1)_\alpha$. Here n_α denotes the αth coordinate of the point of $\gamma\omega$ that corresponds to n. We need to rectify a few problems: 1) h need not be one-to-one, 2) the intersection $h[\mathbb{H}] \cap K$ may be nonempty, and 3) K may be a proper subset of $\bigcap_{n\in\omega} \operatorname{cl} h\bigl[[n,\infty)\bigr]$.

The first two may be solved by introducing one more coordinate: the map $t \mapsto \bigl\langle 2^{-t}, h(t)\bigr\rangle$ from $\mathbb{H}$ to $\mathbb{I} \times \mathbb{I}^\kappa$ does not suffer from problems 1 and 2.

To remedy 3 we introduce the following partial order $\mathbb{P}$: an element of $\mathbb{P}$ is of the form $p = \langle n_p, f_p, \epsilon_p, F_p, U_p\rangle$, where $n_p \in \omega$, U_p is an arcwise connected neighbourhood of K that contains $\omega \setminus n_p$, $f_p : [0, n_p] \to \mathbb{I}^\kappa$ is such that $f_p(n) = n$ for $n \leqslant n_p$ — so in particular $f_p(n_p) \in U_p$, $\epsilon_p > 0$ and $F_p \in [\kappa]^{<\omega}$.

The ordering is defined as follows: $p \leqslant q$ iff $n_p \geqslant n_q$, $\epsilon_p \leqslant \epsilon_q$, $U_p \subseteq U_q$, $F_p \supseteq F_q$, $f_p\bigl[[n_q, n_p]\bigr] \subseteq U_q$ and for all $x \in [0, n_q]$ and for all $\alpha \in F_q$ we have $\bigl|\pi_\alpha(f_p(x)) - \pi_\alpha(f_q(x))\bigr| < \epsilon_q$.

It should be clear that by making F_p larger or ϵ_p and U_p smaller one automatically obtains a stronger condition in $\mathbb{P}$; it follows that the following sets are dense in $\mathbb{P}$:

- $D_{1,\alpha} = \{p : \alpha \in F_p\}$, where $\alpha \in \kappa$;
- $D_{2,n} = \{p : \epsilon_p < 2^{-n}\}$, where $n \in \omega$; and
- $D_{3,U} = \{p : U_p \subseteq U\}$, where U is a basic neighbourhood of K.

We must also consider $D_{4,n} = \{p : n_p \geqslant n\}$, where $n \in \omega$. To see that such a set is dense let n be given. Now if $p \in \mathbb{P}$ and $n_p < n$ then, because U_p is arcwise connected and $\{n_p, \ldots, n\} \subseteq U_p$, we can extend f_p to $g : [0, n] \to \mathbb{I}^\kappa$ with $g\bigl[[n_p, n]\bigr] \subseteq U_p$. It follows that $\langle n, g, \epsilon_p, F_p, U_p\rangle$ is in $D_{4,n}$ and below p.

If G is a filter that meets all of the aforementioned κ many dense sets then the net $\langle f_p\rangle_{p\in G}$ converges uniformly on each compact subset of $\mathbb{H}$ to a map f, which is continuous. For this map f we do have $K = \bigcap_{n\in\omega} \operatorname{cl} f\bigl[[n,\infty)\bigr]$. If we add the extra coordinate to f then we get an embedding $\bar{f} : t \mapsto \bigl\langle 2^{-t}, f(t)\bigr\rangle$ of $\mathbb{H}$ into $\mathbb{I} \times \mathbb{I}^\kappa$ that satisfies $\operatorname{cl} \bar{f}[\mathbb{H}] = \bar{f}[\mathbb{H}] \cup K$.

It remains to show that $\mathbb{P}$ satisfies the ccc. So let Q be an uncountable subset of $\mathbb{P}$. We may and do assume without loss of generality that there are fixed n and ϵ such that $n_p = n$ and $\epsilon_p \geqslant \epsilon$ for all $p \in Q$. We also assume that $\{F_p : p \in Q\}$ is a Δ-system with root F.

With the uniform metric the function space $C\bigl([0,n], \mathbb{R}^F\bigr)$ is separable, hence the set $\{\pi_F \circ f_p : p \in Q\}$ has a complete accumulation point f in it. We therefore assume, again without loss of generality, that for all $p \in Q$ we have $d(f, \pi_F \circ f_p) < \epsilon$.

We show that any two elements of Q are compatible. Indeed, given p and q define $g : [0, n] \to \mathbb{I}^\kappa$ by

$$g_\alpha(x) = \begin{cases} f_\alpha(x) & \text{if } \alpha \in F, \\ f_{p,\alpha}(x) & \text{if } \alpha \in F_p \setminus F, \\ f_{q,\alpha}(x) & \text{if } \alpha \in F_q \setminus F, \text{ and} \\ h_\alpha(x) & \text{otherwise.} \end{cases}$$

The quintuple $\langle n, g, \epsilon, F_p \cup F_q, U_p \cap U_q\rangle$ is below both p and q. □

The Cohen Model. In [10, Chapter 12] Kunen investigated for what cardinals κ the σ-algebra $\mathcal{R}(\kappa)$ generated by the family of rectangles equals the full power set of κ^2, where a *rectangle* is a set of the form $X \times Y$ with $X, Y \subseteq \kappa$.

The following lemma, from [10], gives a convenient sufficient condition for a set to belong to $\mathcal{R}(\kappa)$.

Lemma 5.3. *If $X \subseteq \kappa^2$ and if one can find, for all α, subsets x_α and y_α of ω such that $\langle\alpha,\beta\rangle \in X$ iff $x_\alpha \cap y_\beta$ is infinite, then X belongs to $\mathcal{R}(\kappa)$.*

Proof. Define $X_n = \{\alpha : n \in x_\alpha\}$ and $Y_n = \{\alpha : n \in Y_n\}$. Now observe that $\langle\alpha,\beta\rangle \in X$ iff $(\forall m)(\exists n \geqslant m)(n \in x_\alpha \cap y_\beta)$. Therefore $X = \bigcap_m \bigcup_{n\geqslant m} X_n \times Y_n$. $\square$

Using this lemma Kunen showed that $\mathcal{R}(\omega_1) = \mathcal{P}(\omega_1^2)$ and that MA implies $\mathcal{R}(\kappa) = \mathcal{P}(\kappa^2)$ for every $\kappa < \mathfrak{c}$.

Kunen complemented this by showing that in the model obtained by adding (at least) $\aleph_2$ Cohen reals to a model of CH the set $L = \bigl\{\langle\alpha,\beta\rangle : \alpha < \beta < \omega_2\bigr\}$ does not belong to $\mathcal{R}(\omega_2)$.

Using the results mentioned above it is nearly immediate that in the Cohen model the ordinal space $\omega_2 + 1$ is not a continuous image of ω^*; this is very likely the reason that several authors refer to [10] as the source for this result. Indeed, from a continuous onto mapping one immediately obtains a sequence $\langle x_\alpha : \alpha < \omega_2\rangle$ of subsets of ω such that $x_\beta \subset^* x_\alpha$ whenever $\alpha < \beta$. For each α let y_α be the complement of x_α. It follows that $\alpha < \beta$ iff $x_\alpha \cap y_\beta$ is infinite and so, by Lemma 5.3, $L \in \mathcal{R}(\omega_2)$.

We show that a similar result can be formulated a proved for $\mathbb{H}^*$.

Theorem 5.4. *In the model obtained by adding (at least) $\aleph_2$ Cohen reals to a model of CH the long segment of length ω_2 is not a continuous image of $\mathbb{H}^*$.*

The *long segment* L_{ω_2} of length ω_2 is the set $\omega_2 \times [0,1) \cup \{\omega_2\}$ ordered lexicographically.

We need the following adaptation of Lemma 5.3.

Lemma 5.5. *If there is, with respect to the mod finite order, a strictly increasing ω_2-sequence $\langle f_\alpha : \alpha < \omega_2\rangle$ in $\mathbb{R}^\omega$ then $L \in \mathcal{R}(\omega_2)$.*

Proof. Much as in the proof of Lemma 5.3 define, for $n \in \omega$ and $q \in \mathbb{Q}$, sets $X_{n,q}$ and $Y_{n,q}$ by $X_{n,q} = \{\alpha : q > f_\alpha(n)\}$ and $Y_{n,q} = \omega_2 \setminus X_{n,q}$. It is easily seen that $L = \bigcap_m \bigcup_{n\geqslant m, q\in\mathbb{Q}} X_{n,q} \times Y_{n,q}$. $\square$

To show that in the Cohen model there is no map from $\mathbb{H}^*$ onto L_{ω_2} we show that the existence of such a map entails the existence of sequence $\langle x_\alpha : \alpha < \omega_2\rangle$ of subsets of ω such that $x_\beta \subset^* x_\alpha$ whenever $\alpha < \beta$ or of a sequence $\langle f_\alpha : \alpha < \omega_2\rangle$ in $\mathbb{R}^\omega$ as in Lemma 5.5.

So assume $f : \mathbb{H}^* \to L_{\omega_2}$ is a continuous surjection. Choose, for each $\alpha < \omega_2$, a standard open set U_α in $\beta\mathbb{H}$ that contains $f^\leftarrow\bigl[[\alpha+1,\omega_2]\bigr]$ but whose closure is disjoint from $f^\leftarrow\bigl[[0,\alpha]\bigr]$. Let $\langle I_n : n \in \omega\rangle$ be the sequence of intervals that determines U_0.

For each $\alpha < \omega_2$ we let $J_\alpha = \{n : I_{0,n} \cap U_\alpha \neq \varnothing\}$. Now either $\langle J_\alpha : \alpha < \omega_2\rangle$ is decreasing with respect to $\subset^*$, in which case we are done, or it becomes constant modulo finite sets and then we may as well assume that it is constant and that $J_\alpha = \omega$ for all α.

Now observe that if $\beta < \alpha$ we have $\operatorname{cl} U_\alpha \cap \mathbb{H}^* \subseteq U_\beta$ so there must be an m such that $\operatorname{cl} U_\alpha \setminus U_\beta \subseteq [0,m]$. We can therefore define functions $f_\alpha : \omega \to \mathbb{R}$ by $f_\alpha(n) = \inf U_\alpha \cap I_n$. The sequence $\langle f_\alpha : \alpha < \omega_2\rangle$ is strictly increasing with respect to $<^*$ and we are done.

The Proper Forcing Axiom. The obvious question whether MA is strong enough to imply that every compact space of weight $\mathfrak{c}$ is a continuous image of ω^* has a negative answer, see for example the surveys by Baumgartner [4] and Van Mill [18].

We show, by obvious modifications of the arguments from [4], that PFA implies there is a continuum of weight $\mathfrak{c}$ that is not a continuous image of $\mathbb{H}^*$.

As PFA implies $\mathsf{MA}+\mathfrak{c}=\aleph_2$ we know that there is an $\aleph_2$-saturated ultrafilter u on ω, see [7]. Such an ultrafilter has the property that every ultrapower over it of any countable structure is $\aleph_2$-saturated. We apply this to the linearly ordered set $\mathbb{Q}$ of rational numbers and find an embedding of the lexicographically ordered tree $T=\mathbf{2}^{<\omega_2}$ into $\mathbb{Q}^\omega/u$ and hence into the standard continuum $\mathbb{I}_u$. We show that this standard continuum is not a continuous image of $\mathbb{H}^*$.

Indeed, assume there is a continuous map h from $\mathbb{H}^*$ onto $\mathbb{I}_u$. Let $x\in\mathbf{2}^{\omega_2}$ be such that both $x^\leftarrow(0)$ and $x^\leftarrow(1)$ are cofinal in ω_2; then x determines a $\langle\omega_2,\omega_2^*\rangle$-gap $\bigl\langle l(x),r(x)\bigr\rangle$ in T, where

$$\begin{aligned} l(x)&=\bigl\{s\in T:(\exists\alpha)(\operatorname{dom}s=\alpha+1, s\restriction\alpha=x\restriction\alpha, s(\alpha)=0, x(\alpha)=1)\bigr\}\\ r(x)&=\bigl\{s\in T:(\exists\alpha)(\operatorname{dom}s=\alpha+1, s\restriction\alpha=x\restriction\alpha, x(\alpha)=0, s(\alpha)=1)\bigr\}. \end{aligned}$$

Via the embedding of T into $\mathbb{I}_u$ and the map h we can associate with this gap two sequences $\{U_s:s\in l(x)\}$ and $\{V_s:s\in r(x)\}$ of standard open sets with the properties that

1. if $s\in l(x)$ and $t\in r(x)$ then $U_s\cap V_t=\varnothing$;
2. if $s<t$ in $l(x)$ then $\operatorname{cl}U_s\subseteq U_t$; and
3. if $t<s$ in $r(x)$ then $\operatorname{cl}V_s\subseteq V_t$.

Now the proof, given in [4], that in the Boolean algebra $\mathbb{B}$ there are no $\langle\omega_2,\omega_2^*\rangle$-gaps (assuming PFA) works *mutadis mutandis* to show that no such gaps exist in the lattice $\mathcal{L}$ either. It follows that there must be $L_x\in\mathcal{L}$ such that $U_s\subseteq L_x\subseteq\mathbb{H}^*\setminus V_t$ for all $s\in l(x)$ and $t\in r(x)$.

The assignment $x\mapsto L_x$ is clearly one-to-one; this is impossible however as there are $2^{\aleph_2}$ possible x's and only $\aleph_2$ possible L_x's.

Yet another universal continuum. In the previous subsection we used a $\mathfrak{c}$-saturated ultrafilter u to find a continuum of weight $\mathfrak{c}$, to wit $\mathbb{I}_u$, that is not a continuous image of $\mathbb{H}^*$. The methods used in this paper allows us to show that $\mathbb{I}_u$ is in fact a universal continuum of weight $\mathfrak{c}$.

Indeed, let $\mathcal{I}$ be the family of all finite unions of closed intervals in $\mathbb{I}$ with rational end points. Clearly, $\mathcal{I}$ is a base for the closed sets of $\mathbb{I}$ and so the ultrapower $\mathcal{I}_u=\mathcal{I}^\omega/u$ is a base for the closed sets of $\mathbb{I}_u$; this ultrapower is $\mathfrak{c}$-saturated and hence $\mathfrak{c}^+$-universal. But now the reasoning leading up to Theorem 2.1 can be used to show that $\mathbb{I}_u$ maps onto every continuum of weight $\mathfrak{c}$ or less.

6. Some questions

Of course every result about continuous images of ω^* suggests the possibility of a corresponding one about continuous images of $\mathbb{H}^*$. The more obvious questions are.

Question 6.1 (compare [14]). Is every perfectly normal continuum a continuous image of $\mathbb{H}^*$?

Question 6.2. Is it consistent with $\neg\mathsf{CH}$ (or better still with $\mathsf{MA}+\neg\mathsf{CH}$) that every continuum of weight $\mathfrak{c}$ is a continuous image of $\mathbb{H}^*$?

A more interesting question, raised by G. D. Faulkner and also suggested by the proofs of Theorems 3 and 5.2, is.

Question 6.3. Is every continuum that is a continuous image of ω^* also a continuous image of $\mathbb{H}^*$?

A still more interesting problem is to generalize Parovičenko's characterization of ω^* to the connected case.

Problem 6.4. Find, assuming CH, topological characterizations of the continua $\mathbb{H}^*$ and $\mathbb{I}_u$.

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