LAVER FORCING AND CONVERGING SEQUENCES

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Abstract. We establish that Laver forcing need not add a converging sequence to the Stone space of a Boolean algebra. This shows that it is consistent that there is an Efimov space with weight less than the bounding number
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1. Introduction

The fundamental question that is addressed in this paper is whether Laver forcing necessarily adds a converging sequence. This question arose in joint work with W. Brian [1] in which three cardinal invariants connected to the existence of converging sequences in compact spaces were investigated. These cardinals are the splitting number, $\mathfrak{s}$, the open splitting number $s(\mathbb{R})$, and the cardinal $\mathfrak{z}$. The splitting number is well-known in set-theory and is usually defined in terms of splitting families of infinite subsets of ω which can instead be stated as the minimum cardinal κ such that there is a compact space of weight κ that has an infinite sequence with no converging subsequence. D. Sobota [10] defined $\mathfrak{z}$ to be the minimum weight of an infinite compact space that contained no infinite converging sequences. The cardinal $s(\mathbb{R})$, defined in [1], is the minimum cardinality of a family of open subsets of $\mathbb{R}$ satisfying that for every infinite converging sequence of $\mathbb{R}$ some member of the family meets it in a non-compact subset. In trying to understand the relationship between these cardinals in forcing extensions, the issue of whether a forcing introduces a converging sequence in some compact space with no converging sequences naturally emerged as a fundamental consideration. The simplest way to interpret the statement about adding a converging sequence (at least for proper posets) is to ask if the space of ultrafilters of the Boolean algebra of ground model subsets of $\mathbb{N}$ has a converging sequence in the extension. For a Boolean algebra B we will let $\mathsf{st}(B)$ denote the usual (Stone) space of ultrafilters on B, and for a compact space K we let $\mathsf{CO}(K)$ denote the Boolean algebra of clopen (closed and open) subsets of K.

We answer one of the questions left open in [1] by showing the consistency of $\mathfrak{s} = \mathfrak{z} < s(\mathbb{R})$. It is already known that the Laver model is a model of $\mathfrak{s} = \aleph_1 < \mathfrak{b}$ ([2, 8]) and it was shown in [1] that $\mathfrak{b} \leq s(\mathbb{R}) \leq \mathfrak{c}$ holds in this model.

The question of whether a forcing will add a converging sequence has already been examined for Cohen forcing and random real forcing. S. Koppelberg [5] showed that Cohen forcing does add a converging sequence to the space of ultrafilters

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for every ground model Boolean algebra. This result has found many interesting applications. On the other hand, it was shown in [3] that this was not the case for the Random real forcing. This result arose in the study of the Efimov problem [4]. Indeed many readers will be more interested in the application to the Efimov problem of the results in this paper. More precisely, we prove the consistency of the existence of an Efimov space of weight less than $\mathfrak{b}$. An Efimov space is an infinite compact space that contains no infinite converging sequences and no copies of $\beta\mathbb{N}$ (the Stone-Čech compactification of the integers $\mathbb{N}$). A compact space of weight less than $\mathfrak{c}$ does not contain a copy of $\beta\mathbb{N}$ and so our compact space of weight $\aleph_1 < \mathfrak{b}$ that contains no converging sequences is an Efimov space.

A third result grew out of the methodology and contributed to the proof of the main result. A generic ultrafilter G for a poset P is also an ultrafilter for the canonical complete Boolean algebra associated with P. In the context of this research it is natural to ask if that generic point in the space of ultrafilters is the limit of a converging sequence. Indeed for many standard posets P for adding a real, it is. We show an even stronger negative result holds for Laver forcing. The generic ultrafilter is not a limit point of any countable set of ultrafilters, i.e. G is a weak P-point in the space of ultrafilters on the complete Boolean algebra generated by Laver forcing. Weak P-points of $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ were introduced by Kunen [6].

In the first section of the paper we establish the above mentioned weak P-point result and show how it naturally arose in the investigation and lead to the main result. In the third section of the paper we show that adding a single Laver real will not add a converging sequence. In the final section we present our main result that the same is true, over a ground model of CH, for the ω_2-length countable support iteration of Laver posets. While the final section can be written so as to be independent of the single stage iteration, the author feels the proof for the single stage may be more natural and possibly adaptable to other posets or other problems. For example D. Sobota and L. Zdomskyy communicated interest in generalizing these results to related questions about measure algebras.

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2. Weak P-points of the Laver poset

Let $\mathbb{L}$ denote the Laver poset, defined below, and we will use g to denote the generic filter on $\mathbb{L}$. We let $\mathsf{ro}(\mathbb{L})$ denotes the usual regular open algebra for the poset $\mathbb{L}$ (see [7, II 3.3]). More generally, for any separative poset P we let B_P denote a complete Boolean algebra that has P as a dense subset (constructed from $\mathsf{ro}(P)$ as in [7, II 3.3]).

Proposition 1. *If G is a V-generic filter for a separative poset P, then G generates an ultrafilter on $B_P \cap V$.*

It is natural to ask if a generic filter for P is a limit of a converging sequence in $\mathsf{st}(B_P \cap V)$ and to note the connection to the question of whether converging sequences are introduced to $\mathsf{st}(\mathcal{P}(\mathbb{N}) \cap V)$. In fact this is how we first started investigating the question for $P = \mathbb{L}$.

We record a couple of folklore observations.

Proposition 2. *Every Boolean algebra of cardinality at most $\mathfrak{c}$ can be embedded as a dense subalgebra of the clopen algebra for some closed subset of βN.*

Proof. We proceed topologically. Since $\mathsf{st}(B)$ is a compact 0-dimensional space of weight at most $\mathfrak{c}$, there is a compact subset K of $2^{\mathfrak{c}}$ such that $\mathsf{CO}(K)$ is isomorphic to B. Fix any continuous function φ that maps $\beta\mathbb{N}$ onto $2^{\mathfrak{c}}$. By Zorn's Lemma, there is a closed subset F of $\beta\mathbb{N}$ such that $\psi = \varphi \restriction F$ is an irreducible map onto K. This simply means that no proper closed subset of F will map onto K. It easily follows then that the set $\{\psi^{-1}(U) : U \in \mathsf{CO}(K)\}$ is a dense subset of $\mathsf{CO}(F)$ that is isomorphic to B. $\square$

Proposition 3. *Let P be a poset of cardinality at most $\mathfrak{c}$. If forcing with P adds a converging sequence to $\mathsf{st}(B_P \cap V)$, then forcing with P adds a converging sequence to $\mathsf{st}(\mathcal{P}(\mathbb{N}) \cap V)$.*

Proof. Let $\{\dot{x}_n : n \in \omega\}$ be P-names of a sequence of distinct ultrafilters on B and assume that $p \in P$ forces that the sequence converges. Fix P-names, $\{\dot{U}_n : n \in \omega\}$, for a sequence of elements of $B_P \cap V$ satisfying that, for each $n \neq m \in \omega$, p forces $\dot{U}_n \in \dot{x}_n$ and $\dot{U}_n \wedge \dot{U}_m = 0$. Let B_1 be any subalgebra of B_P of cardinality $\mathfrak{c}$ satisfying that $P \subset B_1$ and $p \Vdash \dot{U}_n \in B_1$ for each $n \in \omega$. It should be clear that p forces that the sequence $\{\dot{x}_n \cap B_1 : n \in \omega\}$ is a non-trivial converging sequence in $\mathsf{st}(B_1)$. By Proposition 2, we may choose a closed subset F of $\beta\mathbb{N}$ so that $\mathsf{CO}(F)$ contains a dense copy, B_2, of B_1. Furthermore, we may fix an isomorphic embedding ι of $\mathsf{CO}(F)$ into B_P satisfying that $\iota(B_2) = B_1$. It again follows that p forces that $\{\dot{x}_n \cap \iota(\mathsf{CO}(F) \cap V) : n \in \omega\}$ is a non-trivial converging sequence. Since ι is an isomorphism, it follows that p forces that $\mathsf{st}(\mathsf{CO}(F) \cap V)$ has a converging sequence. Since $\mathsf{st}(\mathsf{CO}(F) \cap V)$ is a closed subset of $\mathsf{st}(\mathcal{P}(\mathbb{N}) \cap V)$ and this completes the proof. $\square$

For the remainder of the section, let B denote any Boolean subalgebra of $\mathsf{ro}(\mathbb{L})$ that contains $\{[T] : T \in \mathbb{L}\}$. For each $T \in \mathbb{L}$, let $[T]$ denote the associated element in $\mathsf{ro}(\mathbb{L})$. Recall that if $T' < T$, then $[T'] \subset [T]$ and T' forces that the clopen set $[T']$ is an element of the ultrafilter g. Our goal, for the remainder of this section, is to prove:

Theorem 4. *The point in $\mathsf{st}(B)$ generated by the $\mathbb{L}$-generic ultrafilter g is a weak P-point in the Stone space of B.*

As usual, $\omega^{<\omega}$ denotes the set $\bigcup_{n\in\omega} {}^n\omega$ which is a tree when ordered by usual set-inclusion. Conditions $T \in \mathbb{L}$ are infinite downward closed subtrees of $\omega^{<\omega}$ with the property that there is a stem, stem(T), such that no predecessor is branching and each stem$(T) \subseteq t \in T$ has infinitely many immediate successors. The ordering $<$ on $\mathbb{L}$ is given by $T' < T$ providing $T' \subseteq T$.

As usual for $T \in \mathbb{L}$ and $t_T \subseteq t \in T$ (Laver uses t_T to denote the stem or root of T), let T_t denote the subtree of T satisfying that $t_{T_t} = t$. We will let $L(T,t) = \{j : t^\frown j \in T\}$ and $\mathrm{Br}(T) = \{t \in T : t_T \subseteq t\}$.

We fix a canonical order-preserving enumeration $\{t_\ell^{\omega^{<\omega}} : \ell \in \omega\}$ of $\omega^{<\omega}$, and, for $T \in \mathbb{L}$, this transports to a canonical order-preserving enumeration $\{t_\ell^T : \ell \in \omega\}$ of $\{t \in T : t_T \subseteq t\}$. Then, for $T \in \mathbb{L}$, when we write $T' <_n T$ we mean that $T' \in \mathbb{L}$ and that both $T' < T$ and, for all $\ell \leq n$, $t_\ell^{T'} = t_\ell^T$.

Definition 5. *For each $T \in \mathbb{L}$ and $j \leq n \in \omega$, let $T(j,n)$ be the $<_0$-extension of $T_{t_j^T}$ with the property that, for all $i \leq n$, $t_i^T \in T(j,n)$ if and only if $t_i^T \subseteq t_j^T$.*

Proposition 6 (Laver)**.** *For each $T \in \mathbb{L}$, $n \in \omega$, and $\mathbb{L}$-name τ:*

(1) *there is a $T' <_0 T$ such that $T' \Vdash \tau = n$ or $T' \Vdash \tau \neq n$,*
(2) *$\{T(j,n) : j \leq n\}$ is an antichain that is pre-dense below T, in addition, $T = \bigcup\{T(j,n) : j \leq n\}$.*
(3) *If $\{T_n : n \in \omega\} \subset \mathbb{L}$ is a sequence such that, for all $n \in \omega$, $T_{n+1} <_n T_n$, then $T_\omega = \bigcap T_n \in \mathbb{L}$ and satisfies that $T_\omega <_n T_n$ for all $n \in \omega$.*

A sequence as in Proposition 6 (3) is called a fusion sequence. The situation is property (1) is referred to as the assertion that T' decides the statement $\tau = n$. We will use this phrase in more general situations where we say that a forcing condition decides the truth value of a forcing statement. Similarly a condition $p \in P$ forcing a value on a P-name τ will mean that $p \Vdash \tau = \check{v}$ for some element v of the ground model. We will however not use the $\check{v}$ notation for v when the context makes it clear.

Lemma 7. *Let $T \in \mathbb{L}$ and $n \in \omega$. If $\{T'_j : j \leq n\} \subset \mathbb{L}$ satisfies that $T'_j <_0 T(j,n)$ for each $j \leq n$, then $T' = \bigcup_{j \leq n} T'_j$ is in $\mathbb{L}$ and satisfies that $T' \leq_n T$.*

Now we let $\{\dot{x}_n : n \in \omega\}$ be $\mathbb{L}$-names of maximal ultrafilters on B and suppose that $T_0 \Vdash g \notin \{\dot{x}_n : n \in \omega\}$ for some $T_0 \in \mathbb{L}$. The following is proven using a standard fusion argument applying Lemma 7 and Proposition 6 (3).

Lemma 8. *There is a $T <_0 T_0$ satisfying that, for all $j \leq n \in \omega$, if $T_{t^T_j}$ has a $<_0$-extension T' satisfying that $T' \Vdash [T'] \notin \dot{x}_n$, then $T(j,n) \Vdash [T(j,n)] \notin \dot{x}_n$.*

Proof. We define a fusion sequence $\langle T_n : n \in \omega\rangle$ by induction on n so that, for each $j \leq n$, if there is a $T' <_0 T_{n+1}(j,n)$ with $T' \Vdash [T'] \notin \dot{x}_n$, then $T_{n+1}(j,n) \Vdash [T_{n+1}(j,n)] \notin \dot{x}_n$. Suppose we have defined T_n. For each $j \leq n$, we will choose $T'_j <_0 T_n(j,n)$ and set $T_{n+1} = \bigcup_{j \leq n} T'_j$ as in Proposition 6(3). Since, for $j < n$, $T_n(j,n) <_0 T_n(j,n-1))$, it suffices to choose, if possible, $T'_j <_0 T_n(j,n)$ so that $T'_j \Vdash [T'_j] \notin \dot{x}_n$. If there is no such T'_j, then let $T'_j = T_n(j,n)$. To define T'_n, we set $T'_{n,0} = T_n(n,n)$ and we recursively choose $T'_{n,\ell+1} <_0 T'_{n,\ell}$, for $\ell \leq n$ so that, if possible, $T'_{n,\ell+1} \Vdash [T'_{n,\ell+1}] \notin \dot{x}_\ell$. Our definition of T'_n is then $T_{n,n+1}$. Now assume that, for any $j < n$, there is a $T' <_0 T_{n+1}(j,n)$ satisfying that $T' \Vdash [T'] \notin \dot{x}_n$. Since $T_{n+1}(j,n) <_0 T_n(j,n)$, we have that $T' <_0 T_n(j,n)$, and so we did ensure that $T'_j \Vdash [T'_j] \notin \dot{x}_n$. Similarly, assume $j \leq n$ and that there is some $T' <_0 T_{n+1}(n,n)$ such that $T' \Vdash [T'] \notin \dot{x}_j$. Then again, we have that $T' <_0 T_{n,j}$ and so we did ensure that $T'_{n,j} \Vdash [T'_{n,j+1}] \notin \dot{x}_j$. Since $T_{n+1}(n,n) <_0 T'_{n,j+1}$ and $[T_{n+1}(n,n)] \subset [T'_{n,j}]$, we have that $T_{n+1}(n,n) \Vdash [T_{n+1}(n,n)] \notin \dot{x}_j$.

Now we set $T = T_\omega = \bigcap_{n \in \omega} T_n$ and assume that for some $j \leq n \in \omega$, there is a $T' <_0 T_{t^T_j}$ such that $T' \Vdash [T'] \notin \dot{x}_n$. Since $T_{t^T_j} <_0 T(j,n) <_0 T_{n+1}(j,n)$ we have that $T' <_0 T_{n+1}(j,n)$ and so $T_{n+1}(j,n) \Vdash [T_{n+1}(j,n)] \notin \dot{x}_n$. Again, since $[T(j,n)] \subset [T_{n+1}(j,n)]$, we have that $T(j,n) \Vdash [T(j,n)] \notin \dot{x}_n$. □

This next Lemma is one of the basic ideas of the proof in this and later sections. The idea is roughly that if $T' <_n T(j,n)$ then $T'(j,n)$ can not easily manipulate truth value of $[T'(i,n)] \in \dot{x}_m$ because of self-reference, but for any $j \neq i \leq n$, we have many choices for the value of $T'(i,n)$ that are independent of the choice for $T'(j,n)$ and for some of those combination of choices, $T'(j,n)$ can force the failure of $[T'(i,n)] \in \dot{x}_m$.

Lemma 9. *Let $T \in \mathbb{L}$ be chosen to be as in Lemma 8. Let $\tilde{T} \leq T$ and $n, m \in \omega$ be arbitrary and let $j \leq n$. Then there is a $T' <_n \tilde{T}$ such that, for all $j \neq i \leq n$*

$$T'(j,n) \Vdash [T'(i,n)] \notin \dot{x}_m$$

Proof. For each $i \leq n$, choose any sequence $\{\tilde{T}(i,n,\ell) : \ell \in n+2\}$ from $\mathbb{L}$ such that each is a $<_0$-extension of $\tilde{T}(i,n)$ and the intersection of any two is finite. We may choose a $<_n$-extension T' of $\tilde{T}$ satisfying that $T'(i,n) = \tilde{T}(i,n)$ and, by Proposition 6(1), for all $i \neq j \leq n$ and all $\ell \in n+2$,

$$T'(j,n) \Vdash [\tilde{T}(i,n,\ell)] \in \dot{x}_m \quad \text{or} \quad T'(j,n) \Vdash [\tilde{T}(i,n,\ell)] \notin \dot{x}_m \ .$$

In this Lemma, j is fixed. For each $j \neq i \leq n$, choose any $\ell_i \in n+2$ such that $T'(j,n) \Vdash [\tilde{T}(i,n,\ell_i)] \notin \dot{x}_m$. Finish by defining $T'(i,n)$ to be $\tilde{T}(i,n,\ell_i)$. ☐

Corollary 10. *Again, let $T \in \mathbb{L}$ be chosen as in Lemma 8. Then there is a fusion sequence $\langle T_n : n \in \omega\rangle$ with $T_1 = T_0 = T$ satisfying, for all $n > 1$*

$$(\forall i \neq j \leq n)\ (\forall m \leq n)\ T_n(j,n) \Vdash [T_n(i,n)] \notin \dot{x}_m$$

Proof. Having chosen $T_n <_{n-1} T_{n-1}$, we simply choose $T_{n+1} <_n T_n$ by applying Lemma 9, $(n+2)^2$-times, i.e. for each $j, m \leq n$. ☐

Now we complete the proof of Theorem 4. Let T_ω be the limit of the fusion sequence $\langle T_n : n \in \omega\rangle$ from Corollary 10, i.e. $T_\omega = \bigcap_n T_n$. As usual, we have that $T_\omega <_n T_n$ for all $n \geq 1$. Assume, towards a contradiction, that $\tilde{T} \leq T_\omega$ and $m \in \omega$ are such that $\tilde{T} \Vdash [T_\omega] \in \dot{x}_m$. Since $T \Vdash g \neq \dot{x}_m$, we can, by possibly extending $\tilde{T}$, arrange that $\tilde{T} \Vdash [\tilde{T}] \notin \dot{x}_m$ (more carefully: there are $\tilde{T}_1, \tilde{T}_2$ below $\tilde{T}$ such that $\tilde{T}_1 \Vdash [\tilde{T}_2] \in g \setminus \dot{x}_m$, which of course means that $\tilde{T}_1 \leq \tilde{T}_2$, and $[\tilde{T}_1] \subset [\tilde{T}_2]$, hence $\tilde{T}_1 \Vdash [\tilde{T}_1] \notin \dot{x}_m$). There is a unique $j \in \omega$ such that $t_{\tilde{T}} = t^T_j$. By our assumption on T, $T_{t^T_j} \Vdash [T_{t^T_j}] \notin \dot{x}_m$. Choose any $n > \max(j,m)$. We now have that $[T_n(j,n)] \Vdash [T_n(j,n)] \notin \dot{x}_m$, and, for all $j \neq i \leq n$, $T_n(j,n) \Vdash [T_n(i,n)] \notin \dot{x}_m$. That is, $T_n(j,n) \Vdash (\bigcup_{i \leq n}[T_n(i,n)]) \notin \dot{x}_m$. By Lemma 6, this implies that $T_n(j,n) \Vdash [T_n] \notin \dot{x}_m$. Since $[T_\omega] \subset [T_n]$ and $\tilde{T} <_0 T_n(j,n)$, we have the contradictory statement that $\tilde{T} \Vdash [T_\omega] \notin \dot{x}_m$.

3. Adding a Single Laver does not add a converging sequence

In this section we will show that in the forcing extension by $\mathbb{L}$, the space $\mathsf{st}(\mathsf{ro}(2^\omega) \cap V)$ contains no converging sequences. For convenience we choose any $T_0 \in \mathbb{L}$ consisting only of elements t of $\omega^{<\omega}$ that are strictly increasing functions and we assume that T_0 is in the generic filter g. This is just a convenience since there is a dense set of $T \in \mathbb{L}$ that satisfy that $t \setminus t_T$ is strictly increasing.

Since $\mathsf{st}(\mathsf{ro}(2^\omega))$ contains copies of $\beta\mathbb{N}$, it also follows that $\mathbb{L}$ does not add a converging sequence to $\mathsf{st}(\mathcal{P}(\mathbb{N}) \cap V)$. We find it easier to work with $\mathsf{ro}(2^\omega)$. It will be necessary to be more general: if M is a countable elementary submodel of $H(\mathfrak{c}^+)$, then $B = M \cap \mathsf{ro}(2^\omega)$ is a countable atomless Boolean algebra. Since $\mathsf{st}(B)$ is homeomorphic to 2^ω, we can simplify notation by identifying $\mathsf{st}(B)$ with 2^ω (via any suitable homeomorphism). It is similarly true that $\mathsf{ro}(\mathsf{st}(B))$ is equal to the original copy of $\mathsf{ro}(2^\omega)$. Therefore, for an ultrafilter $\mathcal{U}$ on $\mathsf{ro}(2^\omega) \cap V$, we can treat $\mathcal{U} \cap B$ as the filter base of clopen sets for a point in 2^ω under this identification. If U is an open subset of 2^ω and if $\dot{\mathcal{U}}$ is an $\mathbb{L}$-name of an element of $\mathsf{st}(\mathsf{ro}(2^\omega) \cap V)$, we will say that ($T$ forces that) $U \in \dot{\mathcal{U}}$ to mean that U contains some regular open set from $\dot{\mathcal{U}}$. The proof consists of constructing an element W of $\mathsf{ro}(2^\omega)$ and a condition that will force that the clopen subset of $\mathsf{st}(\mathsf{ro}(2^\omega))$ corresponding to W splits a potential converging sequence.

Now we state and prove the main result of this section.

Theorem 11. *In $V[g]$, there is no non-trivial converging sequence in the space $\mathsf{st}(\mathsf{ro}(2^\omega) \cap V)$.*

Proof. Let $\{\dot{\mathcal{U}}_n : n \in \omega\}$ be $\mathbb{L}$-names of distinct ultrafilters on $\mathsf{ro}(2^\omega) \cap V$. Towards a contradiction, assume that (possibly some extension of) T_0 forces that $\langle \dot{\mathcal{U}}_n : n \in \omega\rangle$ converges to the ultrafilter $\dot{\mathcal{U}}$. Choose any countable elementary submodel M of some $H(\theta)$ and assume that $\{\dot{\mathcal{U}}\} \cup \{\dot{\mathcal{U}}_n : n \in \omega\}$ is an element of M and that T_0 is $(M, \mathbb{L})$-generic (see [9, III.2.5]). Let B be the Boolean algebra $M \cap RO(2^\omega)$. Using the identification between $\mathsf{st}(B)$ and 2^ω as discussed above, for each n, let $\dot{x}_n$ be the name of the point of 2^ω that is in every element of $\dot{\mathcal{U}}_n \cap B$. Similarly, let $\dot{x}$ be the element of $\mathsf{st}(B)$ that is in every element of $\dot{\mathcal{U}} \cap B$. By elementarity, T_0 forces that $\dot{x}_n \neq \dot{x}_m$ and $\dot{x}_n \neq \dot{x}$ for all $n \neq m$.

Here is a brief outline of the proof that may help motivate the technical details. We will eventually construct a condition $T <_0 T_0$ (or rather $T_2 <_0 T_0$) together with a hierarchy of *fronts* (see below) $\{S_n : n \in \omega\}$ of T where each $s \in S_n$ satisfies that T_s is able to decide a suitably small neighborhood, here we will call it, W_s of $\dot{x}$. The notion of front ensures that, for each n, it follows that T forces that $W_{S_n} = \bigcup\{W_s : s \in S_n\}$ is a neighborhood of $\dot{x}$. With more work, again using the key idea as explained before Lemma 8, we also ensure that, for each $m \in \omega$, T forces that the regular open algebra complement of some W_{S_n} is an element of $\dot{x}_m$. We can not expect there to be a $<_0$-extension of T that will decide which n has this property however. Instead, we use that the ground model subsets of ω remain splitting and we are able to choose an infinite $I \subset \omega$ and an extension of T forcing that $W_I = \bigcup\{W_{S_n} \setminus \overline{W}_{S_{n+1}} : n \in I\}$ is an element of $\mathsf{ro}(2^\omega) \cap V$ that is forced to split the sequence $\{\dot{\mathcal{U}}_n : n \in \omega\}$ (but not to decide any given n with $W_I \in \dot{\mathcal{U}}_n$ or m with $W_I \notin \dot{\mathcal{U}}_m$).

It follows, from the fact that $\mathbb{L}$ preserves ω-splitting families [2], that we might as well assume that T forces that $\dot{x}$ is not in V. Recall that $L(T,t)$ denotes the set $\{\ell : t^\frown \ell \in T\}$. By a straightforward fusion, we can find $T <_0 T_0$ satisfying the following:

(1) for each $t \in \mathrm{Br}(T)$ and each $n \leq |t|$, there is a point $y(t,n) \in 2^\omega$ such that for each $\ell \in L(T,t)$, $T_{t^\frown \ell} \Vdash y(t,n) \restriction \ell = \dot{x}_n \restriction \ell$,
(2) for each $t \in \mathrm{Br}(T)$, there is a point $y_t \in 2^\omega$ such that for each $\ell \in L(T,t)$, $T_{t^\frown \ell} \Vdash y_t \restriction \ell = \dot{x} \restriction \ell$.

Note that if $T' <_0 T_t$, then what one might designate as $y_t^{T'}$ is simply equal to y_t.

Recall that a set S is a front of $T \in \mathbb{L}$ if $T' \cap S$ is not empty for each $T' < T$. If S is a front of T, then the set of minimal elements of S is also a front. So we will assume that the elements of a front are pairwise incompatible in $\omega^{<\omega}$. Let S_0 be the singleton set consisting of the stem of T. By recursion on n, S_n is the front of elements of T satisfying that there is an $s' \subset s$ with $s' \in S_{n-1}$ such that $y_s \neq y_{s'}$ (S_n is a front because $\dot{x}$ is not in V). For each n, and $s' \in S_{n-1}$, $s \in S_n$ with $s' < s$ let k_s be the minimum of $\{k : y_s \restriction k \neq y_{s'} \restriction k\}$. By a trivial pruning, we can assume that $k_s < \min(L(T,s))$ for all $s \in S_n$. In this proof, $[y \restriction k]$ will denote the standard clopen subset of 2^ω for $y \in 2^\omega$ and $k \in \omega$.

Claim 1. By performing a fusion, we can assume that T satisfies that for incomparable $s, s' \in \bigcup\{S_n : 0 < n \in \omega\}$, $[y_s \restriction k_s]$ and $[y_{s'} \restriction k_{s'}]$ are disjoint, and that if $s, s' \in S_{n+1}$ have the same predecessor in S_n, then $2 < |k_s - k_{s'}|$.

Proof of Claim: The fusion is simply an operation on each S_n at once. When S is any front, we can define a rank function ρ_S on $t \in \mathrm{Br}(T)$ where $\rho_S(t) = 0$ if there is an $s \in S$ with $s \subseteq t$, and for other $t \in \mathrm{Br}(T)$, define $\rho_S(t)$ to be the minimum $\alpha \in \omega_1$ (if one exists) such that there are infinitely many $\ell \in L(T,t)$ such that $\rho_S(t^\frown \ell) < \alpha$. If no such α exists, then $\rho_S(t) = \infty$. If there is a $\bar{t} \in \mathrm{Br}(T)$, such that $\rho_S(\bar{t}) = \infty$, then

$$\bar{T} = \{t \in T : t \subseteq \bar{t}, \text{ or } \bar{t} \subseteq t \text{ and } \rho_S(t) = \infty\}$$

is a condition below T. This condition contradicts that S is a front.

For $n > 0$, define S_n^- to be the set $\{t : \rho_{S_n}(t) = 1\}$. We may assume that T has been pruned to a $<_0$ extension satisfying that, for all $0 < n \in \omega$ and $t \in S_n^-$, $\rho_{S_n}(t^\frown \ell) = 0$ for all $\ell \in L(T,t)$. Also for each $t \in S_n^-$, it follows that $\{k_{t^\frown \ell} : \ell \in L(T,t)\}$ diverges to infinity. Now fix any $s' \in S_{n-1}$ and focus on $T_{s'}$. Therefore, in our desired fusion, we need only remove finitely many immediate successors (and all nodes above them of course) from above each $s' \in S_n^-$ so as to ensure that the map sending $s' \subset s \in S_n$ to k_s is 1-to-1. Therefore, if $s' \subset s_1$ and $s' \subset s_2$ with $s_1, s_2 \in S_n$ and $k_{s_1} < k_{s_2}$, then $y_{s_2} \restriction k_{s_1} = y_{s'} \restriction k_{s_1}$ and $y_{s'} \restriction k_{s_1}$ is incomparable with $y_{s_1} \restriction k_{s_1}$. □

Claim 2. For all $s \in \bigcup_n S_n$, T_s forces that $[y_s \restriction k_s]$ is in $\dot{\mathcal{U}}$.

Since we arranged that $k_s < \min(L(T,s))$, it follows that T_s forces that $y_s \restriction k_s \subset \dot{x}$ (i.e. $T_{s^\frown \ell} \Vdash y_s \restriction \ell \subset \dot{x}$ for all $\ell \in L(T,s)$). It will be useful to note that T has been sufficiently pruned so that the following holds.

Claim 3. For each $n \in \omega$ and $s \in T \cap S_n$, the set $\bigcup\{[y_{\tilde{s}} \restriction k_{\tilde{s}}] : s \subset \tilde{s} \in T \cap S_{n+1}\}$ is a proper regular open subset of $[y_s \restriction k_s]$.

Proof of Claim: For each $k_s \leq j \in \omega$, let $t^s_j \in 2^{j+1}$ be defined by $t^s_j(j) \neq y_s(j)$ and $y_s \restriction j \subset t^s_j$. Note that, for $s \subset s' \in S_{n+1}$, $y_{s'} \restriction k_{s'} = t^s_{k_{s'}}$. The family $\{[t^s_j] : k_s \leq j \in \omega\}$ is a pairwise disjoint family with union equal to $[y_s \restriction k_s] \setminus \{y_s\}$. It follows that the only accumulation point of $\bigcup\{[y_{s'} \restriction k_{s'}] : s \subset s' \in T \cap S_{n+1}\}$ is y_s. By Claim 1, y_s is also an accumulation point of $\bigcup\{[t^s_j] : k_s \leq j \notin \{k_{s'} : s \subset s' \in T \cap S_{n+1}\}\ \}$. It follows that $\bigcup\{[y_{s'} \restriction k_{s'}] : s \subset s' \in T \cap S_{n+1}\}$ is equal to the interior of its closure, i.e. it is regular open. □

Claim 4. For each $n \in \omega$, $t \in S_n$, and $\tilde{T} <_0 T_t$, there is a $T' <_0 \tilde{T}$ satisfying that, for each $m \leq n$, if $T' \Vdash \bigcup\{[y_s \restriction k_s] : t \subset s \in S_{n+1} \cap T'\} \in \dot{\mathcal{U}}_m$, then there exists $s \in S_{n+1}$, with $t \subset s$, such that $T' \Vdash [y_s \restriction k_s] \in \dot{\mathcal{U}}_m$.

Proof of Claim: Let $\{t_m : m \in \omega\}$ be an enumeration of $S^-_{n+1} \cap \tilde{T}$. We perform a fusion $\{\tilde{T}_m : m \in \omega\}$ and ensure that, for all $m \in \omega$,

(1) $S^-_{n+1} \cap \tilde{T} = S^-_{n+1} \cap \tilde{T}_m$,
(2) for each $k \leq m$, the first m elements of $L(\tilde{T}_m, t_k)$ are also in $L(\tilde{T}_{m+1}, t_k)$,

(3) for each $k \leq m$ and each ℓ among the first m elements of $L(\tilde{T}_{m+1}, t_k)$ and for each $j \leq n$, if $(\tilde{T}_{m+1})_{t_k^\frown \ell} \Vdash \bigcup\{[y_s \restriction k_s] : t \subset s \in S_{n+1} \cap \tilde{T}_{m+1}\} \in \dot{\mathcal{U}}_j$ then there is an $s \in S_{n+1}$ such that $t \subset s$ and $(\tilde{T}_{m+1})_{t_k^\frown \ell} \Vdash [y_s \restriction k_s] \in \dot{\mathcal{U}}_j$.

Each inductive step is similar to the construction in the weak P-point section. The first step is to choose $\tilde{T}_0 <_0 \tilde{T}$ so that for all $t \in S^-_{n+1} \cap \tilde{T}_0$, $t^\frown \ell \in S_{n+1}$ for all $\ell \in L(\tilde{T}_0, t)$. The construction of $\tilde{T}_{m+1}$ will require a recursion in which we successively consider a new triple k, ℓ, j as in item (3). It should suffice to just explain the first step for some choice k, ℓ, j. The inductive assumptions (1) and (2) will require that there is a finite subset S' of $S_{n+1} \cap T_m$ that can not be removed and that $t_k^\frown \ell \in S'$. By $<_0$-extending $(T_m)_{t_k^\frown \ell}$, we can arrange that for each $s \in S'$, either $(T_{m+1})_{t_k^\frown \ell} \Vdash [y_s \restriction k_s] \in \dot{\mathcal{U}}_j$ or $(T_{m+1})_{t_k^\frown \ell} \Vdash [y_s \restriction k_s] \notin \dot{\mathcal{U}}_j$. If there is an $s \in S'$ such that we were able to arrange $(\tilde{T}_{m+1})_{t_k^\frown \ell} \Vdash [y_s \restriction k_s] \in \dot{\mathcal{U}}_j$ then this step is complete. Now we suppose that $(\tilde{T}_{m+1})_{t_k^\frown \ell} \Vdash [y_s \restriction k_s] \notin \dot{\mathcal{U}}_j$ for all $s \in S'$. Choose an infinite $S \subset S_{n+1} \setminus S'$ satisfying that for all $t \in S^-_{n+1}$, both sets $\{\ell' \in L(\tilde{T}_m, t) : t^\frown \ell' \in S\}$ and $\{\ell' \in L(\tilde{T}_m, t) : t^\frown \ell' \notin S\}$ are infinite. The set $W_S = \bigcup\{[y_s \restriction k_s] : s \in S\}$ is regular open. We may assume that $(\tilde{T}_{m+1})_{t_k^\frown \ell}$ has been $<_0$-extended so as to force either $W_S \in \dot{\mathcal{U}}_j$ or $W_S \notin \dot{\mathcal{U}}_j$. In the first case, we may construct $\tilde{T}_{m+1}$ so that $S \cap \tilde{T}_{m+1}$ is empty. In the second case we may ensure that $S_{n+1} \cap \tilde{T}_{m+1}$ is contained in $S' \cup S$. With our collective assumptions on $\tilde{T}_{m+1}$, we then have that $(\tilde{T}_{m+1})_{t_k^\frown \ell} \Vdash \bigcup\{[y_s \restriction k_s] : t \subset s \in S_{n+1} \cap \tilde{T}_{m+1}\} \notin \dot{\mathcal{U}}_j$.

Once this recursion is complete, we have by induction hypothesis (1) and (2), that $T' <_0 T_t$ where $T' = \bigcap\{\tilde{T}_m : m \in \omega\}$. It should be clear that the statement of the Claim then follows from induction hypthesis (3). $\square$

Claim 5. There is a $T_1 <_0 T$ satisfying, for each $n \in \omega$ and $t \in S_n \cap T_1$,

$$(T_1)_t \Vdash \bigcup\{[y_s \restriction k_s] : t \subset s \in S_{n+1} \cap T_1\} \in \dot{\mathcal{U}}_m \Rightarrow (\exists t \subset s \in S_{n+1})\ [y_s \restriction k_s] \in \dot{\mathcal{U}}_m$$

for each $m \leq n$.

The proof of Claim 5 is to perform a fusion and repeatedly apply Claim 4 . After obtaining T_1 as in Claim 5, we perform another fusion to obtain the following property. This is the step where we are proceeding as described just before Lemma 8.

Claim 6. There is a $T_2 <_0 T_1$ satisfying that for each $t \in \bigcup_n S_n \cap T_2$ and each $m \leq |t|$, there is an $\tilde{m} \in \omega$ such that, $(T_2)_t \Vdash \dot{x}_m \notin [y_s \restriction k_s]$ for all $s \in S_{\tilde{m}} \cap T_2 \setminus (T_2)_t$.

Proof of Claim: Construct a fusion sequence $\langle T_{1,n} : n \in \omega\rangle$ (i.e. $T_{1,n+1} <_n T_{1,n}$) as follows. Let $T_{1,0} = T_1$ and let $\bar{s}$ be the stem of T_1 (i.e. the unique element of S_0). Assume, by induction, that the statement of the Claim holds for all $t \in \{t_\ell^{T_{1,n}} : \ell < n\} \cap \bigcup_j S_j$ where $T_{1,n}$ is substituted for T_2 in the statement. Suppose that we have chosen $T_{1,n}$. If the element $t_{n+1}^{T_{1,n}}$ is not in $\bigcup_m S_m$, then let $T_{1,n+1} = T_{1,n}$. Otherwise, $t = t_{n+1}^{T_{1,n}}$ is in $\bigcup_m S_m$. We may suppose (by $<_0$-extending) that $(T_{1,n})_t$ decides the statement $\dot{x}_m \in [y_t \restriction k_t]$ for all $m \leq |t|$. Let $\tilde{m} \in \omega$ be chosen so that $\{t_\ell^{T_{1,n}} : \ell \leq n\} \cap \bigcup_j S_j$ is contained in $\bigcup_{j<\tilde{m}} S_j$. For each $m \leq |t|$, let y_m denote the element $y(t, m) \in 2^\omega$ defined above satisfying that $T_{t^\frown \ell} \Vdash y_m \restriction \ell = \dot{x}_m \restriction \ell$ for each $\ell \in L(T, t)$. For those $m \leq |t|$ such that $(T_{1,n})_t \Vdash \dot{x}_m \notin [y_t \restriction k_t]$, we have that $y_t \restriction k_t \not\subset y_m$. For each $m \leq |t|$ such that, for the stem $\bar{s}$, $y_{\bar{s}} \restriction k_{\bar{s}}$ is not an initial segment of y_m, we can assume that, $\tilde{T}$ forces that $y_{\bar{s}} \restriction k_{\bar{s}}$ is not an initial segment of $\dot{x}_m$. Then we have that $\tilde{T}$ forces the statement of the claim holds for t and $\dot{x}_m$. Let A be the set of $m \leq |t|$ such that $y_{\bar{s}} \restriction k_{\bar{s}} \subset y_m$ and $y_t \restriction k_t \not\subset y_m$. The set A is the values of $m \leq |t|$ that $(T_{1,n})_t$ may not yet satisfy the conclusion of the lemma. For each $m \in A$, choose the maximal $s_m \in T_{1,n} \cap \bigcup_{\ell \leq \tilde{m}} S_\ell$ satisfying that $y_{s_m} \restriction k_{s_m} \subset y_m$. Note that $t \notin \{s_m : m \in A\}$. Let $A_1 = \{m \in A : s_m \in S_{\tilde{m}}\}$. By further extending $\tilde{T}$, we can assume that, for each $m \in A_1$, $(\tilde{T})_t$ forces that $y_{s_m} \restriction k_{s_m} \subset \dot{x}_m$. Furthermore, we can arrange that $\tilde{T} \cap \{s_m : m \in A_1\}$ is empty. Also, let $\tilde{T}_{1,n} = T_{1,n} \setminus \bigcup\{(T_{1,n})_{s_m} : m \in A_1\}$. By the definition of $\tilde{m}$, it follows that $\tilde{T}_{1,n} <_n T_{1,n}$.

Let $A_2 = A \setminus A_1$, and for each $m \in A_2$, we can assume that $(\tilde{T})_t$ has decided the statement $\dot{x}_m = y_{s_m}$. For each $m \in A_2$, fix $j_m < \tilde{m}$ so that $s_m \in S_{j_m}$. Then, for any $m \in A_2$ such that $(\tilde{T})_t \Vdash \dot{x}_m = y_{s_m}$, we have that $(\tilde{T})_t \Vdash \dot{x}_m \notin \bigcup\{[y_s \restriction k_s] : s \in S_{j_m+1}\}$, hence $(\tilde{T})_t \Vdash \dot{x}_m \notin \bigcup\{[y_s \restriction k_s] : s \in S_{\tilde{m}}\}$. Let A_3 be the set of $m \in A_2$ such that $\tilde{T}$ forces that $\dot{x}_m \neq y_{s_m}$. For each $m \in A_3$, let τ_m denote the name of the minimum integer such that $\tilde{T} \Vdash \dot{x}_m(\tau_m) \neq y_{s_m}(\tau_m)$. We may similarly assume if some $<_0$-extension of $(\tilde{T})_t$ forces a bound on τ_m, then $(\tilde{T})_t$ decides the value of τ_m. Let $K_m = \{k_{s'} : s_m < s' \in S_{j_m+1} \cap T_{1,n}\}$. If $(\tilde{T})_t$ does force a value, say k, on τ_m, then we check that the maximality of s_m ensures that $k \notin K_m$. Namely, suppose that $k = k_{s'}$. Thus $y_m \notin [y_{s_m} \restriction k_{s'}]$, but $y_m \in [y_{s_m} \restriction k_{s'} - 1]$, i.e., $y_m(k_{s'} - 1) \neq y_{s_m}(k_{s'} - 1)$, and hence $y_m(k_{s'} - 1) = y_{s'}(k_{s'} - 1)$ because there are just two values 0,1 possible. It follows that $y_m \in [y_{s_m} \restriction k_{s'} - 1] = [y_{s'} \restriction k_{s'} - 1]$ and $y_m(k_{s'} - 1) = y_{s'}(k_{s'} - 1)$, and hence $y_m \in [y_{s'} \restriction k_{s'}]$, which contradicts the maximality. Therefore, if $(\tilde{T})_t$ forces a value on τ_m then we would again have that $(\tilde{T})_t \Vdash \dot{x}_m \notin \bigcup\{[y_s \restriction k_s] : s \in T_{1,n} \cap S_{j_m+1}\}$. So we now let A_4 be those $m \in A_3$ such that $(\tilde{T})_t$ does not force a value on τ_m. If A_4 is not empty, we continue with the following construction. For each $m \in A_4$, choose $\ell_m \in \omega$ minimal so that $s_m^\frown j \notin \{t_\ell^{T_{1,n}} : \ell \leq n\}$ for all $j \geq \ell_m$. We may assume that, for each $m \in A_4$, $(\tilde{T})_t$ forces that τ_m is greater than k_s for all $s \in \{t_\ell^{T_{1,n}} : \ell \leq n\}$. Now we proceed by induction on $m \in A_4$ and we recursively define a $<_n$-descending sequence $\{T_{1,n,m} : m \in A_4\}$ such that $T_{1,n,m} <_n \tilde{T}_{1,n}$ for each $m \in A_4$ and that $(T_{1,n,m})_t <_0 \tilde{T}$. We will ensure that $(T_{1,n,m})_t$ forces that $\dot{x}_m \notin \bigcup\{[y_s \restriction k_s] : s_m < s \in T_{1,n,m} \cap S_{j_m+1}\}$. By the assumption on τ_m we have that $(\tilde{T})_t$ forces that $\dot{x}_m \notin V_m = \bigcup\{[y_s \restriction k_s] : s_m < s \in S_{j_m+1} \cap \{t_\ell^{T_{1,n}} : \ell \leq n\}\}$. Let $T_{1,n,-1} = T_{1,n}$ and for $m \in A_4$, let m^- denote the maximum element of $\{-1\} \cup (A_4 \cap m)$. At stage $m \in A_4$, let L be an infinite subset of $L(T_{1,n,m^-}, s_m) \setminus \ell_m$ such that $L(T_{1,n,m^-}, s_m) \setminus L$ is also infinite. It follows that $U_L = \bigcup\{[y_s \restriction k_s] : (\exists \ell \in L)\ s_m^\frown \ell \leq s \in S_{j_m+1} \cap T_{1,n,m^-}\}$ and $W_L = \bigcup\{[y_s \restriction k_s] : (\exists \ell \in L(T_{1,n,m^-}) \setminus L)\ s_m^\frown \ell \subseteq s \in S_{j_m+1} \cap T_{1,n,m^-}\}$ are disjoint. We choose $T_{1,n,m}$ so that $(T_{1,n,m})_t$ decides the statement $\dot{x}_m \in U_L$, and so that $L(T_{1,n,m}, s_m) \setminus \ell_m \subset L$ if $(T_{1,n,m})_t \Vdash \dot{x}_m \notin U_L$, and $L(T_{1,n,m}, s_m) \subset L(T_{1,n,m^-}, s_m) \setminus L$ otherwise. Since $L(T_{1,n,m}, s_m) \setminus \ell_m = L$ implies that $V_m \cup U_L = \bigcup\{[y_s \restriction k_s] : s_m < s \in S_{j_m+1} \cap T_{1,n,m}\}$, and $L(T_{1,n,m}, s_m) = L(T_{1,n,m^-}, s_m) \setminus L$ implies $V_m \cup W_L = \bigcup\{[y_s \restriction k_s] : s_m < s \in S_{j_m+1} \cap T_{1,n,m}\}$, our choice for $T_{1,n,m}$ satisfies the inductive requirement.

We complete the proof of Claim 6, by setting $T_{1,n+1}$ to be $T_{1,n,m}$ for $m = \max(A_4)$ if A_4 is not empty, and otherwise, $T_{1,n+1} <_n T_{1,n}$ satisfies that $(T_{1,n+1})_t = \tilde{T}$. The verification of the inductive hypothesis for this choice is a routine tracking through the construction. $\square$

Now choose any T_2 as in Claim 6. For each $n \in \omega$ and $t \in T_2 \cap S_n$, let $W_t = [y_t \restriction k_t] \setminus \bigcup\{[y_s \restriction k_s] : t < s \in S_{n+1} \cap T_2\}$. We again note, by Claim 3, that W_t and $W_{t'}$ are disjoint for distinct $t, t' \in T_2 \cap \bigcup_n S_n$. For each $n \in \omega$, let W_{S_n} be the join in $\mathsf{ro}(2^\omega)$ of the family $\{W_t : t \in S_n\}$. For each $s \in \bigcup_n S_n$, $[y_s \restriction k_s]$ is contained in $\bigvee_n W_{S_n}$, hence, by Claim 4, $T_2 \Vdash \bigvee_n W_{S_n} \in \dot{\mathcal{U}}$. Let $\dot{J}$ be the name for the set $\{n \in \omega : (\exists m)\ W_{S_n} \in \dot{\mathcal{U}}_m\}$.

Claim 7. T_2 forces that $\dot{J}$ is infinite.

Proof of Claim: We may assume T_2 of Claim 6 forces that $\dot{x}_m \in [y_{\bar{s}} \restriction \bar{s}]$ for all $m > m_0$. Given any $m_0 < m \in \omega$ and $\mathbb{L}$-generic filter G with $T_2 \in G$, there is a $T \in G$ and a k satisfying that $T \Vdash \dot{x} \restriction k \neq \dot{x}_m \restriction k$. By extending T, we can assume that $T < T_2$ and that the stem t of T is in S_n for some n. Now it follows from Claim 7, that there is a $\tilde{m}$ such that $\mathrm{val}_G(\dot{x}_m) \notin W_{S_n}$ for any $n \geq \tilde{m}$. It also follows from Claim 6, that there is an $s \in \bigcup_{\ell < \tilde{m}} S_\ell$ such that $\mathrm{val}_G(\dot{x}_m) \in W_s$, hence $\mathrm{val}_G(\dot{x}_m) \in W_{S_\ell}$ for some ℓ.

Since $\mathbb{L}$ preserves that the ground model is splitting, choose any $I \subset \omega$ and $T_3 < T_2$ such that T_3 forces that $\dot{J} \cap I$ and $\dot{J} \setminus I$ are infinite. Let $U = \bigcup\{U_n : n \in I\}$ and $W = \bigcup\{U_n : n \in \omega \setminus I\}$. Assume (by symmetry) that $T_3 \Vdash U \in \dot{\mathcal{U}}$ and that $T_3 \Vdash U \in \dot{\mathcal{U}}_m$ for all $m > n_0$. Let $t = \mathit{stem}(T_3)$ and assume for convenience that $t \in S_j$ for some $j \in \omega$ and that $|t| > n_0$. Choose $\tilde{m}$ as in Claim 7, that is, so that T_3 forces that $U_n \notin \dot{\mathcal{U}}_m$ for all $n \geq \tilde{m}$ and $m \leq |t|$. Choose any pair $T_4 < T_3$ and $n > \max(n_0, \tilde{m})$ such that $T_4 \Vdash n \in \dot{J} \setminus I$. By possibly extending T_4, we can ensure that there is some m such that $T_4 \Vdash U_n \in \dot{\mathcal{U}}_m$. Since $n > \tilde{m}$, it follows that $m > |t| \geq n_0$. But now we have that T_4 forces that the disjoint sets U_n and $\bigcup\{U_\ell : \ell \in I\}$ are each in $\dot{\mathcal{U}}_m$. □

4. Now add many Laver reals

In this section we prove

Theorem 12 (CH)**.** *In the ω_2-Laver extension, there is an Efimov space of weight $\aleph_1$.*

The proof is at the end of the section as it will require a series of Lemmas and Definitions. We again let B be the Boolean algebra $\mathsf{ro}(2^\omega)$ in the ground model (of CH). We let $\mathbb{P}_\lambda$ be the countable support iteration of the usual Laver poset $\mathbb{L}$. We adopt the convention that elements p of $\mathbb{P}_\lambda$ are functions with domain that is a countable subset of λ and that, implicitly, $p(\alpha)$ is the maximal element of $\mathbb{L}$ for $\alpha \in \lambda \setminus \mathrm{dom}(p)$. With this convention it follows that $\mathbb{P}_\lambda$ is a (complete) subposet of $\mathbb{P}_\mu$ for all $\lambda \leq \mu \leq \omega_2$. When we use the forcing symbol "$\Vdash$" in an expression we will assume that the context makes clear which poset(s) is intended. We prove, by induction on λ that $\mathbb{P}_{\omega_2}$ forces that no $\mathbb{P}_\lambda$-name of an ultrafilter on B is the limit of a converging sequence. By Theorem 11, we may assume that $\lambda > 1$. Since B has cardinality $\aleph_1$ and $\mathbb{P}_{\omega_2}$ has the $\aleph_2$-cc, each $\mathbb{P}_{\omega_2}$-name of an ultrafilter on B is equal to a $\mathbb{P}_\mu$-name for some $\mu < \omega_2$. Let $\dot{\mathcal{U}}$ be a $\mathbb{P}_\lambda$-name of an ultrafilter on B and assume that no condition forces it is equal to a $\mathbb{P}_\delta$-name for any $\delta < \lambda$. Also let $\lambda \leq \mu$ and let $\{\dot{\mathcal{U}}_n : n \in \omega\}$ be $\mathbb{P}_\mu$-names of ultrafilters on B.

The rough outline of the proof is the same as in the previous section except that this time the notion of a front is much more complicated. Indeed, we simply treat

$\mathbb{P}_\mu$ (using Laver's notions) as a poset in much the same was as we did with $\mathbb{L}$. Since the proof relies on $\mathsf{ro}(2^\omega)$ being a complete Boolean algebra, which it fails to be in the forcing extensions, we were unable to formulate an iterable condition and a preservation result to give a more standard iteration proof of the theorem.

Let M be a countable elementary submodel of $H(\aleph_2)$ such that $\dot{\mathcal{U}}$ and $\{\dot{\mathcal{U}}_n : n \in \omega\}$ are elements of M. We assume that 1 forces $\dot{\mathcal{U}} \neq \dot{\mathcal{U}}_n$ for all $n \in \omega$. Let p_0 be an $(M, \mathbb{P}_\mu)$-generic condition. For each $n \in \omega$, fix a $\mathbb{P}_\mu$-name $\dot{U}_n \in M$ such that p_0 forces that $\dot{U}_n \in \dot{\mathcal{U}} \setminus \dot{\mathcal{U}}_n$. If $\dot{U} \in M$ is a $\mathbb{P}_\mu$-name of an element of B, then it may be regarded, for the purposes of Lemma 18 below, as a name of an integer. We make this more precise with this definition.

Definition 13. *Fix an enumeration $\{U(M,\ell) : \ell \in \omega\}$ of B. For each $\mathbb{P}_\mu$-name $\dot{U}$ in M for an element of B, let $\dot{\ell}(\dot{U})$ be a $\mathbb{P}_\mu$-name satisfying that p_0 forces that $\dot{U} = U(M, \dot{\ell}(\dot{U}))$.*

We assume that p_0 forces that every member of $\dot{\mathcal{U}}$ is a member of infinitely many $\dot{\mathcal{U}}_m$ (otherwise $\mathcal{U}$ is not even a limit of the sequence $\{\dot{\mathcal{U}}_m : m \in \omega\}$). We prove that the sequence does not converge.

We record some of Laver's iteration notation.

Definition 14. *For $p \in P_\mu$, finite $F \subset \mathrm{dom}(p)$, $n \in \omega$, and $\sigma : F \to n$, we define $p^{\sigma,n}$ to be the condition where, for $\alpha \in F$, $p^{\sigma,n} \restriction \alpha \Vdash p^{\sigma,n}(\alpha) = (p^{\sigma,n}(\alpha))(\sigma(\alpha), n)$ (as in Definition 5) and for $\alpha \in \mathrm{dom}(p) \setminus F$, $p^{\sigma,n} \restriction \alpha \Vdash p^{\sigma,n}(\alpha) = p(\alpha)$.*

As shown by Laver we have the following.

Proposition 15. *For each finite $F \subset \mathrm{dom}(p)$ and $n \in \omega$, the set $\{p^{\sigma,n} : \sigma : F \to n\}$ is an antichain that is pre-dense below p.*

Definition 16. *For $n \in \omega$, $q \in \mathbb{P}_\mu$, and finite $F \subset \mathrm{dom}(q)$, define $p <_F^n q$ to mean $p < q$ and for all $\alpha \in F$, $p \restriction \alpha \Vdash p(\alpha) <_n q(\alpha)$ (i.e. the stem of $(p(\alpha))(i,n)$ equals that of $(q(\alpha))(i,n)$ for each $i \leq n$.*

Analogous to Lemma 7 we have the following technique for construction of new conditions.

Corollary 17. *Let $p \in \mathbb{P}_\mu$, $n \in \omega$ and let F be a finite subset of $\mathrm{dom}(p)$. Given any set $\{q'(\sigma) : \sigma \in n^F\} \subset \mathbb{P}_\mu$ satisfying, for each $\sigma \in n^F$, $q'(\sigma) <_F^0 p^{\sigma,n}$, then there is a $q <_F^n p$ such that, for each $\sigma \in n^F$, $q^{\sigma,n} = q'(\sigma)$.*

This next lemma is basically due to Laver and condition (3) can be applied to names of the form $\dot{\ell}(\dot{U})$ from Definition 13.

Lemma 18. *Let $\alpha < \delta \leq \mu$ and let $\dot{m}$ be a $\mathbb{P}_\delta$-name of an integer. For any $p \in \mathbb{P}_\delta$, there are $q \in \mathbb{P}_\delta$ and $\dot{n}$ such that*

1. *$q \restriction \alpha = p \restriction \alpha$ and $q <_\alpha^0 p$,*
2. *$\dot{n}$ is a $\mathbb{P}_{\alpha+1}$-name, and*
3. *$q \Vdash \dot{n} = \dot{m}$*

Proof. Let $p \restriction \alpha \in G_\alpha$ be a $\mathbb{P}_\alpha$-generic filter. Let T be $\mathrm{val}_{G_\alpha}(p(\alpha)) \in \mathbb{L}$. In $V[G_\alpha]$, there is a dense set of conditions that decide the value of $\dot{m}$. Let S be the set of mininal $s \in \mathrm{Br}(T)$ such that there is some $q_s \in \mathbb{P}_\delta$ such that $q_s \in G_\alpha$, $q_s(\alpha)$ is a stem preserving extension of T_s, and $q_s < p$ forces a value on $\dot{m}$. The definition of q is that is that $p \restriction \alpha$ forces that $q(\alpha) = \bigcup\{q_s(\alpha) : s \in S\}$ and, for each $s \in S$, $q_s(\alpha)$

forces that $q \restriction [\alpha+1,\delta)$ is $q_s \restriction (\alpha,\delta)$. In other words, $q \restriction (\alpha,\delta)$ is a $\mathbb{P}_{\alpha+1}$-name as described. $\square$

The proof of Lemma 18 motivates the following definitions and corollary.

Definition 19. *For $\alpha < \delta \leq \mu$ and $\mathbb{P}_\delta$-name $\dot{\ell}$ of an integer, we say that $q \in \mathbb{P}_\delta$ forces that a $\mathbb{P}_\alpha$-name $\dot{m}$ is a $\mathbb{P}_\alpha$-name for $\dot{\ell}$ if $q \Vdash \dot{m} = \dot{\ell}$.*

We will say that q strongly forces that $\dot{m}$ is a $\mathbb{P}_{\alpha+1}$-name for $\dot{\ell}$ if q forces that $\dot{m}$ is a $\mathbb{P}_{\alpha+1}$-name for $\dot{\ell}$ and there is a $\mathbb{P}_\alpha$-name $\dot{S}$ such that $q \restriction \alpha$ forces that $\dot{S}$ is a front of $q(\alpha)$ satisfying that, for each $\mathbb{P}_\alpha$-name $\dot{s}$, if $q\restriction\alpha \Vdash \dot{s} \in \dot{S}$, then $q\restriction\alpha^\frown(q(\alpha))_{\dot{s}}$ forces that $\dot{m}$ (and $\dot{\ell}$) is a $\mathbb{P}_\alpha$-name.

Corollary 20. *Let $p \in \mathbb{P}_\mu$, $n \in \omega$, and finite $F \subset \mathrm{dom}(p) \cap \alpha$. For every $\alpha < \delta \leq \mu$ and $\mathbb{P}_\delta$-name $\dot{\ell}$ of an integer, there is a $q <^n_{F\cup\{\alpha\}} p$ and a $\mathbb{P}_{\alpha+1}$-name $\dot{m}$ such that q strongly forces that $\dot{m}$ is a $\mathbb{P}_{\alpha+1}$-name for $\dot{\ell}$.*

Definition 21. *A sequence $\langle p_n, F_n : n \in \omega\rangle$ is a $\mathbb{P}_\mu$ fusion sequence providing*

1. *$\{p_n : n \in \omega\} \subset \mathbb{P}_\mu$,*
2. *for each n, F_n is a finite subset of $\mathrm{dom}(p_n)$ and $p_{n+1} <^n_{F_n} p_n$*
3. *the family $\{F_n : n \in \omega\}$ is increasing*
4. *$\bigcup\{F_n : n \in \omega\}$ is equal to $\bigcup\{\mathrm{dom}(p_n) : n \in \omega\}$.*

Proposition 22 (Laver)**.** *If $\langle p_n, F_n : n \in \omega\rangle$ is a $\mathbb{P}_\mu$ fusion sequence, then there is a condition $p_\omega \in \mathbb{P}_\mu$ satisfying that $p_\omega <^n_{F_n} p_n$ for all $n \in \omega$.*

In this next definition we introduce our tool for analyzing the interaction between the forcing and $\mathsf{st}(B)$.

Definition 23. *For each $p \in \mathbb{P}_\mu$ and $F \in [\mu]^{<\aleph_0}$, let*

$$\dot{\mathcal{U}}^p_F = \{U \in B : (\exists q) q <^0_F p \quad q \Vdash U \in \dot{\mathcal{U}}\}$$

and define, for each $m \in \omega$, $(\dot{\mathcal{U}}_m)^p_F$ similarly. Also let $(\dot{\mathcal{U}}^p_F)^*$ (similarly for $\dot{\mathcal{U}}_m$) denote the set of ultrafilters $\mathcal{W}$ (in whatever model) on B satisfying that $\mathcal{W} \subset \dot{\mathcal{U}}^p_F$.

Remark 24. Since $\dot{\mathcal{U}}$ is a $\mathbb{P}_\lambda$-name, $\dot{\mathcal{U}}^p_F = \dot{\mathcal{U}}^{p\restriction\lambda}_{F\cap\lambda}$.

We omit the proof of these obvious relationships in this next result.

Proposition 25. *If $q <^0_F p$ then $\dot{\mathcal{U}}^q_F \subset \dot{\mathcal{U}}^p_F$ and $(\dot{\mathcal{U}}^q_F)^* \subset (\dot{\mathcal{U}}^p_F)^*$.*

Lemma 26. *For all $p \in \mathbb{P}_\lambda$ and $F \in [\lambda]^{<\aleph_0}$, $(\dot{\mathcal{U}}^p_F)^*$ is a compact subset of $\mathsf{st}(B)$, and for all $U \in \dot{\mathcal{U}}^p_F$, there is a $\mathcal{W}_U$ in $(\dot{\mathcal{U}}^p_F)^*$ with $U \in \mathcal{W}_U$.*

Proof. To show that $(\dot{\mathcal{U}}^p_F)^*$ is compact, we just note that its complement is open. Indeed, if $\mathcal{W} \in \mathsf{st}(B)$ and $W \in \mathcal{W} \setminus \dot{\mathcal{U}}^p_F$, then evidently the clopen set of ultrafilters that include W is disjoint from $(\dot{\mathcal{U}}^p_F)^*$. Now suppose that $U \in \dot{\mathcal{U}}^p_F$ and choose $q <^0_F p$ such that $q \Vdash U \in \dot{\mathcal{U}}$. Assume there is no $\mathcal{W}_U$ as in the Lemma. Then, for all $U \in \mathcal{W} \in \mathsf{st}(B)$ there is some $W_{\mathcal{W}} \in \mathcal{W} \setminus \dot{\mathcal{U}}^p_F$. By the compactness of the clopen set of ultrafilters that include the set U, there is a finite set of such ultrafilters $\{\mathcal{W}_i : i < \ell\}$ such that $\bigcup\{W_{\mathcal{W}_i} : i < \ell\}$ contains a dense subset of U (i.e. the join in B is greater or equal to U). By Lemma 18, there is a $q' <^0_F q <^0_F p$ such that, for each $i < \ell$, q' has decided the truth value of $W_{\mathcal{W}_i} \in \dot{\mathcal{U}}$. Since $q' \Vdash U \in \dot{\mathcal{U}}$, we have our contradiction, since q' must force that $\dot{\mathcal{U}} \cap \{W_{\mathcal{W}_i} : i < \ell\}$ is not empty. $\square$

This next result uses that we are proceeding by induction on $\lambda > 1$.

Lemma 27. *For all $p \in \mathbb{P}_\lambda$ and $F \in [\lambda]^{<\aleph_0}$, the set $(\dot{\mathcal{U}}^p_F)^*$ is infinite.*

Proof. Suppose that $(\dot{\mathcal{U}}^p_F)^*$ is finite and choose $\alpha \in F$ maximal so that, in some $V[G_\alpha]$ (with $p \restriction \alpha \in G_\alpha$), $(\dot{\mathcal{U}}^{p\restriction[\alpha,\lambda)}_{F\setminus\alpha})^*$ is contained in $(\dot{\mathcal{U}}^p_F)^*$. Let $T = \mathrm{val}_{G_\alpha}(p(\alpha))$. For each $t \in T$, let p^t denote the extension of $p \restriction [\alpha, \lambda)$ where $p^t(\alpha) = T_t$ and $p^t(\beta) = p(\beta)$ for $\alpha < \beta \in \mathrm{dom}(p)$. Let $t_T \subseteq t \in T$ and we check that $(\dot{\mathcal{U}}^{p^s}_F)^*$ is contained $(\dot{\mathcal{U}}^{p^t}_{F\setminus\alpha})^*$ for all but finitely many immediate successors $s \in T$ of t. Given any such t assume that there is an infinite set $S \subset T$ of immediate successors of t such that, for each $s \in S$, $(\dot{\mathcal{U}}^{p^s}_F)^*$ is not contained in $(\dot{\mathcal{U}}^{p^t}_{F\setminus\alpha})^*$. For each $s \in S$, choose any $\mathcal{W}_s \in (\dot{\mathcal{U}}^{p^s}_F)^* \setminus (\dot{\mathcal{U}}^{p^t}_{F\setminus\alpha})^*$. Since $(\dot{\mathcal{U}}^p_F)^*$ is finite, and, by induction, the sequence $\{\mathcal{W}_s : s \in S\}$ has no converging subsequences, so there is an infinite $S' \subset S$ and a $U \in B$ (with complement W) such that $U \in \mathcal{W}$ for all $\mathcal{W} \in (\dot{\mathcal{U}}^p_F)^*$ and $W \in \mathcal{W}_s$ for all $s \in S'$. Note that p^t forces that W is not in $\dot{\mathcal{U}}$. For each $s \in S'$ choose $q_s <^0_{F\setminus\alpha} p^s$ such that $q_s \Vdash W \in \dot{\mathcal{U}}$. There is a $q <^0_{F\setminus\alpha} p^t$ such that S' is the set of immediate successors of t in the tree $q(\alpha)$ and, borrowing the notation from p^t, $q^s = q_s$ for all $s \in S'$. Evidently q forces that $W \in \dot{\mathcal{U}}$ which contradicts that p^t forces that $W \notin \dot{\mathcal{U}}$.

Now it follows, by a simple pruning, that we can choose $p_1 <^0_F p$ such that $p_1 \restriction \alpha \in G_\alpha$ and $(\dot{\mathcal{U}}^{p_1^t}_{F\setminus\alpha})^*$ is a subset of $(\dot{\mathcal{U}}^p_F)^*$ for all $t \in T$. Let $G_{\alpha+1} \supset G_\alpha$ be any generic filter with $p_1 \restriction \alpha{+}1 \in G_{\alpha+1}$. By maximality of α, there is some q such that $q \restriction \alpha + 1 \in G_{\alpha+1}$, $q \restriction [\alpha + 1, \lambda) <^0_{F\setminus\alpha+1} p$, and some $U \in B$ such that $q \Vdash U \in \dot{\mathcal{U}}$ and $U \notin \mathcal{W}$ for all $\mathcal{W} \in (\dot{\mathcal{U}}^p_F)^*$. Even if $\alpha = \max(F)$ this makes sense. But now, we can assume there is a $t \in T$ such that $q \restriction \alpha \Vdash t = t^{q(\alpha)}$ and this, by Lemma 26, contradicts that $(\dot{\mathcal{U}}^{p^t}_{F\setminus\alpha})^* \setminus (\dot{\mathcal{U}}^p_F)^*$ is empty since $U \in (\dot{\mathcal{U}}^{p^t}_{F\setminus\alpha})^*$. □

Lemma 28. *For each $p \in \mathbb{P}_\mu$, $n \in \omega$ and $F \in [\mu]^{<\aleph_0}$ and U such that $p \Vdash U \in \dot{\mathcal{U}}$, there is a $W \in B$ and a $q <^n_F p$ such that, $\overline{W} \subset U$, $q \Vdash W \in \dot{\mathcal{U}}$, and for all $m \leq n$ and $\sigma : F \to n$, if $(\dot{\mathcal{U}}^{q^{\sigma,n}}_F)^*$ and $((\dot{\mathcal{U}}_m)^{q^{\sigma,n}}_F)^*$ are disjoint then $q^{\sigma,n} \Vdash W \notin \dot{\mathcal{U}}_m$.*

Proof. Let Σ denote the set of $\sigma : F \to n$. We may assume that, for each $\sigma \in \Sigma$ and $m \in \omega$, either $(\dot{\mathcal{U}}^{p^{\sigma,n}}_F)^*$ and $((\dot{\mathcal{U}}_m)^{p^{\sigma,n}}_F)^*$ are disjoint or for all $q <^0_F p$, $(\dot{\mathcal{U}}^{q^{\sigma,n}}_F)^*$ and $((\dot{\mathcal{U}}_m)^{q^{\sigma,n}}_F)^*$ are not disjoint. By applying Lemma 26 to each $p^{\sigma,n}$, we can ensure that, for each $\sigma \in \Sigma$, there is a $U(\sigma) \in \dot{\mathcal{U}}^{p^{\sigma,n}}_F$ such that $\overline{U(\sigma)} \subset U$. Let U_1 equal $\bigcup\{U(\sigma) : \sigma \in \Sigma\}$ and so long as we choose W for the statement of the lemma so that $W \subset U_1$, we will have that $\overline{W} \subset U$.

For each $\sigma \in \Sigma$, let L_σ be the set of $m < n$ such that $(\dot{\mathcal{U}}^{p^{\sigma,n}}_F)^*$ and $((\dot{\mathcal{U}}_m)^{p^{\sigma,n}}_F)^*$ are disjoint. Let Σ_0 be the set of $\sigma \in \Sigma$ such that L_σ is not empty. Of course if Σ_0 is empty there is nothing to do. Otherwise, choose $\sigma_0 \in \Sigma_0$ and let m be the minimal element of L_{σ_0}. Pick any $\mathcal{W}^{\sigma_0}_m \in ((\dot{\mathcal{U}}_m)^{p^{\sigma_0,n}}_F)^*$ and pick $W^{\sigma_0}_m \in \mathcal{W}^{\sigma_0}_m \setminus \dot{\mathcal{U}}^{p^{\sigma_0,n}}_F$. For each $\sigma_0 \neq \sigma \in \Sigma$, we may, by $<^0_F$-extending $p^{\sigma,n}$, force that some member of $\mathcal{W}^{\sigma_0}_m$ is not in $\dot{\mathcal{U}}$. This uses that $(\dot{\mathcal{U}}^{p^{\sigma,n}}_F)^*$ is infinite (and so not equal to $\mathcal{W}^{\sigma_0}_m$). Shrink $W^{\sigma_0}_m$ so as to be the intersection of all these finitely many elements. Also extend $p^{\sigma_0,n}$ so as to force that $W^{\sigma_0}_m \in \dot{\mathcal{U}}_m$. We now have that p forces that $W^{\sigma_0}_m \notin \dot{\mathcal{U}}$. We can also assume that (the current value of p) has the property that, for all $\sigma \in \Sigma$, $p^{\sigma,n}$ has decided the truth value of $W^{\sigma_0}_m \in \dot{\mathcal{U}}_k$ for all $k < n$. Let m_1 be the smallest value of L_{σ_0} (if there is one) such that $p^{\sigma_0,n}$ forces that $W^{\sigma_0}_m$ is not in $\dot{\mathcal{U}}_{m_1}$. Similarly choose $\mathcal{W}^{\sigma_0}_{m_1} \in ((\dot{\mathcal{U}}_{m_1})^{\sigma_0,n}_F)^*$ and $W^{\sigma_0}_{m_1} \in \mathcal{W}^{\sigma_0}_{m_1} \setminus \dot{\mathcal{U}}^{p^{\sigma_0,n}}_F$ and a $<^n_F$-extension of p that forces $W^{\sigma_0}_{m_1} \notin \dot{\mathcal{U}}$ and $p^{\sigma_0,n} \Vdash W^{\sigma_0}_{m_1} \in \dot{\mathcal{U}}_{m_1}$. Continue this process until we have a W^{σ_0} and p such that $p \Vdash W^{\sigma_0} \notin \dot{\mathcal{U}}$ and $p^{\sigma_0,n} \Vdash W^{\sigma_0} \in \dot{\mathcal{U}}_k$ for all $k \in L_{\sigma_0}$. We also ensure that for all $\sigma \in \Sigma$ and all $k < n$, $p^{\sigma,n}$ decides the truth value of $W^{\sigma_0} \in \dot{\mathcal{U}}_k$. Next choose $\sigma_1 \in \Sigma_0$ such that, for some $k \in L_{\sigma_1}$, $p^{\sigma_1,n} \Vdash W^{\sigma_0} \notin \dot{\mathcal{U}}_k$, and simply continue. In the end, the value of W is the intersection of U_1 with the complement in B of the union of the chosen W^{σ_i}'s for $\sigma \in \Sigma_0$. $\square$

We are ready to construct a fusion sequence that will produce a condition that will force that the sequence $\{\dot{\mathcal{U}}_n : n \in \omega\}$ does not converge to $\dot{\mathcal{U}}$. At the beginning of the section we chose a countable elementary submodel M and an $(M, \mathbb{P}_\mu)$-generic condition p_0. Now choose a countable elementary submodel M_1 of $H(\aleph_2)$ such that M and p_0 are elements of M_1. Fix an enumeration $\{\dot{\ell}_m : m \in \omega\}$ of all the elements of M_1 that are $\mathbb{P}_\mu$-names for integers. Of course if $\dot{\ell} \in M_1$ is a $\mathbb{P}_\delta$-name for an integer for some $\delta < \mu$, then $\dot{\ell}$ is in the list $\{\dot{\ell}_m : m \in \omega\}$.

Lemma 29. *There is a $\mathbb{P}_\mu$ fusion sequence $\{p_n, F_n : n \in \omega\} \subset M_1$ together with a sequence $\{W_n : n \in \omega\} \subset B$ so that, for each $n \in \omega$,*

1. *$\bigcup_{n\in\omega} F_n = M_1 \cap \mu$,*
2. *$W_0 = 2^\omega$ and $p_n \Vdash W_n \in \dot{\mathcal{U}}$,*
3. *$q = p_{n+1}$ and $W = W_{n+1}$ satisfy the conclusion of Lemma 28 for $p = p_n$ and $U = W_n$,*
4. *if $\alpha < \delta$ are successive elements of $F_n \cup \{\mu\}$ and $\dot{\ell}_m$ $(m \le n)$ is a $\mathbb{P}_\delta$-name, then $p_{n+1} \restriction \delta$ strongly forces that $\dot{\ell}_m$ has a $\mathbb{P}_{\alpha+1}$-name (in the list $\{\dot{\ell}_k : k \in \omega\}$),*
5. *if $p^{\sigma,n}_{n+1}$ has a $<^0_{F_n}$-extension forcing a value on $\dot{\ell}_m$ $(m \le n)$, then $p^{\sigma,n}_{n+1}$ forces a value on $\dot{\ell}_m$.*

Also let $p_\omega \in \mathbb{P}_\mu$ be chosen so that $p_\omega <^n_{F_n} p_n$ for all $n \in \omega$.

The proof is a standard fusion argument that can probably be left to the reader.

Lemma 30. *If $q < p_\omega$ forces that $\dot{\ell}(\dot{U}_m) = k_q$, then there are $\bar{q} \le q$, an $n \ge \max(m, k_q)$, and a $\sigma : F_n \to n$ such that $\bar{q} < (p_{n+1})^{\sigma,n}_{F_n}$ and $(p_{n+1})^{\sigma,n}_{F_n}$ also forces that $\dot{\ell}(\dot{U}_m) = k_q$.*

Proof. To prove the Lemma we prove a seemingly more general statement. If $\delta \le \mu$ is in M_1, $m \in \omega$, and $q \in \mathbb{P}_\delta$ satisfies that $q < p_\omega \restriction \delta$ and $q \Vdash \dot{\ell} = \bar{k}$ for some $\mathbb{P}_\delta$-name $\dot{\ell} \in M_1$, then there is an $m < n \in \omega$ and a $\sigma : F_n \to n$ such that q is compatible with $p^{\sigma,n}_{n+1}$ and such that $p^{\sigma,n}_{n+1} \Vdash \dot{\ell} = \bar{k}$. We can prove this statement by induction on $\delta \in M_1$. We skip the trivial argument for the base case $\delta = 1$.

If $1 < \delta < \mu$ we may assume that m is large enough so that $\delta \in F_m$. Now choose any $m < m_0 \in \omega$ so that $\dot{\ell} \in \{\dot{\ell}_k : k \le m_0\}$. Let β_0 be the maximum value of $F_{m_0} \cap \delta$. Then, by Lemma 29, p_{m_0+1} strongly forces that $\dot{\ell}$ has a $\mathbb{P}_{\beta_0+1}$-name. Let $\dot{S}_0 \in M_1$ denote the $\mathbb{P}_{\beta_0}$-name of the front as in Definition 19. By extending q we may assume that there is a $\mathbb{P}_{\beta_0}$-name $\dot{s}_0 \in M_1$ such that $p_{m_0+1} \Vdash \dot{s}_0 \in \dot{S}_0$ and that there is $t_0 \in \omega^{<\omega}$ such that $q \restriction \beta_0$ forces that $t_0 = \dot{s}_0$ and is the stem of $q(\beta_0)$. Note that $q \restriction \beta_0$ forces that t_0 is a branching node of $p_\omega(\beta_0)$. Therefore, we may also assume there is a $k_0 \in \omega$ so that $q \restriction \beta_0$ forces that t_0 is the k_0-th element in the canonical enumeration of the branching nodes of $p_\omega(\beta_0)$. Let $\bar{m}_0$ be chosen so that $\dot{\ell}_{\bar{m}_0}$ is the $\mathbb{P}_{\beta_0}$-name (of an integer) that is forced to equal $\dot{\ell}$ by the condition $p_{m_0+1} \restriction \beta_0^\frown (p_{m_0+1}(\beta_0))_{\dot{s}_0}{}^\frown p_{m_0+1} \restriction (\beta_0, \delta)$.

Now let m_1 equal $\max\{m_0+1, \bar{m}_0, k_0\}$ and let $\beta_1 = \max(F_{m_1} \cap \beta_0)$. Choose any extension q_1 of q so that $q_1 \Vdash q_1(\beta_0) <_0 q(\beta_0)$ and so that there is a $\sigma_1 : F_{m_1} \to m_1$ such that $q_1 < p^{\sigma_1, m_1}_{m_1+1} \restriction \beta_0 + 1$. Since $p_\omega <^{m_1}_{F_{m_1}} p_{m_1+1}$, it follows that $\sigma_1(\beta_0) = k_0$ and that $q_1 \Vdash \dot{\ell}_{\bar{m}_0} = \bar{k}$. Now we apply the induction hypothesis for the pair $q_1 \restriction \beta_0$ and $\dot{\ell}_{\bar{m}_0}$. We choose $n > m_1$ and $\bar{q} < q_1 \restriction \beta_0$ and $\sigma_1 : F_n \to n$ such that $\bar{q} < p^{\sigma_1,n}_{n+1}$ and $p^{\sigma_1,n}_{n+1}$ forces that $\dot{\ell}_{\bar{m}_0} = \bar{k}$. It now follows that $p^{\sigma_1,n}_{n+1}$ is an extension of $p_{m_0+1} \restriction \beta_0{}^\frown (p_{m_0+1}(\beta_0))_{\dot{s}_0}{}^\frown p_{m_0+1} \restriction (\beta_0, \delta)$ and so also forces that $\dot{\ell} = \bar{k}$ as required. □

Corollary 31. *For each $m \in \omega$, p_ω forces that, for some $n \in \omega$, $W_n \in \dot{\mathcal{U}}$ and $W_n \setminus \overline{W}_{n+1}$ is an element of $\dot{\mathcal{U}}_m$.*

Proof. For each $n \in \omega$, $p_n \Vdash W_n \in \dot{\mathcal{U}}$ and, since $p_\omega < p_n$, we have that $p_\omega \Vdash W_n \in \dot{\mathcal{U}}$. Now fix any $m \in \omega$ and arbitrary $q < p_\omega$. By extending q we may assume that $q \Vdash \dot{\ell}(\dot{U}_m) = k$ for some integer k_q. By Lemma 30, we can choose $m < n \in \omega$ and $\sigma : F_n \to n$ so that q is compatible with $p^{\sigma,n}_{n+1}$ and, such that, $p^{\sigma,n}_{n+1} \Vdash \dot{\ell}(\dot{U}_m) = k$. By the choice of the $\mathbb{P}_\mu$-name $\dot{U}_m$, we have that $p^{\sigma,n}_{n+1}$ forces that $U(M,k) \in B$ is an element of $\dot{\mathcal{U}}$ and its regular open complement (also in B) is element of $\dot{\mathcal{U}}_m$. For readability, let q refer to p_{n+1} and let F refer to F_n (as in Lemma 28). It then follows that $U(M,k) \in \mathcal{W}$ for all $\mathcal{W} \in ((\dot{\mathcal{U}})^{p^{\sigma,n}}_F)^*$ and $U(M,k) \notin \mathcal{W}$ for all $\mathcal{W} \in ((\dot{\mathcal{U}}_m)^{p^{\sigma,n}}_F)^*$. Therefore, by the last clause in Lemma 28, it follows from the choice of W_{n+1} in Lemma 29 that $p^{\sigma,n}_{n+1} \Vdash W_{n+1} \notin \dot{\mathcal{U}}_m$. Naturally it follows from this that $p^{\sigma,n}_{n+1}$ forces that there is a $k \le n$ such that $W_k \setminus \overline{W}_{k+1}$ is an element of $\dot{\mathcal{U}}_m$. □

Proof of Theorem 12. Let $\{p_n, F_n, W_n : n \in \omega\}$ and p_ω be the objects as constructed in Lemma 29. Let $\dot{J} = \{n \in \omega : (\exists m) W_n \setminus \overline{W}_{n+1} \in \dot{\mathcal{U}}_m\}$. We have that $p_\omega \Vdash \dot{J} \in [\omega]^{\aleph_0}$ because of Corollary 31 and the fact that p_0 forces that, for each $n \in \omega$ there is an $m \in \omega$, such that $W_n \in \dot{\mathcal{U}}_m$. Choose $I \subset \omega$ and $q < p_\omega$ so that $q \Vdash |\dot{J} \cap I| = |\dot{J} \setminus I|$ (since the ground model reals remain splitting). By symmetry we may assume (additionally) that $q \Vdash U = \bigcup\{W_n \setminus \overline{W}_{n+1} : n \in I\} \notin \dot{\mathcal{U}}$ (notice that $U \in B$ since the sequence $\{W_n \setminus \overline{W}_{n+1} : n \in I\}$ is in V). Since $q \Vdash \dot{J} \cap I$ is infinite, it follows that $\{m : U \in \dot{\mathcal{U}}_m\}$ is infinite. □

References

[1] W. Brian and A. Dow, *Small cardinals and small Efimov spaces* (2019). submitted.

[2] Alan Dow, *Two classes of Fréchet-Urysohn spaces*, Proc. Amer. Math. Soc. **108** (1990), no. 1, 241–247, DOI 10.2307/2047719. MR975638

[3] A. Dow and D. Fremlin, *Compact sets without converging sequences in the random real model*, Acta Math. Univ. Comenian. (N.S.) **76** (2007), no. 2, 161–171. MR2385028

[4] B. Efimov, *The imbedding of the Stone-Čech compactifications of discrete spaces into bicompacta*, Dokl. Akad. Nauk SSSR **189** (1969), 244–246 (Russian). MR0253290

[5] Sabine Koppelberg, *Boolean algebras as unions of chains of subalgebras*, Algebra Universalis **7** (1977), no. 2, 195–203, DOI 10.1007/BF02485429. MR0434914

[6] K. Kunen, *Weak P-points in* $\mathbf{N}^*$, Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978), Colloq. Math. Soc. János Bolyai, vol. 23, North-Holland, Amsterdam-New York, 1980, pp. 741–749. MR588822

[7] Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs. MR597342

[8] Richard Laver, *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), no. 3-4, 151–169, DOI 10.1007/BF02392416. MR422027

[9] Saharon Shelah, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin-New York, 1982. MR675955

[10] D. Sobota, *The dominating number* dom *and convergent sequences.* MathOverflow question (2015) `http://mathoverflow.net/q/206345`.

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