ON THE TIGHTNESS OF G_δ-MODIFICATIONS

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Abstract. The G_δ-modification X_δ of a topological space X is the space on the same underlying set generated by, i.e. having as a basis, the collection of all G_δ subsets of X. Bella and Spadaro recently investigated in [1] the connection between the values of various cardinal functions taken on X and X_δ, respectively. In their paper, as Question 2, they raised the following problem: Is $t(X_\delta) \le 2^{t(X)}$ true for every (compact) T_2 space X? Note that this is actually two questions.

In this note we answer both questions: In the compact case affirmatively and in the non-compact case negatively. In fact, in the latter case we even show that it is consistent with ZFC that no upper bound exists for the tightness of the G_δ-modifications of countably tight, even Fréchet spaces.

1. Introduction

The G_δ-modification X_δ of a topological space X is the space on the same underlying set generated by, i.e. having as a basis, the collection of all G_δ subsets of X. Bella and Spadaro recently investigated in [1] the connection between the values of various cardinal functions taken on X and X_δ, respectively. In their paper, as Question 2, they raised the following problem: Is $t(X_\delta) \le 2^{t(X)}$ true for every (compact) T_2 space X? Note that this is actually two questions. In this note we answer both questions: In the compact case affirmatively and in the non-compact case negatively. Actually, for the compact case we prove something stronger: We show that for every regular Lindelöf space X we have $t(X_\delta) \le 2^{t(X)}$. In the non-compact case we shall show that it is consistent with ZFC that no upper bound exists for

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the tightness of the G_δ-modifications of countably tight, even Fréchet spaces.

We shall use standard notation and terminology from set theory and general topology. In particular, concerning cardinal functions, we follow the notation and terminology of [3] It will be useful to denote by $G_\delta(X)$ the family of all G_δ subsets of a space X. So, as we said, $G_\delta(X)$ is a basis for X_δ. For any $A \subset X$ we shall use $\overline{A}^\delta$ to denote the closure of A in X_δ.

Just like in [1], our proofs will often use elementary submodels of appropriate "initial segments" of the form $H(\lambda)$ of the universe. Most readers will be familiar enough with these notions and for those who are not they are surveyed e.g. in [2].

2. Bounds for the tightness of G_δ-modifications

Theorem 2.1. *If X is a regular Lindelöf space then $t(X_\delta) \le 2^{t(X)}$.*

Proof. Assume that X is a regular Lindelöf space, $p \in X$ and $A \subset X$ are such that $p \in \overline{A}^\delta$. Let us put $\kappa = 2^{t(X)}$ and choose an elementary submodel M of an appropriate $H(\lambda)$ such that $|M| = \kappa$, $M^{t(X)} \subset M$, moreover $\{X, A, p\} \subset M$. We shall show that then $p \in \overline{A \cap M}^\delta$, which by $|A \cap M| \le \kappa$ will complete our proof.

To see this, assume that $p \in H \in G_\delta(X)$, i.e. $H = \bigcap_{n<\omega} V_n$ where each V_n is open in X. We have to prove that $H \cap A \cap M \ne \emptyset$.

For every point $x \in \overline{A \cap M}$ (note that this is the closure in X !) there is a subset $B_x \subset A \cap M$ with $|B_x| \le t(X)$ such that $x \in \overline{B_x}$. Then $M^{t(X)} \subset M$ implies $B_x \in M$. Clearly, if $x \ne p$ then B_x can be chosen so that $p \notin \overline{B_x}$. In this case, by the regularity of X, there are open sets $U_x \supset \overline{B_x}$ and W_x with $p \in W_x$ such that $U_x \cap W_x = \emptyset$, moreover as both $\overline{B_x}$ and p belong to M, we may assume that U_x and W_x also belong to M.

Now, the closed subspace $\overline{A \cap M} \setminus V_n$ of X is Lindelöf for each $n < \omega$, hence there is a countable subset $C_n \subset \overline{A \cap M} \setminus V_n$ such that $\overline{A \cap M} \setminus V_n \subset \bigcup_{x \in C_n} U_x$. This clearly implies that if we put $W_n = \bigcap_{x \in C_n} W_x$ then

$$A \cap M \cap W_n \subset V_n \,.$$

Although we do not know if $C_n \in M$, we do know that $W_n \in M$ because $W_x \in M$ for each $x \in C_n$ and M is countably closed.

It follows then that $W = \bigcap_{n<\omega} W_n \in M \cap G_\delta(X)$, hence $p \in \overline{A}^\delta$ and $p \in W$ imply $W \cap A \neq \emptyset$ and, by elementarity, $W \cap A \cap M \neq \emptyset$ as well. But then we have $\emptyset \neq W \cap A \cap M \subset H \cap A \cap M$, which completes our proof.

□

The one-point compactification of an uncountable discrete space has countable tightness, it is even Fréchet, and its G_δ-modification clearly has tightness ω_1. This, of course, shows that Theorem 2.1 is sharp for countably tight compact spaces under CH. But what happens if the continuum $\mathfrak{c}$ is large? Actually, we do not know the full answer to this question.

However, we happen to have a ready made consistent answer in [4] where a weakening of CH called CH* was introduced. It was shown there that CH* holds in any model obtained by adding any number of Cohen reals to a ground model that satisfies CH. Thus CH* is consistent with $\mathfrak{c}$ being anything it can be.

Let us denote by $\ell_{\omega_1}(A)$ the set of all points obtainable as the limit of a converging ω_1-sequence of points of A in a space X. We permit constant sequences, hence $A \subset \ell_{\omega_1}(A)$. It is obvious that we always have $\ell_{\omega_1}(A) \subset \overline{A}^\delta$. Now, for countably tight compacta, the proof of Theorem 3.2 of [4] actually establishes the following converse of this.

Theorem 2.2. *CH* implies that if X is any countably tight compactum and $A \subset X$ then $\ell_{\omega_1}(A) \supset \overline{A}^\delta$, hence $\ell_{\omega_1}(A) = \overline{A}^\delta$. Consequently, if X_δ is non-discrete then $t(X_\delta) = \omega_1$.*

Although the statement of Theorem 3.2 in [4] is slightly weaker than this, the reader may easily check that actually this is proved there.

This result then leads us to the following natural and intriguing question.

PROBLEM 1. Is it consistent to have a countably tight compactum X for which $t(X_\delta) > \omega_1$?

It turns out by our next two ZFC results that the, somewhat surprising, equality $\ell_{\omega_1}(A) = \overline{A}^\delta$ may occur in other situations as well. It will be useful to introduce the following notation: If X is a space and κ is a cardinal then we write

$$\mathcal{D}_\kappa(X) = \{D \in [X]^\kappa : D \text{ is discrete}\}.$$

A countably tight compact, or just Lindelöf space X contains no uncountable free sequences, i.e. satisfies $F(X) = \omega$. This makes the assumption $F(X) = \omega$ in our following result fitting with the topic of this paper.

Theorem 2.3. *Let X be a regular space such that $F(X) = \omega$ and for every $D \in \mathcal{D}_\omega(X)$ we have $\psi(\overline{D}) \le \omega$. Then for every $A \subset X$ we have $\ell_{\omega_1}(A) = \overline{A}^\delta$.*

Proof. Assume that $p \in X$ and $A \subset X$ are such that $p \in \overline{A}^\delta \setminus A$. By induction on $\alpha < \omega_1$ we shall define closed G_δ sets H_α containing p and points $x_\alpha \in A \cap H_\alpha$ as follows.

Assume that $\{H_\beta : \beta < \alpha\}$ and $Y_\alpha = \{x_\beta : \beta < \alpha\}$ have been defined, moreover Y_α is a free sequence in $X \setminus \{p\}$, hence $Y_\alpha \in \mathcal{D}_\omega(X)$. Then either $p \notin \overline{Y_\alpha}$ or $\psi(p, \overline{Y_\alpha}) \le \omega$. But in both cases there is a closed G_δ set H containing p such that $H \cap (\{p\} \cup \overline{Y_\alpha} = \{p\}$. We then let $H_\alpha = H \cap \bigcap_{\beta<\alpha} H_\beta \in G_\delta(X)$ and use $p \in \overline{A}^\delta$ to pick the point $x_\alpha \in A \cap H_\alpha$.

Thus we have constructed $\{x_\beta : \beta < \omega_1\} \subset A$. The sequence $\{x_\beta : \beta < \omega_1\}$ is free in $X \setminus \{p\}$ because for every $\alpha < \omega_1$ we have $\overline{Y_\alpha} \cap H_\alpha \subset \{p\}$ and $\overline{\{x_\beta : \beta \ge \alpha\}} \subset H_\alpha$.

If U is any open set containing p then $\{x_\alpha : x_\alpha \notin U\}$ is free in X, hence it is countable. In other words, U contains a tail of $\{x_\alpha : \alpha < \omega_1\}$, i.e. the ω_1-sequence $\{x_\alpha : \alpha < \omega_1\} \subset A$ indeed converges to x. □

Clearly, the condition $\psi(\overline{D}) \le w(\overline{D}) = \omega$ is satisfied for any countable subset D of the Σ-product $\Sigma(\kappa)$ taken inside the Tychonov cube of weight κ. Also, compact subspaces of such Σ-products, i.e. Corson-compacta are Fréchet, hence do not contain uncountable free sequences. Thus we immediately obtain the following corollary of Theorem 2.3:

Corollary 2.4. *For every subset A of a Corson-compact space X we have $\ell_{\omega_1}(A) = \overline{A}^\delta$.*

To facilitate the formulation of our next result, we introduce the notation $CAP(\kappa)$ to denote the class of all spaces in which every subset of cardinality κ has a complete accumulation point.

Theorem 2.5. *Assume that $X \in CAP(\omega_1)$ is a countably tight regular space such that $\psi(\overline{S}) \le \omega_1$ for every countable subset $S \subset X$. Then for every $A \subset X$ we have $\ell_{\omega_1}(A) = \overline{A}^\delta$.*

Proof. Consider any point $p \in \overline{A}^\delta \setminus A$ and then choose an ω_1-chain $\langle N_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels of an appropriate $H(\lambda)$ such that
(i) $\{X, A, p\} \subset N_0$;
(ii) for every $\beta < \omega_1$ we have $\langle N_\alpha : \alpha < \beta \rangle \in N_\beta$. Let $N = \bigcup_{\alpha<\omega_1} N_\alpha$.

Since X has countable tightness, for every $\alpha < \omega_1$ we have $p \in \overline{N_\alpha \cap A}$, hence $\psi(p, \overline{N_\alpha \cap A}) \le \omega_1$. It follows that there is a family $\mathcal{U}_\alpha \in N_{\alpha+1}$ of open sets with $|\mathcal{U}_\alpha| \le \omega_1$ such that

$$\{p\} = \bigcap \mathcal{U}_\alpha \cap \overline{N_\alpha \cap A}.$$

Note that then $\mathcal{U}_\alpha \in N_{\alpha+1} \subset N$ and $|\mathcal{U}_\alpha| \le \omega_1$ imply $\mathcal{U}_\alpha \subset N$. Consequently, for all $\alpha < \omega_1$ we have

$$\{p\} = \bigcap \{U \in N \cap \tau(X) : p \in U\} \cap \overline{N_\alpha \cap A},$$

where $\tau(X)$ denotes the topology of X. Since X has countable tightness, we also have $\overline{N \cap A} = \bigcup_{\alpha<\omega_1} \overline{N_\alpha \cap A}$, hence

$$\{p\} = \bigcap \{U \in N \cap \tau(X) : p \in U\} \cap \overline{N \cap A}.$$

Let us now put

$$H_\alpha = \bigcap \{U \in N_\alpha \cap \tau(X) : p \in U\}.$$

Then H_α is a G_δ set that we claim is closed in X. Indeed, this is because for every $U \in N_\alpha \cap \tau(X)$ with $p \in U$ there is, by the regularity of X and by elementarity, some $V \in N_\alpha \cap \tau(X)$ with $p \in V$ such that $\overline{V} \subset U$.

Since $N \cap \tau(X) = \bigcup_{\alpha<\omega_1} N_\alpha \cap \tau(X)$, it follows that

$$\{p\} = \bigcap_{\alpha<\omega_1} H_\alpha \cap \overline{N \cap A}.$$

Now $p \in \overline{A}^\delta$ and $p \in H_\alpha \in N$ imply that for every $\alpha < \omega_1$ we can pick a point

$$x_\alpha \in N \cap A \cap H_\alpha.$$

We claim that the sequence $S = \{x_\alpha : \alpha < \omega_1\} \subset A$ converges to p. Indeed, if $q \ne p$ then there is a $\beta < \omega_1$ with $q \notin H_\beta \cap \overline{N \cap A}$. Then

$X \setminus H_\alpha \cap \overline{N \cap A}$ is a neighborhood of q that misses the final segment $\{x_\alpha : \beta \le \alpha < \omega_1\}$, hence q is not a complete accumulation point of S. But $X \in CAP(\omega_1)$ then implies that p is the unique complete accumulation point of S, and hence S indeed converges to p.

□

3. No bound for the tightness of the G_δ-modifications of Fréchet spaces

Theorem 3.1. *Assume that S is a non-reflecting stationary set of ω-limits in an uncountable regular cardinal κ. Then there is a 0-dimensional Fréchet topology τ on $\kappa + 1 = \kappa \cup \{\kappa\}$ such that for the space $X = (\kappa + 1, \tau)$ we have $t(X_\delta) = \kappa$.*

Proof. Let us denote by $\mathcal{V}$ the family of all subsets V of κ having the property that for every $\alpha \in S$ there is $\beta < \alpha$ with $(\beta, \alpha) = \alpha \setminus \beta \subset V$. We define τ to be the topology on $\kappa + 1$ for which all points in κ are isolated and $\{V \cup \{\kappa\} : V \in \mathcal{V}\}$ forms a neighborhood base for the point κ. In other words, $\tau = \mathcal{P}(\kappa) \cup \{V \cup \{\kappa\} : V \in \mathcal{V}\}$. It is simple to verify that τ is indeed a 0-dimensional topology.

To see that τ is Fréchet, observe that if $A \subset \kappa$ accumulates to the point κ then there is an $\alpha \in S$ such that $\sup(A \cap \alpha) = \alpha$. But then there is an increasing ω-sequence $B = \{\beta_n : n < \omega\} \subset A \cap \alpha$ with $\sup B = \alpha$ and clearly B converges to the point κ.

Since S is stationary, it is an immediate consequence of Fodor's theorem that every set $V \in \mathcal{V}$ includes a final segment of κ. By $cf(\kappa) > \omega$ it follows then that every G_δ set containing the point κ also includes a final segment of κ, hence the set κ accumulates to the point κ in the space X_δ. Thus, since κ is regular, to prove $t(X_\delta) = \kappa$ it will suffice to show that no proper initial segment of κ accumulates to the point κ in the space X_δ.

This, of course, is equivalent to showing that for each $\eta < \kappa$ the initial segment η is an F_σ set in X. We do this by transfinite induction on $\eta < \kappa$. Thus assume that $\eta < \kappa$ and we know that for every $\zeta < \eta$ the initial segment η is an F_σ set X, i.e. $\zeta = \cup_{n<\omega} F_{\zeta,n}$ with each $F_{\zeta,n}$ closed. Of course, if $cf(\eta) \le \omega$ then it is trivial that η is also an F_σ.

So, assume that $cf(\eta) > \omega$ and, using that S is non-reflecting, fix a closed unbounded subset C in η with $0 \in C$ that is disjoint from S. For every $\gamma \in C$ let γ^+ denote the least member of C above γ.

Let us now define for each $n < \omega$ the set $F_{\eta,n}$ as follows:

$$F_{\eta,n} = \bigcup \{F_{\gamma^+,n} \cap [\gamma, \gamma^+) : \gamma \in C\}\,.$$

Then it is obvious that every $F_{\eta,n}$ is closed in X, i.e. does not contain the point κ in its closure, and their union equals η, hence it is indeed an F_σ.

□

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References

[1] A. Bella and S. Spadaro, Cardinal Invariants for the G_δ topology arXiv:1707.04871

[2] Alan Dow, An introduction to applications of elementary submodels to topology, Topology Proc. 13 (1988), no. 1, 17–72.

[3] I. Juhász, Cardinal functions ten years later, Math. Centre Tract 123 (1980), Amsterdam.

[4] I. Juhász, L. Soukup, and Z. Szentmiklóssy, What is left of CH after you add Cohen reals ?, Topology and its Appl., 85 (1998), pp 165–174.

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