# WHAT I LEARNED ABOUT $\beta X$, C(X) AND PRODUCTS FROM WIS COMFORT

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The first task one faces when setting out to review the impact of Wis Comfort's work on the theory of the Stone-Cech compactification is to select which of the 85 or more, some would say, odd papers and books to discuss. Favouring the older and those closest to my own, much narrower, interests has left me with a lengthy list of papers involving a very complete investigation of the relationships between products, pseudocompact spaces, realcompact spaces and the ring of continuous real-valued functions. In light of the truly exceptional surveys, and of course, books by Wis it would be folly to presume to compose a traditional style survey that could have anything to add to what Wis himself has already written in an infinitely more polished and readable form. Rather, we will begin with some largely unnecessary remarks which recall just how major and important Wis' influence has been in establishing the modern field of set-theoretic topology and then proceed to review our selected topics. One does not have to look beyond the singularly influential and prominent book, *The theory of ultrafilters*, to realize how profound the impact has been. The majority of the table of contents of this book continues to figure prominently in the literature today, some 25 years later.

Of course to Wis' first doctoral student, S. Negrepontis, we willingly bestow equal credit for the impact of this book. Also to Wis' third doctoral student, N. Hindman, we can attribute the dictum that one must always start with the definition of an ultrafilter. However I was fascinated by the 'definition' in *The theory of ultrafilters*:

> An *ultrafilter* is a truth value assignment to the family of subsets of a set, and a method of convergence to infinity. From the first (logical) property arises its connection with two-valued logic and model theory; from the second (convergence) property arises its connection with topology and set-theory. Both these descriptions of an ultrafilter are connected with compactness. The model-theoretic property finds its expression in the construction of the ultraproduct and the compactness type of theorem of Loś; and the convergence property leads to ... the Stone-Čech compactification. ... it is reasonable to expect that a study of ultrafilters from these points of view will yield results and methods which can be fruitfully cross-bred.

This passage all on its own amply makes my main point concerning Wis' impact on the fields, namely that he played a very major role in bringing about the marriage of set-theory and logic with General Topology. This marriage, (or, at least, a mutually agreeable co-habitational arrangement) may be something that is so well-established today that it is easy to overlook how difficult a task it most certainly was for the pioneering matchmakers (I inadvertently typed 'mathmakers' here). Of course there are a few other very notable and amazing people who played, in some cases, an even larger role in this transition; in the actual area of Stone-Cech compactifications, Comfort and Negrepontis are unsurpassed. The book still stands as a clear testimony to their bold and visionary approach to the subject. To borrow another poetic quote from the master [6]:

"*Satisfied and content in the familiarity of the status quo we may admire the courage of a crusader; but we applaud only rarely and we follow more rarely still.*"

Many of us are still applauding, for, to summarize, what is surely the most important aspect of Wis' influence in the theory of Stone-Cech compactifications is his extremely early recognition of the huge role that set-theory and model theory would come to play in the topological study of ultrafilters, his pivotal role in maintaining the lines of communication between the fields for their mutual benefit, and his patient nurturing of a generation of hybrids.

There are many possible reasons why one might find it difficult to write this review. One obvious reason, as I mentioned, is the sheer quantity of the work produced. The second is that Wis' impact is so pervasive that it is a daunting task to do it any justice in a short analysis of the subject. However a completely unexpected phenomenon was actually the most powerful obstruction. The papers were just too interesting to simply read and report on. I would get too absorbed in the material and then side-tracked pursuing the problems discussed. Well come to think of it, this must fairly be treated as testimony to just one more of the ways in which Wis has influenced the field. Falling victim to the charms of his papers we are reduced to writing some self-indulgent remarks on the topics covered in these few papers. Totally contrary to our goal of tracking the influence on "today's" mathematics we found ourselves rather absorbed in topics that have temporarily moved to the backburner.

## 1. Preliminaries

Well we have already introduced an ultrafilter (simply a truth value assignment): i.e. the conjuction of finitely many true things is again true and everything is either true or false; it is more normal to write that $\mathfrak{U}$ is an ultrafilter of, or on, a lattice of sets, $\mathcal{L}$, if $U_1 \cap U_2 \in \mathfrak{U}$ for all $U_1, U_2 \in \mathfrak{U} \subset \mathcal{L}$ and if $V \in \mathcal{L} \setminus \mathfrak{U}$, then there is some $U \in \mathfrak{U}$ which is disjoint from $V$.

For a (Tychonoff) space $X$, $C(X)$ is the set of continuous real-valued functions on $X$, while $C^*(X)$ is the set of members of $C(X)$ which are bounded. The Stone-Čech compactification of $X$, $\beta X$, has several presentations: the space of maximal ideals on $C^*(X)$; the closure of the embedded copy of $X$ in $\mathbb{R}^{C^*(X)}$ under the evaluation map $x \mapsto \langle f(x) : f \in C^*(X)\rangle$; the unique compactification of $X$ to which every member of $C^*(X)$ extends; the space of ultrafilters on $\mathcal{Z}(X) = \{Z \subset X : Z$ is a zero-set of $X\}$; the unique compactification of $X$ in which disjoint zero-sets of $X$ have disjoint closures. For many of us, $\beta X$ is $X$ union the set of free zero-set ultrafilters and the topology is determined by the assertion that the closure of a zero-set is obtained by attaching all the free ultrafilters to which the zero-set belongs.

Given that $\beta X$ is so intimately tied to $C^*(X)$ it is natural to ask about the relationship between $|C^*(X)|$ and the properties of $X$. Comfort and Hagar investigate this in [8] and they find that the weight of $X$ raised to the power of something they call the weak covering number (perhaps weak Lindelof number is an appropriate name as well) is about the best general formula. Of course, $|C(X)| = |C^*(X)|$, but $C(X)$ has another important role to play.

A space $X$ is said to be *realcompact* if $X$ can be embedded as a closed subspace of a power of the real line. The Hewitt realcompactification, $\upsilon X$, of a space $X$ is the smallest subspace of $\beta X$ which is realcompact. Hewitt introduced it during the investigation of those maximal ideals, $\mathfrak{M}$, on $C(X)$ which have the property that the quotient ring $C(X)/\mathfrak{M}$ is non-Archimedean and proved that $\upsilon X$ is precisely this set of maximal ideals. It also turns out to be those ultrafilters on $\mathcal{Z}(X)$ which have the countable intersection property. A final useful equivalence is that $p \in \beta X \setminus X$ if there is a nowhere vanishing $f \in C^*(X)$ such that $\beta f(p) = 0$. A space $X$ is realcompact iff $\upsilon X = X$.

Of course the product of compact spaces is compact and it is immediate that a product of realcompact spaces is again realcompact. As Comfort remarks in [6], "an overly enthusiastic view" of these facts "can lead one to the belief that $\beta(X \times Y) = \beta X \times \beta Y$ and $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$". It is certainly now well-known that neither is true, but the fascinating search for some understanding of when it is true is the topic of many of our papers of interest.

In some sense complementary to the notion of realcompact is that of pseudocompact. A space $X$ is pseudocompact if it carries no unbounded real-valued continuous function (i.e. $C(X) = C^*(X)$). Pseudocompactness has an important place in the theory of topological groups since, as Comfort and Ross showed [17], these are the groups whose Stone-Cech compactification is again a group. It also gained in prominence as a result of Glicksburg's ground breaking solution of the question of when $\beta(X \times Y) = \beta X \times \beta Y$: precisely when $X \times Y$ is pseudocompact. Whenever you are looking for a particularly weird relationship between $X$ and $\beta X$, the space $X$ will most probably be pseudocompact e.g. [2, 11, 3, 14].

If $X$ is not pseudocompact, then $\beta X$ is not so weird because, in this case, $\beta X \setminus X$ contains an infinite subset homeomorphic to $\beta N$ (as every Stone-Cech compactification should). A space $X$ is an $F$-space if every finitely generated ideal in $C(X)$ is principle. If Gillman and Henriksen's definition does not give you much insight into their structure, you may prefer to know that this means that every cozero set is $C^*$-embedded. For compact spaces, this is equivalent to the property of disjoint cozero sets having disjoint closures. If we drop the compactness assumption, we come to the definition of $F'$-spaces introduced(?) in [10] in the investigation of their behaviour in products. $P$-spaces have proven to be quite relevant to the study of extending continuous real-valued functions defined on products. A space $X$ is a $P$-space if every prime ideal of $C(X)$ is maximal; more frequently formulated as the property that every zero set is open.

## 2. Remainders, $F$-spaces and products

Products are very bad for $F$-spaces. Indeed, in [16] among other things we find the following result.

**Proposition 2.1.** *If $\Pi_n\{X_n : n \in \omega\}$ is an $F$-space, then all but finitely many of the $X_n$ are singleton spaces.*

If infinitely many of the $X_n$'s are not singletons, then it is easily seen that the product contains a non-trivial converging sequence. However it is now well-known that $F$-spaces do not contain converging sequences since every countable discrete subset is $C^*$-embedded (see [10]).

Even the product of two $F$-spaces is rarely an $F$-space. Again from [10]:

**Proposition 2.2.** *If $X \times Y$ is an $F$-space then one of them is a $P$-space.*

Note that even $\beta N \times \beta N$ then is not an $F$-space.

*Proof.* If neither $X$ nor $Y$ is a $P$-space, then we may choose points $x, y$ in $X, Y$ respectively together with cozero sets $C, D$ of $X$ and $Y$ respectively so that $x$ is in $\overline{C} \setminus C$ and $y \in \overline{D} \setminus D$. Let $f \in C^*(X)$ and $g \in C^*(Y)$ be such that $C = Cz(f) = X \setminus f^{-1}(0)$ and $D = Cz(g)$. Clearly, for each $n$, $\left(X \setminus f^{-1}(-\frac{1}{2^n}, \frac{1}{2^n})\right) \times \{y\}$ is completely separated from $\{x\} \times Y$ and, in addition $\{x\} \times \left(Y \setminus g^{-1}(-\frac{1}{2^n}, \frac{1}{2^n})\right)$ is completely separated from $X \times \{y\}$. Therefore there is a function $h_n : X \times Y \to [-\frac{1}{2^n}, \frac{1}{2^n}]$ which is identically $\frac{1}{2^n}$ on $\left(X \setminus f^{-1}(-\frac{1}{n}, \frac{1}{n})\right) \times \{y\}$, non-negative on $X \times \{y\}$, identically $-\frac{1}{2^n}$ on $\{x\} \times \left(Y \setminus g^{-1}(-\frac{1}{n}, \frac{1}{n})\right)$, and, finally, non-positive on $\{x\} \times Y$. If we define $h = \Sigma h_n$, then $h \in C^*(X \times Y)$, and $C \times \{y\} \subset h^{-1}(0, \infty)$ and $D \subset h^{-1}(-\infty, 0)$. Hence $(x, y)$ is a limit point of two disjoint cozero sets. $\square$

This next result was proven some time ago by Fine and Gillman but Negrepontis discovered a beautiful proof.

**Proposition 2.3.** *If $X$ is locally compact and $\sigma$-compact, then $\beta X \setminus X$ is a compact $F$-space.*

Clearly $\beta X \setminus X$ is a compact space, hence every cozero subset is $\sigma$-compact. Now if $C \subset \beta X \setminus X$ is $\sigma$-compact, then $X \cup C$ is also $\sigma$-compact, thus is normal. If $f$ is a bounded continuous real-valued function on $C$, then Tietze's extension theorem implies that $f$ extends to all of $X \cup C$ since $C$ is closed in $X \cup C$ by the local compactness of $X$. Any function defined on all of $X$ has a unique continuous extension to $\beta X$, thus completing the proof that $C$ is $C^*$-embedded.

In particular, of course, if $K$ is a compact space, then the remainder of $\omega \times K$ is an $F$-space. It is natural to wonder if this is the case for the remainder of $\omega_1 \times K$ when $\omega_1$ is given the discrete topology. It happens that if $K$ is any space which is not an $F$-space and $X = \omega_1 \times K$, then $\beta X \setminus X$ is not an $F$-space. (If you are wondering about the ordinal topology on $\omega_1$, van Douwen showed that if $K$ is compact, then the remainder of $\omega_1 \times K$ is $K$).

Since $K$ is not an $F$-space, there is a function $f \in C^*(K)$ such that $f^{-1}[(-1, 0)]$ is not completely separated from $f^{-1}[(0, 1)]$. Clearly the function $h(\alpha, k) = f(k)$ is a member of $C^*(X)$ which has a continuous extension, $H$, to all of $\beta X$. We will prove that $H^{-1}[(-1, 0)] \setminus X$ and $H^{-1}[(0, 1)] \setminus X$ have a common limit point in $\beta X \setminus X$.

Now, for each $\alpha \in \omega_1$, the closure of $\{\alpha\} \times K$ in $\beta X$ can be identified with $\{\alpha\} \times \beta K$. In addition, there is a point $y \in \beta K$, such that $y$ is a limit point of both $f^{-1}[(-1, 0)]$ and $f^{-1}[(0, 1)]$. Therefore, for each $\alpha$, following our identification, the point $(\alpha, y)$ is a common limit of $h^{-1}[(-1, 0)]$ and $h^{-1}[(0, 1)]$. While $(\alpha, y)$ may be in X, there is certainly a complete accumulation point $z$ of $\{(\alpha, y) : \alpha \in \omega_1\}$ which is not in $X$. We will prove that $z$ is the desired common limit point. Fix any $\beta X$-neighbourhood $W$ of $z$ . By the definition of $z$, there are uncountably many $\alpha$ such that $(\alpha, y) \in W$. For each such $\alpha$, pick a point $x_\alpha \in X$, such that $f(x_\alpha) > 0$ and $(\alpha, x_\alpha) \in W$. It follows that there is some $\epsilon > 0$ such that $h(\alpha, x_\alpha) > \epsilon$ for uncountably many $\alpha$, in which case $h(p) \geq \epsilon$ for any limit point of these uncountably many points. Completely symmetrically, the closure of $W$ also contains points of $\beta X \setminus X$ at which $h$ is less than 0.

Is there a $\sigma$-compact space whose remainder is not an $F$-space? Is $\beta\mathbb{Q} \setminus \mathbb{Q}$ an $F$-space? Well Yes and No. For example, for each $n \in \omega$, there are cozero subsets of $\beta\mathbb{Q}$, $C_n$ and $D_n$, such that $C_n \cap \mathbb{Q} = (2n-1, 2n) \cap \mathbb{Q}$ and $D_n \cap \mathbb{Q} = (2n, 2n+1) \cap \mathbb{Q}$. It follows that $C = \bigcup_n C_n$ and $D = \bigcup_n D_n$ are disjoint cozero subsets of $\beta\mathbb{Q}$ each of which contains all limit points of $\{2n : n \in \omega\}$ in their closure. Since $\mathbb{Q}$ is nowhere locally compact, $\beta\mathbb{Q} \setminus \mathbb{Q}$ is dense in $\beta\mathbb{Q}$ which ensures that $C \setminus \mathbb{Q}$ and $D \setminus \mathbb{Q}$ are disjoint cozero subsets of $\beta\mathbb{Q} \setminus \mathbb{Q}$ which do not have disjoint closures.

As we said, for strange behaviour, use pseudocompact $X$.

For example, for the ordinal space $\omega_1$, $\beta\omega_1$ is just $\omega_1 + 1$ since every $f \in C(X)$ is eventually constant. Using this same idea, for any space $Y$, we can take the ordinal space $\kappa = |Y|^+$ and so for $f \in C(\kappa \times Y)$ there will be an $\alpha < \kappa$ and a $g \in C(Y)$ so that $f((\alpha, \kappa) \times \{y\}) = g(y)$ for all $y \in Y$. Therefore $(\kappa + 1) \times \beta Y$ sits naturally in $\beta(\kappa \times Y)$. This "plank" construction is a frequently used tool.

Are there $X$ such that $\beta X \setminus X$ is homeomorphic to $X$? In [14], we are searching for $Y$ with $X \subset Y \subset \beta X$ ($X$ first countable realcompact, i.e. *nice* $X$) such that $\beta Y \setminus Y$ is homeomorphic to $Y$. It turns out that $X$ is discrete and $Y$ is pseudocompact. To see that such a $Y$ exists is quite simple but elegant; quite reminiscent of the Schroeder-Bernstein proof.

Following the approach in [14], suppose we have a compact space $K$ such that $K$ contains a nowhere dense copy of itself, $K_1$. Clearly then, we have a descending sequence $K_n$ ($n \in \omega$) of copies of $K$, such that $K_0 = K$ and, for each $n$, there is a homeomorphism carrying $K$ to $K_n$ such that the image of $K_1$ is $K_{n+1}$. It is not difficult to observe that $\bigcup_n [K_{2n} \setminus K_{2n+1}]$ is homeomorphic to $\bigcup_n [K_{2n+1} \setminus K_{2n+2}]$. Of course there is some of $K$ left over, namely $H = \bigcap_n K_n$.

So let us take us start with two copies of $K$ by using $K \times \{0, 1\}$ and set

$$Y = \bigcup \{[K_{2n} \setminus K_{2n+1}] \times \{0, 1\} : n \in \omega\} \cup H \times \{0\}$$

then $Y$ is dense in $K \times 2$ and homeomorphic to $K \times \{0, 1\} \setminus Y = H \times \{1\} \cup \bigcup \{[K_{2n+1} \setminus K_{2n+2}] \times \{0, 1\} : n \in \omega\}$ . This can be seen by taking the map sending $[K_{2n} \setminus K_{2n+1}] \times \{e\}$ to $[K_{2n+1} \setminus K_{2n+2}] \times \{1 - e\}$ in the obvious way, and then sending $H \times \{0\}$ to $H \times \{1\}$.

If we can just arrange that $Y$ is $C^*$-embedded in $K \times 2$, then we have an example. More specifically, if there is a first-countable realcompact $X \subset Y$ such that $K = \beta X$. Clearly all that is required is to start with any $X$ such that $\beta X$ embeds into $\beta X \setminus X$. The main results in [14] are that, if there are no measurable cardinals, then this will hold iff $X$ is discrete (as we have seen above $\beta X \setminus X$ rarely contains converging sequences).

If realcompact is dropped then there is a non-discrete metric space $X$ with such a $Y$. This latter fact offers a chance to illustrate the benefits of the marriage. E. van Douwen and A. Kato, independently supplied the authors of [14] with the following construction.

Let $\mu$ be a measurable cardinal. This means that there is a $\mu$-complete ultrafilter, $\mathcal{U}$, on the discrete space $\mu$. If we fix any compact first countable space $K$, and let $X$ be the product of the discrete space $\mu$ with $K$, then of course $X$ is a first-countable (non-realcompact) space. We would just like to prove that $\mu \times K$ will embed into $\beta(\mu \times K) \setminus (\mu \times K) = (\mu \times K)^*$. It is easy to establish that it is sufficient to show that $K$ embeds into $(\mu \times K)^*$.

It turns out that the closed set

$$K_{\mathcal{U}} = \bigcap\{\overline{(U \times K)} : U \in \mathcal{U}\}$$

is a copy of $K$. This intersection along an ultrafilter is quite a useful idea. To better illustrate the idea we will make the unnecessary but simplifying assumption that $K$ is 0-dimensional. This will permit us to work with Stone duality. Thus $\beta(\mu \times K)$ is just the Stone space of the Boolean algebra, $B = (CO(K))^{\mu}$; namely the product of $\mu$ many copies of the algebra of clopen subsets of $K$. Since $K_{\mathcal{U}}$ is a subspace, we should identify its algebra of clopen sets as a quotient of $B$. Indeed, if we set $\mathcal{I}$ equal to the dual ideal of $\mathcal{U}$, then it is quite straightforward to show that $a, b \in B$ meet $K_{\mathcal{U}}$ identically iff $\{\alpha \in \mu : a \wedge (\{\alpha\} \times K) \neq b \wedge (\{\alpha\} \times K)\}$ is a member of $\mathcal{I}$. Well this is just another way of saying that $CO(K_{\mathcal{U}})$ is equal to the ultrapower of $CO(K)$ by the ultrafilter $\mathcal{U}$, i.e. $(CO(K))^{\mu}/\mathcal{U}$. The map which sends $a \in CO(K)$ to the equivalence class $\langle a, a, a, a, \ldots\rangle/\mathcal{U}$ is an isomorphic embedding of $CO(K)$ into $CO(K_{\mathcal{U}})$. So far this has nothing to do with $\mu$ being measurable. These interesting ultrapowers are studied in great detail in [13]. In this case we have that the intersection of $\mathfrak{c}$ members of $\mathcal{U}$ is again a member of $\mathcal{U}$ and that $|CO(K)| \leq \mathfrak{c}$. This implies that every member of $(CO(K))^{\mu}/\mathcal{U}$ is simply equivalent to $\langle a, a, a, \ldots a\rangle/\mathcal{U}$ for some $a \in CO(K)$. That is to say, that the above mentioned isomorphic embedding is a surjection. Therefore $CO(K_{\mathcal{U}})$ is equal to $CO(K)$.

**Proposition 2.4** ([2]). *If there is a retraction $r$ from $\beta X$ onto $\beta X \setminus X$, then $X$ is locally compact and pseudocompact.*

Of course an easy example in which there is such a retraction is $X = \omega_1 \times K$ for the ordinal space $\omega_1$ and compact $K$.

This lovely little result gives us an opportunity to examine a little more closely the role of countable discrete sets in $F$-spaces. Obviously being a continuous image of $\beta X$, $X^*$ is compact and so $X$ is locally compact. If $X$ is not pseudocompact, then it must contain a copy of $N$ which is closed and $C^*$-embedded (pick one point out of each member of a locally finite family). For each $n$, let $z_n = r(n)$ and note that for each limit point $p$ of $N$, $r(p)$ is a limit point of $\{z_n : n \in \omega\}$. It is even the case that if $p$ is the $\mathcal{U}$-limit of $N$, then $r(p)$ is the $\mathcal{U}$-limit of $\{r(n) : n \in \omega\}$. This actually contradicts Frolík's results concerning *types*. However we do not have to be so sophisticated here. We will produce a $p \in \mathrm{cl}(N)$ which is not a limit point of $r(N)$.

Since $N$ is $C^*$-embedded in $\beta X$, its closure is homeomorphic to $\beta N$ and its closure intersected with $\beta X \setminus X$ is homeomorphic to $N^*$. Let $A = \{n : z_n \notin \mathrm{cl}(N)\}$. For each $n \in A$, fix a closed neighbourhood $W_n$ of $z_n$ such that $W_n \cap N$ is empty. For each $n \in N$, fix a compact neighbourhood $V_n \subset X$ of $n$ so that $V_n \cap W_k = \emptyset$ for all $k \leq n$. By shrinking $V_n$, we may assume that $\{V_n : n \in \omega\}$ is locally finite. For each $n$, fix a function, $f_n : X \mapsto [0,1]$ such that $f_n(n) = 1$ and $f_n[X \setminus V_n] = 0$. By the local finiteness, the function $f = \Sigma f_n \in C^*(X)$ and extends to all of $\beta X$. It is simple to check that $Cz(f) \cap W_n \subset \bigcup_{k<n} V_k$ (for each $n \in A$) and so has compact closure. Therefore $f(z_n) = 0$ for all $z \in Z$ and $f(p) \geq 1$ for all $p \in \mathrm{cl}(N)$. Therefore the closure of $\{z_n : n \in A\}$ is disjoint from $N^*$, while $\{z_n : n \notin A\}$ is a countable, hence, nowhere dense subset of $N^*$.

One of Wis' earliest papers, [1], is the novel observation that $\beta X$ can have density less than that of $X$. The example is to take $X$ to be the set of points in $[0,1]^{\mathfrak{c}}$ which are non-zero only countably often. This set is clearly dense, well-known not to be separable, but surprisingly, it is $C^*$-embedded; hence its Stone-Čech compactification is simply its closure. To see this, suppose that $f \in C^*(X)(= C(X))$ and let $z \in [0,1]^{\mathfrak{c}}$. For any countable $A \subset \mathfrak{c}$, let $z(A) \in X$ denote the element which agrees with $z$ on $A$ and is 0 otherwise. Inductively, choose a strictly increasing sequence $\{A_\alpha : \alpha \in \lambda\}$ of countable sets so that for each $\alpha \in \lambda$, $f(z(A_{\alpha+1}))$ is distinct from $f(z(A_\alpha))$. If this continues to $\lambda = \omega_1$, then there is a pair of rationals $p < q$ so that there is an uncountable set $S$ so that for $\alpha \in S$, $f(z(A_\alpha)) < p$ and $q < f(z(A_{\alpha+1}))$ (or vice-versa). But if $\beta$ is any limit such that $S \cap \beta$ is cofinal in $\beta$, then both $\{z(A_\alpha) : \alpha \in \beta \cap S\}$ and $\{z(A_{\alpha+1}) : \alpha \in \beta \cap S\}$ converge to $z(\cup\{A_\alpha : \alpha \in \beta\})$ which would imply that $f$ is not continuous. Therefore $\lambda$ is countable, hence there is some countable set $A$ such that $f(z(B)) = f(z(A))$ for all countable $B \supset A$. Define $f(z)$ to be equal to $f(z(A))$. To see that $f$ is continuous at $z$, suppose that there is some $\epsilon > 0$ such that each neighbourhood of $z$ contains a point $y \in X$ such that $f(y) > f(z) + \epsilon$. Inductively choose a sequence, $\{y_n : n \in \omega\} \subset X$ and an increasing sequence $\{B_n : n \in \omega\}$ of finite subsets of $\mathfrak{c}$ so that, $A \subset B = \bigcup_n B_n$, for each $n$ and $\gamma \notin \bigcup_n B_n$, $y_n(\gamma) = 0$, and for each $\gamma \in B_n$, $|y_n(\gamma) - z(\gamma)| < \frac{1}{n}$, and, finally, $f(y_n) > f(z) + \epsilon$. It follows easily that $\{y_n : n \in \omega\}$ converges to $z(B)$, hence $f(z(B)) > f(z(A)) + \epsilon$ – a contradiction.

Given my belief that bizarre spaces are pseudocompact, I was moved to ask "But what if $X$ is realcompact?". I was able to construct a non-separable realcompact space $X$ such that $\beta X$ is separable with the assistance of $\diamondsuit$ but I do not know the general answer: Is the density of $\beta X$ equal to the density of $\upsilon X$.

Recall that $\diamondsuit$ is the statement that there is a sequence $\{A_\alpha : \alpha \in \omega_1\}$ such that for each $A \subset \omega_1$ and each closed and unbounded set $C \subset \omega_1$, there is an $\alpha \in C$ such that $A \cap \alpha = A_\alpha$. Fix such a $\diamondsuit$ sequence. Fix a bijection $h$ from $\omega_1$ to $\omega \times \omega_1 \times \omega$. For each $\alpha$, $h[A_\alpha] \subset \omega \times \omega_1 \times \omega$. For each $n \in \omega$, let $z(n,\alpha) = \{(\beta, m) : (n, \beta, m) \in h[A_\alpha]\}$. Let $L$ be the set of all $\alpha$ such that it just happens that, for each $n$, $z(n,\alpha)$ is a member of the product space $\alpha^\omega$; $\diamondsuit$ actually ensures this will happen quite frequently. Indeed, a standard "closing off" argument (see [13]) implies that for any countable set $\{z_n : n \in \omega\} \subset (\omega)^{\omega_1}$ and each closed and unbounded set $C \subset \omega_1$, there will be $\alpha \in C$ such that $z(n,\alpha) = z_n \restriction \alpha$ for each $n$ – hence $\alpha \in L$.

One can choose two sequences: $\{x_n : n \in \omega\} \subset (\omega+1)^{\omega_1}$ and $\{y_\xi : \xi \in \omega_1\} \subset \omega^{\omega_1}$ by defining $\{x_n \restriction \alpha : n \in \omega\}$ and $\{y_\xi \restriction \alpha : \xi < \alpha\}$ by induction on $\alpha$. The plan is to ensure that $\{x_n : n \in \omega\}$ and $\{y_\xi : \xi \in \omega_1\}$ have the same closure in $(\omega+1)^{\omega_1}$ (which is therefore separable). However we will ensure that the closure of $\{y_\xi : \xi \in \omega_1\}$ in $\omega^{\omega_1}$ is not separable. This latter closure is then a realcompact non-separable space which has a separable compactification.

The inductive hypotheses are:

1. for each limit $\alpha < \omega_1$ and each $\beta < \alpha$, $\{y_\xi \restriction \alpha : \beta < \xi < \alpha\}$ and $\{x_n \restriction \alpha : n \in \omega\}$ have the same closure in $(\omega+1)^\alpha$;
2. for each $n$ and each limit $\alpha$, $x_n^{-1}(\omega)$ is cofinal in $\alpha$;
3. if $\alpha \in L$ then there is a proper dense open subset $U_\alpha$ of $\omega^\alpha$ containing $\{z(n,\alpha) : n \in \omega\}$ such that
   (a) $\{x_n \restriction \alpha : n \in \omega\} \setminus U_\alpha$ is infinite;
   (b) $x_n(\alpha) = 0$ iff $x_n \restriction \alpha \notin U_\alpha$.

The only difficulty to the construction would appear to be choosing $U_\alpha$. This is actually quite easy though. Note that $\{y_\xi \restriction \alpha : \xi < \alpha\}$ has no isolated points (in a situation where there are disjoint sets with the same closure) and so has uncountable closure in $\omega^\alpha$. Therefore $\{x_n \restriction \alpha : n \in \omega\}$ contains an infinite subset which converges to some point of $\omega^\alpha$ which is not a member of $\{z(n,\alpha) : n \in \omega\}$. Next choose $y_{\alpha+m} \restriction \alpha+\omega$ and $x_n(\alpha+m)$ for $n, m \in \omega$ and lastly define $y_\xi \restriction [\alpha, \alpha+\omega)$ for $\xi < \alpha$.

We will leave the details of the proof to the reader but roughly the idea is that if $\{z_n : n \in \omega\}$ is any subset of the closure of $\{y_\xi : \xi \in \omega_1\}$, there will be an $\alpha \in \omega_1$ such that $z_n \restriction \alpha = z(n,\alpha)$ for each $n$. Now clearly $z_n$ is not a limit point of $\{x_n : n \in \omega\} \setminus U_\alpha$, hence it is a limit point of $\{x_n : n \in \omega\} \cap U_\alpha$. Therefore, for each $n$, $z_n(\alpha) > 0$. It follows that $\{z_n : n \in \omega\}$ is not dense.

Speaking of density: an interesting challenge is raised in [5, 7], it is a call for an easy proof that the density of $C(X)$ with the uniform metric is no larger than that of $C^*(X)$. (Actually I'm taking liberties here since it was in fact requested that it should be deduced from the case when $X$ is connected). We will show that the density of $C(X)$ is bounded by the density of $C^*(X)$ times the weight of $\beta X$; which suffices since as we learn in [5], Smirnov proved that the density of $C^*(X)$ is equal to the weight of $\beta X$ (for infinite $X$ of course). Well I think this proof qualifies as easy (enough to be interesting). Just put together these facts.

1. If $d(C^*(X))^\omega = d(C^*(X))$, then $d(C(X)) = d(C^*(X))$.
   There is an instructive topological proof of this fact but surprisingly it's not really a topological statement. If the density of a metric space is an $\omega$ power, then the density is equal to the cardinality. Therefore this case follows from the fact, mentioned earlier, that $|C(X)| \le |C^*(X)|$.
2. If $\mathcal{A}$ is a locally finite, pairwise completely separated family in $X$, then $w(\beta X)$ is at least as large as $\Pi\{w(\mathrm{cl}_{\beta X}(A)) : A \in \mathcal{A}\}$.
   This is rather straightforward to prove.
3. For each $f \in C(X)$, there is an $n$ such that $w\left(\mathrm{cl}_{\beta X}(X \setminus f^{-1}[-n,n])\right)^\omega \le w(\beta X)$.
   Well this is really the main point, but it follows immediately from 2 and the observation that $\{f^{-1}([2n, 2n+1]) : n \in \mathbb{Z}\}$ and $\{f^{-1}([2n-1, 2n]) : n \in \mathbb{Z}\}$ are locally finite.

So this is how to put it together. Let $\kappa$ denote the weight of $\beta X$. Fix a base $\mathcal{B}$ for $\beta X$ of cardinality $\kappa$. For each $B \in \mathcal{B}$ such that $w(\beta X \setminus B)^\omega \le \kappa$, it is easy enough to see, by 1 and 2, that

$$|\{f \restriction (X \setminus B) : f \in C(X)\}| \le \kappa\ .$$

Therefore the density of

$$\{f \in C(X) : f \restriction (B \cap X) \in C^*(B \cap X)\} \text{ is at most } \kappa\ .$$

By item 3, the union of all the $\kappa$ many families is dense.

## 3. General remarks on pseudocompact and realcompact spaces

The issue of when a product of spaces is pseudocompact certainly gained prominence after Glicksberg's landmark result that this corresponds to the Stone-Čech compactification of the product being equal the product of the Stone-Čech compactifications of the factors. Glicksberg had shown that a product $\Pi_\alpha X_\alpha$ is pseudocompact iff every countable subproduct is. In [3], Wis puts the theory of ultrafilters to good use to prove that this is not true for finite subproducts.

**Proposition 3.1.** *There is a space $X$ such that $X^\omega$ is not pseudocompact and yet $X^n$ is pseudocompact for each integer $n$.*

*Fact 1.* The space $[\omega \cup S]^n \subset [\beta\omega]^n$ is pseudocompact iff for each $\{f_0, \ldots, f_{n-1}\} \subset {}^\omega\omega$, there is a $p \in \omega^*$ such that $\{f_0(p), f_1(p), f_2(p), \ldots, f_{n-1}(p)\} \subset S$.

*Proof.* There are two key ideas to the proof. Any family of pairwise disjont open sets can be refined to a family of basic open sets. Basic open sets can be viewed as functions from $n$ into $\omega$. An indexed subfamily can be found such that on each coordinate they all differ or they are all the same. Each $f_i$ is then defined to enumerate the $i$-th coordinate projection of the ranges according to the index of the family. The converse implication is similar: for each $m$, let $[(f_0(m), f_1(m), \ldots, f_{n-1}(m))]$ denote the naturally associated basic clopen set. This family is easily seen to be locally finite in $(\omega \cup S)^n$ if there is no $p$ as hypothesized. □

*Proof of Proposition 3.1.* For each $i$ and $n$, let $g_i(n) = n + i$. For each $n$, construct a subset $S_n$ of $\omega^*$ such that for each $\{f_i : i < n\} \subset {}^\omega\omega$, there is a $p \in \omega^*$ such that $\{f_i(p) : i < n\} \subset (\omega \cup S_n)$ and for each $q \in S_n$, there is an $i \leq n$ such that $g_i(q) \notin S_n$. It follows from this latter assumption that, for all $p \in \omega^*$, $\{g_i(p) : i \leq n\}$ is not contained in $\omega \cup S_n$; in particular $(\omega \cup S_n)^{n+1}$ is not pseudocompact. Also ensure (since $\omega^*$ is so big) that for each $n < k$ and each $f \in {}^\omega\omega$, $f(S_n) \cap S_k$ is empty.

Now it follows that the family $\{[(n, n+1, n+2, \ldots, n+n)] : n \in \omega\}$ is locally finite in the space $(\omega \cup \bigcup_n S_n)^\omega$. Clearly, for each $n$, $(\omega \cup \bigcup_n S_n)^n$ is pseudocompact since it has the dense pseudocompact subspace $(\omega \cup S_n)^n$. □

Well, products of pseudocompact spaces need not be pseudocompact but they are Baire; hence, clearly, each pseudocompact space is Baire. This quite interesting fact was proven in [11]. There are two interesting aspects to the paper, one is, as the title suggests, the Baire property of $X$ can be regarded as a ring-theoretic property of $C(X)$ and, as the authors state, "the point of departure is" to investigate which properties are passed to $G_\delta$ dense subsets. A subset $Y$ of $X$ is $G_\delta$-dense if it meets every non-empty $G_\delta$-subset of $X$. Sometimes the $G_\delta$ topology on $X$ is useful, i.e. the $G_\delta$ sets form a base for a new ($P$-space) topology on $X$. As remarked in [11], it is a straightforward observation that a $G_\delta$-dense subset of a space with the Baire property again is Baire. Well, $X$ is always $G_\delta$-dense in $\upsilon X$, hence if $\upsilon X = \beta X$, then $X$ is Baire. It is also not too difficult to see ([11, 1.3]) that $\Pi\{X_\alpha : \alpha \in \Lambda\}$ is $G_\delta$-dense in $\Pi\{\beta X_\alpha : \alpha \in \Lambda\}$ if each $X_\alpha$ is pseudocompact.

In the paper [15], the $G_\delta$ topology, and more generally, the $G_\kappa$ topology for larger $\kappa$, is investigated in the context of realcompactness and Herrlich's more general notion of $\alpha$-compactness (with $\omega_1$-compact = realcompact). A detailed discussion of these notions would be too involved but the restriction, to the notion of realcompact, of two of the results therein are certainly of quite general interest. In these restricted forms the first is originally due to Frolik, the second to Juhasz.

**Proposition 3.2.** *If $X$ is realcompact, then the $G_\delta$ topology is again realcompact.*

*Proof.* Here's an embedding of the $G_\delta$-topology on $\mathbb{R}^\kappa$ into $\mathbb{R}^I$ as a closed subset. Let $I = \kappa \cup [\kappa \times \mathbb{R}]^\omega$. For each $\vec{x} \in \mathbb{R}^\kappa$, define $e(\vec{x}) \in \mathbb{R}^I$ according to the rule, $e(\vec{x})(\alpha) = \vec{x}(\alpha)$ and

$$e(\vec{x})(\langle(\alpha_n, r_n)\rangle_{n\in\omega}) = \begin{cases} 0 & \text{if } (\forall n)\ \vec{x}(\alpha_n) = r_n \\ \\ n + \dfrac{1}{|r_n - \vec{x}(\alpha_n)|} & \text{for minimal } \ n \text{ such that } \vec{x}(\alpha_n) \neq r_n \end{cases}.$$

Assume that $f \in \mathbb{R}^I$ is not in $e[\mathbb{R}^\kappa]$. Set $\vec{x} = f \restriction \kappa$ and fix any coordinate, $i = \langle(\alpha_n, r_n) : n \in \omega\rangle$, at which $f$ and $e(\vec{x})$ disagree. If $e(\vec{x})(i) > 0$, then there is a minimal $n$ such that $\vec{x}(\alpha_n) \neq r_n$, and we can choose a small enough neighbourhood around $\langle f(\alpha_m) : m \leq n\rangle$ so as to ensure that for every $e(\vec{y})$ in that neighbourhood, either $e(\vec{y})(i) \geq 2f(i)$ or $e(\vec{y})(i)$ is closer to $e(\vec{x})(i)$ than to $f(i)$ (i.e. we can be sure that $y(\alpha_n) \neq r_n$ and if $y(\alpha_m) \neq r_m$ for some smaller $m$ we can ensure that the difference is very small). Conversely if $e(\vec{x}(i)) = 0$, then $f(i) \neq 0$ and we can choose some $n$ so that $|f(i)| < n$. Now it is an easy matter to choose a neighbourhood of $\langle f(\alpha_m) : m \leq n\rangle$ so that for any $e(\vec{y})$ in this neighbourhood, either $e(\vec{y})(i) = 0$, or $e(\vec{y})(i) \geq n$. In either case, $f$ has a neighbourhood missing $e[\mathbb{R}^\kappa]$. □

This next result says essentially that there are no *large* realcompact spaces which are locally *small*.

**Proposition 3.3.** *If a realcompact space $X$ has the property that its $G_\mu$-topology, for a measurable cardinal $\mu$, is discrete, then $|X| < \mu$.*

*Proof.* Let $\mathcal{U}$ be a $\mu$-complete ultrafilter on the set $\mu$ . Assume that $\{x_\alpha : \alpha < \mu\}$ are distinct points of $X$. Clearly $\mathcal{Z} = \{Z \text{ a zero set in } \ X : \{\alpha < \mu : x_\alpha \in Z\} \in \mathcal{U}\ \}$ is a countably complete, in fact $\mu$-complete, filter on $X$. Let us verify that it is an ultrafilter. Let $f \in C^*(X)$ be arbitrary and assume that $Z(f) \cap Z \neq \emptyset$ for all $Z \in \mathcal{Z}$. For each $n$, let $U_n = \{\alpha \in \mu : |f(x_\alpha)| < \frac{1}{n}\}$. Since $X \setminus f^{-1}[(-\frac{1}{n}, \frac{1}{n})]$ is a zero set disjoint from $Z(f)$, it follows that $U_n \in \mathcal{U}$. Therefore $\bigcap_n U_n$ is in $U$, hence $Z(f) \in \mathcal{Z}$ as required to show that $\mathcal{Z}$ is an ultrafilter. Since $X$ is realcompact, there is some $x \in X$ which is a member of $Z$ for all $Z \in \mathcal{Z}$. Since $\mathcal{Z}$ is $\mu$-complete, it follows that $x$ is not isolated in the $G_\mu$-topology. □

## 4. $\beta X$ and products

Certainly for quite a period of time the study of extending functions in products was a principal focus for Wis' research; and rightly so, it is an interesting and difficult topic. Generally the issue is, if we have $f \in C^*(X \times Y)$, can we extend $f$ continuously to certain subsets of $\beta X \times \beta Y$? There are two main approaches. The first, originating with Glicksberg, is to examine the function from $X$ into $C^*(Y)$ induced by $f$, i.e. $x \mapsto f_x$ where $f_x(y) = f(x, y)$. Of course, if $f$ does extend continuously, then for each $x \in X$ we have no choice but to take the natural fiberwise extension of $f \restriction \{x\} \times Y$ to all of $\{x\} \times \beta Y$, i.e. $f_x$ extends to $\beta Y$. The second approach, from Tamano, is to examine the image of zero sets of $X \times Y$ under the projection maps. To list only these two names in this connection constitutes a major omission but the subject is thoroughly reviewed and analyzed in [9].

**Theorem 4.1.** *If $f \in C^*(X \times Y)$ then the obvious extension of $f$ to $X \times \beta Y$ is continuous iff the mapping $x \mapsto f_x$ from $X$ into $C^*(Y)$, with the uniform metric topology, is continuous.*

We will omit the proof since it is very straightforward; the novelty is in the approach not in the difficulty of the proof.

**Theorem 4.2.** *The following are equivalent for infinite spaces $X$, $Y$:*

1. $\beta(X \times Y) = \beta X \times \beta Y$*;*
2. $X \times Y$ *is pseudocompact;*
3. $X$ *is pseudocompact and for each* $f \in C^*(X \times Y)$, $x \mapsto f_x$ *is a continuous map from* $X$ *into* $C^*(Y)$*;*
4. $X$ *is pseudocompact and* $\pi_X$ *carries zero sets of* $X \times Y$ *to closed sets in* $X$.

In some sense we already know the implication (1) implies (2) since the remainder of a countable locally finite family of $X \times Y$ will be contain a large subspace which is an $F$-space and $F$-space subspaces of $\beta X \times \beta Y$ are somewhat more scarce. However it is too difficult to make this precise. The most revealing implication is (2) implies (4) in the sense that the special role of the product structure is most apparent (i.e. a continuous open mapping with pseudocompact domain will not, in general, carry zero sets to closed sets). Once a mapping has been lifted to $X \times \beta Y$, we are then in the situation that one factor is pseudocompact and the other is compact. This is much easier to handle (with the same techniques) thus lifting us all the way to $\beta X \times \beta Y$.

*Proof.* For (1) implies (2), fix a locally finite family, $\{U_n \times V_n : n \in \omega\}$ of products of cozero sets. It is not difficult to argue that we may assume that both families, $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$, are pairwise disjoint; e.g. if simply refining them is not possible then one of them contains an infinite locally finite subfamily and any family from the other coordinate will suffice. For each $n$, choose a function $f_n : X \times Y \to [0,1]$ such that $Cz(f) \subset U_n \times V_n$ and $f(x_n, y_n) = 1$ for some point $(x_n, y_n)$ . The sum $f = \Sigma_n f_n$ is bounded and continuous and takes value 0 at any point $(x, y)$ which is not in $\bigcup_n U_n \times V_n$ . Fix a neighbourhood, $U \times V$, of any limit point, $(p, q) \in \beta X \times \beta Y$, of $\{(x_n, y_n) : n \in \omega\}$. There are $n < k$ such that $\{(x_n, y_n), (x_k, y_k)\} \in U \times V$. Therefore $(x_n, y_k) \in (U \times V) \cap f^{-1}(0)$ which shows that $f$ cannot be continuously extend to $(p, q)$ .

(2) implies (4): Assume that $Z$ is a zero set in $X \times Y$ and that $x \notin \pi_X(Z)$ is a limit point of $\pi_X(Z)$. Since $X \times Y$ is pseudocompact, so are $Y$ and $\{x\} \times Y$. Fix a function $f \in C^*(X \times Y)$ such that $Z = Z(f)$ and $f$ is non-negative. Since $\{x\} \times Y$ is pseudocompact and $f$ is non-zero on this set, $f$ is even bounded away from 0 on it. With no loss of generality, assume that $f(\{x\} \times Y) = 1$. We choose a sequence of basic open sets $\{U_n \times V_n : n \in \omega\}$ and $\{U'_n \times V_n : n \in \omega\}$ as follows. First of all, let $U_0 = X$ and then inductively: $x$ will be a member of $U_{n-1}$, hence $U_{n-1} \times Y$ will meet $Z$. We pick a point $(x_n, y_n) \in Z \cap (U_{n-1} \times Y)$ and then slide over to the point $(x, y_n)$ for which $f(x, y_n) = 1$. Now we choose three open sets: $x_n \in U'_n \subset U_{n-1}$, $x \in U_n \subset U_{n-1}$ and $y_n \in V_n$ so that $f(U'_n \times V_n) \subset [0, \frac{1}{3}]$ and $f(U_n \times V_n) \subset [\frac{2}{3}, 2]$. The points $(x_n, y_n)$ are just devices to make the choices of the open sets. Since $X \times Y$ is pseudocompact, the family $\{U'_n \times V_n : n \in \omega\}$ has a limit point $(\bar{x}, \bar{y})$. The remarkable thing now is that the combination of ensuring, for $n < k$, that $U_n \supset U'_k$ and the fact that we are in a product space, implies that $(\bar{x}, \bar{y})$ is also a limit point of $\{(U_n, V_n) : n \in \omega\}$, i.e. if $U \times V$ meets $U'_n \times V_n$ and $U'_k \times V_k$, with $n < k$, then $U \times V$ also meets $U_n \times V_n$ . This immediately implies that $f$ is not continuous at $(\bar{x}, \bar{y})$.

(4) implies (3): Fix $f \in C^*(X \times Y)$, $x \in X$, and $\epsilon > 0$. Define the function $h \in C^*(X \times Y)$ by $h(x', y) = |f(x', y) - f(x, y)|$; clearly $h$ is constantly 0 on $\{x\} \times Y$. Since $\pi_X$ carries zero sets to closed sets, there is a neighbourhood $W$ of $x$ such that $W \times Y$ is disjoint from $h^{-1}([\epsilon/2, \infty))$. Therefore, for each $x' \in W$, $\rho(f_{x'}, f_x) \leq \epsilon/2$ where $\rho$ is the uniform metric on $C^*(Y)$. This proves that $x \mapsto f_x$ is continuous.

(3) implies (1): Let $f \in C^*(X \times Y)$. By Theorem 4.1, we have that $f$ extends to all of $X \times \beta Y$. Again by Theorem 4.1, we have that $x \mapsto f_x^\beta \in C(\beta Y)$ is a continous function. Since $X$ is pseudocompact, $\{f_x^\beta : x \in X\}$ is a pseudocompact subset of the metric space $C(\beta Y)$ and, so, is compact. Therefore this extends continuously to a mapping from $\beta X$ into $C(\beta Y)$ . Applying 4.1 yet again completes the proof. $\square$

A closer examination of the previous proof reveals that if each of the projection maps carry zero sets to closed sets then each $f \in C^*(X \times Y)$ will extend continuously to $(X \times \beta Y) \cup (\beta X \times Y)$, i.e. $(X \times \beta Y) \cup (\beta X \times Y)$ sits canonically in $\beta(X \times Y)$. The converse is also true. Such a pair, $X, Y$ is labelled as a $C^*$-pair in the influential paper [12] and the problem discussed, naturally, is what is the structure of *proper* $C^*$-pairs, i.e. a $C^*$-pair in which $\beta(X \times Y) \neq \beta X \times \beta Y$. It is rather immediate that pairs of infinite discrete spaces are proper $C^*$-pairs. The starting point is the following result which was also discovered independently by Hagar and Mrowka.

**Lemma 4.3.** *If $\pi_X : X \times Y \to X$ carries zero sets to closed sets then either $X$ is a $P$-space or $Y$ is pseudocompact.*

*Proof.* Assume that $x \in X$ is not a $P$-point and $\{V_n : n \in \omega\}$ is a locally finite family of zero sets of $Y$. Fix a cozero set $C$ of $X$ such that $x \in \overline{C} \setminus C$ and let $\{Z_n : n \in \omega\}$ be an increasing family of zero sets of $X$ whose union is $C$. Since the union of a locally finite family of zero sets is itself a zero set, it follows that $Z = \bigcup\{Z_n \times V_n : n \in \omega\}$ is a zero set of $X \times Y$. Clearly $C = \pi_X(Z)$. $\square$

Pseudocompact $P$-spaces are of course finite (countable subsets of $P$-spaces are closed discrete). Therefore, for $C^*$-pairs, we have the following dichotomy: both factors are pseudocompact or both factors are $P$-spaces. By Theorem 4.2, if either is pseudocompact, then it is not a proper $C^*$-pair. Another interesting fact is that $\upsilon X$ is a $P$-space for each $P$-space $X$. This naturally leads to the following refinement of the proper $C^*$-pair question:

If $(X, Y)$ is a proper $C^*$-pair, is $X \times \beta Y$ nonetheless $C^*$-embedded in $\upsilon X \times \beta Y$?

The next result, largely taken from [12], helps illustrate the usefulness of Theorem 4.2 and settles this question completely. Since every open cover of a metric space has a $\sigma$-discrete refinement (e.g. this is also true of the dual ideal of cozero sets whose complements are in a given zero set ultrafilter), a metric space is realcompact if and only if it has cardinality less than $\mu$, the least measurable cardinal (if it exists). Therefore the mention of the realcompactness of $C^*(K)$ is, to some extent a disguise or rather a very attractive costume. One says that a space is pseudo-$\kappa$-compact if it contains no family of open sets which is locally finite and of cardinality $\kappa$.

**Proposition 4.4.** *Let $K$ be a compact space.*

1. *If $C^*(K)$ is realcompact, then $\upsilon X \times K \subset \beta(X \times K)$ for all $X$.*
2. *If $X$ is pseudo-$\mu$-compact, then $\upsilon X \times K \subset \beta(X \times K)$ for all compact $K$.*
3. *If $C^*(K)$ is not realcompact and $X$ is not pseudo-$\mu$-compact, then $X \times K$ is not $C^*$-embedded in $\upsilon X \times K$.*

*Proof.* 1. Let $f$ be any member of $C^*(X\times K)$ and recall that we should consider the continuous mapping $x\mapsto f_x$ from $X$ into $C^*(K)$. Since $C^*(K)$ is realcompact, this map extends continuously to all of $\upsilon X$. That is, for each $p\in\upsilon X$, we have an associated $f_p\in C^*(K)$. The naturally defined $f:\upsilon X\times K\to\mathbb{R}$ is continuous by Proposition 4.1.

2. This part is quite similar. Assume that $X$ is pseudo-$\mu$-compact and let $f\in C^*(X\times K)$. Again the mapping $x\mapsto f_x$ sends $X$ continuously into the metric space $C^*(K)$. The image of $X$ is a pseudo-$\mu$-compact metric space, and so is realcompact. Therefore, this map again lifts to $\upsilon X$ and we proceed as in the first case.
3. In this case we fix a locally finite family $\{U_\alpha:\alpha\in\mu\}$ of cozero subsets of $X$. Also, since $C^*(K)$ is not realcompact, we may choose a closed discrete subset $\{f_\alpha:\alpha\in\mu\}\subset C^*(K)$. Now, for each $\alpha\in\mu$, fix any point $x_\alpha\in U_\alpha$ and a function $g_\alpha\in C^*(X)$ such that $g_\alpha(x_\alpha)=1$ and $g_\alpha(X\setminus U_\alpha)=0$. The function
$$f(x,y)=\Sigma\{g_\alpha(x)\cdot f_\alpha(y):\alpha\in\mu\}$$
is easily seen to be a member of $C^*(X\times Y)$. Clearly $f_{x_\alpha}=f_\alpha$ for each $\alpha\in\mu$. We leave it to the reader to observe that there must be some $p\in\upsilon X\setminus X$ which is a limit point of $\{x_\alpha:\alpha\in\mu\}$ and thus $f$ cannot be extended to all of $\{p\}\times K$, since this would entail the existence of an $f_p\in C^*(K)$ which is a limit of $\{f_\alpha:\alpha\in\mu\}$.

□

This leads us naturally into the final section.

## 5. $\upsilon X$ and products

With 20-20 hindsight let us begin this section with the following discouraging theorem (due to Hušek).

**Theorem 5.1.** *There are spaces $X,Y,X',Y'$ so that*
1. *$X\times Y$ is homeomorphic to $X'\times Y'$;*
2. *$\upsilon(X\times Y)=\upsilon X\times\upsilon Y$;*
3. *$\upsilon(X'\times Y')\neq\upsilon X'\times\upsilon Y'$.*

We say discouraging since it finally establishes that, in sharp contrast to Glicksberg theorem, there simply is no topological property of $X\times Y$ which is equivalent to the assertion that $\upsilon(X\times Y)=\upsilon X\times\upsilon Y$. It does not, however, preclude some characterization which involves the projection maps.

It is surprisingly easy to prove Theorem 5.1. Since $\upsilon X$ is obtained by attaching only the countably complete ultrafilters, it is quite easy to prove that $\upsilon(\omega\times Z)$ is equal to $\omega\times\upsilon Z$ and thus equal to $\upsilon\omega\times\upsilon Z$ for any $Z$.

More generally, as Comfort establishes ([4] and see Theorem 5.3 below), this is true for any locally compact realcompact $D$ of nonmeasurable cardinality in place of $\omega$.

Now take any pair of spaces, say $G$ and $L$, such that $\upsilon(G\times L)$ is not equal to $\upsilon G\times\upsilon L$. For example, we can take $G$ and $L$ to be pseudocompact such that $G\times L$ fails to be pseudocompact (as discussed in Proposition 3.1).

Set $X=\omega$ and $Y=(G\times L)$. Also set $X'=(\omega\times G)$ and $Y'=L$. Carrying out the simple computations we have that $\upsilon(X\times Y)=\omega\times\upsilon Y=\upsilon X\times\upsilon Y$, and $\upsilon(X'\times Y')=\omega\times\upsilon(G\times L)$, while $\upsilon X'\times\upsilon Y'=(\omega\times\upsilon G)\times\upsilon L$. It is easy to arrange that $\upsilon(X' \times Y')$ is not homeomorphic to $\upsilon X' \times \upsilon Y'$ (I can't imagine that they ever could be homeomorphic) but it is not necessary, the assertion is simply that $X' \times Y' = \omega \times G \times L$ is not $C^*$-embedded in $\omega \times \beta G \times \beta L$ which follows immediately from the fact that $G \times L$ is not $C^*$-embedded in $\beta G \times \beta L$.

Therefore this situation makes it even more important to search for general criteria which will ensure that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ does hold.

Comfort establishes in [4] the following quite general result.

**Theorem 5.2.** *If $\upsilon X$ is locally compact and $Y$ is a $k$-space, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$.*

Recall also, from [4] and attributed there to [12], the following ($\mu$ denotes the least measurable cardinal). It follows from Theorem 4.4 (with compact subsets of $X$ playing the role of $K$).

**Theorem 5.3.** *If $X$ has cardinality less than $\mu$ and is locally compact, then $X \times \upsilon Y \subset \upsilon(X \times Y)$ for all $Y$, i.e. $X \times Y$ is $C^*$-embedded in $X \times \upsilon Y$.*

Here's a slight generalization, which illustrates how one simply tries to utilize Glicksberg's theorem as much as possible. In particular, note that each first-countable space $X$ satisfies the hypothesis of the theorem.

**Theorem 5.4.** *If every point of $X$ is contained in a compact set of countable character of cardinality less than $\mu$ and if $Y$ has property TBA, then $X \times Y$ is $C^*$-embedded in $X \times \upsilon Y$.*

*Proof.* Let $(x,p) \in X \times \upsilon Y$ and $f \in C^*(X \times Y)$. For each $x' \in X$, there is continuous extension of $f \restriction (\{x'\} \times Y)$ to all of $\{x'\} \times \upsilon Y$; for convenience let us use $f(x',q)$ to denote this canonical value for each $q \in \upsilon Y$. Let $Z$ be a compact set of countable character that contains $x$. By Theorem 4.4, $f \restriction (Z \times \upsilon Y)$ is continuous. Towards showing that $f$ is continuous at $(x,p)$, fix any positive real $\epsilon$ and assume, for convenience, that $f(x,p) = 0$ and $f$ is everywhere non-negative. By the continuity of $f \restriction (Z \times \upsilon Y)$, there is a zero set neighbourhood, $W$, of $x$ in $X$ and an element $Y_0$ of $p$ such that $f \restriction ((W \cap Z) \times \upsilon Y)$ is everywhere less than $\epsilon$. Fix a descending neighbourhood base, $\{U_n : n \in \omega\}$, in $X$ for $W \cap Z$. For each $n$, choose, if possible, an $x_n \in U_n$ so that $f(x_n, p) \geq \epsilon$ and let $\tilde{x} \in Z$ be any limit point of $\{x_n : n \in \omega\}$. Now we use that $p$ is a countably complete ultrafilter: for each $n > 0$, choose a zero set $Y_n$ in the ultrafilter $p$ so that $f(x_n, y) = f(x_n, p)$ for all $y \in Y_n$. In addition, let $\tilde{Y}$ be a member of $p$ such that $f(\tilde{x}, y) = f(\tilde{x}, p)$ for all $y \in \tilde{Y}$. Clearly this situation contradicts the continuity of $f$ at each point of the nonempty set $\{\tilde{x}\} \times \bigcap\{\tilde{Y} \cap Y_n : n \in \omega\}$.

So what we have is that there is some $n$ such that $f(x',p) < \epsilon$ for all $x' \in U_n$. Since we have assumed that $Y$ is TBA, this completes the proof. $\square$

The previous result is, of course, quite unusual (to say the least). However I began the proof believing that TBA = Tychonof and thought it would be amusing to leave it (as a problem).

The most fruitful approach seems to be via the projection maps, but the condition that zero sets are carried to closed sets is much too strong. It seems that what makes Glicksberg's theorem "go" is that we can utilize all "directions" of convergence (although the absence of a characterization, in terms of the factors, of when a product is pseudocompact is also indicative of the difficulty to be encountered as we move to $\upsilon$). It is easy to see that, for a point $p \in \upsilon Y$, $X \times Y$ is not $C^*$-embedded in $X \times (Y \cup \{p\})$ is equivalent to the existence of an $x$ in $X$ and a zero set $Z$ of $X \times Y$ such that $Z \cap (\{x\} \times Y)$ is empty and $(x, p) \in \mathrm{cl} Z$ . The more I chip away at this problem the more discouraged I become. I will close with a few remarks that do not fill me with optimism that a reasonable characterization can be found.

**Theorem 5.5.** *For all $Y$, $\upsilon(\mathbb{Q} \times Y) = \mathbb{Q} \times \upsilon Y$ iff each locally finite countable family of cozero subsets of $Y$ is also locally finite in $\upsilon Y$.*

*Proof.* This is much the same as the previous (non) Theorem. Fix any function $f \in C^*(\mathbb{Q} \times Y)$ and $(q, p) \in \mathbb{Q} \times \upsilon Y$. We may as well assume that $f$ is identically 0 on $\{q\} \times Y$ and that it is enough to show that $(q, p)$ is not a limit point of $f^{-1}(2)$. For each rational $r$, let $C_r$ denote the cozero subset of $Y$ such that $f^{-1}((1, \infty)) \cap \{r\} \times Y = \{r\} \times C_r$. By our analysis in Theorem 5.4, we know there is an neighbourhood of $q$ such that $f_r(p) < 1$ for each $r$ in that neighbourhood; assume this is the case for all $r$. Now for each $n \in \omega$, let $U_n = \bigcup\{C_r : |r - q| < \frac{1}{n+1}\}$. The family $\{U_n : n \in \omega\}$ is locally finite at $y$ iff $(q, y)$ is a limit point of $f^{-1}((1, \infty))$. Therefore the family is locally finite in $Y$, and by our condition on $Y$, also locally finite at $p$.

For the converse, assume that $\{U_n : n \in \omega\}$ is locally finite in $Y$ but not at $p \in \upsilon Y$, each $U_n$ is a cozero set. For each $n$, fix an increasing sequence $\{Z_{n,m} : m \in \omega\}$ of zero sets in $Y$ whose union is $U_n$. Let $\{q(n, m) : n, m \in \omega\} \subset \mathbb{Q}$ be such that, for all $n$, $\{q(n, m) : m \in \omega\} \subset (\frac{\pi}{n+2}, \frac{\pi}{n+1})$ converges to $\frac{\pi}{n+1}$. It is easily checked that $Z = \bigcup_{n,m}\{q(n, m)\} \times Z_{n,m}$ is a zero set in $\mathbb{Q} \times Y$ which has $(0, p)$ as a limit. As mentioned above, this shows that $\mathbb{Q} \times Y$ is not $C^*$-embedded in $\mathbb{Q} \times (Y \cup \{p\})$ . □

The search for an example $Y$ failing to have the property mention above led to the following admittedly messy (perhaps necessarily) construction. We denote the collection of cozero subsets of a space $X$ by $Cz(X)$.

**Theorem 5.6.** *For each space $X$ which is not locally compact, there is a Tychonoff space $Y$ such that $X \times Y$ is not $C^*$-embedded in $X \times \upsilon Y$.*

*Proof.* Since $X$ is not locally compact, $\beta X \setminus X$ is not compact. Fix $A = \{a_\alpha : \alpha < \kappa\} \subset \beta X \setminus X$ of minimum cardinality such that $A$ has a limit in $X$. We can choose $A$ to be discrete: (this is well-known: fix an open cover of $\beta X \setminus X$ with no finite subcover of minimum cardinality, inductively choose $a_\alpha$ together with a finite subcover of $\{a_\beta : \beta \leq \alpha\}$ so that $a_\alpha$ is not in any of the open sets chosen so far). Since $A$ is discrete, $\overline{\overline{A}}$ is nowhere dense.

We start with a big plank (the interested reader, whose existence may depend on measurable cardinals, may wish to draw a picture):
$(\kappa^+ + 1) \times (A \cup \{a_\kappa\}) \setminus \{(\kappa^+, a_\kappa)\}$ where $\kappa^+ + 1$ has the ordinal topology. The topology on $(A \cup \{a_\kappa\})$ for the sake of our plank, is that $A$ is discrete and neighbourhoods of $a_\kappa$ take the form $\{a_\alpha : \beta < \alpha \leq \kappa\}$ . Next we attach a great big discrete space, $A \times (X \times Cz(X))$, which we glue to the plank as follows: for each $\alpha < \kappa$, neighbourhoods of $(\kappa^+, a_\alpha)$ reach into $\{a_\alpha\} \times (X \times Cz(X))$: i.e. each neighbourhood of $(\kappa^+, a_\alpha)$ contains some

$$W(\alpha, U) = \{a_\alpha\} \times \{(x, C) : x \in C \subset U\}$$

where $U$ is an open subset of $\beta X$ with $a_\alpha \in U$. So our space $Y$ is this topology on the base set

$$((\kappa^+ + 1) \times A) \cup \{\kappa^+\} \times \{a_\kappa\} \cup A \times (X \times Cz(X)) .$$

The usual properties of planks ensure that the corner point $(\kappa^+, a_\kappa)$ is a member of $\upsilon Y$ and if $f \in C^*(\upsilon Y)$ is 0 at $(\kappa^+, a_\kappa)$, then there will be a $\beta < \kappa$ such that $f$ is less than $\frac{1}{2}$ at $(\kappa^+, a_\alpha)$ for all $\beta < \alpha \leq \kappa$.

Now we define a locally finite family $\mathcal{U}$ of cozero subsets of $X \times Y$. For each $x \in X \setminus \overline{A}$ select any $C_x \in Cz(X)$ such that $x \in C_x$ and $\overline{C_x} \cap \overline{A}$ is empty.

Define

$$U_x = \{(\tilde{x}, (a_\alpha, (x, C))) : \alpha \in \kappa,\ \{\tilde{x}, x\} \subset C \subset C_x\}\ .$$

We must show that $U_x$ is a cozero subset of $X \times Y$. Fix any function $f_x \in C(X)$ such that $C_x$ is the cozero set for $f_x$ . Define $F_x((\tilde{x}, (a_\alpha, (x, C)))) = f_x(\tilde{x})$ for each point in $U_x$ and set $F_x$ to be 0 for all other points of $X \times Y$. For each point $(x, C) \in X \times Cz(X)$, $C \times (A \times \{(x, C)\})$ is a cozero set of $X \times Y$ on which $F_x$ is clearly continuous. The only other points at which continuity of $F_x$ is perhaps not immediate are points of the form $(\tilde{x}, (\kappa^+, a_\alpha))$. Such a point has a neighbourhood whose intersection with $A \times (X \times Cz(X))$ is contained in $W(\alpha, U)$ where $U$ is a cozero subset of $\beta X$ containing $a_\alpha$ and not containing $x$. Since any point $(a_\alpha, (x, C))$ in $W(\alpha, U)$ has the property that $C \subset U$, we have immediately that $(a_\alpha, (x, C))$ is not in $U_x$ since $x \in C \setminus U$. Thus $F_x$ is constantly 0 on a neighbourhood of $(\tilde{x}, (\kappa^+, a_\alpha))$.

The family $\mathcal{U} = \{U_x : x \in X \setminus \overline{A}\}$ is the desired family. We check that it is locally finite in $X \times Y$. We leave to the reader to check all points other than those that are of the form $(x, (\kappa^+, a_\alpha))$ . For one of these, pick a pair of disjoint open subsets of $\beta X$, $V, U$ such that $x \in V$ and $a_\alpha \in U$. Suppose that $(\tilde{x}, (a_\alpha, (x_1, C))) \in U_{x_1} \cap (V \times W(\alpha, U))$. Clearly, $\tilde{x} \in V$ and $\{\tilde{x}, x_1\} \subset C \subset U$; i.e. $\tilde{x} \in U \cap V$, a contradiction.

Well, this actually does it. Take the function $F$ on $X \times Y$ which is the sum of the locally finitely supported family of functions $\{F_x : x \in X \setminus \overline{A}\}$. Fix any $x_0 \in \overline{A} \cap X$ and note that $F$ is constantly 0 on $\{x_0\} \times Y$. It was not relevant to the discussion above but now it is: we can assume that $f_x(x) = 1$ for each $x \in X \setminus \overline{A}$.

We finish by proving that $(x_0, (\kappa^+, a_\kappa))$ is a limit point of the set

$$\{(x, (a_\alpha, (x, C))) : x \notin \overline{A}, \quad x \in C \subset C_x\}$$

which will suffice since $F$ is equal to 1 at each of these points. Let $V$ be any cozero subset of $\beta X$ which contains $x_0$. Consider an arbitrary neighbourhood, $\mathcal{W}$, of $(\kappa^+, a_\kappa)$ in $Y$. We do not care about much of the neighbourhood, except that there will be a $\beta < \kappa$ such that for each $\alpha$ with $\beta < \alpha < \kappa$, the point $(\kappa^+, a_\alpha)$ will be in $\mathcal{W}$ . Therefore, for each such $a_\alpha$ there is a cozero set $U_\alpha$ of $\beta X$ containing $a_\alpha$ such that $W(\alpha, U_\alpha)$ is contained in $\mathcal{W}$. Since $x_0$ is a limit point of $\{a_\alpha : \beta < \alpha < \kappa\}$, there is an $\alpha$ such that $a_\alpha \in V$ and $(\kappa^+, a_\alpha) \in \mathcal{W}$ . Fix any $x \in (V \cap U_\alpha) \setminus \overline{A}$ (remember that $\overline{A}$ is nowhere dense) and set $C = C_x \cap V \cap U_\alpha$ . Well, unravel it all and you will find that $(x, (a_\alpha, (x, C)))$ is a member of $U_x \cap (V \times \mathcal{W})$ . □

One can use a space $X$ with only one non-isolated point very much like the subspace $\{(\kappa^+, a_\alpha)\} \cup (\{a_\alpha\} \times (X \times Cz(X)))$ of our space $Y$ from the previous theorem to prove the following simple fact.

**Theorem 5.7.** *For each non-realcompact space $Y$, there is a space $X$ (of cardinality $2^{|Y|}$) such that $X \times Y$ is not $C^*$-embedded in $X \times \upsilon Y$.*

Finally, even our old friend, the plank $(\omega_1 + 1) \times \omega_1$ is big trouble. For a subset $S$ of the ordinal space $\omega_1$, let $S^+$ denote $S \cup \{\omega_1\}$. Note that $S^+ = \upsilon S$ iff $S$ is stationary. I'm not quite sure how a natural purely topological characterization involving the projection maps could detect when $S \cap T$ is stationary.

**Theorem 5.8.** *For uncountable subsets $S, T$ of the ordinal space $\omega_1$, $S^+ \times T$ is $C^*$-embedded in $S^+ \times T^+$ iff $S \cap T$ is stationary.*

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