TWO TO ONE CONTINUOUS IMAGES OF $\mathbb{N}^*$

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Abstract. A function is two-to-one if every point in the image has exactly two inverse points. We show that every two-to-one continuous image of $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$ when the continuum hypothesis is assumed. We also prove that there is no irreducible two-to-one continuous function whose domain is $\mathbb{N}^*$ under the same assumption.

1. Introduction

A function $f\colon X \to Y$ is *two-to-one* if for each $y \in Y$, there are exactly two points of X that map to y. All spaces considered are Tychonoff. For some spaces X, there does not exist a two-to-one continuous function $f\colon X \to Y$ for any choice of Y. For example, Harrold [Ha39] showed that there is no two-to-one continuous function $f\colon [0,1] \to Y$ and Mioduszewski [Mi61] proved that there is no two-to-one continuous function $f : \mathbb{R} \to Y$. In fact, Heath [He86] later showed that every two-to-one continuous function $f\colon \mathbb{R} \to Y$ has infinitely many discontinuities.

Another situation is when there are two-to-one continuous functions $f\colon X \to Y$ defined on a space X, but given any such function the image space Y is determined up to a homeomorphism. For example, if $f\colon [0,1) \to S^1$, where S^1 is the unit circle $\{p \in \mathbb{R}^2 : ||p|| = 1\}$, is defined by $f(x) = exp(4\pi ix)$, then f is a two-to-one continuous function. Mioduszewski [Mi61] proved that if

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$f\colon [0,1) \to Y$ is a two-to-one continuous function, then Y is homeomorphic to S^1.

In this paper we investigate the behavior of two-to-one continuous maps defined on $\mathbb{N}^*$, the remainder $\beta\mathbb{N} \setminus \mathbb{N}$ of the Stone-Čech compactification of the space $\mathbb{N}$ of natural numbers. We give partial answers to questions recently raised by R. Levy [Le04]. In particular, we show that every two-to-one continuous image of $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$ when the continuum hypothesis (CH) is assumed.

There are two-to-one continuous functions defined on $\mathbb{N}^*$. For example, the space $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^* \times 2$, so the projection map $f\colon \mathbb{N}^* \times 2 \to \mathbb{N}^*$ given by $f(x,k) = x$ is a two-to-one continuous function on $\mathbb{N}^*$ and the image is $\mathbb{N}^*$. Such an example would be called trivial. A continuous two-to-one function $f : \mathbb{N}^* \to \mathbb{N}^*$ is *trivial* if there is a clopen subset C of $\mathbb{N}^*$ such that the restrictions $f \restriction_C$ and $f \restriction_{(\mathbb{N}^* \setminus C)}$ are homeomorphisms. In [Do04], the first author proved that all maps on $\mathbb{N}^*$ that are two-to-one are trivial, in the above sense, under the presence of the Proper Forcing Axiom (PFA). Eric van Douwen [vD93] has also produced a surprising answer to a similar question raised by R. Levy. He showed that the space $\mathbb{N}^*$, which is a compact space and very far from being separable, can be mapped onto a compact separable space by a $\leq$two-to-one continuous function. We are concerned with the question of whether every exactly two-to-one continuous image of $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$.

It is well known that if a space Y is homeomorphic to $\mathbb{N}^*$, then Y is a *Parovičenko space*, that is, a compact zero-dimensional F-space of weight $\mathfrak{c}$ which has no isolated points and with the property that every nonempty

G_δ-set has infinite interior. Therefore, if we are interested in whether or not two-to-one continuous images of $\mathbb{N}^*$ are homeomorphic to $\mathbb{N}^*$, we should investigate which of these six properties are satisfied by the spaces which are two-to-one continuous images of $\mathbb{N}^*$.

If Y is a two-to-one continuous image of $\mathbb{N}^*$, obviously Y is compact since $\mathbb{N}^*$ is compact and a continuous image of a compact space is compact, and Y has no isolated points since two-to-one continuous maps preserve the property of having no isolated points. Ronnie Levy [Le04] showed that Y has the property that countable discrete subsets are C^*- embedded and Y contains a copy of $\mathbb{N}^*$ and so Y has weight $\mathfrak{c}$. We include his proof for the reader's convenience.

Theorem 1 (Levy)**.** *Let X be a space such that every countable discrete subset of X is C^*-embedded in X. If $f : X \to Y$ is a two-to-one continuous function, then every countable discrete subset of Y is C^*-embedded in Y.*

Proof. Let C be a countable discrete subset of Y. Since f is two-to-one and C is countable discrete, $f^{-1}(C)$ is a countable discrete subset of X. Therefore, $f^{-1}(C)$ is C^*-embedded in X. We must show that disjoint subsets of C have disjoint closures in Y. Let $A, B \subseteq C$ such that $A \cap B = \emptyset$. Assume that there exists $p \in Cl_Y A \cap Cl_Y B$.

For each $a \in A$ let $f^{-1}(\{a\}) = \{a_1, a_2\}$ and for each $b \in B$, let $f^{-1}(\{b\}) = \{b_1, b_2\}$. Let $A_1 = \{a_1 : a \in A\}$, $A_2 = \{a_2 : a \in A\}$, $B_1 = \{b_1 : b \in B\}$, $B_2 = \{b_2 : b \in B\}$. These four sets are pairwise disjoint subsets of $f^{-1}(C)$ and therefore their closures are also pairwise disjoint since $f^{-1}(C)$ is C^*- embedded. By the continuity of f, each of the four sets $Cl_X A_1$, $Cl_X A_2$, $Cl_X B_1$, $Cl_X B_2$ contains an element of the fiber $f^{-1}(\{p\})$.

Since these sets are pairwise disjoint, $|f^{-1}(\{p\})| \geq 4$. This contradicts the fact that f is two-to-one. $\square$

Then Levy asked in the same paper whether Y has the remaining three properties. We show that Y actually has these three properties under CH, that is, Y is a zero-dimensional F-space in which every nonempty G_δ-set has infinite interior. A consequence then is that two-to-one continuous images of $\mathbb{N}^*$ are homeomorphic to $\mathbb{N}^*$ when CH is assumed since $\mathbb{N}^*$ is the only *Parovičenko space* under CH [vM84].

2. Irreducible Maps

A mapping f of X onto Y is *irreducible* if no proper closed subset of X maps onto Y. Thus, the image of an open set by a closed irreducible mapping will have interior. It follows easily from Zorn's lemma [Wa74] that if X and Y are compact spaces and f is a continuous function from X onto Y, then there is a closed subspace F of X such that $f\restriction_F$ is an irreducible map from F onto Y. Levy [Le04] asked if there exists an irreducible two-to-one continuous function whose domain is $\mathbb{N}^*$. Under CH, we will show that there is no such function.

Notation . For a map $f : X \to Y$ and $A \subseteq X$, let $J_A = f^{-1}(f[X \backslash A]) \cap A$ and for a point x, x' will denote a point $x' \neq x$ such that $f(x) = f(x')$.

Lemma 2. *Let $f : X \to Y$ be an irreducible continuous closed map. If A is an open subset of X, then J_A is nowhere dense in X.*

Proof. Suppose that J_A is not nowhere dense in X. Then $Int\bar{J}_A$ is a nonempty open subset of X. Then clearly $f[A \cap Int\bar{J}_A] \subseteq f[A]$ and $\bar{J}_A \subseteq$

$f^{-1}(f[X \setminus A])$ since $J_A \subseteq f^{-1}(f[X \setminus A])$ and $f[X \setminus A]$ is closed. Thus $f[\bar{J}_A] \subseteq f[X \setminus A]$ and in particular $f[A \cap Int\bar{J}_A] \subseteq f[X \setminus A]$. Therefore, $X \setminus (A \cap Int\bar{J}_A)$ is a proper closed subset of X since $A \cap Int\bar{J}_A$ is a nonempty open set contained in A, and $f[X \setminus (A \cap Int\bar{J}_A)] = Y$ since $f[A \cap Int\bar{J}_A] \subseteq f[A]$ and $f[A \cap Int\bar{J}_A] \subseteq f[X \setminus A]$. This is a contradiction since f is irreducible. $\square$

The following result is the main ingredient in our analysis of the structure of two-to-one continuous functions.

Theorem 3 (CH)**.** *Let X be a compact space of weight $\mathfrak{c}$ and suppose that countable discrete subsets of X are C^*-embedded. If $f : X \to K$ is a two-to-one continuous function and Z is a closed subset of X such that $f\restriction_Z$ is irreducible and maps Z onto K, then for every nonempty open set $W \subseteq K$ there exists an open set B in X such that $\overline{B} \cap Z = \emptyset$ and $Int(f[B]) \cap W \neq \emptyset$. Furthermore, if X is zero-dimensional, then B can be chosen to be clopen.*

Proof. Let W be a nonempty open subset of K. Seeking a contradiction, suppose that $f[B] \cap W$ is nowhere dense in K for each open set $B \subseteq X$ with closure disjoint from Z . For each open subset $B \subseteq X \setminus Z$, let $I_B = f^{-1}(\overline{f[B] \cap W}) \cap Z$. Then I_B is nowhere dense in Z since f is closed irreducible and $f[I_B] \subseteq \overline{f[B] \cap W}$ which is nowhere dense. Since f is continuous, $f^{-1}(W) \cap Z$ has nonempty interior in Z. For each open subset $A \subseteq f^{-1}(W) \cap Z$, J_A is nowhere dense in Z by Lemma 2.

We construct, by induction, a family $\{ A_\alpha : \alpha < \omega_1 \}$ which is a filter base of cozero subsets of $f^{-1}(W) \cap Z$ such that $\bigcap_{\alpha < \omega_1} A_\alpha$ is a singleton $\{x\}$ so that

$f^{-1}(f(x)) = \{x\}$. This yields a contradiction as f is an exactly two-to-one function.

Let $\{B_\alpha : \alpha < \omega_1\}$ enumerate all the cozero sets B in X such that $\overline{B} \cap Z = \emptyset$ and let $\{ C^0_\alpha, C^1_\alpha : \alpha < \omega_1 \}$ enumerate all pairs of cozero sets in Z such that $Z = C^0_\alpha \cup C^1_\alpha$. We construct $\{ A_\alpha : \alpha < \omega_1 \}$ such that for each $\alpha < \omega_1$:

(1) $\bigcap_{\beta \leq \alpha} A_\beta$ is nonempty;
(2) $\overline{A}_\alpha \subseteq C^0_\alpha$ or $\overline{A}_\alpha \subseteq C^1_\alpha$; and
(3) If $\alpha = \beta + 1$, then $\overline{A}_\alpha \subseteq A_\beta \setminus (J_{A_\beta} \cup I_{B_\beta})$.

We show how to define the first two cozero sets A_0 and A_1: Let $x \in f^{-1}(W) \cap Z$. Then choose a cozero set neighborhood A_0 of x such that $\overline{A}_0 \subseteq f^{-1}(W) \cap Z$ and either $\overline{A}_0 \subseteq C^0_0$ or $\overline{A}_0 \subseteq C^1_0$.

Then J_{A_0} and I_{B_0} are nowhere dense in Z, hence $A_0 \setminus (J_{A_0} \cup I_{B_0}) \neq \emptyset$. Let x, possibly different from the previous x, be a point in $A_0 \setminus (J_{A_0} \cup I_{B_0})$. Then choose a cozero set neighborhood A_1 of x such that $\overline{A}_1 \subseteq A_0 \setminus (J_{A_0} \cup I_{B_0})$ and either $\overline{A}_1 \subseteq C^0_1$ or $\overline{A}_1 \subseteq C^1_1$. For each $n \in \omega$ we can define A_n in the same manner.

Suppose that $\alpha \geq \omega$ and we have constructed the family $\{ A_\beta : \beta < \gamma \}$ for all $\gamma < \alpha$. If $\alpha = \beta + 1$, then $\bigcap_{\gamma \leq \beta} A_\gamma \neq \emptyset$ by the induction assumption. If α is a limit, then the induction hypothesis (1) ensures that $G_\alpha = \bigcap_{\beta < \alpha} A_\beta$ is not empty since Z is compact and $A_\beta \subseteq \overline{A}_{\beta+1}$ for each $\beta < \alpha$ so that $\bigcap_{\beta < \alpha} A_\beta \supseteq \bigcap_{\beta < \alpha} \overline{A}_{\beta+1}$.

If α is a limit and $x \in G_\alpha = \bigcap_{\beta < \alpha} A_\beta$, we choose a cozero set neighborhood A_α of x such that $\overline{A}_\alpha \subseteq C^0_\alpha$ if $x \in C^0_\alpha$ or $\overline{A}_\alpha \subseteq C^1_\alpha$ otherwise. If $\alpha = \beta + 1$ with $\beta \geq \omega$, we must define a cozero set A_α so that $\overline{A}_\alpha \subseteq A_\beta \setminus [J_{A_\beta} \cup I_{B_\beta}]$.

Let λ be the largest limit ordinal less than α. We enumerate $\lambda \cup \{\beta\}$ by: $\beta = \beta_0,\ \beta_1,\ \beta_2,\ \beta_3,\ \cdots$. We now consider the cases when G_α has nonempty interior and G_α is nowhere dense.

If G_α has nonempty interior, then $G_\alpha \setminus [J_{A_\beta} \cup I_{B_\beta}] \neq \emptyset$ since $J_{A_\beta} \cup I_{B_\beta}$ is nowhere dense. Then pick a point $x \in G_\alpha \setminus [J_{A_\beta} \cup I_{B_\beta}]$ and a cozero set A_α containing x such that $\overline{A}_\alpha \subseteq G_\alpha$ and either $\overline{A}_\alpha \subseteq C^0_\alpha$ or $\overline{A}_\alpha \subseteq C^1_\alpha$.

If G_α is nowhere dense, $J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha$ is nowhere dense and $A_\beta \setminus [J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha] \neq \emptyset$. Let $x_0 \in A_\beta \setminus [J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha]$. Then choose $i_1 > 1$ large enough such that $x_0 \notin A_{\beta_{i_1}}$, which we may do since $x_0 \notin G_\alpha$. Then either $x_0^{'} \in A_{\beta_{i_1}}$, $x_0^{'} \in J_{A_{\beta_{i_1}}}$, or $x_0^{'} \notin Z$. In any case, $x_0^{'} \notin \overline{A}_{\beta_{i_1}+1}$ since $\overline{A}_{\beta_{i_1}+1} \subseteq A_{\beta_{i_1}} \setminus J_{A_{\beta_{i_1}}}$ by construction. So there is an $i_1^{'} > 1$ such that $\beta_{i_1^{'}} = \beta_{i_1} + 1$ and $x_0^{'} \notin \overline{A}_{\beta_{i_1^{'}}}$. Thus $x_0, x_0^{'} \notin G_\alpha$.

Similarly, pick $x_1 \in [A_\beta \cap A_{\beta_1} \cap \cdots \cap A_{\beta_{i_1}}] \setminus [J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha]$ and choose $i_2 > i_1^{'}, i_1 > 1$ large enough so that $x_1 \notin A_{\beta_{i_2}}$. Then $x_1^{'} \notin \overline{A}_{\beta_{i_2}+1}$ by construction. So there is an $i_2^{'} > i_1^{'}, i_1 > 1$ such that $\beta_{i_2^{'}} = \beta_{i_2} + 1$ and $x_1^{'} \notin \overline{A}_{\beta_{i_2^{'}}}$. Thus $x_1, x_1^{'} \notin G_\alpha$. Continuing this process, for every integer n, pick $x_n \in [A_\beta \cap A_{\beta_1} \cap \cdots \cap A_{\beta_{i_n}}] \setminus [J_{A_\beta} \cup I_{B_\beta} \cup G_\alpha]$ and choose an integer $i_{n+1} > i_n, i_n^{'} > \cdots > i_2, i_2^{'} > i_1^{'}, i_1 > 1$ large enough so that $x_n \notin A_{\beta_{i_{n+1}}}$. Then there exists an integer $i_{n+1}^{'} > i_n^{'}, i_n$ such that $x_n^{'} \notin \overline{A}_{\beta_{i_n^{'}}}$. Thus $x_n, x_n^{'} \notin G_\alpha$.

Hence, we get a countable set $\{x_n\}_n \cup \{x_n^{'}\}_n$. We remark that the x_n's and $x_n^{'}$'s can be chosen from some dense subset of $f^{-1}(W) \cap Z$. From the construction of the x_n's and $x_n^{'}$'s

$$\forall i > n \quad x_i, x_i^{'} \in f^{-1}(f[\overline{A}_\beta \cap \overline{A}_{\beta_1} \cap \cdots \cap \overline{A}_{\beta_{i_n}}])$$

$$\forall j \leq n \quad x_j, x_j^{'} \notin f^{-1}(f[\overline{A}_\beta \cap \overline{A}_{\beta_1} \cap \cdots \cap \overline{A}_{\beta_{i_n+2}}])$$

and since $f^{-1}(f[\overline{A}_\beta \cap \overline{A}_{\beta_1} \cap \cdots \cap \overline{A}_{\beta_k}])$ is closed for all k, the set $\{x_n\}_n \cup \{x_n^{'}\}_n$ is discrete. Therefore $\overline{\{x_n\}_n} \cap \overline{\{x_n^{'}\}_n} = \emptyset$ since $\{x_n\}_n \cap \{x_n^{'}\}_n = \emptyset$ and countable discrete subsets of X are C^*-embedded. We have $f(\overline{\{x_n\}_n}) = f(\overline{\{x_n^{'}\}_n})$ by continuity of f and the fact that $f(\{x_n\}_n) = f(\{x_n^{'}\}_n)$. We also have $\overline{\{x_n\}_n} \setminus \{x_n\}_n \neq \emptyset$ and $\overline{\{x_n^{'}\}_n} \setminus \{x_n^{'}\}_n \neq \emptyset$ since every infinite subset of a compact set has a limit point. By the construction of the x_n's we see that $\overline{\{x_n\}_n} \setminus \{x_n\}_n \subseteq G_\alpha$.

If $\{z \in Z : z^{'} \in Z\} \cap f^{-1}(W)$ is dense in $f^{-1}(W) \cap Z$, we can choose the $x_n^{'}$'s in $f^{-1}(W) \cap Z$ so that we also have $\overline{\{x_n^{'}\}_n} \setminus \{x_n^{'}\}_n \subseteq G_\alpha$. In this case if we choose $x \in \overline{\{x_n\}_n} \setminus \{x_n\}_n$, then $x^{'} \in \overline{\{x_n^{'}\}_n} \setminus \{x_n^{'}\}_n$ and $x \neq x^{'}$ since $\overline{\{x_n\}_n} \cap \overline{\{x_n^{'}\}_n} = \emptyset$. Moreover $x \notin J_{A_\beta}$ since $x \in J_{A_\beta}$ implies $x^{'} \notin A_\beta$ which contradicts $x^{'} \in G_\alpha \subseteq A_\beta$. We also have $x \notin I_{B_\beta}$, that is, $x^{'} \notin B_\beta$ since $x_n^{'} \notin B_\beta$ for all n and B_β is a cozero set. Thus we have found an $x \in G_\alpha$ such that $x \notin J_{A_\beta} \cup I_{B_\beta}$.

If $\{z \in Z : z^{'} \in Z\} \cap f^{-1}(W)$ is nowhere dense in $f^{-1}(W) \cap Z$, find a cozero set $A \subseteq A_\beta \subseteq f^{-1}(W) \cap Z$ such that $f\restriction_Z$ is one-to-one on all points of $\overline{A}$, that is, $f^{-1}(f[\overline{A}])$ meets Z in $\overline{A}$. In this case we can choose the x_n's so that $\{x_n\}_n \subseteq A$ and hence $\{x_n^{'}\}_n \subseteq X \setminus Z$. Then $\overline{\{x_n\}_n} \setminus \{x_n\}_n \subseteq \overline{A}$. Choose an $x \in \overline{\{x_n\}_n} \setminus \{x_n\}_n$. Then $x \in \overline{A}$ and so $x^{'} \notin Z$; in particular $x^{'} \notin Z \setminus A_\beta$ and so $x \notin J_{A_\beta}$. It is also true that $x \notin I_{B_\beta}$, that is, $x^{'} \notin B_\beta$ since $x_n^{'} \notin B_\beta$ for all n and B_β is a cozero set. Therefore $x \in G_\alpha$ and $x \notin J_{A_\beta} \cup I_{B_\beta}$.

We choose a cozero set A_α containing x such that $\overline{A}_\alpha \subseteq A_\beta \setminus [J_{A_\beta} \cup I_{B_\beta}]$ and either $\overline{A}_\alpha \subseteq C^0_\alpha$ or $\overline{A}_\alpha \subseteq C^1_\alpha$. Then A_α satisfies all the induction assumptions and this completes the inductive construction.

But now $\bigcap_{\alpha<\omega_1} A_\alpha \neq \emptyset$ since $\bigcap_{\beta<\omega_1} A_\beta \supseteq \bigcap_{\beta<\omega_1} \overline{A}_{\beta+1}$ and X is compact. Moreover the fact that $\bigcap_{\alpha<\omega_1} A_\alpha$ is a singleton is easily seen by the induction hypothesis (2). Let $\bigcap_{\alpha<\omega_1} A_\alpha = \{x\}$.

Claim 1. $f^{-1}(f(x)) = \{x\}$.

Proof of Claim. Suppose that for some $x' \neq x$, we have $f(x) = f(x')$. If $x' \in Z$, then $x' \notin A_\alpha$ for some $\alpha < \omega_1$. This implies $x \in J_{A_\alpha}$ and so $x \notin A_{\alpha+1}$. This is a contradiction. If $x' \notin Z$, then $x' \in B_\alpha$ for some $\alpha < \omega_1$. This implies $x \in I_{B_\alpha}$ and so $x \notin A_{\alpha+1}$. This is also a contradiction. Therefore $f^{-1}(f(x)) = \{x\}$. □

This contradicts f being exactly two-to-one function. Moreover the zero-dimensional case follows immediately from the general case. If X is zero-dimensional, and B is an open set such that $\overline{B} \cap Z = \emptyset$, then there is a clopen set disjoint from Z containing B. □

Corollary 4 (CH)**.** *Let X^* be the Stone-Čech remainder of a locally compact separable metric space X. If $f\colon X^* \to K$ is a two-to-one continuous function, then f is not irreducible. In particular, if $f\colon \mathbb{N}^* \to K$ or $f\colon \mathbb{R}^* \to K$ is two-to-one and continuous, then f is not irreducible.*

Proof. Suppose that f is irreducible. Taking $X = Z = X^*$ and $W = K$ in Theorem 3 we get a nonempty subset B of the empty set $X^* \setminus Z$. □

Corollary 5 (CH)**.** *If $f\colon \mathbb{N}^* \to K$ is a two-to-one continuous function, then K is not ccc.*

Proof. Let W be a nonempty open subset of K. By Zorn's Lemma [Wa74], there is a closed subset Z of $\mathbb{N}^*$ such that $f\restriction_Z\colon Z \to K$ is irreducible. Then, by Theorem 3, there exists a nonempty clopen set $B \subseteq \mathbb{N}^*\backslash Z$ such that $\mathrm{Int}(f[B]) \cap W \neq \emptyset$ since $\mathbb{N}^*$ is zero-dimensional. Then $f\restriction_B$ is a closed one-to-one function and so B is homeomorphic to $f[B]$. Thus, $f[B]$ has no open ccc subset since $\mathbb{N}^*$ has no open ccc subset. Therefore, K is not ccc. $\square$

3. Examples of Nontrivial Two-to-One Maps

A two-to-one function $f\colon X \to Y$ will be called *trivial* if there exist disjoint clopen sets A and B such that $X = A \cup B$ and $f[A] = f[B] = Y$. In [Do04] the first author proved that all functions defined on $\mathbb{N}^*$ that are two-to-one continuous are trivial under PFA. In this section we will give some nontrivial examples of two-to-one continuous functions defined on $\mathbb{N}^*$ when CH is assumed.

A point is called a *P-point* if the family of its neighborhoods is closed under countable intersections. A subset of a space is a P-set if the family of its neighborhoods is closed under countable intersections. CH implies that $\mathbb{N}^*$ has P-points and contains a nowhere dense closed P-set which is homeomorphic to $\mathbb{N}^*$ [vM84].

Example 1 (CH): We give an example of a nontrivial two-to-one continuous function $f\colon \mathbb{N}^* \to \mathbb{N}^*$ such that f is locally one-to-one at every point of $\mathbb{N}^*$ except for two P-points.

Consider two copies of $\mathbb{N}^*$: $\mathbb{N}_1^*$, $\mathbb{N}_2^*$. Let $p_1 \in \mathbb{N}_1^*$ and $p_2 \in \mathbb{N}_2^*$ be P-points. There is a homeomorphism $g : \mathbb{N}_1^* \to \mathbb{N}_2^*$ such that $g(p_1) = p_2$ under CH [vM84]. Then $g^{-1} : \mathbb{N}_2^* \to \mathbb{N}_1^*$ is also a homeomorphism and $g^{-1}(p_2) = p_1$. The free union of the two copies of $\mathbb{N}^*$: $\mathbb{N}_1^* \cup \mathbb{N}_2^*$ is homeomorphic to $\mathbb{N}^*$. Let $h_1 : \mathbb{N}_1^* \cup \mathbb{N}_2^* \to \mathbb{N}_1^* \cup \mathbb{N}_2^*$ be defined by $h_1 = g \cup g^{-1}$. Then ${h_1}^2 = id$. In a similar manner define $h_2 : \mathbb{N}_3^* \cup \mathbb{N}_4^* \to \mathbb{N}_3^* \cup \mathbb{N}_4^*$ so that $h_2 = \tilde{g} \cup \tilde{g}^{-1}$ and ${h_2}^2 = id$, where $\mathbb{N}_3^*$ and $\mathbb{N}_4^*$ are other copies of $\mathbb{N}^*$ with corresponding P-points p_3 and p_4 and $\tilde{g} : \mathbb{N}_3^* \to \mathbb{N}_4^*$ is a homeomorphism with $\tilde{g}(p_3) = p_4$. The quotient spaces

$$(\mathbb{N}_1^* \cup \mathbb{N}_2^*)/p_1 \equiv p_2 \qquad (\mathbb{N}_3^* \cup \mathbb{N}_4^*)/p_3 \equiv p_4 \qquad (\mathbb{N}_1^* \cup \mathbb{N}_4^*)/p_1 \equiv p_4$$

identifying p_1 and p_2, p_3 and p_4, p_1 and p_4, as single P-points in their respective spaces, are homeomorphic to $\mathbb{N}^*$ [vM84]. The free union $(\mathbb{N}_1^* \cup \mathbb{N}_2^*)/p_1 \equiv p_2 \oplus (\mathbb{N}_3^* \cup \mathbb{N}_4^*)/p_3 \equiv p_4$ is also homeomorphic to $\mathbb{N}^*$. Now define f on this space by

$$f : \; [(\mathbb{N}_1^* \cup \mathbb{N}_2^*)/p_1 \equiv p_2 \; \oplus \; (\mathbb{N}_3^* \cup \mathbb{N}_4^*)/p_3 \equiv p_4] \to (\mathbb{N}_1^* \cup \mathbb{N}_4^*)/p_1 \equiv p_4$$

$$f(x) = \begin{cases} h_1(x) & if \quad x \in \mathbb{N}_2^* \setminus \{p_2\} \\ h_2(x) & if \quad x \in \mathbb{N}_3^* \setminus \{p_3\} \\ x & if \quad x \in \mathbb{N}_1^* \setminus \{p_1\} \;\cup\; \mathbb{N}_4^* \setminus \{p_4\} \\ p_1 \equiv p_4 & if \quad x \in \{p_1 \equiv p_2, \;\; p_3 \equiv p_4\} \end{cases}$$

Then f is a continuous and exactly two-to-one function. Moreover, the image of f is homeomorphic to $\mathbb{N}^*$.

We now introduce some notation that will be used in our future discussions about this kind of two-to-one continuous maps. Let

$$X_0 = (\mathbb{N}_1^* \cup \mathbb{N}_2^*)/p_1 \equiv p_2 \ \oplus \ (\mathbb{N}_3^* \cup \mathbb{N}_4^*)/p_3 \equiv p_4$$

$$I_0 = \{A \subseteq X_0 : \ A = A_0 \dot\cup A_0^{'}, \ A_0, A_0^{'} \ - \ clopen, \ f[A_0] = f[A_0^{'}]\}$$

So I_0 is a family of clopen sets A in X_0 such that $A = f^{-1}(f[A])$, i.e., saturated, and f is locally one-to-one on A. Let $\mathcal{U}_0$ denote the union of all the A's in I_0,

$$\mathcal{U}_0 = \bigcup_{A \in I_0} A$$

and in this example $\mathcal{U}_0 = X_0 \setminus \{p_1 \equiv p_2, p_3 \equiv p_4\}$. Thus f is locally one-to-one except at the two P-points $p_1 \equiv p_2$ and $p_3 \equiv p_4$. Let $X_1 = X_0 \backslash U_0$ which is again for this example given by

$$X_1 = \{p_1 \equiv p_2, p_3 \equiv p_4\}$$

Then I_1 is the analogous set in X_1 but the points in X_1 are not in $\mathcal{U}_0$ because as can be seen above f is not locally one-to-one at the points $p_1 \equiv p_2$ and $p_3 \equiv p_4$.

Using a similar construction to Veličković's poset [Ve93], Example 1 can be done consistent with $MA+\neg CH$. But $MA+\neg CH$ is not by itself enough to do the construction because of the first author's PFA result [Do04].

Example 2 (CH): We give an example of a nontrivial two-to-one continuous function $f : \mathbb{N}^* \to \mathbb{N}^*$ such that f is locally one-to-one at every point of $\mathbb{N}^*$ except for two P-sets. We extend the first example by considering nowhere dense closed P-sets instead of P-points.

Consider two copies of $\mathbb{N}^*$: $\mathbb{N}^*_1$, $\mathbb{N}^*_2$. Let $P_1 \subseteq \mathbb{N}^*_1$ and $P_2 \subseteq \mathbb{N}^*_2$ be two different closed P-sets such that P_1 is homeomorphic to P_2. There is a homeomorphism $g_{12} : \mathbb{N}^*_1 \to \mathbb{N}^*_2$ such that $g_{12}(P_1) = P_2$ under CH [vM84]. Therefore $g_{12}^{-1} : \mathbb{N}^*_2 \to \mathbb{N}^*_1$ is also a homeomorphism and $g_{12}^{-1}(P_2) = P_1$. The free union of the two copies of $\mathbb{N}^*$: $\mathbb{N}^*_1 \cup \mathbb{N}^*_2$ is homeomorphic to $\mathbb{N}^*$. Let

$$h_1 : \mathbb{N}^*_1 \cup \mathbb{N}^*_2 \to \mathbb{N}^*_1 \cup \mathbb{N}^*_2 \quad \text{be defined by} \quad h_1 = g_{12} \cup g_{12}^{-1}$$

Then $h_1(x) \neq x$ for each x and ${h_1}^2 = id$. In a similar manner define $h_2 : \mathbb{N}^*_3 \cup \mathbb{N}^*_4 \to \mathbb{N}^*_3 \cup \mathbb{N}^*_4$ so that $h_2 = g_{34} \cup g_{34}^{-1}$ and ${h_2}^2 = id$, where $\mathbb{N}^*_3$ and $\mathbb{N}^*_4$ are other copies of $\mathbb{N}^*$ with corresponding homeomorphic P-sets P_3 and P_4. The adjunction spaces

$$\mathbb{N}^*_1 \cup_{g_1} \mathbb{N}^*_2 \qquad \mathbb{N}^*_3 \cup_{g_2} \mathbb{N}^*_4 \qquad \mathbb{N}^*_1 \cup_{g_3} \mathbb{N}^*_4$$

where we identify the P-sets P_1 with P_2, P_3 with P_4, and P_1 with P_4 are homeomorphic to $\mathbb{N}^*$ [vM84]. The free union

$$(\mathbb{N}^*_1 \cup_{g_1} \mathbb{N}^*_2) \ \oplus \ (\mathbb{N}^*_3 \cup_{g_2} \mathbb{N}^*_4)$$

is also homeomorphic to $\mathbb{N}^*$.

Let $\varphi : P_2 \to P_4$ be a homeomorphism. Now let us define

$$f : [(\mathbb{N}^*_1 \cup_{g_1} \mathbb{N}^*_2) \ \oplus \ (\mathbb{N}^*_3 \cup_{g_2} \mathbb{N}^*_4)] \to \mathbb{N}^*_1 \cup_{g_3} \mathbb{N}^*_4$$

by

$$f(x)=\begin{cases} h_1(x) & if \quad x\in N_2^*\backslash P_2 \\ h_2(x) & if \quad x\in N_3^*\backslash P_3 \\ x & if \quad x\in N_1^*\backslash P_1 \ \cup \ N_4^* \\ \varphi(x) & if \quad x\in P_2 \end{cases}$$

Then f is a continuous and exactly two-to-one function and the image is homeomorphic to $\mathbb{N}^*$. Let us find the sets $I_0, \mathcal{U}_0$, X_1, $I_1, \mathcal{U}_1$, and X_2 for this example which are introduced in Example 1.

$$X_0=\mathbb{N}_1^*\cup_{g_1}\mathbb{N}_2^* \ \oplus \ \mathbb{N}_3^*\cup_{g_2}\mathbb{N}_4^*$$

$$I_0=\{A\subseteq X_0:\ A=A_0\dot{\cup}A_0^{'},\ A_0,A_0^{'}\ -\ clopen,\ f[A_0]=f[A_0^{'}]\}$$

$$\mathcal{U}_0=\bigcup_{A\in I_0}A=X_0\setminus(P_2\cup P_4)$$

$$X_1=X_0\backslash\mathcal{U}_0=P_2\cup P_4$$

The function f is not locally one-to-one in the nowhere dense closed sets P_2 and P_4. But $f\restriction_{X_1}$ is a continuous two-to-one function. I_1 is the analogous set in X_1, $\mathcal{U}_1=\bigcup_{A\in I_1}A=X_1$, and $X_2=X_1\backslash\mathcal{U}_1=\emptyset$.

Example 3 (CH): We give an example of a nontrivial two-to-one continuous function $f\colon\mathbb{N}^*\to\mathbb{N}^*$ which is locally one-to-one at every point of $\mathbb{N}^*$ except for two P-sets and with the property that $X_2\neq\emptyset$ and $X_3=\emptyset$. We know that $\mathbb{N}^*$ can be embedded as a nowhere dense P-set in $\mathbb{N}^*$ assuming CH [vM84]. Consider two copies of $\mathbb{N}^*$: $\mathbb{N}_5^*$, $\mathbb{N}_6^*$. Embed $\mathbb{N}_1^*\cup_{g_1}\mathbb{N}_2^*$ and $\mathbb{N}_3^*\cup_{g_2}\mathbb{N}_4^*$ in Example 2 as nowhere dense P-sets P_5 and P_6 in $\mathbb{N}_5^*$ and $\mathbb{N}_6^*$, respectively:

$$\mathbb{N}_1^*\cup_{g_1}\mathbb{N}_2^*\hookrightarrow\mathbb{N}_5^* \qquad \text{and} \qquad \mathbb{N}_3^*\cup_{g_2}\mathbb{N}_4^*\hookrightarrow\mathbb{N}_6^*$$

In a similar fashion to Example 1, let $g : \mathbb{N}_5^* \to \mathbb{N}_6^*$ be a homeomorphism such that $g(P_5) = P_6$. Therefore $g^{-1} : \mathbb{N}_6^* \to \mathbb{N}_5^*$ is also a homeomorphism and $g^{-1}(P_6) = P_5$. Let

$$h_1 : \mathbb{N}_5^* \cup \mathbb{N}_6^* \to \mathbb{N}_5^* \cup \mathbb{N}_6^* \quad \text{be defined by} \quad h_1 = g \cup g^{-1}$$

Then ${h_1}^2 = id$. Let $\mathbb{N}_7^* \cup \mathbb{N}_8^*$ be another copy of $\mathbb{N}_5^* \cup \mathbb{N}_6^*$. Suppose that $h_2 : \mathbb{N}_7^* \cup \mathbb{N}_8^* \to \mathbb{N}_7^* \cup \mathbb{N}_6^*$ is defined similarly so that ${h_2}^2 = id$.

The adjunction spaces $\mathbb{N}_5^* \cup_{g_5} \mathbb{N}_6^*$, $\mathbb{N}_7^* \cup_{g_6} \mathbb{N}_8^*$, $\mathbb{N}_5^* \cup_{g_7} \mathbb{N}_8^*$ where we identify the P-sets P_5 with P_6, P_7 with P_8, and P_5 with P_8 are homeomorphic to $\mathbb{N}^*$ [vM84]. The free union $(\mathbb{N}_5^* \cup_{g_5} \mathbb{N}_6^*) \oplus (\mathbb{N}_7^* \cup_{g_6} \mathbb{N}_8^*)$ is also homeomorphic to $\mathbb{N}^*$. Let $\varphi : P_6 \to P_8$ be the two-to-one function defined in Example 2. Define f

$$f : [(\mathbb{N}_5^* \cup_{g_5} \mathbb{N}_6^*) \oplus (\mathbb{N}_7^* \cup_{g_6} \mathbb{N}_8^*)] \to \mathbb{N}_5^* \cup_{g_7} \mathbb{N}_8^*$$

by

$$f(x) = \begin{cases} h_1(x) & if \quad x \in \mathbb{N}_6^* \backslash P_6 \\ h_2(x) & if \quad x \in \mathbb{N}_7^* \backslash P_7 \\ x & if \quad x \in \mathbb{N}_5^* \backslash P_5 \cup \mathbb{N}_8^* \backslash P_8 \\ \varphi(x) & if \quad x \in P_6 \cup P_8 \end{cases}$$

Let us find the sets I_0, $\mathcal{U}_0$, X_1, I_1, $\mathcal{U}_1$, X_2, I_2, $\mathcal{U}_2$, X_3 in this example.

$$X_0 = (\mathbb{N}_5^* \cup_{g_5} \mathbb{N}_6^*) \oplus (\mathbb{N}_7^* \cup_{g_6} \mathbb{N}_8^*)$$

$$I_0 = \{A \subseteq X_0 : \ A = A_0 \dot{\cup} A_0', \ A_0, A_0' - clopen, \ f[A_0] = f[A_0']\}$$

$$\mathcal{U}_0 = \bigcup_{A \in I_0} A = X_0 \setminus (P_6 \cup P_8)$$

$$X_1 = X_0 \backslash U_0 = P_6 \cup P_8$$

The function f is not locally one-to-one in the nowhere dense closed sets P_6 and P_8. Now $f\restriction_{X_1}$ is an exactly two-to-one continuous function which is the same as the function in Example 2. Therefore,

$$I_1 = \{A \subseteq X_1 : \ A = A_1 \dot{\cup} A_1^{'}, \ A_1, A_1^{'} - \mathit{clopen}, \ f[A_1] = f[A_1^{'}]\}$$

$$\mathcal{U}_1 = \bigcup_{A \in I_1} A = X_1 \setminus (P_2 \cup P_4)$$

$$X_2 = X_1 \backslash \mathcal{U}_1 = P_2 \cup P_4$$

$$I_2 = \{A \subseteq X_2 : \ A = A_2 \dot{\cup} A_1^{'}, \ A_2, A_2^{'} - \mathit{clopen}, \ f[A_2] = f[A_2^{'}]\}$$

$$\mathcal{U}_2 = \bigcup_{A \in I_2} A = X_2$$

$$X_3 = X_2 \backslash \mathcal{U}_2 = \emptyset.$$

It is clear that we can continue this process for any finite number of steps in the following sense: If $f : \mathbb{N}^* \to K$ is a two-to-one continuous function and $X_0 = \mathbb{N}^*$, then for each integer n

$$I_n = \{A \subseteq X_n : \ A = A_n \dot{\cup} A_n^{'}, \ A_n, A_n^{'} - \mathit{clopen}, \ f[A_n] = f[A_n^{'}]\}$$

$$\mathcal{U}_n = \bigcup_{A \in I_n} A$$

$$X_{n+1} = X_n \backslash \mathcal{U}_n.$$

Then for each integer n there is an f so that $X_n \neq \emptyset$ while $X_{n+1} = \emptyset$.

4. Zero-dimensional Spaces

A space X is called *zero-dimensional* if it has a base consisting of clopen sets, that is, if for every point $x \in X$ and for every neighborhood U of x there exists a clopen subset $C \subseteq X$ such that $x \in C \subseteq U$. $\mathbb{N}^*$ is a zero-dimensional space and in this section we show that every two-to-one continuous image of $\mathbb{N}^*$ is zero-dimensional under CH.

Suppose that $f : \mathbb{N}^* \to K$ is a two-to-one continuous function. As in the examples given in section 3, let

$$X_0 = \mathbb{N}^*,\ K_0 = K,$$

$$I_0 = \{A \subseteq X_0 :\ A = A_0 \,\dot\cup\, A_0^{'},\ A_0, A_0^{'} - \ clopen,\ f[A_0] = f[A_0^{'}]\}, \text{and}$$

$$\mathcal{U}_0 = \bigcup_{A \in I_0} A.$$

Claim 2. $I_0 \neq \emptyset$

Proof of claim. By Theorem 3, there is a clopen set $B \subseteq \mathbb{N}^*$ such that $f\restriction_B$ is one-to-one and $\mathrm{Int} f[B] \neq \emptyset$. Therefore, $f[B]$ is homeomorphic to B and hence there is a clopen set $B' \subseteq \mathrm{Int} f[B]$ and $f^{-1}[B']$ is clopen since $f\restriction_B$ is one-to-one and it can be written as a union of two disjoint clopen sets $f^{-1}[B'] = A_0 \,\dot\cup A_0^{'}$ such that $f[A_0] = f[A_0^{'}]$. Therefore, $f^{-1}[B'] \in I_0$. This shows I_0 is nonempty and $f[\mathcal{U}_0]$ is dense in K_0.

$\square$

Now let $X_1 = X_0 \backslash \mathcal{U}_0$ and $K_1 = K_0 \backslash f(\mathcal{U}_0)$. Then X_1 is a closed subset of X_0. If $X_1 \neq \emptyset$, then $f\restriction_{X_1}: X_1 \to K_1$ is an exactly two-to-one continuous function. In a similar way as before let

$$I_1 = \{A \subseteq X_1 :\ A = A_1 \dot\cup A_1',\ A_1, A_1' \ -\ clopen,\ f[A_1] = f[A_1']\}$$

$$\mathcal{U}_1 = \bigcup_{A \in I_1} A \qquad X_2 = X_1 \backslash \mathcal{U}_1 \qquad K_2 = K_1 \backslash f(\mathcal{U}_1)$$

If $X_1 \neq \emptyset$, then $I_1 \neq \emptyset$ by Theorem 3. If $X_2 \neq \emptyset$, then $f \restriction_{X_2}: X_2 \to K_2$ is an exactly two-to-one continuous function. Continuing in a similar fashion, for each n we define

$$I_n = \{A \subseteq X_n :\ A = A_n \dot\cup A_n',\ A_n, A_n' \ -\ clopen,\ f[A_n] = f[A_n']\}$$

$$\mathcal{U}_n = \bigcup_{A \in I_n} A \qquad X_{n+1} = X_n \setminus \mathcal{U}_n \qquad K_{n+1} = K_n \setminus f(\mathcal{U}_n)$$

Then $X_\omega = \bigcap_n X_n$ and $K_\omega = \bigcap_n K_n$. Recall that we showed in section 3 that X_n may be nonempty for any given natural number n. Therefore, the next result is quite a surprise.

Theorem 6 (CH)**.** $X_\omega = \emptyset$ *and* $K_\omega = \emptyset$.

Proof. Suppose $X_\omega \neq \emptyset$. Then $I_\omega \neq \emptyset$ where

$$I_\omega = \{A \subseteq X_\omega : A = A_\omega \dot\cup A_\omega',\ A_\omega, A_\omega' - clopen,\ f[A_\omega] = f[A_\omega']\}$$

Therefore, there exist two nonempty disjoint clopen sets $A_\omega, A_\omega' \subseteq X_\omega$ such that $f[A_\omega] = f[A_\omega']$. Since X_ω is compact and X_0 is zero-dimensional there are disjoint clopen sets $B_0, B_0' \subseteq X_0$ such that $B_0 \cap X_\omega = A_\omega$ and $B_0' \cap X_\omega = A_\omega'$. Thus $X_0 - (B_0 \cup B_0')$ is clopen in X_0 and

$$(A_\omega \cup A_\omega') \cap f^{-1}\Big(f(X_0 - (B_0 \cup B_0'))\Big) = \emptyset$$

by the definition of A_ω and A'_ω. But

$$\begin{aligned} A_\omega \cup A'_\omega &= \left(B_0 \cup B'_0\right) \cap X_\omega \\ &= \left(B_0 \cup B'_0\right) \cap \left(\bigcap_n X_n\right) \\ &= \bigcap_n \left(B_0 \cup B'_0\right) \cap X_n \end{aligned}$$

Therefore $f^{-1}\left(f[X_0 \setminus (B_0 \cup B'_0)]\right) \cap \left((B_0 \cup B'_0) \cap X_m\right) = \emptyset$ for some m.

Claim 3. $\exists\, n_0 > m$ such that $\forall\, n > n_0 \quad f^{-1}\left(f[B_0]\right) \supseteq X_n \cap B'_0$.

Proof of claim. Otherwise $\forall n > n_0 \quad \exists\, x_n, x'_n \in \left(X_n \cap B'_0\right) \setminus f^{-1}\left(f[B_0]\right)$ such that $f(x_n) = f(x'_n)$. Then $\{x_n\} \cup \{x'_n\}$ is a discrete subset of $\mathbb{N}^*$ and therefore $\overline{\{x_n\}} \cap \overline{\{x'_n\}} = \emptyset$. Moreover $\overline{\{x_n\}} \setminus \{x_n\} \cup \overline{\{x'_n\}} \setminus \{x'_n\} \subseteq X_\omega \cap B'_0 = A'_\omega$ and $\overline{\{x_n\}} \setminus \{x_n\} \cup \overline{\{x'_n\}} \setminus \{x'_n\}$ is nonempty since every infinite discrete set in a compact space has a limit point.

But then, there are elements $x \in \overline{\{x_n\}} \setminus \{x_n\}$ and $x' \in \overline{\{x'_n\}} \setminus \{x'_n\}$ such that $f(x) = f(x')$. This is a contradiction since $f \restriction_{A'_\omega}$ is one-to-one. Therefore $\exists\, n_0 > m$ such that $\forall\, n > n_0 \quad f^{-1}\left(f[B_0]\right) \supseteq X_n \cap B'_0$. $\square$

By symmetry $\exists k_0 > m$ such that $\forall n > k_0 \quad f^{-1}\left(f[B_0]\right) \supseteq X_n \cap B'_0$. Let $k = \max\{k_0, n_0\}$. Then $f(B_0 \cap X_{k+1}) = f(B'_0 \cap X_{k+1})$. This implies $A_\omega \subseteq U_{k+1}$ and $A'_\omega \subseteq U_{k+1}$. This is a contradiction since $A_\omega, A'_\omega \subseteq X_\omega \subseteq X_{k+1} \setminus U_{k+1}$. Hence $X_\omega = \emptyset \quad$ and $\quad K_\omega = \emptyset$. $\square$

Lemma 7. *If $A \subseteq X_1$ is clopen with $f^{-1}(f[A]) = A$ and $U \subseteq X_0$ is open with $A \subseteq U$, then there is a clopen set $A' \subseteq U$ in X_0 such that $A' \cap X_1 = A$ and $f^{-1}(f[A']) = A'$.*

Proof. Since A is clopen in X_1 and X_1 is a subspace of X_0, there is a clopen set $B \subseteq X_0$ such that $B \cap X_1 = A$ and $B \subseteq U$. Then $f[X_0 \setminus B] \cap f[A] = \emptyset$.

Let $A' = B \backslash f^{-1}(f[X_0 \setminus B])$. We now show that A' is the clopen subset of X_0 we are looking for. Clearly A' is open in X_0, $A' \subseteq U$, $A' \cap X_1 = A$, and $f^{-1}(f[A']) = A'$. It remains to show that A' is closed in X_0. This is equivalent to showing that $B \cap f^{-1}(f[X_0 \setminus B])$ is open.

Let $x \in B \cap f^{-1}(f[X_0 \setminus B])$ and let $x' \in X_0$ such that $f(x) = f(x')$. This implies $x,\ x' \in \mathcal{U}_0 = X_0 \setminus X_1$. Therefore, by the definition of $\mathcal{U}_0$, there are disjoint clopen sets $A_0, A_0' \subseteq \mathcal{U}_0$ in X_0 such that $x \in A_0$, $x' \in A_0'$, and $f[A_0] = f[A_0']$. Now shrink A_0 and A_0' to clopen sets B_0 and B_0', respectively, so that $x \in B_0 \subseteq B$, $x' \in B_0' \subseteq f^{-1}(f[X_0 \setminus B])$, and $f[B_0] = f[B_0']$. Then $x \in B_0 \subseteq B \cap f^{-1}(f[X_0 \setminus B])$.

Therefore, $B \cap f^{-1}(f[X_0 \setminus B])$ is open and A' is closed in X_0. $\square$

Lemma 8. *If $A \subseteq X_{n+1}$ is clopen with $f^{-1}(f[A]) = A$ and $U \subseteq X_n$ is open with $A \subseteq U$, then there is a clopen set $A' \subseteq U$ in X_n such that $A' \cap X_{n+1} = A$ and $f^{-1}(f[A']) = A'$.*

Proof. The proof is similar to the proof of Lemma 7 with X_{n+1} and X_n playing the roles of X_1 and X_0, respectively. $\square$

Lemma 9. *If $A \subseteq X_n$ is clopen with $f^{-1}(f[A]) = A$ and $U \subseteq X_0$ is open with $A \subseteq U$, then there is a clopen set $A' \subseteq U$ in X_0 such that $A' \cap X_n = A$ and $f^{-1}(f[A']) = A'$.*

Proof. This follows from Lemma 7 and Lemma 8 by induction. $\square$

Theorem 10 (CH). *If $f : \mathbb{N}^* \to K$ is a two-to-one continuous function, then K is zero-dimensional.*

Proof. Let $y \in V$ where V is an open subset of K. Then $y \in K_n \setminus K_{n+1}$ since $K_\omega = \emptyset$ by Theorem 6. This implies $y \in f(\mathcal{U}_n) = K_n \setminus K_{n+1}$. Therefore, $y \in f[A_n] \subseteq f(\mathcal{U}_n) = K_n \setminus K_{n+1}$ for some clopen set $A_n \subseteq X_n$ such that $f[A_n]$ is clopen and $f \restriction_{A_n}$ is one-to-one. This is by the definition of $\mathcal{U}_n$. Then $f[A_n]$ is homeomorphic to A_n and so there is a clopen set $B \subseteq \operatorname{Int} f[A_n]$ containing y. Shrink B so that $y \in B \subseteq V \cap K_n$.

Let $A = f^{-1}(B)$ and $U = f^{-1}(V)$. Then $A \subseteq X_n$ is clopen with $f^{-1}(f[A]) = A$ and $U \subseteq X_0$ is open with $A \subseteq U$. By Lemma 7, $\exists\ A' \subseteq U$ clopen in X_0 such that $A' \cap X_n = A$ and $f^{-1}(f[A']) = A'$. Then $y \in f[A'] \subseteq V$ and $f[A']$ is clopen in K since $f^{-1}(f[A']) = A'$. Hence K is zero-dimensional.

□

5. F-spaces

A space is called an *F-space* if every pair of disjoint cozero subsets are completely separated. It is well known that $\mathbb{N}^*$ is an F-space [Wa74] and in this section we show that every two-to-one continuous image of $\mathbb{N}^*$ is also an F-space under CH.

Theorem 11 (CH)**.** *If* $f : \mathbb{N}^* \to K$ *is a two-to-one continuous function, K is an F-space.*

Proof. Let C_1 and C_2 be two disjoint cozero sets in K. Then $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are disjoint cozero sets in $\mathbb{N}^*$. Since $\mathbb{N}^*$ is an F-space we have $\overline{f^{-1}(C_1)} \cap \overline{f^{-1}(C_2)} = \emptyset$. We must show that $\overline{C_1} \cap \overline{C_2} = \emptyset$. It is sufficient to show that for any $y \in \overline{C_1}$ there are two elements $x, x' \in \overline{f^{-1}(C_1)}$ such that $f(x) = y = f(x')$. This shows that $y \notin \overline{C_2}$. Otherwise, if $y \in$

$\overline{C_2} \subseteq f\left[\overline{f^{-1}(C_2)}\right]$, there exists an $x'' \in \overline{f^{-1}(C_2)}$ such that $f(x'') = y$ and $x'' \neq x, x'$ since $\overline{f^{-1}(C_1)} \cap \overline{f^{-1}(C_2)} = \emptyset$ and $x, x' \in \overline{f^{-1}(C_1)}$. So three different points x, x', x'' mapped to y. This is a contradiction to the fact that the function f is exactly two-to-one.

Let $y \in \overline{C_1}$. Then $y \in \overline{C_1} \subseteq f[\overline{f^{-1}(C_1)}]$ since $C_1 = f[f^{-1}(C_1)] \subseteq f[\overline{f^{-1}(C_1)}]$. This implies there exists an $x \in \overline{f^{-1}(C_1)}$ such that $f(x) = y$. By Theorem 6 $K_\omega = \emptyset$ and so $y \in K_n \setminus K_{n+1}$ for some integer n. Let $m \leq n$ be maximal such that $y \in \overline{C_1 \cap K_m}$. Then $y \notin \overline{C_1 \cap K_{m+1}}$ and so there is a cozero set $C_y \subseteq K_m$ such that $y \in C_y$ and $C_y \cap \overline{C_1 \cap K_{m+1}} = \emptyset$. Thus $C_y \cap C_1 \cap K_{m+1} = \emptyset$.

Therefore, without loss of generality, we can assume that $C_1 \cap K_{m+1} = \emptyset$ and C_1 is a cozero set in K_m since we can take C_1 to be the cozero set $C_y \cap C_1$. Then $f^{-1}(C_1)$ is a cozero set in X_m and $f^{-1}(C_1) \cap X_{m+1} = \emptyset$, that is, $f^{-1}(C_1) \subseteq \mathcal{U}_m$ where $\mathcal{U}_m$ is defined as in section 4 by

$\mathcal{U}_m = \bigcup_{A \in I_m} A$ and

$I_m = \{A \subseteq X_m : A = A_m \dot\cup A_m',\ A_m, A_m' - \mathit{clopen}, f[A_m] = f[A_m']\}$

Since $f^{-1}(C_1)$ is a cozero set in X_m and X_m is compact zero-dimensional F-space, it can be written as a countable union of disjoint clopen sets in such a way that $f^{-1}(C_1) = \dot{\bigcup}_{n=0}^{\infty} \left[A_n \cup A_n'\right]$ where each A_n and A_n' are disjoint clopen sets in X_m and $f[A_n] = f[A_n']$. Therefore, $f^{-1}(C_1)$ can be written as a union of two disjoint sets

$$f^{-1}(C_1) = \left(\dot{\bigcup_{n=0}^{\infty}} A_n\right) \cup \left(\dot{\bigcup_{n=0}^{\infty}} A_n'\right)$$

and by the definition of A_n and $A_n^{'}$ we get

$$f\left[\bigcup_{n=0}^{\infty} A_n\right] = f\left[\bigcup_{n=0}^{\infty} A_n^{'}\right].$$

Thus

$$\overline{f^{-1}(C_1)} = \overline{\bigcup_{n=0}^{\infty} A_n} \quad \cup \quad \overline{\bigcup_{n=0}^{\infty} A_n'}.$$

The sets $\bigcup\limits_{n=0}^{\infty} A_n$ and $\bigcup\limits_{n=0}^{\infty} A_n^{'}$ are cozero sets since a countable union of clopen sets is a cozero set and they are disjoint by construction. Therefore, since X_m is an F-space we get

$$\overline{\bigcup_{n=0}^{\infty} A_n} \quad \cap \quad \overline{\bigcup_{n=0}^{\infty} A_n'} = \emptyset.$$

Now since

$$x \in \overline{f^{-1}(C_1)} = \overline{\bigcup_{n=0}^{\infty} A_n} \cup \overline{\bigcup_{n=0}^{\infty} A_n'}$$

we assume, without loss of generality, that

$$x \in \overline{\bigcup_{n=0}^{\infty} A_n} \quad \text{and} \quad x \notin \overline{\bigcup_{n=0}^{\infty} A_n'}.$$

By continuity of f and the fact that

$$f\left[\bigcup_{n=0}^{\infty} A_n\right] = f\left[\bigcup_{n=0}^{\infty} A_n^{'}\right]$$

we get

$$f\left[\overline{\bigcup_{n=0}^{\infty} A_n}\right] = f\left[\overline{\bigcup_{n=0}^{\infty} A_n'}\right]$$

and

$$y \in \overline{C_1} \subseteq f\left[\overline{f^{-1}(C_1)}\right] = f\left[\overline{\bigcup_{n=0}^{\infty} A_n}\right] = f\left[\overline{\bigcup_{n=0}^{\infty} A_n'}\right].$$

Therefore, there exists an $x' \in \overline{\bigcup_{n=0}^{\infty} A'_n}$ such that $f(x') = y$.

Now $x' \neq x$ because $x' \in \overline{\bigcup_{n=0}^{\infty} A'_n}$, $x \in \overline{\bigcup_{n=0}^{\infty} A_n}$, and

$$\overline{\bigcup_{n=0}^{\infty} A_n} \cap \overline{\bigcup_{n=0}^{\infty} A'_n} = \emptyset.$$

Thus, there are two different points $x, x' \in \overline{f^{-1}(C_1)}$ such that

$$f(x) = y = f(x')$$

Hence K is an F-space. □

6. Nonempty G_δ-sets

The intersection of countably many open sets is called a G_δ*-set.* Nonempty G_δ-sets on $\mathbb{N}^*$ have nonempty interiors. In this section we prove that two-to-one continuous images of $\mathbb{N}^*$ have the same property.

Theorem 12 (CH)**.** *If $f : \mathbb{N}^* \to K$ is a two-to-one continuous function, then nonempty G_δ-sets in K have nonempty interior.*

Proof. Suppose that $\{b_n : n \in \omega\}$ is a descending sequence of clopen subsets of K with $b_0 = K$. It suffices to deal with clopen sets since we have shown that K is zero-dimensional. Assume to the contrary that $\bigcap_n b_n$ is nowhere dense. Let $Z \subseteq \mathbb{N}^*$ be such that $f \restriction_Z$ is irreducible. This is possible by Zorn's Lemma.

For each n let $a_n = f^{-1}(b_n) \cap Z$. Then $\{a_n : n \in \omega\}$ is a descending sequence of clopen subsets of Z and $a_0 = Z$. Therefore, by Theorem 3, for each n pick clopen sets $e_n \subseteq a_n \setminus a_{n+1}$ and $e_n^{'} \subseteq \mathbb{N}^* \setminus Z$ such that $f[e_n] = f[e_n^{'}]$. Then, in Z, $\overline{\bigcup_n (a_n \setminus (e_n \cup a_{n+1})} \cap \overline{\bigcup_n e_n} = \emptyset$ since Z is an F-space.

Since we assumed $\bigcap_n b_n$ is nowhere dense in K, $f^{-1}(\bigcap_n b_n) \cap Z$ is nowhere dense and so

$$Z = \overline{\bigcup_n (a_n \setminus (e_n \cup a_{n+1})} \cup \overline{\bigcup_n e_n}.$$

Thus $\overline{\bigcup_n e_n}$ is clopen in Z. Clearly $\overline{\bigcup_n (a_n \setminus a_{n+1})} \cap \bigcup_n e_n^{'} = \emptyset$ since Z is closed and $e_n^{'} \subseteq \mathbb{N}^* \setminus Z$.

Let us show that $\bigcup_n (a_n \setminus a_{n+1}) \cap \overline{\bigcup_n e_n^{'}} = \emptyset$. For each n, $f^{-1}(K \setminus b_n)$ is clopen in $\mathbb{N}^*$ since b_n is clopen and f is continuous. By construction $\bigcup_{m \geq n} e_m^{'} \cap f^{-1}(K \setminus b_n) = \emptyset$ and $\bigcup_{m \in \omega} e_m^{'} \cap f^{-1}(K - b_n)$ is clopen in $\mathbb{N}^*$ since it is a finite intersection of clopen sets. Because

$$a_n \setminus a_{n+1} \subseteq f^{-1}(K \setminus b_{n+1}) \text{ and } [a_n \setminus (a_{n+1} \cup e_n)] \cap \bigcup_{n \in \omega} e_n^{'} = \emptyset$$

we have $\bigcup_n (a_n \setminus a_{n+1}) \cap \overline{\bigcup_n e_n^{'}} = \emptyset$. Thus $\overline{\bigcup_n (a_n \setminus a_{n+1})} \cap \overline{\bigcup_n e_n^{'}} = \emptyset$ since $\mathbb{N}^*$ is an F-space. Therefore $\overline{\bigcup_n e_n} \cap \overline{\bigcup_n e_n^{'}} = \emptyset$ and hence f is a two-to-one function on $\overline{\bigcup_n e_n} \cup \overline{\bigcup_n e_n^{'}}$.

Now since $f\left[\overline{\bigcup_n e_n}\right] = f\left[\overline{\bigcup_n e_n^{'}}\right]$ we have $K = f\left[\overline{\bigcup_n e_n}\right] \cup f\left[Z \setminus \overline{\bigcup_n e_n}\right]$ and $f\left[\overline{\bigcup_n e_n}\right] \cap f\left[Z \setminus \overline{\bigcup_n e_n}\right] = \emptyset$. Therefore $f\left[\overline{\bigcup_n e_n}\right] \subseteq K$ is clopen in K. But then

$$f^{-1}\left(f\left[\overline{\bigcup_n e_n^{'}}\right]\right) = \overline{\bigcup_n e_n} \cup \overline{\bigcup_n e_n^{'}}$$

is clopen. This is a contradiction since $\overline{\bigcup_n e_n} \cup \overline{\bigcup_n e'_n}$ is not clopen by the fact that in $\mathbb{N}^*$ nonempty G_δ-sets have nonempty interior. □

If $f : \mathbb{N}^* \to K$ is a two-to-one continuous function, Levy [Le04] proved that countable discrete subsets of K are C^*-embedded and the weight of K is $\mathfrak{c}$. This completes everything needed to prove that a two-to-one continuous image of $\mathbb{N}^*$ is $\mathbb{N}^*$ up to a homeomorphism.

Corollary 13 (CH)**.** *If $f : \mathbb{N}^* \to K$ is a two-to-one continuous function, then K is homeomorphic to $\mathbb{N}^*$.*

Proof. Follows from Theorem 1, 10, 11, and 12. □

Open Problems

Our results are all under the set theoretic assumption CH. Is it possible to eliminate CH? In particular, if $f : \mathbb{N}^* \to K$ is a two-to-one continuous function:

(1) Is it true that f is not irreducible?
(2) Is K homeomorphic to $\mathbb{N}^*$?
(3) Is every countable subset of K C^*-embedded ?
(4) Can K be separable or ccc ? (Levy question [vD93])
(5) If f is n-to-one continuous with $n > 2$, is K homeomorphic to $\mathbb{N}^*$ under CH or ZFC?

References

[vD93] Eric K. van Douwen, *Applications of Maximal Topologies*, Topology and Its Appl. **51** (1993), no. 2, 125–139.

[Do04] Alan Dow, *Two to One Images and PFA*, Preprint, 2004.

[Ha39] O.G. Harrold, *The non-existence of a certain type of continuous transformation*, Duke Math. J. 5, (1939). 789–793

[He86] J. Heath, *Every exactly 2-to-1 function on the reals has an infinite set of discontinuities*, Proc. Amer. Math. Soc. 98 (1986), no. 2, 369–373.

[Le04] Ronnie Levy, *The Weight of Certain Images of ω^**, Preprint, 2004.

[vM84] Jan van Mill, *An Introduction to $\beta\omega$* , Handbook of Set-Theoretic Topology (K.Kunen and J.E. Vaughan, eds), Elsevier Science Publishers BV, North-Holland, Amsterdam, 1984, pp. 503–567.

[Mi61] J. Mioduszewski, *On Two-to-One Continuous Functions*, Rozprawy Matematyczne 24, Warszawa, 1961, 43 pp.

[Ve93] Boban Veličković, *OCA and Automorphisms of $P(\omega)/fin$*, Topology and Its Appl. **49** (1993), 1–13.

[Wa74] Russell C. Walker, *The Stone-Čech Compactification*, Springer-Verlag, New York, 1974, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 83.

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