UNDER CH, A COMPACT SPACE X IS METRIZABLE IF AND ONLY IF $X^2 \setminus \Delta$ IS DOMINATED BY THE IRRATIONALS

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Abstract. In this note we partially answer a question of Cascales, Orihuela and Tkachuk by proving that under CH a compact space X is metrizable provided $X^2 \setminus \Delta$ can be covered by a family of compact sets $\{K_f : f \in \omega^\omega\}$ satisfying that $K_f \subset K_h$ whenever $f \leq h$.

Keywords: Compact space, $\mathbb{P}$-dominated space, Small diagonal

1. Introduction

Definition 1.1. A space X has a $\mathbb{P}$-diagonal if $X^2 \setminus \Delta$ is covered (dominated) by a family of compact sets $\{K_f : f \in \omega^\omega = \mathbb{P}\}$ satisfying that $K_f \subset K_h$ whenever $f \leq h$ coordinatewise.

In [2], B. Cascales, J. Orihuela and V. Tkachuk, using different terminology, proved in ZFC, among many other things, that any compact space with a $\mathbb{P}$-diagonal and countable tightness is metrizable. In the same paper, assuming $\omega_1 < \mathfrak{d}$, they proved that every compact space with a $\mathbb{P}$-diagonal has a small diagonal, hence it is countably tight, and therefore, $\omega_1 < \mathfrak{d}$ implies that every compact space with a $\mathbb{P}$-diagonal is metrizable.

Then they ask if the same conclusion can be obtained in ZFC for every compact space with a $\mathbb{P}$-diagonal.

Clearly the case left to consider is $\omega_1 = \mathfrak{d}$. It is not difficult to show that if there is a non-metrizable compact space with a $\mathbb{P}$-diagonal then its weight cannot exceed ω_1 (see [3])

We will show that, at least, under CH we have a positive result.

2. $\mathbb{P}$-diagonal and converging sequences

Theorem 2.1. *CH implies that every compact space with a $\mathbb{P}$-diagonal is metrizable.*

As mentioned in the introduction, the following ZFC result is proved in [2]. We prove it here to introduce the ideas applied to obtain the later results.

Proposition 2.2. *If X has countable tightness, then X is metrizable.*

Proof. For each $t \in \omega^{<\omega}$, let $K(t) = \bigcup\{K_f : t \subset f\}$. Observe that if $s \geq t$, and $\mathrm{dom}(s) \subset \mathrm{dom}(t)$, then $K(s) \supset K(t)$. This is simply because if $t \subset f$, then $s \oplus f = s \cup f \restriction [\mathrm{dom}(s), \omega)$ satisfies that $K_{s \oplus f} \supset K_f$.

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For each $h \in \omega^\omega$, let $C(h) = \bigcap\{K(h \restriction n) : n \in \omega\}$. We show that the closure of $C(h)$ is disjoint from Δ. Since X^2 has countable tightness, it suffices to consider any sequence $\{y_n : n \in \omega\} \subset C(h)$. Recursively choose $\langle h_n : n \in \omega\rangle$ with $h = h_0 \leq h_1 \leq \cdots$ so that $h_n \restriction n \subset h_{n+1}$, and so that $y_n \in K_{h_{n+1}}$. To do so, observe that since $K(h_n \restriction n+1) \supset K(h \restriction n+1)$, we have that $y_n \in K(h_n \restriction n+1)$. Therefore there is an h_{n+1} with $y_n \in K_{h_{n+1}}$ as required. Let $h_\omega = \bigcup_n h_n \restriction n$ and notice that $\{y_n\}_n \subset K_{h_\omega}$.

Now consider any open $U \subset X^2$ such that the closure of $C(h)$ is contained in U and $U \cap \Delta$ is empty. We claim there is an n such that the closure of $K(h \restriction n)$ is contained in U. Otherwise, perform a similar recursion: choosing $h_n \geq h \restriction n$ and $x_n \in K_{h_{n+1}} \setminus U$. For each n, let $h_\omega(n) = \max\{h_k(n) : k \leq n\}$. We obtain that $\{x_n\}_n \subset K_{h_\omega} \setminus U$. More importantly, we have that for each n, the set $\{x_k\}_{k>n} \subset K_{h\restriction n \oplus h_\omega}$, and so all the limit points are contained in $K(h \restriction n)$. This contradicts that U contains $C(h)$.

It now follows that X has a G_δ-diagonal, since $X^2 \setminus \Delta$ is covered by the collection of all $\overline{K(t)}$ which are disjoint from Δ. □

Now suppose X is a compact space with $\mathbb{P}$-diagonal and uncountable tightness so it contains a convergent free sequence of lenght ω_1 (see [4]) $\{x_\alpha : \alpha \in \omega_1\}$. We may assume that $\{x_\alpha : \alpha \in \omega_1\}$ is dense in X. This means that there is a continuous map from X onto $\omega_1 + 1$. We now show that X also maps continously onto $[0,1]^{\omega_1}$. To do so we will apply some ideas from the investigations into the Moore-Mrowka problem, especially Eisworth's paper [1] on hereditary countable π-character.

Theorem 2.3. *If φ maps a compact space X continuously onto $\omega_1 + 1$, and if X does not map onto $[0,1]^{\omega_1}$, then X does not have $\mathbb{P}$-diagonal.*

Proof. We will work in the subspace $Y = X \setminus \varphi^{-1}(\omega_1) = \varphi^{-1}([0,\omega_1))$. For a subset H of Y, define σH to be the $\aleph_0$-bounded closure of H, that is $\sigma H = \bigcup\{\overline{H_0} : H_0 \in [H]^\omega\}$. Let $\mathcal{F}$ denote any maximal filter of $\aleph_0$-bounded sets such that the family $\{\varphi^{-1}([\alpha,\omega_1)) : \alpha \in \omega_1\}$ is contained in $\mathcal{F}$. Such a filter exists simply by Zorn's Lemma. It is easy to verify that $\mathcal{F}$ is closed under countable intersections.

We say that $H \in \mathcal{F}^+$ providing $H \cap F$ is not empty for all $F \in \mathcal{F}$. Notice that if $H \in \mathcal{F}^+$, then $\sigma H \in \mathcal{F}$. We will now explore how the members of $\mathcal{F}$ interact with the family $\{K_f : f \in \omega^\omega\}$. Let π_2 denote the projection map from $Y \times Y$ onto the second coordinate – thus we will be focussing on the upper triangle in Y^2.

For $F \in \mathcal{F}$ and $t \in \omega^{<\omega}$, define

$$F(t) = \{x \in F : \sigma(\pi_2\ [K(t) \cap (\{x\} \times F)]) \in \mathcal{F}\}$$

For each $t \in \omega^{<\omega}$ choose, if possible, $F_t \in \mathcal{F}$ so that $F_t(t) \notin \mathcal{F}^+$. Let $F_0 \in \mathcal{F}$ be contained in each such F_t.

Now choose any countable elementary submodel $M \prec H(\theta)$, where θ is any sufficiently large regular cardinal and $H(\theta)$ denotes the family of sets which are hereditarily of cardinality less than θ. Sufficiently large just means here that X is based on some ordinal λ and $|\mathcal{P}(\mathcal{P}(\lambda))| < \theta$. We of course want that $\varphi, X, \mathcal{F}$ and $\{K_f : f \in \omega^\omega\}$ are all elements of M. One can assume that F_0 is also in M or simply carry out the selection of the F_t's within M.

Now we define Z to be $\bigcap\{\overline{F \cap M} : F \in \mathcal{F} \cap M\}$.

Choose any $z \in Z$ and $y \in F_0 \cap M$. Notice that $z \notin M$ and so $(y, z) \in X^2 \setminus \Delta$. Choose any $h_0 \in \omega^\omega$ so that $(y, z) \in K_{h_0}$. What is important is the properties of h_0.

Choose any $t \geq h_0 \upharpoonright \mathrm{dom}(t)$ (hence $(y, z) \in K(t)$). Now let $H_y = \pi_2[K(t) \cap (\{y\} \times F)]$ and notice that H_y and σH_y are in M. If $\sigma H_y \notin \mathcal{F}$, then there is an $F_2 \in \mathcal{F} \cap M$ such that $\sigma H_y \cap F_2$ is empty. However, $z \in \sigma(F_2 \cap M) \subset F_2$ and also $z \in H_y$ which cannot happen, thus $\sigma H_y \in \mathcal{F}$. Also, we must have that F_t did not exist, otherwise F_0 is contained in it and H_y is even smaller than $\pi_2(K(t) \cap (\{y\} \times F_t)$ and so F_t existing means that $\sigma(H_y)$ is not in $\mathcal{F}$– but it is!

Let us check that even more is true about $t \geq h_0 \upharpoonright \mathrm{dom}(t)$:
Choose any $F \in \mathcal{F} \cap M$ and any open $W \subset X$ such that $W \cap Z$ is not empty. Choose any $z_1 \in W \cap Z$. Then we claim there is a $y_1 \in W \cap F \cap M$ such that $W \cap (H_{y_1} \cap M)$ is also not empty. As before, we may assume that $F \subset F_0$ and we know that F_t did not exist. This means that $F(t) \in \mathcal{F}^+$, and so $\sigma F(t)$ is in $\mathcal{F}$. This of course means that Z is contained in the closure of $M \cap \sigma F(t)$. By elementarity, $M \cap \sigma F(t)$ is contained in $\sigma(M \cap F(t))$. So we may choose some $y_1 \in W \cap M \cap F(t)$.
Again let $H_{y_1} = \pi_2[K(t) \cap (\{y_1\} \times F)]$ and it is easily shown that z is in the closure of $M \cap \sigma H_{y_1}$. But again, by elementarity, it follows that z_1 is in the closure of $M \cap H_{y_1}$, and we have that $W \cap M \cap H_{y_1}$ is not empty, as required.

The conclusion we want is that if $t \geq h_0 \upharpoonright \mathrm{dom}(t)$, $F \in \mathcal{F} \cap M$, and open an W meets Z, then there is point $(y_1, y_2) \in K(t) \cap M \cap (W \cap F)^2$.

Since X does not map onto $[0,1]^{\omega_1}$ we may assume that every closed subset K of X contains a point which has countable π-character in K (see ??)
So, let us now choose a point $x \in Z$ which has countable π-character in Z. Let $\{U_n, W_n : n \in \omega\}$ be open subsets of X satisfying that, for each n, $\overline{W_n} \subset U_n$, and such that the family $\{U_n \cap Z : n \in \omega\}$ is a local π-base for x in Z. Also ensure that $W_n \cap Z$ is non-empty for each n. For convenience, we assume that each pair U_n, W_n is listed infinitely many times.
Begin our (by now) standard recursive construction of a sequence of functions $\{h_n : n \in \omega\}$ so that $h_{n+1} \geq h_n$ and $h_{n+1} \supset h_n \upharpoonright n$. Also, let $\{F_n : n \in \omega\}$ be an enumeration for a descending base for $M \cap \mathcal{F}$. Choose h_{n+1} so that there is a pair $(y_1^n, y_2^n) \in K(h_n \upharpoonright n) \cap M \cap (W_n \cap F_n)^2$ as discussed above. Let $h_\omega = \bigcup_n h_n \upharpoonright n$, hence $h_\omega \geq h_n$ for all n.

Consider a pair U_k, W_k which was listed infinitely often. Let $L_k = \{n : (U_n, W_n) = (U_k, W_k)\}$. The sequence $\{(y_1^n, y_2^n) : n \in L_k\}$ accumulates to some point (z_1^k, z_2^k) which is in $(\overline{W_n} \cap Z)^2$. To see this, it is enough to notice that every limit point of the entire set $\{y_1^n, y_2^n : n \in \omega\}$ is in Z because a cofinite subset of it is contained in $F_\ell \cap M$ for each ℓ. Notice then that $(z_1^k, z_2^k) \in (U_k \cap Z)^2$. Since the family $\{(U_k \cap Z)^2 : k \in \omega\}$ is a local π-base at (x, x), we have that (x, x) is in the closure.
Now we have a contradiction since $\{(y_1^n, y_2^n) : n \in \omega\}$ is contained in K_{h_ω}.

□

We now prove that $\beta\omega$ does not have a $\mathbb{P}$-diagonal.

Theorem 2.4. *A compact space with a $\mathbb{P}$-diagonal must contain a converging sequence.*

Proof. Suppose we have a compact space X with no converging sequences. And we assume that $\{K_f : f \in \omega^\omega\}$ is a compact cover of $X^2 \setminus \Delta$.

First notice that for all $x \in X$ and infinite compact $J \subset X$, there is an f so that $K_f \cap (\{x\} \times J)$ is infinite. To see this, simply fix any uncountable $\{y_\alpha : \alpha \in \omega_1\} \subset J \setminus \{x\}$. For each α, choose f_α so that $(x, y_\alpha \in K_{f_\alpha}$. There is an $h \in \omega^\omega$ so that for each n, there is an α_n such that $h \restriction n \subset f_{\alpha_n}$. Now define $f \in \omega^\omega$ so that for each n, $f(n) \geq \max\{f_{\alpha_k}(n) : k \leq n\}$. Of course this means that $f_{\alpha_n} \leq f$ for all n. In which case $(x, y_{\alpha_n}) \in K_f$ for all n.

Similarly (but now using the hypothesis), for each infinite compact $J \subset X$, there is an f so that K_f contains $J_0 \times J_1$ for some infinite compact $L_0, L_1 \subset J$.

To see this, let $\{x_\alpha : \alpha \in \omega_1\}$ be any subset of J (which must be uncountable because it has no converging sequences). By recursion, we choose a descending sequence $\{J_\alpha : \alpha \in \omega_1\}$ of compact infinite sets with $J_0 = J$. We require that, for each α, there is an f_α so that K_{f_α} contains $\{x_\alpha\} \times J_{\alpha+1}$. If J_α is compact infinite, then the existence of f_α and infinite compact $J_{\alpha+1}$ follows from the first claim. For limit α, $J_\alpha = \bigcap\{J_\beta : \beta < \alpha\}$ is infinite because X contains no converging sequences. Now again choose any f so that there is an infinite increasing sequence $\{\alpha_n : n \in \omega\}$ with $f_{\alpha_n} \leq f$ for all n. Let J_0 denote the (infinite) set of limit points of $\{x_{\alpha_n} : n \in \omega\}$, and let $L_1 = \bigcap\{J_{\alpha_n} : n \in \omega\}$. We have that $L_0 \times L_1$ is contained in K_f.

Now specify any indexing $\{t_k : k \in \omega\}$ of $\omega^{<\omega}$. We may as well assume that $t_k \subset t_j$ implies $k < j$. By a countable recursion, choose a descending sequence $\{J_k : k \in \omega\}$ of infinite closed subsets of X with $J_0 = X$. Having chosen J_k, we consider t_k. If there exists some infinite compact $J \subset J_k$ so that for all $f \supset t_k$, we have that K_f does not contain any product $J^0 \times J^1$ with J^0, J^1 being infinite compact subsets of J, then choose J_{k+1} to be such a set. Otherwise, let $J_{k+1} = J_k$, and notice then that there is no such J contained in J_{k+1}.

When this recursion is complete, set $J = \bigcap_k J_k$, and again note that J is an infinite compact subset of X. Choose any h_0 so that K_{h_0} contains $L_0 \times L_1$ for infinite compact L_0, L_1 contained in J. Of course we now know that for any $t_k \geq h_0 \restriction \mathrm{dom}(t_k)$, and every J_{k+1} there is not any $J \subset J_k$ such that for all $f \supset t_k$, we have that K_f does not contain any product $J^0 \times J^1$ with J^0, J^1 being infinite compact subsets of J. So, we recursively choose $h_1 \geq h_0$ with $h_0 \restriction 1 \subset h_1$ and so that there are L_2, L_3 contained in L_0 with $L_2 \times L_3 \subset K_{h_1}$. Continue recursively with $L_{2k+2}, L_{2k+3} \subset L_{2k}$ and $L_{2k+2} \times L_{2k+3} \subset K_{h_{k+1}}$, and $h_k \restriction k \subset h_{k+1}$. Choosing $h \geq h_k$ for all k, we show that K_h will hit Δ. For each k choose $y_k \in L_{2k+1}$ and let y be any limit point of $\{y_k : k \in \omega\}$. Of course $\{y_\ell : \ell > 2k\} \subset L_{2k}$ and so $y \in L_{2k}$ for all k. Similarly, $(y, y_{k+1}) \in L_{2k+2} \times L_{2k+3}$ for all k, which implies that $(y, y) \in K_h$. □

From the previous result we can conclude that a compact space with $\mathbb{P}$-diagonal cannot contain a copy of $\beta\omega$ and therefore it cannot be mapped continuously onto $[0,1]^{\mathfrak{c}}$. We can now use CH and Theorem 2.3 to conclude that Theorem 2.1 is proved.

References

[1] Todd Eisworth, *Countable compactness, hereditary π-character, and the continuum hypothesis*, Topology Appl. **153** (2006), no. 18, 3572–3597, DOI 10.1016/j.topol.2006.03.021. MR2270606 (2008b:03082)

[2] B. Cascales, J. Orihuela, and V. V. Tkachuk, *Domination by second countable spaces and Lindelöf Σ-property*, Topology Appl. **158** (2011), no. 2, 204–214, DOI 10.1016/j.topol.2010.10.014. MR2739891 (2011j:54018)

[3] D. Guerrero Sánchez, *Domination in products*, Topology Appl.
To appear.

[4] I. Juhsz and Z. Szentmiklssy, *Convergent free sequences in compact spaces*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 1153–1160. MR1137223 (93b:54024)

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