Weak covering properties of weak topologies

A. Dow, H. Junnila and J. Pelant[1]

ABSTRACT. We study covering properties of weak topologies of Banach spaces, especially of weak or point–wise topologies of function spaces $C(K)$, for K compact, and we answer questions posed by A.V. Arkhangel'skij, S.P. Gul'ko and R.W. Hansell. Our main results are the following:

– If the weight of K is at most ω_1, then $C_w(K)$ and $C_p(K)$ are hereditarily metaLindelöf.
– Let $\bar{T}$ be the one–point compactification of a tree T. Then $C_p(\bar{T})$ is hereditarily σ–metacompact.
– If T is an infinitely branching full tree of uncountable height and of cardinality bigger than $\mathbf{c}$, then the weak topology of the unit sphere of $C(\bar{T})$ is not σ–fragmented by any metric.
– $C_p(\beta\omega_1)$ is neither metaLindelöf nor σ–relatively metacompact.
– $C_p(\beta\omega_2)$ is not σ–relatively metaLindelöf.
– ($\Diamond$) There exists a scattered K such that $C_p(K)$ is not σ–relatively metacompact.
– ($\boxed{\Diamond}$) There exists a scattered K such that $C_p(K)$ is not σ–relatively metaLindelöf.

AMS Classification 54C35 46B20 54E20 54D30

Introduction. In this paper we present examples to solve some problems on weak covering properties of function spaces. In [Ar$_2$] and [Ar$_3$], A.V. Arkhangel'skii attributed the following problem to S.P. Gul'ko: Is $C_p(K)$ hereditarily metaLindelöf for every compact space K ? In [Han$_1$], R.W. Hansell asked whether $C_p(K)$ is hereditarily "weakly θ–refinable" (i.e., "weakly submetacompact") for every compact space K. We show that the space $C_p(\beta\omega_1)$ is a counterexample to both questions. We show that the space $C_p(\beta\omega_2)$ does not even have the property of "weak $\delta\theta$–refinability" (i.e., "weak submetaLindelöfness"), and we show that it is consistent with ZFC that there exists a scattered compact space K such that $C_p(K)$ fails to have this property.

Besides counterexamples, we also give positive results dealing with weak covering properties. We show that for every Banach space B of density at most ω_1, the weak topology of B is hereditarily metaLindelöf; similarly, for every compact space K of weight at most ω_1, the space $C_p(K)$ is hereditarily metaLindelöf. We study spaces $C_p(\bar{T})$, where $\bar{T}$ is the one–point compactification of a tree T with its usual "tree topology". We show that the topology of such a space $C_p(\bar{T})$ has a "σ–sheltering base" in the sense

[1] Partially supported by the grant GAČR 201/94/0069 and the grant of Czech Acad. Sci 119 401.

of P.Nyikos [Ny]; it follows from this result that $C_p(\bar{T})$ is hereditarily $\sigma-$ metacompact. Using a particular tree, we give an example of a compact scattered space K such that $C_p(K)$ is hereditarily σ–metacompact even though it is not "σ–fragmented", in the sense of J. Jayne, I. Namioka and C. Rogers [JaNaRo$_1$], by any metric.

Notation and terminology. Ordinals are sets of smaller ordinals, and cardinals are "initial ordinals". For a cardinal κ, we also let κ stand for the discrete topological space whose ground–set is κ, and we let $\kappa+1$ stand for the compact space obtained when $\kappa+1$ is equipped the order topology. For a completely regular topological space X, βX denotes the maximal (Čech–Stone) compactification of X, and X^* denotes the remainder $\beta X \setminus X$ of X in βX. Thus $\beta\kappa$ and κ^* denote the Čech–Stone compactification and the remainder of the discrete space on κ.

A family $\mathcal{N}$ of subsets of a topological space X is a *network* for the topology of X provided that every open subset of X can be written as the union of some subfamily of $\mathcal{N}$. Note that a network is a base for the topology if, and only if, it consists of open sets. The *weight* $w(X)$ of a topological space X is the least cardinal of a base for the topology of X and the *density* $d(X)$ of X is the least cardinal of a dense subset of X. Note that $d(X) \leq w(X)$ and, if X is metric (in particular, if X is a Banach space with norm topology), then $d(X) = w(X)$.

If B is a Banach space, we denote by B_w the set B equipped with the weak topology of B. For a compact Hausdorff space K, $C(K)$ denotes the Banach space that one obtains by giving the "supremum norm" for the linear space consisting of all real–valued continuous functions on K. We denote by $C_w(K)$ the set $C(K)$ equipped with the weak topology of $C(K)$ and by $C_p(K)$ the set $C(K)$ equipped with the topology of pointwise convergence (i.e., the topology inherited from the space $\mathbb{R}^K$). Note that the topology of pointwise convergence is weaker than the weak topology of $C(K)$. We shall also need later the following known inequality: $d(C(K)) \leq w(K)$.

A family $\mathcal{G}$ of sets is *point–finite* (*point–countable*) provided that, for every $x \in \bigcup \mathcal{G}$, the family $(\mathcal{G})_x = \{G \in \mathcal{G} : x \in G\}$ if finite (countable). A topological space is *metacompact* (*metaLindelöf*) provided that every open cover of the space has a point–finite (point–countable) open refinement. If a family of sets can be written as the union of countably many point–finite subfamilies, then we say that the family is σ–point–finite. A topological space is *σ–metacompact* provided every open cover of the space has a σ–point–finite open refinement.

Let $\mathcal{L}$ be a family of subsets of a topological space X. We say that $\mathcal{L}$ is a *relatively open* family in X if every $L \in \mathcal{L}$ is open in the subspace $\bigcup \mathcal{L}$ of X. Further, we say that $\mathcal{L}$ is *σ–relatively open* if $\mathcal{L}$ can be written as the union of countably many relatively open families. Note that a family $\mathcal{N}$ of subsets of X is disjoint and relatively open if, and only if, $\mathcal{N}$ is a relatively discrete family (i.e., $\mathcal{N}$ is a discrete family in the subspace $\bigcup \mathcal{N}$). Also note that if $\bigcup_{k \in \omega} \mathcal{N}_k$ is a cover of X such that all the families $\mathcal{N}_k$ are disjoint (point–finite, point countable) and relatively open, then the family $\bigcup_{k \in \omega} \mathcal{N}'_k$, where $\mathcal{N}'_k = \{N \setminus \bigcup_{l<k} \bigcup \mathcal{N}_l : N \in \mathcal{N}_k\}$ for each $k \in \omega$, is a disjoint (point–finite, point–countable) σ–relatively open cover of X.

We introduce the following terminology: we say that a topological space X is *σ–relatively metacompact* (*σ–relatively metaLindelöf*) if every open cover of X has a point–finite (point–countable) σ–relatively open refinement. The reader can easily check that σ–relatively metacompact spaces coincide with the spaces that have been earlier called either "weakly θ–refinable" or "weakly submetacompact" while σ–relatively metaLindelöf spaces coincide with the so–called "weakly $\delta\theta$–refinable" or "weakly submetaLindelöf" spaces. We introduce the new terminology because it seems to be self–explanatory and therefore easier to memorize than the earlier terminology.

We refer the reader to [En], [Ar_1], [Bu], [Ku] and [DeGoZi] for general background and for definitions of terms and conceps used below without definition.

1. Positive results. We begin by mentioning some known results dealing with weak covering properties of function spaces with pointwise topology and Banach spaces with weak topology. Hansell showed in [Han_1] that if a Banach space B admits an equivalent "Kadeč norm" (i.e., a norm $||\cdot||$ such that the norm topology of B and the weak topology of B coincide in the set $\{b \in B : ||b|| = 1\}$), then the weak topology of B has a σ–relatively discrete network, and therefore the space B_w is hereditarily σ–relatively metacompact. One class of Banach spaces admitting Kadeč norms are the spaces $C(K)$ where K is a "Valdivia compact" space; hence for such a space K, the space $C_w(K)$ is hereditarily σ–relatively metacompact; however, as noted by Hansell, it is not known whether $C_p(K)$ is hereditarily σ–relatively metacompact for each Valdivia compact space K. Another result on weak covering properties on function spaces that we want to mention is the recent result of S.P. Gul'ko ([Gu_2]) that for every Corson compact space K, the space $C_p(K)$ is hereditarily σ–metacompact; according to a well known earlier result of Gul'ko ([Gu_1]) and K. Alster and R. Pol ([AlPo]), the space $C_p(K)$ is Lindelöf whenever K is

Corson compact. Not only the spaces of type $C_p(K)$ or $C_w(K)$ but also certain subspaces of such spaces have been considered from the covering–point–of–view; for instance, N.N. Yakovlev showed in [Ya] that every compact subset of a $C_p(K)$–space is hereditarily σ–metacompact, and G. Gruenhage obtained a converse to this result when he proved in [Gru] that a compact space C can be embedded in a $C_p(K)$–space if the space $C \times C$ is hereditarily σ–metacompact.

In our first result, we indicate one class of Banach spaces which are hereditarily metaLindelöf in their weak topologies. We need the following auxiliary result.

1.1 Lemma *Let τ and π be two topologies on X such that $\tau \subset \pi$ and the topology π is metrizable. Assume that there exists a cover $\{F_\alpha : \alpha \in \omega_1\}$ of X such that the following conditions hold for every $\alpha \in \omega_1$:*
(i) F_α is τ–closed.
(ii) F_α is π–separable.
(iii) $\bigcup_{\beta<\alpha} F_{\beta+1}$ is a π–dense subset of F_α.
Then the topology τ is hereditarily metaLindelöf.

Proof. Note that the conditions of the lemma are hereditary with respect to τ–open sets, and that to prove the lemma it therefore suffices to show that τ is metaLindelöf. Let d be a metric on X which induces the topology π; we may assume that d is bounded. For all $x \in X$ and $r > 0$, denote by $B(x,r)$ the d–ball of radius r with center x.

Let $\mathcal{O}$ be an open cover of (X,τ). For every $x \in X$, let

$$r_x = \sup\{r : \text{ there exists } O \in \mathcal{O} \text{ such that } B(x,r) \subset O\},$$

and note that $r_x > 0$, because $\tau \subset \pi$; further, let $O_x \in \mathcal{O}$ be such that $B(x, \frac{3}{4}r_x) \subset O_x$.

For every $\alpha \in \omega_1$, let C_α be a countable π–dense subset of $F_{\alpha+1} \setminus F_\alpha$. We show that the family

$$\mathcal{U} = \{O_z \setminus F_\alpha : \alpha \in \omega \text{ and } z \in C_\alpha\}$$

is a point–countable open refinement of $\mathcal{O}$ in (X,τ). Clearly, $\mathcal{U}$ is a point–countable family, and for every $U \in \mathcal{U}$, the set U is open and it is contained in some set of the family $\mathcal{O}$; hence we need only to show that $\mathcal{U}$ covers X. Let $x \in X$. Denote by η_x the least $\eta \in \omega_1$ such that $x \in F_\eta$. We show that there exists $\alpha < \eta_x$ such that $(F_{\alpha+1} \setminus F_\alpha) \cap B(x, \frac{1}{4}r_x) \neq \emptyset$. If η_x is a successor, let α be the predecessor of η_x, and note that then $x \in (F_{\alpha+1} \setminus F_\alpha) \cap B(x, \frac{1}{4}r_x)$. If η_x is a limit ordinal, then it is a consequence of

condition (iii) that there exists a minimal ordinal $\alpha < \eta_x$ such that $F_{\alpha+1} \cap B(x, \frac{1}{4}r_x) \neq \emptyset$; it now follows from minimality of α that we have $(F_{\alpha+1} \setminus F_\alpha) \cap B(x, \frac{1}{4}r_x) \neq \emptyset$. Since C_α is dense in $F_{\alpha+1} \setminus F_\alpha$, it follows from the foregoing that there exists $z \in C_\alpha \cap B(x, \frac{1}{4}r_x)$. We show that $x \in O_z$. Since $z \in B(x, \frac{1}{4}r_x)$ and $B(x, \frac{3}{4}r_x) \subset O_x$, we have that $B(z, \frac{1}{2}r_x) \subset O_x$ and hence that $r_z \geq \frac{1}{2}r_x$; further, since $B(z, \frac{3}{4}r_z) \subset O_z$, we have that $B(z, \frac{3}{8}r_x) \subset O_z$ and hence that $x \in O_z$. Since $\alpha < \eta_x$, we have that $x \notin F_\alpha$. It follows that $x \in O_z \setminus F_\alpha \in \mathcal{U}$.
$\square$

Remark. Note that if the topologies τ and π and the family $\{F_\alpha : \alpha \in \omega_1\}$ satify the conditions of the lemma, then for every topology τ' lying between τ and π, the topologies τ' and π and the family $\{F_\alpha : \alpha \in \omega_1\}$ satisfy the conditions; as a consequence, every such topology τ' is hereditarily metaLindelöf.

The above lemma, together with well–known results, implies hereditary metaLindelöfness of the weak topologies of certain "small" spaces.

1.2 Theorem *(A) Let B be a Banach space of density at most ω_1. Then the space B_w is hereditarily metaLindelöf.*
(B) Let K be a compact Hausdorff space of weight at most ω_1. Then the spaces $C_w(K)$ and $C_p(K)$ are hereditarily metaLindelöf.

Proof. (A) Denote by π the norm topology of B and by τ the weak topology. Let $\{x_\alpha : \alpha < \omega_1\}$ be a dense subset of (B, π). For each $\alpha < \omega_1$, let F_α be the π–closed linear subspace generated by the set $\{x_\beta : \beta < \alpha\}$. It is a well–known consequence of the Hahn–Banach Theorem that each F_α is τ–closed, and it is easy to check that τ, π and $\{F_\alpha : \alpha \in \omega_1\}$ satisfy also the other conditions of Lemma 1.1; by the lemma, the topology τ is hereditarily metaLindelöf.
(B) Denote by π the norm topology of $C(K)$ and by τ the pointwise topology. We have that $d((C(K), \pi)) = w(K) \leq \omega_1$. Let $\{f_\alpha : \alpha < \omega_1\}$ be a dense subset of $(C(K), \pi)$ such that f_0 is a non–zero constant function. For each $\alpha < \omega_1$, let F_α be the π–closed subalgebra of $C(K)$ generated by the set $\{f_\beta : \beta < \alpha\}$. It is a well–known consequence of the Stone–Weierstrass Theorem that each F_α is τ–closed, and it is easy to check that τ, π and $\{F_\alpha : \alpha \in \omega_1\}$ satisfy also the other conditions of Lemma 1.1; by the lemma, the topology τ is hereditarily metaLindelöf. Since the weak topology of $C(K)$ lies between the topologies τ and π, it follows by the remark made after Lemma 1.1 that also the space $C_w(K)$ is hereditarily metaLindelöf. $\square$

Remark. Note that Theorem 1.2 establishes hereditarily metaLindelöfness of, for example, the spaces $C_w((\omega_1+1)^{\omega_1})$ and $C_p((\omega_1+1)^{\omega_1})$ and, under the Continuum Hypothesis (CH), of the spaces $C_w(K)$ and $C_p(K)$, where K is the "two arrows space".

Next we exhibit a class of $C_p(K)$–spaces which satisfy a covering property even stronger than hereditary metaLindelöfness. To this end, we consider trees T in their usual "tree topology", i.e., in the topology generated by the intervals $(y, x]$, where $x \in T$ and $y < x$. In order to get Hausdorff spaces, we restrict our considerations to "Hausdorff trees", that is, to trees in which every bounded chain has a supremum. From now on, "tree" means a Hausdorff tree. The tree T equipped with its tree topology is a locally compact Hausdorff space, and hence its one–point compactification is a compact Hausdorff space; we denote by $\bar{T}$ the one point compactification of the space T and by ∞ the "point at infinity"; hence $\bar{T} = T \cup \{\infty\}$. We extend the ordering $<$ of T to $\bar{T}$ by letting $t < \infty$ for every $t \in T$.

R. Haydon has made a deep study of Banach spaces $C(\bar{T})$; he has shown (see [Hay$_2$] and [Hay$_3$]) that these spaces serve as important examples and counterexamples in the renorming theory of Banach spaces. We are going to show that all spaces $C_p(\bar{T})$ share a base property which implies that these spaces are hereditarily σ–metacompact. To define the base property, let A be a subset of a set X, and let $\{R_a : a \in A\}$ be an indexed family of subsets of X. We say that the family $\{R_a : a \in A\}$ is a *shelter* (for A in X) provided that $a \in R_a$ for every $a \in A$ and for every $B \subset A$, if $\bigcap_{b\in B} R_b \neq \emptyset$, then there exists a finite set $E \subset B$ such that $B \subset \bigcup_{b\in E} R_b$. Further, we say that an indexed collection $\{\mathcal{R}_a : a \in A\}$ of non-empty families of subsets of X is a *sheltering collection* in X provided that every indexed family $\{R_a : a \in A\}$, where $R_a \in \mathcal{R}_a$ for every $a \in A$, is a shelter for A in X. A *sheltering base* for a topological space X is a sheltering collection $\{\mathcal{B}_x : x \in X\}$ of neighbourhood bases for the points of X. A σ*–sheltering base* is a countable collection $\big\{\{\mathcal{B}_{n,x} : x \in A_n\} : n \in \omega\big\}$ of sheltering collections such that, for every $x \in X$, the family $\bigcup\{\mathcal{B}_{n,x} : n \in \omega \text{ and } x \in A_n\}$ is a neighbourhood base of x.

We mention some examples of spaces with sheltering bases. If X has only one non–isolated point, then X has a sheltering base; in particular, the one–point compactification of a discrete space has a sheltering base. A less trivial example is the following: if X is a metacompact σ–space (i.e., a metacompact space with a σ–discrete closed network), in particular, if X is a stratifiable space, then X has a sheltering base; P.Nyikos observed in [Ny] that every metacompact σ–space has a σ–sheltering base, and the argument outlined by him can be easily adapted to produce a sheltering base.

Sheltering bases and σ–sheltering bases were introduced by Nyikos in [Ny]. He observed that every space with a (σ–)sheltering base is hereditarily (σ–)metacompact; these facts are consequences of the following simple lemma:

1.3 Lemma *([Ny]) Let $\{R_a : a \in A\}$ be a shelter for A in X. Then there exists $B \subset A$ such that the family $\{R_a : a \in B\}$ covers A and is point–finite in X.*

Proof. Let $\prec$ be a well–ordering of A. Use induction on $\prec$ to define the set B by the following condition: for $a \in A$, if $B \cap \{d \in A : d \prec a\}$ has been defined, then put $a \in B$ if we have $a \notin R_b$ whenever $b \prec a$ and $b \in B$; otherwise put $a \notin B$. $\square$

Covering properties, compactness excluded, are usually badly behaved in product spaces; therefore the main motivation for the introduction of the above two classes of spaces is the following result of Nyikos [Ny]: if I is finite (countable) and, for each $i \in I$, X_i has a (σ–)sheltering base, then $\prod_{i \in I} X_i$ has a (σ–)sheltering base. This result is a consequence of the following lemma:

1.4 Lemma *([Ny]) Let $\{R_a : a \in A\}$ be a shelter of A in X and let $\{S_b : b \in B\}$ be a shelter for B in Y. Then $\{R_a \times S_b : (a, b) \in A \times B\}$ is a shelter for $A \times B$ in $X \times Y$.*

That spaces with σ–sheltering bases are relevant in the study of weak topologies, can be seen from the following characterization of Eberlein compact spaces (recall that a compact space is *Eberlein compact* provided that it can be embedded in a Banach space equipped with weak topology).

1.5 Theorem *A compact Hausdorff space is Eberlein compact if, and only if, the space has a σ–sheltering base.*

Proof. *Necessity.* In [Ju_1], it is shown that the topology of an Eberlein compact space has a "σ–point–additively Noetherian base". Nyikos [Ny] has shown that if a space X has a σ–point–additively Noetherian base, then X has a σ–sheltering base.
Sufficiency. If X has a σ–sheltering base, then the results of [Ny] mentioned above imply that $X \times X$ is hereditarily σ–metacompact. Gruenhage has shown in [Gru] that a compact Hausdorff space X is Eberlein compact provided that $X \times X$ is hereditarily σ–metacompact. $\square$

Note that the one point–compactification $A(\kappa)$ of the discrete space on κ is homeomorphic with a space $\bar{T}$, where T is a tree (of "height" ≤ 2). It is a well–known consequence of results of Amir, Lindenstrauss and Grothendieck that every weakly compact subset K of a Banach space can be embedded in some space $C_p(A(\kappa))$; hence K can be embedded in some space $C_p(\bar{T})$. These considerations show that the existence of a σ–sheltering base for an Eberlein compact space is also a consequence of our next theorem.

1.6 Theorem *Let T be a tree. Then $C_p(\bar{T})$ has a σ–sheltering base.*

Proof. If T is finite, then $C_p(\bar{T})$ is metrizable and the result follows. Assume that T is infinite. Then a standard argument shows that $C_p(\bar{T})$ is homeomorphic with its subspace $C_p(\bar{T})_\infty$ consisting of those functions which vanish at infinity (i.e., $f(\infty)=0$); we denote this subspace by C and we shall prove that C has a σ–sheltering base.

Denote by $\mathcal{I}$ the countable collection consisting of all finite families $\mathcal{P}$ of open intervals in $\mathbb{R}$ with rational endpoints such that no two members of $\mathcal{P}$ have common points or common endpoints. For $\mathcal{P}\in\mathcal{I}$, denote by $m(\mathcal{P})$ the *mesh* $\max\{q-p:(p,q)\in\mathcal{P}\}$ of $\mathcal{P}$ and by $\delta(\mathcal{P})$ the minimum of distances between two sets in $\mathcal{P}$. For each $\mathcal{P}\in\mathcal{I}$, let $F(\mathcal{P})=\{f\in C: f(\bar{T})\subset\bigcup\mathcal{P}\}$. Note that for every $f\in C$, since $\bar{T}$ is scattered, the subset $f(\bar{T})$ of $\mathbb{R}$ is scattered and hence nowhere dense; as a consequence, for each $\epsilon>0$, there exists $\mathcal{P}\in\mathcal{I}$ such that $f\in F(\mathcal{P})$ and $m(\mathcal{P})<\epsilon$ (cf. with Lemma 1 of [Hay$_1$]).

Let $\mathcal{P}\in\mathcal{I}$ and $f\in F(\mathcal{P})$. Then the family $\{f^{-1}(P):P\in\mathcal{P}\}$ is a disjoint clopen cover of $\bar{T}$. It follows that there exists a finite subset $Z(f,\mathcal{P})$ of $\bar{T}$ such that $Z(f,\mathcal{P})$ is closed under infimums and the following holds: for every $P\in\mathcal{P}$ and for each $a\in f^{-1}(P)$, there exists $z\in Z(f,\mathcal{P})$ such that $a\le z$ and $\{b\in\bar{T}:a\le b\le z\}\subset f^{-1}(P)$.

For all $\mathcal{P}\in\mathcal{I}$, $f\in F(\mathcal{P})$ and $A\in[\bar{T}]^{<\omega}$, we set

$$B_{\mathcal{P}}(f,A)=\{g\in C: g(a)\in \mathrm{St}(f(a),\mathcal{P}) \text{ for every } a\in A\}\,.$$

Further, for all $\mathcal{P}\in\mathcal{I}$ and $f\in F(\mathcal{P})$, we set

$$\mathcal{B}_{\mathcal{P}}(f)=\{B_{\mathcal{P}}(f,A): A\in[\bar{T}]^{<\omega} \text{ and } Z(f,\mathcal{P})\subset A\}\,.$$

It is easy to see that, for every $f\in C$, the collection $\bigcup\{\mathcal{B}_{\mathcal{P}}(f):\mathcal{P}\in\mathcal{I} \text{ and } f\in F(\mathcal{P})\}$ is a neighbourhood base of f in C. To complete the proof, let $\mathcal{P}\in\mathcal{I}$. We show that the collection $\mathcal{B}_{\mathcal{P}}=\{\mathcal{B}_{\mathcal{P}}(f):f\in F(\mathcal{P})\}$ is sheltering. Assume that this is not true. Then we can find $g\in C$ and, for $k\in\omega$, $f_k\in F(\mathcal{P})$ and $A_k\in[\bar{T}]^{<\omega}$ with $Z(f_k,\mathcal{P})\subset A_k$ such that $g\in\bigcap_{k\in\omega}B_{\mathcal{P}}(f_k,A_k)$ and, for every $k\in\omega$, $f_k\notin\bigcup_{i<k}B_{\mathcal{P}}(f_i,A_i)$. Note that, for all $i<k<\omega$, there exists $a\in A_i$ such that $f_k(a)\notin\mathrm{St}(f_i(a),\mathcal{P})$; using finiteness of the family $\mathcal{P}$ and the sets A_k, we may assume, without loss of generality, that there are $a_i \in A_i$, for $i \in \omega$, and $Q, R \in \mathcal{P}$ such that $Q \neq R$ and, for all $i < k < \omega$, $f_i(a_i) \in Q$ and $f_k(a_i) \in R$. For every $i \in \omega$, since $g \in B_{\mathcal{P}}(f_i, A_i)$, we have that $g(a_i) \in Q$. Set $\delta = \delta(\mathcal{P})$. We can write $\bar{T}$ as a finite union $\bigcup \mathcal{E}$ such that, for each $E \in \mathcal{E}$, the diameter of $g(E)$ is less that δ and E is either an interval of T or the complement, in $\bar{T}$, of finitely many "initial parts" of T (i.e., of intervals $[a, x] \subset T$, where a is the root of T). We may assume that there exists $E \in \mathcal{E}$ such that $a_i \in E$ for every $i \in \omega$. Now we consider two cases. Assume first that E is an interval of T. Then there exist $i, k \in \omega$ such that $i < k$ and $a_i < a_k$. We have that $a_i \in f_k^{-1}(R)$ and it follows that there exists $z \in Z(f_k, \mathcal{P})$ such that $z \geq a_i$ and $f_k(x) \in R$ whenever $a_i \leq x \leq z$. Since $Z(f_k, \mathcal{P})$ is closed under infima, the infimum of a_k and z is in $Z(f_k, \mathcal{P})$; denote this infimum by b and note that it is a consequence of the inequalities $a_i \leq b \leq a_k$ that $b \in E$. We have that $a_i \leq b \leq z$ and hence that $f_k(b) \in R$; it follows, since $g \in B_{\mathcal{P}}(f_k, A_k)$ and $b \in Z(f_k, \mathcal{P}) \subset A_k$, that $g(b) \in R$. It follows from the foregoing that $g(b)$ and $g(a_i)$ are in different sets of the family $\mathcal{P}$; this however, is a contradiction, since $a_i, b \in E$ and $\operatorname{diam} g(E) < \delta = \delta(\mathcal{P})$. To deal with the other case, assume that E is the complement, in $\bar{T}$, of finitely many initial parts of T. Let i and k be any natural numbers satisfying the inequality $i < k$. We have that $a_i \in f_k^{-1}(R)$ and it follows that there exists $z \in Z(f_k, \mathcal{P})$ such that $z \geq a_i$ and $f_k(x) \in R$ whenever $a_i \leq x \leq z$. Note that, in the case at hand, it follows from the condition $z \geq a_i \in E$ that $z \in E$; hence we can set $b = z$ and proceed as in the first case to derive a contradiction. □

1.7 Corollary *Let T be a tree. Then $C_p(\bar{T})$ is hereditarily σ–metacompact.*

Note that we cannot expect to improve the result of the corollary in any obvious way because every Eberlein compact space can be embedded in some space $C_p(\bar{T})$ and there are examples of Eberlein compact spaces which are not hereditarily metacompact (see [Ya] or [Ju$_2$]) or hereditarily screenable (see [BeSt]).

We do not know whether the results corresponding to Theorem 1.6 and its corollary hold for the weak topology of $C(\bar{T})$ instead of the pointwise topology. However, since every tree–space is scattered and since the relative weak and pointwise topologies coincide on every bounded subset of $C(K)$ whenever K is compact and scattered, we see from Corollary 1.7 that every space $C_w(\bar{T})$ is the union of countably many hereditarily σ–metacompact subspaces; therefore we get the following result.

1.8 Corollary *Let T be a tree. Then $C_w(\bar{T})$ is hereditarily σ–relatively metacompact.*

Remark. The last corollary and Examples 2.2 and 2.3 below were combined in [HoPe] with well–known measure–theoretic results to produce Banach spaces which are Radon but not σ–fragmentable.

2. Counterexamples. We are going to present examples of function spaces, whose pointwise topologies have "bad" covering properties. However, first we study some particular spaces from the class of "good" spaces considered above: we give examples of trees whose function spaces with the pointwise topology, even though hereditarily σ–metacompact, fail to have good fragmentability properties.

Recall that a topology τ of a set X or the topological space (X,τ) is said to be *fragmented* by a metric d defined on $X\times X$ provided that, for every $\epsilon>0$, and for every nonempty subset A of X, there exists a nonempty relatively τ– open subset B of A such that the d–diameter of B is at most ϵ ([JaRo]). The topology τ or the space (X,τ) is said to be *σ–fragmented* by the metric d provided that, for every $\epsilon>0$, there exists a countable cover $\{A_n : n\in\omega\}$ of X such that for every $n\in\omega$ and for every nonempty subset A of A_n, there exists a nonempty relatively open subset B of A such that the d–diameter of B is at most ϵ ([JaNaRo$_1$]). Note that if ρ is a separable metric on X, then ρ is σ–fragmenting for every topology of X. Sometimes some extra conditions are put on the fragmenting or σ–fragmenting metric: in Banach space theory, the main interest is in the situation when the weak topology of a Banach space or of some subset of the space is *norm–fragmented* or *norm σ–fragmented*, i.e., fragmented or σ–fragmented by the metric associated with the norm of the space; note that in this situation, the metric topology is finer than the topology under consideration. Hansell ([Han1]) says that a topology τ is *properly* (σ–)fragmented if it is (σ–)fragmented by a metric d such that the metric topology is finer than τ. Another condition on the (σ–)fragmenting metric that has turned out to be useful (see e.g., [JaNaRo$_2$]) is that the metric is lower semicontinuous.

Note that, for a Banach space $(B,||\cdot||)$, norm (σ–)fragmentability of B_w is independent of the norm in the following sense: if B_w is norm (σ–)fragmentable with respect to $||\cdot||$, then it is (σ–)fragmentable with respect to every norm of B which is equivalent to $||\cdot||$. We mention two results of Jayne, Namioka and Rogers on norm σ–fragmentability: the space $(\ell^\infty)_w$ is not σ–fragmented by the norm of ℓ^∞ ([JaNaRo$_3$]); if B admits a Kadeč norm, then B_w is norm σ–fragmented ([JaNaRo$_2$]).

To verify the non–fragmentability properties of some tree spaces considered below, we use the following topological characterization of (σ–)fragmentability. The result below is an easy consequence of results of N.K. Ribarska. To state the result, we need the following terminology: a family $\mathcal{L}$ of subsets of a set X is *point–separating* provided that for any two distinct points x and y of X, the families $\{L \in \mathcal{L} : x \in L\}$ and $\{L \in \mathcal{L} : y \in L\}$ are different; the family $\mathcal{L}$ is *well–monotone* if $\mathcal{L}$ is well ordered by inclusion.

2.1 Lemma *([Ri$_1$]) A topology τ of a set X is (σ–)fragmented by some metric if, and only if, X has a point–separating family which is the union of countably many well–monotone (relatively) τ–open families.*

With terminology introduced by J. Chaber in [Ch$_1$], the above characterization of fragmentability can also be expressed as follows: a topology τ of X is fragmented by some metric if, and only if, the space (X, τ) is a primitive $\sigma^{\#}$–space.

Note that if the topology of a T_0–space X is not σ–fragmented by any metric, then it follows from the above lemma that X does not have a σ–relatively discrete point–separating family; in particular, X does not have a σ–relatively discrete network.

Let T be a (Hausdorff) tree. For each $t \in T$, let $\downarrow t = \{s \in T : s \le t\}$ and $\uparrow t = \{s \in T : s \ge t\}$. Note that, for each $t \in T$, the set $\downarrow t$ is a non–empty clopen subset of $\bar{T}$ and hence its characteristic function $\chi_{\downarrow t}$ belongs to the unit sphere of $C(\bar{T})$; Haydon [Hay$_3$] calls the topology of T that we obtain by identifying T with the subspace $\{\chi_{\downarrow t} : t \in T\}$ of $C_p(\bar{T})$, the *reverse topology* of the tree T; note that in the reverse topology, a neighbourhood base for an element t of T is formed by the sets $\uparrow t \setminus \bigcup_{s \in A} \uparrow s$, where A is a finite set of immediate successors of t in T (we shall refer to these sets as the "basic reverse neighbourhoods" of t). Note that T, in the reverse topology, embeds not only in $C_p(\bar{T})$ but also in $C_w(\bar{T})$ because the pointwise and the weak topologies agree on the bounded set $\{\chi_{\downarrow t} : t \in T\}$ of $C_p(\bar{T})$. Hence we can study both the spaces $C_p(\bar{T})$ and $C_w(\bar{T})$ by considering the space obtained when T is equipped with its reverse topology (from now on, we shall refer to such a topological space as a *reverse tree*, and we shall denote a reverse tree T by T^r).

For a "complete" tree, the reverse topology is similar to but, in general, not equal with the Lawson topology of the tree (see [Ju$_1$]). Reverse trees have been used as counterexamples in topology, for instance by C.E. Aull in [Au]. As noted implicitly by Aull, the reverse trees are hereditarily ultraparacompact (i.e., every open family has a disjoint open refinement). We shall now show that, even though reverse trees always satisfy strong covering properties, we can find a tree T whose reverse topology is not σ–fragmented by any metric; hence the weak topology of the unit sphere of $C(\bar{T})$ cannot be σ–fragmented by any metric. Note that if no conditions are put on the σ–fragmenting metric, then the existence of a σ–fragmenting metric is always guaranteed for spaces of small cardinality: a T_1–space X is σ–fragmented by a separable metric if, and only if, $|X| \leq \mathbf{c}$; in particular, each Banach space of density at most $\mathbf{c}$ is σ–fragmented by some separable metric. Hence to find the example mentioned above, we have to look at trees of cardinality bigger than $\mathbf{c}$.

All our examples on trees are the kind of "full" trees considered by Haydon in [Hay$_2$] and [Hay$_3$]. Let λ and κ be cardinals. A *full λ–branching tree of height κ* is a tree such that every element has exactly λ immediate successors and every maximal chain is of type κ. Note that any two full full λ– branching trees of height κ are isomorphic. It is convenient to choose fixed representives for these trees. For each ordinal α, let $L_{\alpha,\lambda} = \{f : f \text{ is a mapping } \alpha \to \lambda\}$, and let $T_{\kappa,\lambda} = \bigcup_{\alpha<\kappa} L_{\alpha,\lambda}$. Then the poset $(T_{\kappa,\lambda}, \subset)$ is a full λ–branching tree of height κ; we denote this tree simply by $T_{\kappa,\lambda}$.

2.2 Example Let T be an infinitely branching full tree of uncountable height and of cardinality bigger than $\mathbf{c}$. Then the weak topology of the unit sphere of $C(\bar{T})$ is not σ–fragmented by any metric.

Proof. We may assume that $T = T_{\kappa,\lambda}$ for some uncountable κ and infinite λ. We show that the reverse topology of T is not σ–fragmented by any metric. Note that for every cardinal $\mu < \kappa$, the space $T^r_{\mu,\lambda}$ is a subspace of the space $T^r_{\kappa,\lambda} = T^r$. Hence to show that the T^r is not σ–fragmented by any metric, it suffices to show that, for some $\mu < \kappa$, $T^r_{\mu,\lambda}$ is not σ–fragmented by any metric. This observation allows us to assume that the cofinality of κ is uncountable: otherwise we would have $\kappa_n \nearrow \kappa$ for some regular uncountable cardinals κ_n, $n \in \omega$, and thus $T_{\kappa,\lambda} = \bigcup_{n\in\omega} T_{\kappa_n,\lambda}$ and it would follow that $|T_{\kappa_n,\lambda}| > \mathbf{c}$ for some n, and we could then use $T_{\kappa_n,\lambda}$ instead of T.

According to Lemma 2.1, to show that T^r is not σ–fragmented by any metric, it suffices to show that T^r does not have any σ–well–monotone σ–relatively open point–separating family. Let $\mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n$ be a family of subsets of T^r where each $\mathcal{L}_n$ is well–monotone and relatively open. We assume that $\mathcal{L}$ is point–separating, and we show that this is impossible.

For each $t \in T$, we choose two distinct elements t^* and $t^\#$ from $\uparrow t$ such that we have $\{n \in \omega : t^* \in \bigcup \mathcal{L}_n\} = \{n \in \omega : t^\# \in \bigcup \mathcal{L}_n\}$; this is possible because we have $|\uparrow t| = |T| > \mathbf{c}$; since $\mathcal{L}$ is point–separating, we can find $n_t \in \omega$ and $L(t) \in \mathcal{L}_{n_t}$ such that the set $L(t)$ contains exactly one of the points t^* and $t^\#$; note that we have both $t^* \in \bigcup \mathcal{L}_{n_t}$ and $t^\# \in \bigcup \mathcal{L}_{n_t}$. Also note that, since $L(t)$ is an open neighbourhood, in $\bigcup \mathcal{L}_{n_t}$, of one of the points t^* and $t^\#$, there exists $\varphi(t) \in \uparrow t^* \cup \uparrow t^\#$ such that $\bigcup \mathcal{L}_{n_t} \cap \uparrow \varphi(t) \subset L(t)$.

By transfinite induction, we can define a sequence $\langle t_\alpha \rangle_{\alpha \in \omega_1}$ in T such that, for every $\alpha \in \omega_1$, if $\alpha = \beta + 1$, then $t_\alpha = \varphi(t_\beta)$ and, if α is a limit ordinal, then $t_\alpha = \sup\{t_\beta : \beta < \alpha\}$. Let $n \in \omega$ be such that the set $E = \{\alpha < \omega_1 : n_{t_\alpha} = n\}$ is infinite. Using the well–monotoneness of $\mathcal{L}_n$, we can find $\alpha, \gamma \in E$ such that $\alpha < \gamma$ and $L(t_\alpha) \subset L(t_\gamma)$. Now we have that

$$\{t_\gamma^*, t_\gamma^\#\} \subset \bigcup \mathcal{L}_n \cap \uparrow t_\gamma \subset \bigcup \mathcal{L}_n \cap \uparrow \varphi(t_\alpha) \subset L(t_\alpha) \subset L(t_\gamma)$$

and this is a contradiction since the set $L(t_\gamma)$ was supposed to contain only one of the points t_γ^* and $t_\gamma^\#$. $\square$

Remarks. (i) Note that all three assumptions "infinitely branching", "of uncountable height" and "of cardinality bigger than $\mathbf{c}$" are necessary in the above example: if the tree T is either finitely branching ([Hay$_3$]) or of countable height ([HayRo]), then the Banach space $C(\bar{T})$ has a Kadeč norm and is therefore norm σ–fragmented; if $|T| \leq \mathbf{c}$, then $C(\bar{T})$ is σ–fragmented by some separable metric.
(ii) We can take above, for example, either $T = T_{\mathbf{c}^+,\omega}$ or $T = T_{\omega_1,\mathbf{c}^+}$. On the other hand, it depends on cardinal arithmetic whether we can for take $T = T_{\mathbf{c},\omega}$ or $T = T_{\mathbf{c},\mathbf{c}}$: if $2^\kappa > \mathbf{c}$ for some $\kappa < \mathbf{c}$, then $T_{\mathbf{c},\omega}$ satisfies the conditions of Example 2.2 and therefore the unit sphere of $C(\bar{T}_{\mathbf{c},\omega})$ is not σ–fragmented by any metric in the weak topology, but if $2^\kappa \leq \mathbf{c}$ for every $\kappa < \mathbf{c}$, then $|T_{\mathbf{c},\mathbf{c}}| \leq \mathbf{c}$ and therefore the weak topology of $C(\bar{T}_{\mathbf{c},\mathbf{c}})$ is σ–fragmented by some (separable) metric.

The above example has stronger non–fragmentability properties than an earlier example from [HanJaTa] (see also [JaNaRo$_3$], p. 219): in that example, a Banach subspace B of $l^\infty_{\omega_1}$ was given such that the unit ball of B is not fragmented by any metric. Using trees, we can give a very easy example of such a Banach space; unlike in our previous example, here we can use a "small" tree to obtain the desired Banach space.

2.3 Example For the tree $T = T_{\omega_1,\omega}$, the weak topology of the unit sphere of $C(\bar{T})$ is not fragmented by any metric.

Proof. Note that T^r is exactly the space considered by Aull in [Au]. Chaber [Ch$_2$] has shown that Aull's space is not a primitive $\sigma^{\#}$–space and hence the conclusion follows by the remark made after Lemma 2.1. $\square$

The space $C(\bar{T})$ of the preceding example as well as the space $B \subset l^{\infty}_{\omega_1}$ mentioned before the example are Banach spaces of cardinality $\mathbf{c}$. It follows that both admit separable, hence σ–fragmenting, metrics. However, Ribarska [Ri$_2$] has shown that if a space is σ–fragmented by some lower semicontinuous metric, then the space is fragmented by some other metric. Hence the weak topologies of the unit spheres of the Banach spaces $C(\bar{T})$ and B are not σ–fragmented by any lower semicontinuous metrics. In particular, the weak topologies of the unit spheres are not norm–σ–fragmented.

We record here an interesting property of reverse full trees; with the help of this property, we can use full trees to to solve a problem from [Ar$_3$].

2.4 Lemma *Let $T = T_{\kappa,\lambda}$, where the cardinal κ is infinite and regular. Then every dense subspace of T^r contains a homeomorphic copy of T^r.*

Proof. Let $\alpha < \kappa$, and let $f \in L_{\alpha,\lambda}$. We define $\delta_f \in \lambda$ as follows: if α is a successor, say $\alpha = \beta + 1$, then we set $\delta_f = f(\beta)$; if α is a limit, then we set $\delta_f = 0$. For every $\gamma \in \lambda$, we denote by $f^{(\gamma)}$ the extension of f to a mapping $\alpha + 1 \to \lambda$ defined by setting $f^{(\gamma)}(\alpha) = \gamma$.

Note that it follows from regularity of κ that if F is any chain of T with $|F| < \kappa$, then $\sup F \in T$. This observation allows us to use recursion to define an order–isomorphism φ from T onto a subset of D: for the root $\emptyset$ of T, we let $\varphi(\emptyset)$ be any element of D; if $f \in T$ is such that $\varphi(g)$ has been already defined for all predecessors g of f in T, then we set $h = \sup_{g<f} \varphi(g)$ and we choose $\varphi(f)$ to be an element of the intersection of open set $\uparrow h^{(\delta_f)}$ with the dense set D.

We leave it to reader to verify that the order–isomorphism φ is also a homeomorphism with respect to the reverse topology. $\square$

Arkhangel'skij asks in Problem 6 of [Ar$_3$] whether every Baire–subspace of a $C_p(K)$–space has a dense set of G_δ–points. We can now easily show that the question has a strong negative answer.

2.5 Example There exists a compact Hausdorff space $\bar{T}$ and a Baire–subspace Y of $C_p(\bar{T})$ such that every dense subspace of Y has a non–G_δ–point.

Proof. Let $T = T_{\omega_1,\omega_1}$. As shown by Haydon in [Hay$_2$] and [Hay$_3$], the space T^r is a Baire space. Since $C_p(\bar{T})$ has a subspace Y homeomorphic with T^r, it suffices to note that every dense subspace of T^r has a non–G_δ–point: this follows from Lemma 2.4, because T^r itself has no G_δ–points. $\square$

Note that it is not a mere coincidence that reverse trees could be used in both the previous two examples. It is easy to see that the following result holds true: if a Baire space is fragmented (properly fragmented) by some metric, then the space has a dense submetrizable (metrizable) subspace.

Now we turn to consider Banach spaces whose weak (or pointwise) topologies have "bad" covering properties. Our first examples of such Banach spaces are spaces l^∞_κ, where κ is an uncountable cardinal; these spaces can also be represented as function spaces $C(\beta\kappa)$, and we shall use this representation below. We must consider uncountable κ's, because it is consistent with the usual axioms of set–theory (ZFC) that for countable κ, the spaces to be considered do not yield the promised counterexamples:

2.6 Example (CH) For every closed subspace K of $\beta\omega$, the spaces $C_w(K)$ and $C_p(K)$ are hereditarily metaLindelöf.

Proof. Under CH, the space $\beta\omega$ has weight ω_1, and hence the result follows from Theorem 1.2.(B). $\square$

In the last problem of [A$_2$] (Problem 76, attributed to Gul'ko), Arkhangel'skij asks whether every $C_p(K)$–space is (hereditarily) metaLindelöf. We shall solve the problem by showing that the space $C_p(\beta\omega_1)$ is not metaLindelöf. It is very easy to see (by using the Δ–System Lemma and the fact that no continuous mapping $\beta\omega_1 \to \omega_1^*$ fixes all uniform ultrafilters of ω_1) that $C_p(\beta\omega_1)$ is not *hereditarily* metaLindelöf: the relatively discrete subset $S = \{\chi_{\{\alpha\}} : \alpha \in \omega_1\}$ has no "point–countable open expansion" (in other words, there is no assignment $\{U(f) : f \in S\}$ of neighbourhoods for the points of S in the space $C_p(\beta\omega_1)$ such that, for every $g \in C_p(\beta\omega_1)$, the set $\{f \in S : g \in U(f)\}$ is countable). A little extra work gives the conclusion that $C_p(\beta\omega_1)$ is not metaLindelöf, because it has a *closed* discrete subset which fails to have a point–countable open expansion.

In the following two examples, Cl denotes closure in the space $\beta\omega_1$, and $U(\omega_1)$ denotes the set of all "uniform" ultrafilters on ω_1.

2.7 Example The space $C_p(\beta\omega_1)$ is not metaLindelöf.

Proof. We show that the closed subspace $X = C_p(\beta\omega_1, \{0,1\})$ of $C_p(\beta\omega_1)$ is not meta-Lindelöf. This follows once we exhibit a closed discrete subspace F of X which has no point–countable open expansion in X.

For every $\alpha < \omega_1$, let f_α be the extension to a continuous mapping $\beta\omega_1 \to \{0,1\}$ of the characteristic function of the subset α of ω_1. Let $F = \{f_\alpha : \alpha \in \omega_1\}$. Note that a function $f : \beta\omega_1 \to \{0,1\}$ belongs to F if, and only if, is the restriction of f to ω_1 is non–increasing; it follows from this characterization that F is a closed subset of X. To see that F is discrete, let $\alpha \in \omega_1$. We shall define a neighbourhood V of f_α in X such that V contains no other points of F. If $\alpha = 0$, then we set $V = \{g \in X : g(0) = 0\}$. If α is a successor, say $\alpha = \beta + 1$, then we set $V = \{g \in X : g(\beta) = 1 \text{ and } g(\alpha) = 0\}$. Finally, if α is a limit ordinal, then we choose a point s from the subset $\mathrm{Cl}\,\alpha \setminus \bigcup_{\beta<\alpha} \mathrm{Cl}\,\beta$ of $\beta\omega_1$ (that this set is non–empty follows from compactness of $\mathrm{Cl}\,\alpha$), and we set $V = \{g \in X : g(s) = 1 \text{ and } g(\alpha) = 0\}$. It is easy to see that, in all the cases considered, $V \cap F = \{f_\alpha\}$.

To show that F has no point–countable open expansion, let $\{U_\alpha : \alpha \in \omega_1\}$ be an open collection in X such that, for each α, $f_\alpha \in U_\alpha$. Define the map $\varphi : \beta\omega_1 \setminus U(\omega_1) \to \omega_1$ by letting $\varphi(p)$ be the least $\alpha \in \omega_1$ such that $p \in \mathrm{Cl}\,\alpha$. For every $\alpha \in \omega_1$, since U_α is an open neighbourhood of f_α in the subspace X of $C_p(\beta\omega_1)$, there exists $E_\alpha \in [\beta\omega_1]^{<\omega}$ such that for every $g \in X$, if $g|E_\alpha = f_\alpha|E_\alpha$, then $g \in U_\alpha$. Let $L \in [\omega_1]^{\omega_1}$ be such that the family $\{\varphi(E_\alpha \setminus U(\omega_1)) : \alpha \in L\}$ is a Δ–system, with root R; for each $\alpha \in L$, let $E'_\alpha = \{p \in E_\alpha \setminus U(\omega_1) : \varphi(p) \notin R\}$ and $S_\alpha = \varphi(E'_\alpha)$. Without loss of generality, we may assume that the set L has been chosen so that $\min L > \max R$; note that it then follows that $f_\alpha(r) = 1$ for all $r \in R$ and $\alpha \in L$.

By induction on $<$, define $\gamma(\alpha) \in L$ and $\beta(\alpha) \in \omega_1$, for $\alpha \in \omega_1$, so that $\beta(0) = \max R$ and the following conditions hold for every α:

1^o $\min S_{\gamma(\alpha)} > \beta(\alpha) + 1$.

2^o $\max S_{\gamma(\alpha)} \leq \beta(\alpha + 1)$.

3^o $\beta(\alpha) \geq \sup\{\beta(\delta) : \delta < \alpha\}$.

For every α, let D_α be the interval $[\beta(\alpha)+1, \beta(\alpha+1)]$ of ω_1 and note that $E'_{\gamma(\alpha)} \subset \mathrm{Cl} D_\alpha$. Let $\mathcal{Q} \subset [\omega_1]^{\omega_1}$ be such that $|\mathcal{Q}| = \omega_2$ and $\mathcal{Q}$ is "almost disjoint" in the sense that for any two members P and Q of $\mathcal{Q}$, we have that $|P \cap Q| \leq \omega$. For each $Q \in \mathcal{Q}$, let $D_Q = \bigcup_{\alpha \in Q} D_\alpha$. Note that also the family $\{D_Q : Q \in \mathcal{Q}\}$ is almost disjoint. It follows that the collection $\mathcal{H} = \{\mathrm{Cl}(D_Q) \cap U(\omega_1) : Q \in \mathcal{Q}\}$ is a disjoint family. Since $|\mathcal{H}| = \omega_2$ and since the set $E = \bigcup\{E_\alpha : \alpha \in \omega_1\}$ has cardinality at most ω_1, we can find $P \in \mathcal{Q}$ such that $\mathrm{Cl}(D_P) \cap E \cap U(\omega_1) = \emptyset$.

We now define a function $h : \omega_1 \to \{0,1\}$ as follows. If $\eta \leq \beta(0)$, then we set $h(\eta) = 1$. Next we define h in the set $D_P = \bigcup_{\alpha \in P} D_\alpha$ as follows: for every $\alpha \in P$, we find disjoint neighborhoods G_p in the subspace $\mathrm{Cl}\, D_\alpha$ for the elements p of the finite set $E'_{\gamma(\alpha)}$, and we define $h(\eta)$, for $\eta \in D_{\alpha,n}$, so that $h(\eta) = f_{\gamma(\alpha)}(p)$ if $\eta \in G_p$ and $h(\eta) = 0$ if η belongs to no set G_p. We complete the definition of h by setting $h(\eta) = 0$ if $\eta > \beta(0)$ and $\eta \notin D_P$.

Let g be the extension of h to a continuous mapping $\beta\omega_1 \to \{0,1\}$. We show that $g \in U_{\gamma(\alpha)}$ for every $\alpha \in P$; note that it follows from this that the family $\mathcal{U}$ is not point–countable at g. Let $\alpha \in P$ and let $s \in E_\alpha$. We show that $g(s) = f_{\gamma(\alpha)}(s)$. Assume that $s \in U(\omega_1)$. Note that then $f_{\gamma(\alpha)}(s) = 0$: since $s \in E_{\gamma(\alpha)} \subset E$, we have that $s \notin \mathrm{Cl}(D_P)$, and it follows from this that $g(s) = 0 = f_{\gamma(\alpha)}(s)$. Assume that $s \notin U(\omega_1)$. If $\varphi(s) \in R$, then $\varphi(s) \leq \beta(0)$ and hence $s \in \mathrm{Cl}\, \beta(0)$ so that $g(s) = 1 = f_{\gamma(\alpha)}(s)$. If $\varphi(s) \notin R$, then $s \in E'_{\gamma(\alpha)}$ and hence $s \in \mathrm{Cl}\,(G_s \cap \omega_1) \subset \mathrm{Cl}\, D_\alpha$; since $h(\eta) = f_{\gamma(\alpha)}(s)$ for every $\eta \in G_s \cap \omega_1$, we have that $g(s) = f_{\gamma(\alpha)}(s)$. We have shown that g agrees with $f_{\gamma(\alpha)}$ in the set $E_{\gamma(\alpha)}$; it follows that $g \in U_{\gamma(\alpha)}$. $\square$

Note that it follows from the above example that some set–theoretic assumption like CH is necessary in Example 2.6: this follows because under the condition $2^{\omega_1} = \mathbf{c}$, the space $\beta\omega_1$ can be embedded in the space $\beta\omega$ (see [vM], p.558); see also the remark following Example 2.9 below.

Next we shall give examples of the failure of σ–relative metacompactness and σ–relative metaLindelöfness in $C_p(K)$–spaces. We do not know whether it is true in ZFC that $\beta\omega$ has a subspace K such that the space $C_p(K)$ is not σ–relatively metacompact, but it is a consequence of the next example that, under CH, this holds for $K = \omega^*$.

We denote by $\beta\omega_1 / U(\omega_1)$ the quotient space obtained when all the uniform ultrafilters of ω_1 have been identified to one point.

2.8 Example The space $C_p\,(\beta\omega_1 / U(\omega_1))$ is not σ–relatively metacompact.

Proof. It is easy to see that the space $C_p\,(\beta\omega_1 / U(\omega_1))$ is (linearly) homeomorphic with the subspace $X = \{f \in C(\beta\omega_1) : f(p) = 0 \text{ for every } p \in U(\omega_1)\}$ of $C_p(\beta\omega_1)$; hence it suffices to show that X is not σ–relatively metacompact. For each $\alpha \in \omega_1$, let

$$F_\alpha = \{f \in C(\beta\omega_1) : f(\gamma) = 1 \text{ for each } \gamma \leq \alpha \text{ and } f(p) = 0 \text{ for every } p \in U(\omega_1)\},$$

and further, let $O_\alpha = \{X \setminus F_\alpha : \alpha \in \omega_1\}$. We show that the open cover $\mathcal{O} = \{O_\alpha : \alpha \in \omega_1\}$ of X has no point–finite σ–relatively open refinement. Assume on the contrary that the cover has a point–finite refinement $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ where each $\mathcal{U}_n$ consists of relatively open subsets of $\bigcup \mathcal{U}_n$. Let

$$A = \{n \in \omega : \nexists \alpha \in \omega_1 \text{ such that } \bigcup \mathcal{U}_n \cap F_\alpha = \emptyset\},$$

and let $\alpha_0 \in \omega_1$ be such that, for every $n \in \omega \setminus A$ we have that $\bigcup \mathcal{U}_n \cap F_{\alpha_0} = \emptyset$. For all $\alpha \in \omega_1$ and $n \in A$, let $U_{\alpha,n} \in \mathcal{U}_n$, $f_{\alpha,n} \in U_{\alpha,n} \cap F_\alpha$, and $E_{\alpha,n} \in [\beta\omega_1 \setminus U(\omega_1)]^{<\omega}$ be such that for every $g \in \bigcup \mathcal{U}_n$, if $g|E_{\alpha,n} = f_{\alpha,n}|E_{\alpha,n}$, then $g \in U_{\alpha,n}$; without loss of generality, we may assume that $U_{\alpha,n} \neq U_{\beta,n}$ whenever $\alpha \neq \beta$. Define the map $\varphi : \beta\omega_1 \setminus U(\omega_1) \to \omega_1$ by letting $\varphi(p)$ be the least $\alpha \in \omega_1$ such that $p \in \mathrm{Cl}\,\alpha$. For every $n \in A$, let $L_n \in [\omega_1]^{\omega_1}$ be such that the family $\{\varphi(E_{\alpha,n}) : \alpha \in L_n\}$ is a Δ–system, with root R_n; for each $\alpha \in L_n$, let $E'_{\alpha,n} = \{p \in E_{\alpha,n} : \varphi(p) \notin R_n\}$ and $S_{\alpha,n} = \varphi(E'_{\alpha,n})$. Let $\tilde{\alpha}_0 = \max(\alpha_0, \sup \bigcup_{n\in\omega} R_n)$.

Denote by $\prec$ the lexicographical order on $\omega_1 \times A$. By induction on $\prec$, define $\gamma(\alpha,n) \in L_n$ and $\beta(\alpha,n) \in \omega_1$, for $(\alpha,n) \in \omega_1 \times A$, so that $\beta(\alpha,n) \geq \tilde{\alpha}_0$ and the following conditions hold for every (α,n):

1^o $\min S_{\gamma(\alpha,n),n} > \beta(\alpha,n) + 1$.

2^o $\beta(\alpha,n+1) \geq \max S_{\gamma(\alpha,n),n}$.

3^o $\beta(\alpha,n) \geq \sup\{\beta(\delta,k) : (\delta,k) \prec (\alpha,n)\}$.

For every (α,n), let $D_{\alpha,n} = [\beta(\alpha,n)+1, \beta(\alpha,n+1)]$ and note that $E'_{\gamma(\alpha,n),n} \subset \mathrm{Cl}\,D_{\alpha,n}$. Further, for each α, let $D_\alpha = \bigcup_{n\in\omega} D_{\alpha,n}$.

Let $\mathcal{Q} \subset [\omega_1]^{\omega_1}$ be such that $|\mathcal{Q}| = \omega_2$ and $\mathcal{Q}$ is "almost disjoint" in the sense that for any two members P and Q of $\mathcal{Q}$, we have that $|P \cap Q| \leq \omega$. For each $Q \in \mathcal{Q}$, let $D_Q = \bigcup_{\alpha\in Q} D_\alpha$. Note that also the family $\{D_Q : Q \in \mathcal{Q}\}$ is almost disjoint. It follows that the collection $\mathcal{H} = \{\mathrm{Cl}(D_Q) \cap U(\omega_1) : Q \in \mathcal{Q}\}$ is a disjoint family. Since $|\mathcal{H}| = \omega_2$ and since the set $E = \bigcup\{E_{\alpha,n} : (\alpha,n) \in \omega_1 \times A\}$ has cardinality ω_1, we can find $P \in \mathcal{Q}$ such that $\mathrm{Cl}(D_P) \cap E \cap U(\omega_1) = \emptyset$.

Choose $\iota_k \in P$, for $k \in \omega$, so that $\iota_k < \iota_{k+1}$ for every $k \in \omega$. Define a function $h : \omega_1 \to \{0,1\}$ as follows. For $\eta \leq \tilde{\alpha}_0$, set $h(\eta) = 1$. Next define h in the set $D = \bigcup_{k\in\omega} D_{\iota_k}$ as follows: for every $(k,n) \in \omega \times A$, find disjoint neighborhoods G_p in the subspace $\mathrm{Cl}\,D_{\iota_k,n}$ for the elements p of the finite set $E'_{\gamma(\iota_k,n),n}$, and define $h(\eta)$, for $\eta \in D_{\iota_k,n}$, so that $h(\alpha) = f_{\gamma(\iota_k,n),n}(p)$ if $\eta \in G_p$ and $h(\eta) = 0$ if η belongs to no set G_p. Finally set $h(\eta) = 0$ if $\eta \in \omega_1 \setminus (\tilde{\alpha}_0 \cup D)$.

Let g be the extension of h to a continuous mapping on $\beta\omega_1$. Note that $g \in X$. Also note that, since $h(\alpha) = 1$ for every $\alpha \leq \tilde{\alpha}_0$, we have that $g \in F_{\tilde{\alpha}_0} \subset F_{\alpha_0}$. Let

$n \in \omega$ be such that the family $(\mathcal{U}_n)_g = \{U \in \mathcal{U}_n : g \in U\}$ is nonempty. We show that the family $(\mathcal{U}_n)_g$ is infinite. Note that since $g \in F_{\alpha_0}$, we have that $n \in A$. To show that the family $(\mathcal{U}_n)_g$ is infinite, it suffices to show that $g \in U_{\gamma(\iota_k,n),n}$ for every $k \in \omega$; this follows once we show that $g|E_{\gamma(\iota_k,n),n} = f_{\gamma(\iota_k,n),n}|E_{\gamma(\iota_k,n),n}$ for every k. Let $k \in \omega$, and let $p \in E_{\gamma(\iota_k,n),n}$. If $\varphi(p) \in R_n$, then $p \in \mathrm{Cl}\,\alpha_0$ and it follows, since $g \in F_{\tilde{\alpha}_0} \subset F_{\alpha_0}$ and $f_{\gamma(\iota_k,n),n} \in F_{\iota_k} \subset F_{\alpha_0}$, that $g(p) = 1 = f_{\gamma(\iota_k,n),n}(p)$. Assume that $\varphi(p) \notin R_n$. Then $p \in E'_{\gamma(\iota_k,n),n}$. Note that, for the neighbourhood G_p of p in $D_{\iota_k,n}$, we have that $p \in \mathrm{Cl}(G_p \cap \omega_1)$. Since g is continuous and $h(\alpha) = f_{\gamma(\iota_k,n),n}(p)$ for every $\alpha \in G_p \cap \omega_1$, we have that $g(p) = f_{\gamma(\iota_k,n),n}(p)$. We have shown that $g(p) = f_{\gamma(\iota_k,n),n}(p)$ for every $p \in E_{\gamma(\iota_k,n),n}$. □

2.9 Example (CH) $C_p(\omega^*)$ is not σ–relatively metacompact.

Proof. The space $\beta\omega_1/U(\omega_1)$ has weight $\mathbf{c}$; hence, under CH, $\beta\omega_1/U(\omega_1)$ is a continuous image of ω^* (see [vM], Corollary 1.3.4), and therefore $C_p(\beta\omega_1/U(\omega_1))$ embeds as a closed set in $C_p(\omega^*)$. □

Remark. Relevant results on continuous images of ω^* were given in [Ba] and [BaFrZb]. The authors showed that the statement "each compact 0-dimensional space of weight at most $\mathbf{c}$ is a continuous image of ω^*" is consistent with "the continuum is arbitrarily large + MA(σ–linked) holds". Hence CH fails in their model nevertheless the spaces $\beta\omega_1$, $\beta\omega_1/U(\omega_1)$ and, if $\mathbf{c} > \omega_2$, then also the space $\beta\omega_2/U(\omega_2)$, are continous images of ω^*. It follows that it is consistent with ZFC+$\neg$CH that $C_p(\omega^*)$ is neither σ–relatively metacompact nor metaLindelf. The last result and Example 2.6 show that the statement "$C_p(\omega^*)$ is metaLindelöf" is consistent with and independent from ZFC. It is consistent with ZFC that $C_p(\omega^*)$ is not even σ–relatively metaLindelöf; this follows from Example 2.10 below and the result on $\beta\omega_2/U(\omega_2)$ mentioned above.

Problem. Is it true in ZFC that $C_p(\beta\omega)$ and $C_p(\omega^*)$ are not σ–relatively metacompact?

The same method of proof as in Example 2.8 gives the following example of a $C_p(K)$–space which fails to be σ–relatively metaLindelöf.

2.10 Example $C_p(\beta\omega_2/U(\omega_2))$ is not σ–relatively metaLindelöf.

Remark. In Examples 2.8 – 2.10 we can replace the pointwise topology by the weak topology.

Hansell has asked whether $C_p(H)$ is hereditarily σ–relatively metacompact for every scattered compact space H. We shall now give consistent examples of scattered compact spaces K for which $C_p(K)$ fails to be σ–relatively metacompact. Note that such a scattered space K cannot be very simple: we know that it cannot be the one–point compactification of a tree, and by the result of Hansell mentioned above, the Banach space $C(K)$ cannot have an equivalent Kadeč norm and hence not an equivalent LUR norm; therefore another restriction on K follows from a result of Haydon and Rogers [HayRo]: the "scattered height" of K must be at least ω_1. The first space that we shall construct is in one respect as simple as possible: it admits a finite–to–one continuous map onto the ordinal space $\omega_1 + 1$ and therefore its scattered height has the minimum possible value $\omega_1 + 1$.

To construct our first space K_1, we will use the set–theoretic principle $\diamondsuit$ which is valid in the constructible universe. Recall that $\diamondsuit$ is the following statement:
There are sets $A_\alpha \subset \alpha$, for $\alpha \in \omega_1$, such that

$$\text{for every } A \subset \omega_1,\ \{\alpha < \omega_1 : A \cap \alpha = A_\alpha\} \text{ is stationary .}$$

We refer to [Ku] for basic information on $\diamondsuit$.

2.11 Example ($\diamondsuit$) There exists a compact scattered space K_1 such that $C_p(K_1)$ is not σ–relatively metacompact.

Proof. We assume that axiom $\diamondsuit$ holds, and we use it to construct a compact topology on $\omega_1 + 1$. The axiom $\diamondsuit$ will enable us to mimique the reasoning used in Example 2.8. Intuitively, we construct the space from the linearly ordered space $\omega_1 + 1$ by "splitting" some limit ordinals $\alpha < \omega_1$ and their neighbourhoods into two. For technical reasons, we do not want to introduce any new points to $\omega_1 + 1$, and therefore we use α and $\alpha + 1$ as the "new" points. To construct the new topology, we first define by induction a set $W_\alpha \subset \alpha + 1$ for each $\alpha \le \omega_1$; then we shall use these sets to define the topology.

Let $\langle t_\gamma : \gamma < \omega_1 \rangle$ enumerate all finite functions $t \subset \omega_1 \times \{0, 1\}$.

Let $\delta \le \omega_1$ and let $\gamma_\iota \in \omega_1$ for each $\iota < \delta$. We say that the transfinite sequence $\langle \gamma_\iota : \iota < \delta \rangle$ is an "orderly sequence" provided that for all $\iota < \nu < \delta$, we have that

$$\max(\{\gamma_\iota\} \cup \operatorname{dom} t_{\gamma_\iota}) < \min(\{\gamma_\nu\} \cup \operatorname{dom} t_{\gamma_\nu}) \quad .$$

We put $\omega_1^{(0)} = \{\alpha < \omega_1 : \alpha = 0 \text{ or } \alpha \in \lim(\omega_1)\}$ and, for $n > 0$, we put $\omega_1^{(n)} = \{\alpha + n : \alpha \in \omega_1^{(0)}\}$. For every $A \subset \omega_1$, we define sets $A^{(n)}$, for $n < \omega$ as follows: we let $\langle \alpha_\iota : \iota < \delta \rangle$ be an increasing enumeration of the elements of A, and then we put $A^{(n)} = \{\alpha_\iota : \iota < \delta \text{ and } \iota \in \omega_1^{(n)}\}$.

The limit ordinals which will be "split" are those elements α of ω_1 for which there exists an orderly sequence $\langle \sigma_n : n \le \omega \rangle$ such that $\sup_{n \in \omega} \sigma_n = \alpha$ and $\sigma_k \in A_\alpha^{(k)}$ for every $k \in \omega$; we denote by S the set consisting of all such ordinals α.

We now define inductively the sets W_α. If there is no $\delta \in S$ such that either $\alpha = \delta$ or $\alpha = \delta + 1$, then we set $W_\alpha = \alpha + 1$. If $\alpha = \delta + 1$ for some $\delta \in S$, then we set $W_\alpha = (\alpha + 1) \setminus W_\delta$. It remains to define W_α when α belongs to the set S. For this purpose, let $\langle \sigma_n : n \le \omega \rangle$ be an orderly sequence with the properties mentioned above. Denote by $t^{(\alpha)}$ the function $\bigcup_{n \in \omega} t_{\sigma_n}$. Let $\{\rho_n : n \in \omega\}$ be an increasing enumeration of $\operatorname{dom} t^{(\alpha)}$. Define $\nu_0 = \rho_{-1} = -1$ and, for $n > 0$, define ν_n as follows: put $\nu_n = \rho_{n-1}$ if $\rho_{n-1} \notin S$ and if $\rho_{n-1} \in S$, then let ν_n be the largest element of $\{\rho_{n-1}+1, \rho_{n-2}+1\} \setminus \{\rho_n\}$. Finally, set

$$W_\alpha = \{\alpha\} \cup \bigcup \{W_{\rho_n} \cap (\nu_n, \rho_n] : n \in \omega \text{ and } t^{(\alpha)}(\rho_n) = 1\} .$$

Note that it follows from the above construction that we have $t^{(\alpha)} \subset \chi_{W_\alpha}$.

By induction, it is easy to verify the following property of the sets W_α: for all $\alpha, \beta \in \omega_1$, if $\beta \in W_\alpha$, then there exists $\gamma \in \beta \setminus S$ such that $W_\beta \cap (\gamma, \beta] \subset W_\alpha$. Using this property, we see that there exists a topology τ on the set $\omega_1 + 1$ such that, for every $\alpha \le \omega_1$, the family $\{W_\alpha \cap (\gamma, \alpha] : \gamma \in \alpha \setminus S\}$ is a clopen neighbourhood base of α in $(\omega_1 + 1, \tau)$. Let us denote the space $(\omega_1 + 1, \tau)$ by K_1. Note that, since $W_\alpha \subset \alpha + 1$ for every α, the space K_1 is scattered. Clearly, K_1 is a Hausdorff space. From the definition of the sets W_α it follows that the finite–to–one mapping f from K_1 onto $\omega_1 + 1$, defined by setting $f(\alpha) = \alpha$ if α is a limit and $f(\alpha) = \alpha - 1$ if α is a successor, is closed and continuous.As a consequence, the space K_1 is compact.

We show that the space $C_p(K_1)$ is not σ–relatively metacompact. Put

$$P = \{\chi_G : G \text{ is clopen in } K_1\, , 0 \in G \text{ and } \omega_1 \notin G\}$$

and note that P is a closed subspace of $C_p(K_1)$. It suffices to show that P is not σ–relatively metacompact.

For each $\alpha \in \omega_1$, let $F_\alpha = \{f \in P : f(\gamma) = 1 \text{ for each } \gamma \le \alpha\}$, and further, let $O_\alpha = \{P \setminus F_\alpha : \alpha \in \omega_1\}$. Note that the family $\mathcal{O} = \{O_\alpha : \alpha \in \omega_1\}$ is an open cover of P. We show that $\mathcal{O}$ has no point–finite σ–relatively open refinement. Let $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ be a refinement of $\mathcal{O}$ such that each $\mathcal{U}_n$ consists of relatively open subsets of $\bigcup \mathcal{U}_n$. To show that $\mathcal{U}$ is not point–finite, let

$$H = \{n \in \omega : \bigcup \mathcal{U}_n \cap F_\alpha \neq \emptyset \text{ for every } \alpha \in \omega_1\},$$

and let $\alpha_0 \in \omega_1$ be such that, for every $n \in \omega \setminus H$ we have that $\bigcup \mathcal{U}_n \cap F_{\alpha_0} = \emptyset$. Write H as $\{h_k : k \in \omega\}$ so that each element of H is repeated infinitely many times. For all $\alpha \in \omega_1$ and $k \in \omega$, let $U_{\alpha,k} \in \mathcal{U}_{h_k}$, $f_{\alpha,k} \in U_{\alpha,k} \cap F_\alpha$, and $E_{\alpha,k} \in [\omega_1]^{<\omega}$ be such that for every $g \in \bigcup \mathcal{U}_{h_k}$, if $f_{\alpha,k}|E_{\alpha,k} \subset g$, then $g \in U_{\alpha,k}$; without loss of generality, we may assume that $U_{\alpha,k} \neq U_{\beta,j}$ whenever $(\alpha, k) \neq (\beta, j)$. For every $k \in \omega$, let $L_k \in [\omega_1]^{\omega_1}$ be such that the family $\{E_{\alpha,k} : \alpha \in L_k\}$ is a Δ–system, with root R_k. Let $\tilde{\alpha}_0 = \sup(\{\alpha_0\} \cup \bigcup_{k\in\omega} R_k) + 2$. Note that we have $f_{\alpha,k}(\beta) = 1$ whenever $\alpha \in L_k \setminus \tilde{\alpha}_0$ and $\beta \in R_k$. For all $k \in \omega$ and $\alpha \in L_k$, let $t_{\alpha,k} = f_{\alpha,k}|(E_{\alpha,k} \setminus R_k)$.

By induction, we construct an orderly sequence $\langle \eta_\iota : \iota < \omega_1\rangle$ such that $\eta_0 > \tilde{\alpha}_0$ and $t_{\eta_\iota} \in \{t_{\alpha,k} : \alpha \in L_k\}$ whenever $\iota \in \omega_1^{(k)}$. Let $E = \{\eta_\iota : \iota < \omega_1\}$ and denote the cub $\omega_1 \cap E'$ by F. By the principle $\diamondsuit$, we know that there will be an $\alpha \in F$ such that $A_\alpha = E \cap \alpha$. Note that $\alpha \in S$ since $\alpha \in E'$.

The set $G = (\tilde{\alpha}_0) \cup W_\alpha$ is clopen in K_1. We shall use the characteristic function of G to show that the family $\mathcal{U}$ is not point–finite. Let $m \in \omega$ be such that $\chi_G \in \bigcup \mathcal{U}_m$. Note that we have $\chi_G \in F_{\tilde{\alpha}_0} \subset F_{\alpha_0}$ and therefore $m \in H$. Denote by J the infinite set $\{j \in \omega \setminus \{0\} : h_j = m\}$. We shall find distinct sets $U_j \in \mathcal{U}_m$, for $j \in J$, such that $\chi_G \in \bigcap_{j\in J} U_j$; from this it of course follows that $\mathcal{U}_m$ is not point–finite at χ_G. Let $j \in J$. Let $\langle \sigma_n : n \leq \omega\rangle$ be that orderly sequence which was used in the definition of W_α. Note that since $A_\alpha = E \cap \alpha$, we have that $A_\alpha^{(j)} \subset E^{(j)} = \{\eta_\iota : \iota \in \omega_1^{(j)}\}$. Consequently, we have that $\sigma_j \in A_\alpha^{(j)} \subset \{\eta_\iota : \iota \in \omega_1^{(j)}\}$ and it follows from this that there exists $\delta \in L_j$ such that $t_{\sigma_j} = t_{\delta,j}$. We now have

$$t_{\delta,j} \subset \bigcup_{i\in\omega} t_{\sigma_i} = t^{(\alpha)} \subset \chi_{W_\alpha}.$$

Since $\tilde{\alpha}_0 \cap \operatorname{dom} t_{\delta,j} = \emptyset$, it follows from the foregoing that $t_{\delta,j} \subset \chi_G$; from this it follows further, since $t_{\delta,j} = f_{\delta,j}|(E_{\delta,j} \setminus R_j)$ and $R_j \subset G$, that $f_{\delta,j}|E_{\delta,j} \subset \chi_G$; this in turn implies that $\chi_G \in U_{\delta,j}$. Let $U_j = U_{\delta,j}$ and note that $U_j \in \mathcal{U}_{h_j} = \mathcal{U}_m$. Finally note that the construction guarantees that the sets U_j , $j \in J$, will all be distinct. $\square$

Note that it follows from Theorem 1.2 that the space $C_p(K_1)$ is hereditarily metaLindelöf. We do not know if it is possible to construct from $\diamondsuit$ a scattered compact space

H such that $C_p(H)$ is not hereditarily metaLindelöf, but we are able to produce such a space under a related set–theoretic principle, $\boxed{\Diamond}$. This principle was first considered by Ch. Gray in [Gra], and later used in [AbShSo] and [Ve].

Here is the statement of the principle $\boxed{\Diamond}$:

There are sequences $\langle C_\alpha : \alpha < \omega_2\,,\ \lim(\alpha)\ \rangle$ and $\langle S_\alpha : \alpha < \omega_2\,,\ \lim(\alpha)\ \rangle$ such that the following conditions hold:

(i) $\langle C_\alpha : \alpha < \omega_2\,,\ \lim(\alpha)\ \rangle$ is a $\Box_{\omega_2}$ sequence, i.e.

- a) $C_\alpha \subset \alpha$, C_α is closed unbounded in α
- b) if $\mathrm{cof}(\alpha) < \lambda$ then $|C_\alpha| < \lambda$
- c) if $\gamma < \alpha$ is a limit point of C_α then $C_\gamma = C_\alpha \cap \gamma$

(ii) for every limit ordinal $\alpha < \omega_2$, $S_\alpha \subseteq \alpha$,

(iii) if β is a limit of C_α, then $S_\beta = S_\alpha \cap \beta$,

(iv) for every $S \subseteq \omega_2$ and cub $C \subset \omega_2$, there is an $\alpha < \omega_2$ of cofinality ω_1 such that $S_\alpha = S \cap \alpha$ and $C_\alpha \subset C$.

2.12 Example ($\boxed{\Diamond}$) There exists a compact scattered space K_2 such that $C_p(K_2)$ is not σ–relatively metaLindelöf.

Proof. The construction of the space and the verification of its properties are similar to those used in the previous example, and therefore we shall only indicate the changes needed in the construction.

Now we use axiom $\boxed{\Diamond}$ to construct a compact topology on $\omega_2 + 1$. Again, the new space is intuitively obtained from an ordinal space by "splitting" some limit ordinals and their neighbourhoods into two. First, sets $W_\alpha \subset \alpha + 1$, for $\alpha \leq \omega_2$, will be defined and then these sets are used to define the topology.

Let $\langle t_\gamma : \gamma < \omega_2 \rangle$ enumerate all the finite functions $t \subset \omega_2 \times \{0,1\}$. Define "orderly sequence" similarly as above.

The limit ordinals which will be "split" are in the closure (with respect to the order topology) $\overline{S}$ of a set S consisting of ordinals of countable cofinality. We now define S: first, we set

$$A = \{\alpha \in \omega_2 : \lim(\alpha) \text{ and } \alpha \notin C''_\alpha\}$$

and for each $\alpha \in A$, we denote by $\psi(\alpha)$ the largest ordinal of the set $\{0\} \cup (C'_\alpha \cap \alpha)$. Then we define S to be the set consisting of all those elements α of A for which there exists an orderly sequence $\langle \sigma_n : n \le \omega \rangle$ such that $\min \operatorname{dom} t_{\sigma_0} > \psi(\alpha)$, $\sup_{n\in\omega} \sigma_n = \alpha$ and $\sigma_k \in S_\alpha^{(k)}$ for every $k \in \omega$.

Next we define inductively the sets W_α. If there is no $\delta \in \overline{S}$ such that either $\alpha = \delta$ or $\alpha = \delta + 1$, then we set $W_\alpha = \alpha + 1$. If $\alpha = \delta + 1$ for some $\delta \in \overline{S}$, then we set $W_\alpha = (\alpha + 1) \setminus W_\delta$. For $\alpha \in S$, we define $t^{(\alpha)}$ and W_α in the same way as in the preceding example. If $\alpha \in \overline{S} \setminus S$, then we set

$$W_\alpha = \{\alpha\} \cup \bigcup \{W_\gamma : \gamma \in C'_\alpha \cap \overline{S}\} \ .$$

For $\alpha \in S$, it follows from the construction that $t^{(\alpha)} \subset \chi_{W_\alpha}$; from this it follows further that $t^{(\alpha)} \subset \chi_{W_\beta}$ whenever $\beta \in \overline{S}$ and $\alpha \in C'_\beta$.

The following property holds for the sets W_α: for all $\alpha, \beta \in \omega_2$, if $\beta \in W_\alpha$, then there exists $\gamma \in \beta \setminus \overline{S}$ such that $W_\beta \cap (\gamma, \beta] \subset W_\alpha$. As a consequence, there exists a topology τ on the set $\omega_2 + 1$ such that, for every $\alpha \le \omega_2$, the family $\{W_\alpha \cap (\gamma, \alpha] : \gamma \in \alpha \setminus \overline{S}\}$ is a clopen neighbourhood base of α in $(\omega_2 + 1, \tau)$. Denote the space $(\omega_2 + 1, \tau)$ by K_2, and note that K_2 is a scattered compact Hausdorff space.

We leave it to the reader to modify the argument used in the preceding example to show, with the help of the axiom $\boxed{\diamondsuit}$, that the space $C_p(K_2)$ is not σ–relatively metaLindelf. $\square$

We close this paper with an open problem. In all our examples of $C_p(K)$–spaces, when $C_p(K)$ is not σ–relatively metacompact, then it is not norm σ–fragmented (one can show that also the space $C_p(K_1)$, under $\diamondsuit$, fails to be norm σ–fragmented). This observation leads us to the following question.

Problem. If a space B_w or $C_p(K)$ is norm σ–fragmented, is the space then (hereditarily) σ–relatively metacompact?

A positive solution to the hereditary version of the above problem would be a pleasant result for the generalized theory of descriptive sets (see [Han$_1$], [Han$_2$]).

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Department of Mathematics
York University
North York ON
Canada
dowa@mathstat.yorku.ca

Department of Mathematics
University of Helsinki
Helsinki
Finland
hjkjunnila@cc.helsinki.fi

Department of Mathematics
Czech Academy of Sciences
Prague
Czech Republic
pelant@earn.cvut.cz