PRODUCTIVITY OF CELLULAR-LINDELÖF SPACES

ALAN DOW AND R.M. STEPHENSON, JR.

Abstract. The main purpose of this note is to prove that the product of a cellular-Lindelöf space with a space of countable spread need not be cellular-Lindelöf.

1. Introduction

All hypothesized spaces are Hausdorff, and by *Lindelöf* (*of countable spread*) we shall mean any space in which every open cover has a countable subcover (every discrete subspace is countable); otherwise, the terminology and notation not defined here generally agree with that in [3] or [5].

A. Bella and S. Spadaro in [1] defined a space X to be *cellular-Lindelöf* provided that for every family $\mathcal{U}$ of pairwise disjoint nonempty open subsets of X there is a Lindelöf subspace $L \subset X$ such that $L \cap U \neq \emptyset$ for every $U \in \mathcal{U}$. In that article and also in [2] they derived some properties of cellular-Lindelöf spaces. Likewise, W. Xuan and Y. Song studied cellular-Lindelöf spaces in [15] and several other articles referred to in [15]. A similar family of spaces, the cellular-compact spaces, were introduced and studied extensively by V.V. Tkachuk and R.G. Wilson in [13]. A space X is said to be *cellular-compact* provided that for every family $\mathcal{U}$ of pairwise disjoint nonempty open subsets of X there is a compact subspace $K \subset X$ such that $K \cap U \neq \emptyset$ for every $U \in \mathcal{U}$.

Among the nice properties of these two cellular concepts, most of which were noted or shown in several of the articles above, are: each is preserved by regular closed subsets, extensions, continuous images, and finite unions (and for the case cellular-Lindelöf, countable unions and open F_σ-subsets).

In [15] and especially in [13] its authors proved that dense subspaces of compact product spaces, such as Σ-product subspaces, provide a useful source of examples of non-Lindelöf spaces illustrating properties of cellular-compact or cellular-Lindelöf spaces. In [15] its authors presented several theorems, examples and questions concerning products of cellular-Lindelöf spaces. We list some of these results next, but first note one obvious necessary condition for a product space to be cellular-compact or cellular-Lindelöf, according to the invariance under continuous maps property.

Theorem 1.1. *If a product space is cellular-compact or cellular-Lindelöf, then each of its factor spaces must have the same property.*

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In [10] A.D. Rojas-Sánchez and A. Tamariz-Mascarúa provided a very clever proof that there exist two Lindelöf spaces whose product contains an uncountable clopen discrete subspace D. The authors of [15] referred the reader to [10], re-described that example, re-derived the properties of D, and noted that it establishes the following:

Theorem 1.2. *The product of two Tychonoff Lindelöf spaces need not be cellular-Lindelöf.*

They then obtained the following result.

Theorem 1.3. ([15], Theorem 3.9) *The product of a separable space and a cellular-Lindelöf space is cellular-Lindelöf.*

Proof. We sketch a different proof from the one in [15]: It is immediate that the product of a countable space and a cellular-Lindelöf space is cellular-Lindelöf, and hence any extension of such a space must also be cellular-Lindelöf. □

Noting that this theorem implies the product of a compact metrizable space with a cellular-Lindelöf space is cellular-Lindelöf, the authors of [15] asked:

Question 1.4. *Is the product of a compact space and a cellular-Lindelöf space cellular-Lindelöf?*

In Section 2 we shall provide a negative answer to this question, and we shall answer an analogous question not raised in [13] or [15] about whether or not the property cellular-compact is productive. It will be shown that the product of a convergent sequence with a cellular-compact space need not be cellular-compact.

The authors of [15] stated in their Theorem 3.12 that the product of a cellular-Lindelöf space with a space of countable spread, such as a hereditarily Lindelöf space, is cellular-Lindelöf. However, as noted in the Abstract, the main purpose of this note is to provide a proof that this assertion is not true. In Section 3 the focus will be on properties possible counterexamples will need to possess. A proof will be given in Section 4 showing it is consistent that the product of a cellular-Lindelöf space with a hereditarily Lindelöf space need not be cellular-Lindelöf. Then in Section 5 a construction of Justin Moore will be used to produce very different counterexamples within ZFC.

2. Products with one factor compact

Let us first recall some needed terminology and a theorem. A family $\mathcal{V}$ of nonempty open subsets of a space X is called a *π-base for X* (a *local π-base for a point $x \in X$*) provided that for every nonempty open subset T of X (containing x) there exists $V \in \mathcal{V}$ such that $V \subset T$. The *π-weight* $\pi(X)$ of the space X (π-character $\pi\chi(x, X)$ of the space X at the point x) is the minimal cardinality of a π-base for X (a local π-base at x). The *density* $\mathrm{d}(X)$ of X is the minimal cardinality of a dense subset of X. A point $p \in \beta X \setminus X$ is called a *remote point of a Tychonoff space* X if $p \notin \mathrm{cl}_{\beta X}(Y)$ for any nowhere dense set $Y \subset X$. Finally, (Theorem 3.13 of [13]) if X is a regular cellular-compact space and p is a non-isolated point of X, then the space $X \setminus \{p\}$ is cellular-compact if and only if there exists no pairwise disjoint local π-base at the point p in X. We shall use only a special case of a corollary to this theorem (Corollary 3.14 of [13]), a case which by itself has an immediate proof, and so we give both here.

Corollary 2.1. ([13]) *If K is a compact space, p is a non-isolated point of K, and there exists no pairwise disjoint local π-base at the point p in K, then the space $X = K \setminus \{p\}$ is cellular-compact.*

Proof. Assume $\mathcal{U}$ is a family of pairwise disjoint nonempty open subsets of K. Choose an open neighborhood W of p which does not contain any member of $\mathcal{U}$. Then $K \setminus W = X \setminus W$ is a compact set which meets every member of $\mathcal{U}$. □

Theorem 2.2. *Let $X = \beta\mathbb{Q} \setminus \{p\}$, where p is any remote point of $\mathbb{Q}$, and let $Y = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$. Then the space X is cellular-compact, but the product space $X \times Y$ is not cellular-compact.*

Proof. To prove that X is cellular-compact, it suffices by Corollary 2.1 for us to prove that there is no pairwise disjoint local π-base for p in $\beta\mathbb{Q}$. Let $\mathcal{V}$ be an arbitrary family of pairwise disjoint nonempty open sets in $\beta\mathbb{Q}$. Proceeding as in Example 3.21 of [13], we may (and do) choose, for each $V \in \mathcal{V}$, a rational number $q_V \in V$. Then $D = \{q_V : V \in \mathcal{V}\}$ is discrete, and so the remote point $p \notin \mathrm{cl}_{\beta\mathbb{Q}}(D)$. Hence $\beta\mathbb{Q} \setminus \mathrm{cl}_{\beta\mathbb{Q}}(D)$ is an open neighborhood of p which contains no member of $\mathcal{V}$. Therefore, $\mathcal{V}$ cannot be a local π-base for p in $\beta\mathbb{Q}$.

Next, again using the denseness of $\mathbb{Q}$ in $\beta\mathbb{Q}$, let $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ be a π-base for $\beta\mathbb{Q}$, and for each $n \in \mathbb{N}$, let $U_n = (W_n \setminus \{p\}) \times \{n\}$. Then $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a family of pairwise disjoint nonempty open subsets of $X \times Y$.

Finally, assume K is any compact subset of $X \times Y$. To complete the proof, we need to show that $K \cap U_n = \emptyset$ for some $n \in \mathbb{N}$. Since the projection $\mathrm{pr}_X(K)$ is a compact subset of $\beta\mathbb{Q}$ not containing p, the set $T = \beta\mathbb{Q} \setminus \mathrm{pr}_X(K)$ is an open neighborhood of p in $\beta\mathbb{Q}$ which is disjoint from $\mathrm{pr}_X(K)$. As $\mathcal{W}$ is a π-base for $\beta\mathbb{Q}$, T must contain some W_n. Hence, $K \cap U_n \subset K \cap ((W_n \setminus \{p\}) \times Y) \subset K \cap (T \times Y) = \emptyset$. □

We turn next to producing a negative answer to Question 1.4, and we shall do this by finding a cellular-Lindelöf (actually, another cellular-compact) space X having the form $X = K \setminus \{p\}$, where K is compact, such that the product space $X \times (\kappa+1)$, where $\kappa = \pi\chi(p, X)$, is not cellular-Lindelöf.

Lemma 2.3. *There exists a compact space K having a P-point p such that K has cellularity $\mathfrak{c} < \pi\chi(p, K)$.*

Proof. Consider the P-space Y obtained by endowing the product space $2^{\mathfrak{c}^+}$ with the G_δ-topology. Let p be the constant 0 function in Y and $K = \beta Y$.

We verify that the P-point p of K has $\pi\chi(p, K) = \mathfrak{c}^+$. Since K is an extension space of Y which is regular, it will be enough to check that $\pi\chi(p, Y) = \mathfrak{c}^+$.

Let $\mathcal{A}$ be the family of all countable subsets of $\mathfrak{c}^+$. For each $A \in \mathcal{A}$ let $N_A = \{y \in Y : y(\alpha) = 0 \text{ for all } \alpha \in \mathrm{A}\}$. Then $\{N_A : A \in \mathcal{A}\}$ is a local base for p in Y, so $\pi\chi(p, Y) \leq \pi(p, Y) \leq |\mathcal{A}| = \mathfrak{c}^+$. Assume next that $\mathcal{V}$ is an arbitrary family of basic clopen G_δ-subsets of Y such that $|\mathcal{V}| \leq \mathfrak{c}$. For each $V \in \mathcal{V}$, let D_V be the countable set of restricted coordinates of V, and let $\mathcal{D} = \{D : V_D \in \mathcal{V}\}$. Then $\mathcal{V}$ cannot be a local π-base for p in Y, since for any $\alpha \in \mathfrak{c}^+ \setminus \bigcup \mathcal{D}$, ${\pi_\alpha}^{-1}(0)$ is a neighborhood of p in Y which contains no member of $\mathcal{V}$. Thus, $\pi\chi(p, Y) \geq \mathfrak{c}^+$.

Suppose next that $\{f_\alpha : \alpha < \mathfrak{c}^+\}$ are countable partial functions from $\mathfrak{c}^+$ to 2 coding basic clopen G_δ-sets $\{[f_\alpha] : \alpha < \mathfrak{c}^+\}$ of Y. Let this family be an element of $M \prec H(\mathfrak{c}^+)$, where M has cardinality $\mathfrak{c}$ and $M^\omega \subset M$. Let $M \cap \mathfrak{c}^+ \subset \delta \in \mathfrak{c}^+$, and let $f = f_\delta \restriction M$. Since the domain of f is a countable subset of M, f is in M by our choice of M. Thus there is an $\alpha \in M$ such that $f \subset f_\alpha$. It follows that $f_\alpha \cup f_\delta$ is a function, and therefore that $[f_\alpha] \cap [f_\delta]$ is not empty. This proves that K has cellularity equal to $\mathfrak{c} < \pi\chi(p, K)$. □

Theorem 2.4. *Let X be any space having the form $X = K \setminus \{p\}$, where K is a compact space, p is a P-point of K, and K has cellularity $\mathfrak{c} < \pi\chi(p, K) = \kappa$. Topologize the set $\kappa + 1$ so that it is the one-point compactification of the discrete space κ. Then the space X is cellular-compact, but the product space $X \times (\kappa + 1)$ is not cellular-Lindelöf.*

Proof. It follows from Corollary 2.1 (or Corollary 3.17 of [13]) and the cellularity inequality satisfied by K that the space X is cellular-compact.

Now fix a local π-base $\{U_\alpha : \alpha < \kappa\}$ for p in K. We may assume that $p \notin U_\alpha$ for all $\alpha < \kappa$. For each $\alpha \in \kappa$, the set $W_\alpha = U_\alpha \times \{\alpha\}$ is open in $X \times (\kappa + 1)$, and $\mathcal{W} = \{W_\alpha : \alpha < \kappa\}$ is a pairwise disjoint family of nonempty open sets.

Assume that Y is a subspace of $X \times (\kappa + 1)$ such that $Y \cap W_\alpha \neq \emptyset$ for all $\alpha \in \kappa$. We prove that Y cannot be Lindelöf. For each $y \in Y$, choose a neighborhood W_y of y and subset V_y of X such that $W_y \subset V_y \times (\kappa + 1)$ and the point $p \notin \overline{V_y}$. If there is a countable subcover $\{W_{y_n} : n \in \omega\}$ of Y, then since p is a P-point, we may choose $\alpha < \kappa$ such that $U_\alpha \cap V_{y_n} = \emptyset$ for all n. This implies that Y is then disjoint from W_α. □

Remark 2.5. *For any space X and P-point p of X, if p has character $\aleph_1$, then there is a pairwise disjoint local π-base at p, and hence one can show the space $X \setminus \{p\}$ cannot be cellular-Lindelöf.*

One corollary to Theorems 2.2 and 2.4 which illustrates a difference between properties of the spaces studied here and those in [10] is the following.

Corollary 2.6. *The property cellular-Lindelöf (cellular-compact) is not an inverse invariant of open perfect surjections.*

3. Seeking a counterexample of countable spread

We consider next some properties spaces X and Y would need to have in order to be counterexamples to the assertion that if X is cellular-Lindelöf and Y is of countable spread, then $X \times Y$ is cellular-Lindelöf.

As noted earlier, neither space can be separable. In addition, the following theorem, a corollary to one due to B. Šapirovskiĭ in [12], shows the space Y must have a dense hereditarily Lindelöf subspace.

Theorem 3.1. ([4], Proposition 5.6, or [12]) *Every non-separable space of countable spread has a dense subspace that is hereditarily Lindelöf.*

A tool often found useful in producing counterexamples is the *Alexandroff double* $\mathbb{D}(X)$ *of a topological space* X. We recall that $\mathbb{D}(X) = X \times \{0,1\}$, topologized so that a subset T of $\mathbb{D}(X)$ is defined to be open if and only if for each point $(x,0) \in T$, there is an open subset U of the space X such that $(x,0) \in ((U \times \{0,1\}) \setminus \{(x,1)\}) \subset T$. Three well-known, easily verified properties are: $\mathbb{D}(X)$ is Lindelöf if and only if X is Lindelöf; and since $X \times \{1\}$ is an open discrete subspace, $\mathbb{D}(X)$ is separable (hereditarily Lindelöf) if and only if X is countable.

Theorem 3.2. *Suppose X and Y are regular spaces such that $\mathbb{D}(X) \times Y$ is cellular-Lindelöf, Y has π-weight $\pi(Y) = \kappa > \aleph_0$, and for every nonempty open subset T of X, $\mathrm{d}(T) = \pi(T) = \kappa$. Then $\mathbb{D}(X) \times Y$ has a Lindelöf subspace which is dense in $(X \times \{0\}) \times Y$.*

Proof. A recursive construction shows that there is a family $\{D_\alpha : \alpha < \kappa\}$ of pairwise disjoint dense subsets of X, each having cardinality κ.

Let $\{U_\alpha : \alpha \in \kappa\}$ enumerate a π-base for Y. Set $W_{\alpha,d} = \{(d,1)\} \times U_\alpha$ for all $\alpha < \kappa$ and $d \in D_\alpha$, and $\mathcal{W} = \{W_{\alpha,d} : \alpha \in \kappa \text{ and } d \in D_\alpha\}$. Then $\mathcal{W}$ is a cellular family, so assume that L is a Lindelöf subspace of $\mathbb{D}(X) \times Y$ such that $L \cap W \neq \emptyset$ for all $W \in \mathcal{W}$. We shall prove that L is dense in $(X \times \{0\}) \times Y$.

Fix any nonempty regular closed sets $B \subset X$ and $R \subset Y$. We show that $L \cap ((B \times \{0\}) \times R) \neq \emptyset$.

Since $\{U_\alpha : \alpha \in \kappa\}$ is a π-base for Y, there exists $\beta < \kappa$ such that $U_\beta \subset \mathrm{int}(R)$. As D_β is a dense subset of X and $d(\mathrm{int}(B)) = \kappa$, it follows that $L \cap ((B \times \{0,1\}) \times R)$ meets $W_{\beta,d}$ for κ-many points $d \in D_\beta$. For each such d, pick $((d,1), y_d) \in L \cap W_{\beta,d}$, and note that $((d,1), y_d) \in L \cap ((B \times \{0,1\}) \times R)$, Moreover, $L \cap ((B \times \{0,1\}) \times R)$ is Lindelöf. As this subset of points $((d,1), y_d) \in L \cap ((B \times \{0,1\}) \times R)$ has cardinality $\kappa > \aleph_0$, it must have a limit point in $L \cap ((B \times \{0\}) \times R)$, for every point of $L \cap ((B \times \{1\}) \times R)$ has a neighborhood meeting that subset in at most a single point.

□

Based on the preceding results, we seek as counterexamples spaces X and Y, each nowhere separable and of π-weight $\aleph_1$, so that X is Lindelöf, Y is hereditarily Lindelöf, and $\mathbb{D}(X) \times Y$ has no Lindelöf subspace which is dense in $(X \times \{0\}) \times Y$.

4. Consistent product space counterexamples

Let S be a full-branching Souslin tree (a subtree of $2^{<\omega_1}$). In this section we show how to get X and Y from S. We begin by giving some notation and terminology that will be used.

Let $s, t \in S$ and $A \subset S$. Then $s \leq t$ is assigned its usual meaning (as on p. 68 of [5]), namely, $s \leq t$ if and only if $s \subset t$ if and only if the sequence t extends s, and likewise $s \vee t$, $s \wedge t$, and $s \perp t$ have their usual meanings with respect to the relation $\leq$. In addition, $o(s) \in \omega_1$ denotes the domain of s, $[s]$ denotes $\{u \in S : s \leq u\}$, and A is said to be *cofinal above* s if for each $u \in [s]$, $A \cap [u] \neq \emptyset$. The set A is said to be *cofinal in* S if A is cofinal above s for every $s \in S$. If $o(s) = \xi$ and $i, j \in \{0,1\}$, then $s^\frown i$ denotes the sequence $s \cup \{(\xi, i)\}$ whose domain is $\xi + 1$, and similarly, $s^\frown ij$ denotes

$(s^\frown i)^\frown j$. In case $s \neq t$, we define $\Delta(s,t) = \min\{\alpha \in o(s) \cup o(t) : s(\alpha) \neq t(\alpha)\}$ (using the convention that $s(\alpha) \neq t(\alpha)$ if $\alpha \in (o(s) \setminus o(t)) \cup (o(t) \setminus o(s))$), and denote by $\prec$ the following order: $s \prec t$ if and only if $s(\Delta(s,t)) = 0$ or $t(\Delta(s,t)) = 1$.

The reader familiar with the procedures M.E. Rudin developed on pp. 1116–1118 of [11] may notice that the relation $\prec$ is an example of a general type of relation presented and studied in [11] for the purpose of demonstrating how to use a "normalized and pruned" Souslin tree and Dedekind completion to produce a Souslin line. We include for later reference statements and proof outlines illustrating the properties of $\prec$ needed for our examples. Some of these properties were derived or pointed out in [11].

Claim 4.1. *The relation $\prec$ is a dense total order on S.*

Proof. First we prove $\prec$ is transitive. Let s, t, and $u \in S$ satisfy $s \prec t$ and $t \prec u$, and let $\alpha_0 = \Delta(s,t)$ and $\beta_0 = \Delta(t,u)$. We consider three cases.

Suppose $\alpha_0 = \beta_0$. Then $s \restriction \alpha_0 = t \restriction \alpha_0 = u \restriction \alpha_0$, and the only possible values for s, t, and u at α_0 in this case are $s(\alpha_0) = 0$, $t(\alpha_0)$ is not defined, and $u(\alpha_0) = 1$, which together imply $s \prec u$.

Suppose $\alpha_0 < \beta_0$. Then $s \restriction \alpha_0 = t \restriction \alpha_0 = u \restriction \alpha_0$. If $s(\alpha_0) = 0$ or is undefined, then $t(\alpha_0) = 1$, and $t \restriction \beta_0 = u \restriction \beta_0$ implies $u(\alpha_0) = 1$, and so $s \prec u$.

Assume $\beta_0 < \alpha_0$. Then $s \restriction \beta_0 = t \restriction \beta_0$, $\beta_0 \in o(t)$, and $s(\beta_0) = t(\beta_0)$. Since also $t \prec u$, we must have $t(\beta_0) = 0$. Then $s(\beta_0) = 0$, $s \restriction \beta_0 = u \restriction \beta_0$, and since $t \prec u$, either $u(\beta_0) = 1$ or $u(\beta_0)$ is not defined. Hence $s(\beta_0) = 0 \neq u(\beta_0)$, and $s \prec u$.

Obviously the partial order $\prec$ is a total order. To prove it is a dense order, let s and t be as above. We find a sequence $m \in S$ such that $s \prec m \prec t$, depending on which of the three possibilities occurs.

If $s(\alpha_0) = 0$ and $t(\alpha_0) = 1$, let $m = s \restriction \alpha_0$.

Suppose $s(\alpha_0) = 0$ and $t(\alpha_0)$ is not defined. Find the first $\beta \geq \alpha_0 + 1$ where $s(\beta) = 0$, and set $m = (s \restriction \beta)^\frown 1$; and if no such β exists, set $m = s^\frown 1$.

Assume $s(\alpha_0)$ is not defined and $t(\alpha_0) = 1$. Find the first $\beta \geq \alpha_0 + 1$ where $t(\beta) = 1$, and set $m = (t \restriction \beta)^\frown 0$; and if no such β exists, let $m = t^\frown 0$. $\square$

Before stating our next claim, some additional notation needed is, for any $t \in S$: let $t^\dagger = t$ if t is on a limit level of S; and let $t^\dagger$ denote the "other" $<$-successor of $t \restriction \alpha$ in S if $o(t) = \alpha + 1$.

Claim 4.2. *Let $s, t \in S$ with $o(s) + 1 < o(t)$.* (i) *If $s \prec t$ then $s \prec ([t] \cup [t^\dagger])$, and* (ii) *if $t \prec s$ then $([t] \cup [t^\dagger]) \prec s$.*

Proof. Let $s, t \in S$, and assume the hypothesis of (i) holds. Then whether $s(\Delta(s,t))$ is not defined or equals 0, since $o(s) + 1 < o(t)$, it follows that $t(\Delta(s,t)) = 1$, $t^\dagger(\Delta(s,t)) = 1$, and $\Delta(s,t) = \Delta(s,t^\dagger)$. Thus, for any $u \in S$, if $u \supset t$ or $u \supset t^\dagger$ then $\Delta(s,u) = \Delta(s,t)$, $u(\Delta(s,u)) = t(\Delta(s,t)) = 1$, and hence $s \prec u$. The statement (ii) is proved similarly. $\square$

For the remainder of this section, S is endowed with the *$\prec$-order topology*, and the *open interval notation* (s,t) will refer to $\prec$, where $s \prec t$ (except when it is used to denote a point in the product space $S \times S$). Thus, *topological terms referring to*

S should be understood to be referring to the topology induced on S by $\prec$ (but *not* to the one induced on it by the tree order $<$).

Claim 4.3. *For each $s \in S$, $[s]$ is a clopen subset of S, and the family $\{[s] : s \in S\}$ is a π-base for the topology on S.*

Proof. First, we observe that each $[s]$ is an open set: for any $t \geq s$, $t^\frown 0 \prec t \prec t^\frown 1$, and so $I = (t^\frown 0, t^\frown 1)$ is an open interval containing t, and one can check that $s \leq t \leq u$ for any $u \in I$, and hence $I \subset [s]$.

To see that $[s]$ is a closed set, consider any point $t \in S$ such that $s \not\leq t$. Let $\alpha = \Delta(s,t)$. Then $\alpha \in o(s) \cap o(t)$ (if $s \perp t$) or $\alpha \in o(s) \setminus o(t)$. If $\alpha \in o(t)$, then $[t]$ is an open neighborhood of t disjoint with $[s]$. If $\alpha \notin o(t)$, then $[t^\frown(1 - s(\alpha))]$ is an open neighborhood of t disjoint with $[s]$.

Next, let us prove that the family $\{(v_1, v_2) : v_1 \prec v_2 \text{ and } v_1 \perp v_2\}$ is a π-base. Fix any $s \prec t$ with $s \neq t$, and we find $(v_1, v_2) \subset (s,t)$ from this π-base. If $s \perp t$, then $v_1 = s$ and $v_2 = t$ suffice. So assume $s \not\perp t$, and we consider the remaining possibilities.

Assume $s \leq t$, i.e., $s \subset t$. Then $s(\Delta(s,t))$ is not defined and $t(\Delta(s,t)) = 1$, since $s \prec t$. Thus $s^\frown 1 \subset t$, and we define $v_1 = t^\frown 00$ and $v_2 = t^\frown 01$.

Assume $t \leq s$, i.e., $t \subset s$. Then $s(\Delta(s,t)) = 0$ and $t(\Delta(s,t))$ is not defined, since $s \prec t$. Hence $t^\frown 0 \subset s$, and we define $v_1 = s^\frown 10$ and $v_2 = s^\frown 11$.

In either of these definitions, it follows easily that $v_1 \perp v_2$ and $s \prec v_1 \prec v_2 \prec t$.

To finish the proof, we just prove that for each $v_1 \prec v_2$ with $v_1 \perp v_2$, there is some s with $v_1 \prec [s] \prec v_2$. Evidently $\Delta(v_1, v_2) \in o(v_1) \cap o(v_2)$, so $v_1(\Delta(v_1, v_2)) = 0 < 1 = v_2(\Delta(v_1, v_2))$. Let $s = v_1{}^\frown 1$. Hence $v_1 \prec [s]$. Furthermore, if $s \subset t$, then $t(\Delta(v_1, v_2)) = s(\Delta(v_1, v_2)) = v_1(\Delta(v_1, v_2)) = 0$, and $t \upharpoonright \Delta(v_1, v_2) = s \upharpoonright \Delta(v_1, v_2) = v_1 \upharpoonright \Delta(v_1, v_2) = v_2 \upharpoonright \Delta(v_1, v_2)$. Thus $t \prec v_2$, which shows $[s] \prec v_2$. $\square$

Claim 4.4. *The space S is hereditarily Lindelöf.*

Proof. Since S is a linearly ordered space (with respect to the $\prec$-order), to prove that it is hereditarily Lindelöf, it suffices by Theorem 2.2 in [6] to prove that it satisfies the countable chain condition (*ccc*). It was shown in [11] that the Dedekind completion of any such space is *ccc*. Using the same method, we apply Claim 4.3 here and observe that if $\mathcal{A} = \{[s_\alpha] : \alpha \in A\}$ is a pairwise disjoint listing of the members of a family of elements of the π-base for S, then $\{s_\alpha : \alpha \in A\}$ is an antichain in the Souslin tree $(S, \leq)$, and so $\mathcal{A}$ must be countable. $\square$

Claim 4.5. *Let $\{(u_\xi, v_\xi) : \xi < \omega_1\}$ be a subset of $S \times S$ such that for all $\xi < \omega_1$, there is a t_ξ with $o(t_\xi) \geq \xi$, $t_\xi \leq u_\xi$, and $t_\xi \neq t_\xi^\dagger \leq v_\xi$ (hence every $o(t_\xi)$ is a successor). Then this set co-countably converges to the diagonal in $S \times S$.*

Proof. Any open neighborhood of the diagonal Δ in $S \times S$ will contain a set of the form W=$\bigcup\{(s^-, s^+) \times (s^-, s^+) : s \in S\}$, where for all $s \in S$, $s^- \prec s \prec s^+$. Since S (and hence Δ) are Lindelöf and S is a Souslin tree, there is a $\delta < \omega_1$ such that the collection $\{(s^-, s^+) : s \in S \text{ and } o(s) < \delta\}$ is a countable cover of S. Let $\delta \leq \beta < \omega_1$ be large enough so that $\max\{o(s^-), o(s^+)\} < \beta$ for all $s \in S$ with $o(s) < \delta$. Choose any $\xi < \omega_1$ such that $\beta + 1 < \xi$, and note that $\beta + 1 < o(t_\xi)$. Since $t_\xi \neq t_\xi^\dagger$, we can let

$\bar{t}_\xi$ be the member of S so that t_ξ and $t_\xi^\dagger$ are its immediate successors. Choose $s \in S$ with $o(s) < \delta$ such that $s^- \prec \bar{t}_\xi \prec s^+$. Since $o(\bar{t}_\xi) > \beta \geq \max\{o(s^-), o(s^+)\} + 1$, it follows from Claim 4.2 that $[\bar{t}_\xi] \subset (s^-, s^+)$, i.e., $s^- \prec [\bar{t}_\xi]$ and $[\bar{t}_\xi] \prec s^+$. This proves that $(u_\xi, v_\xi) \in W$. □

Now we are ready to construct the example. Let S^+ denote the set of all $s \in S$ such that $o(s)$ is a successor, U denote the set of all $s \in S^+$ such that for $o(s) = \alpha+1$, $s(\alpha) = 0$, and $V = S^+ \setminus U$. Then note that $V = \{u^\dagger : u \in U\}$, each of U and V is cofinal in S (which means, as defined at the beginning of this section, with respect to the tree order $\leq$), and thus U and V are disjoint dense subsets of S. We let $X = \mathbb{D}(U)$, the Alexandroff double of U.

Theorem 4.6. *The space $X = \mathbb{D}(U)$ is Lindelöf, and the space V is hereditarily Lindelöf, but $X \times V$ is not cellular-Lindelöf..*

Proof. It suffices to prove that $X \times V$ is not cellular-Lindelöf.

Let $\{t_\xi : \xi \in \omega_1\}$ be an enumeration of a subset of U such that for each $\xi < \omega_1$, $o(t_\xi) \geq \xi$. For each $\xi \in \omega_1$, choose any $u_\xi \in U$ and $w_\xi \in V$ such that $t_\xi \leq u_\xi$, $u_\xi \neq u_\eta$ for any $\eta < \xi$, and $t_\xi^\dagger < w_\xi$. Also choose $w_\xi^- \prec w_\xi \prec w_\xi^+$ in $[t_\xi^\dagger]$, and note that either $t_\xi \prec w_\xi^-$ or $w_\xi^+ \prec t_\xi$. Now, for each $\xi < \omega_1$, the set $U_\xi = \{(u_\xi, 1)\} \times (V \cap (w_\xi^-, w_\xi^+))$ is a nonempty open subset of $X \times V$, and $\{U_\xi : \xi < \omega_1\}$ is pairwise disjoint.

Let $Y \subset X \times V$, and assume that $Y \cap U_\xi$ is nonempty for all $\xi < \omega_1$. We prove that Y is not Lindelöf by proving that it has an uncountable subset R with no complete accumulation point in $X \times V$. For each $\xi \in \omega_1$ choose $v_\xi \in V \cap (w_\xi^-, w_\xi^+)$ so that $((u_\xi, 1), v_\xi) \in U_\xi \cap Y$, and let $R = \{(u_\xi, 1), v_\xi) : \xi < \omega_1\}$. Note that by construction, each $v_\xi \in [t_\xi^\dagger]$.

Fix any $(u, v) \in U \times V$. Since $\{(u, 1)\} \times V$ is a neighborhood of the point $((u, 1), v)$ which intersects R in at most one point, $((u, 1), v)$ is not a limit point of R. Let us prove that neither is $((u, 0), v)$ a complete accumulation point of R. Choose $u^- \prec u \prec u^+$ and $v^- \prec v \prec v^+$ (from S) so that $(u^-, u^+) \cap (v^-, v^+)$ is empty, and so that the closure of $(u^-, u^+) \times (v^-, v^+)$ is disjoint from the diagonal in $S \times S$. By Claim 4.5, we may choose $\eta < \omega_1$ so that $(u_\xi, v_\xi) \notin ((u^-, u^+) \times (v^-, v^+))$ for all $\eta < \xi < \omega_1$. It follows easily that $((u_\xi, 1), v_\xi) \notin ((u^-, u^+) \times \{0, 1\}) \times (v^-, v^+)$ for all $\eta < \xi < \omega_1$, and that completes the proof. □

Remark 4.7. *Another way to prove Theorem 4.6 would be to prove that there is no Lindelöf subspace of $\mathbb{D}(U) \times V$ which is dense in $(U \times \{0\}) \times V$, and then appeal to Theorem 3.2.*

5. ZFC product space counterexamples

In this section we show that Moore's L space contains a pair of nowhere separable hL spaces whose product has no dense Lindelöf subspace.

Moore defines his L space after developing the key properties of a very special combinatorial object *Osc*. The object *Osc* is a function from the ordered pairs $\alpha < \beta < \omega_1$ into the finite subsets of α. Similarly, the function $osc(\alpha, \beta)$ is defined as the cardinality of the set $Osc(\alpha, \beta)$. These definitions depend on the choice of a family $\vec{\mathcal{C}} = \langle C_\alpha : \alpha \in \omega_1 \rangle$ where

Definition 5.1. *A family $\vec{\mathcal{C}} = \langle C_\alpha : \alpha \in \omega_1 \rangle$ is a C-sequence on ω_1 provided*

(1) $C_0 = \{0\}$,
(2) *for each $0 < \alpha < \omega_1$, $0 \in C_\alpha$ is a cofinal subset of α (in case $\alpha = \beta + 1$, this just means $\beta \in C_\alpha$),*
(3) *for all $\gamma < \alpha$, $C_\alpha \cap \gamma$ is finite.*

We will be using properties of the function *osc* and Moore's L space from both papers [8] and [9]. In each paper, it is stated that any choice of a C-sequence will suffice. We mention this here because in order to establish our main result, we will impose an additional restriction on the C-sequence that we use.

Here is the definition of Moore's L space from [8]. First of all, $\mathbb{T}$ denotes the unit circle in the complex plane, and for $z = e^{i\,r} \in \mathbb{T}$ and $k \in \mathbb{N}$, of course $z^k = e^{i(kr)}$ is also an element of $\mathbb{T}$. The topology on $\mathbb{T}$ is the one inherited as a subspace of the complex plane. We let $1 \in \mathbb{T}$ denote the element $(1, 0)$.

Definition 5.2. *Let $\{z_\alpha : \alpha \in \omega_1\}$ be a rationally independent set. For all $\beta \in \omega_1$, define $w_\beta \in \mathbb{T}^{\omega_1}$ with the formula*

$$w_\beta(\xi) = \begin{cases} z_\xi^{osc(\xi,\beta)+1} & \xi < \beta \\ 1 & \beta \leq \xi. \end{cases}$$

Theorem 5.3 ([8], 7.11)**.** *For all uncountable $X \subset \omega_1$, $\mathcal{L}_X = \{w_\beta \restriction X : \beta \in X\}$ is hereditarily Lindelöf.*

For each $X \subset \omega_1$, $\mathcal{L}[X] = \{w_\beta : \beta \in X\}$. It is also noted in [8] that the closures of countable subsets of $\mathcal{L} = \mathcal{L}[\omega_1]$ are countable, and therefore we have the following

Lemma 5.4. *For all uncountable $X \subset \omega_1$, $\mathcal{L}[X]$ is a nowhere separable hereditarily Lindelöf space.*

For convenience we make some assumptions about the family $\{z_\alpha : \alpha \in \omega_1\}$. We choose a rationally independent set $\{r_\alpha : \alpha < \omega_1\}$ of reals in the interval $(0, \frac{1}{2\pi})$, and for each $\alpha < \omega_1$ we let z_α be the point in $\mathbb{T}$ obtained by rotating 1 by $2\pi r_\alpha$ radians. For convenience we adopt the following metric on $\mathbb{T}$. For points z, w of $\mathbb{T}$ we let $\rho(z, w)$ be the smallest value $0 \leq r < 2\pi$ so that $w \in \{e^{-2\pi r\,i}z, e^{2\pi r\,i}z\}$. In other words, $\rho(z, w)$ is $\frac{1}{2\pi}$ times the arc length within $\mathbb{T}$ between z and w. Note that (since $\mathbb{T}$ is a group) for any $u \in \mathbb{T}$, $\rho(z, w) = \rho(u \cdot z, u \cdot w)$.

For any point $w \in \mathbb{T}^{\omega_1}$, let $U[w; F, \epsilon]$ (for finite $F \subset \omega_1$ and real $\epsilon > 0$) denote the basic open set consisting of all points $w' \in \mathbb{T}^{\omega_1}$ satisfying that $\rho(w'(\xi), w(\xi)) < \epsilon$ for all $\xi \in F$.

Let Λ denote the set of limit ordinals in ω_1, and then let Λ' denote those members δ of Λ such that $\Lambda \cap \delta$ is cofinal in δ (i.e., Λ' is the set of limit points of Λ).

Lemma 5.5. *There is a choice of the C-sequence $\vec{\mathcal{C}} = \langle C_\alpha : \alpha \in \omega_1 \rangle$ so that there are disjoint uncountable subsets X and Y of ω_1 satisfying that for all $\delta \in \Lambda'$ and $\delta < \alpha \in X$ (respectively Y) and $\alpha < \beta \in Y$ (respectively X), $|\,Osc(\alpha, \beta) \cap \delta| > 1$.*

We will prove Lemma 5.5 later. For now we fix the pair of uncountable sets X and Y. One of the key properties of osc is proven in Lemma 4.4 of [8]. This is improved in Lemma 8 of [9] by Peng and Wu. We will just need a minor variation of a special case of this Lemma 8 that we record here. The set of two-element subsets of ω_1 is denoted by $[\omega_1]^2$. For $a \in [\omega_1]^2$, we let $a(0) = \min(a)$ and $a(1) = \max(a)$. For $a, b \in [\omega_1]^2$, we let $a < b$ denote the relation that $a(1) < b(0)$. We will apply this Lemma to pairs $a \in [\omega_1]^2$ that have an element in each of X and Y from Lemma 5.5. It follows that the integer c in the statement of the next lemma would then be at least 2.

Lemma 5.6. *Suppose that $\mathcal{A} \subset [\omega_1]^2$ is an uncountable family of pairwise disjoint sets. Then there are an uncountable collection $\mathcal{A}' \subset \mathcal{A}$, an integer c, and a $\delta \in \Lambda'$ satisfying that for any $a < b$, both in $\mathcal{A}'$, $|Osc(a(0), a(1)) \cap \delta| = c$ and*

$$osc(a(0), b(0)) + c - 1 \leq osc(a(0), b(1)) \leq osc(a(0), b(0)) + c.$$

We defer the proofs of Lemma 5.5 and Lemma 5.6 until after we prove that the main result is a consequence.

Theorem 5.7. *If X and Y are the disjoint uncountable subsets as in Lemma 5.6, then $\mathcal{L}[X] \times \mathcal{L}[Y]$ does not contain a dense Lindelöf subset.*

Proof. Assume that $D \subset X \times Y$ and that $\tilde{D} = \{(w_\beta, w_\gamma) : (\beta, \gamma) \in D\}$ is dense in $\mathcal{L}[X] \times \mathcal{L}[Y]$. We will produce an uncountable subset of $\tilde{D}$ that has no complete accumulation point in $\mathcal{L}[X] \times \mathcal{L}[Y]$.

By Lemma 5.4, we have that, for all $\delta \in \omega_1$,

$$\tilde{D}_\delta = \{(w_\beta, w_\gamma) : (\beta, \gamma) \in D \cap (\delta \times \omega_1 \cup \omega_1 \times \delta)\}$$

is not dense. Therefore we may choose an uncountable subset $D_1 \subset D$ that has the property that if $d_1, d_2 \in D_1$ are distinct, then there is a $\delta \in \omega_1$ such that (wlog) $d_1 \in \delta \times \delta$ and both coordinates of d_2 are greater than δ. By passing to a subset of D_1 we can assume that (wlog) for each $d \in D_1$, the first coordinate of d ($d(0)$ in X) is less than the second coordinate of d ($d(1)$ which is in Y). Let $\mathcal{A}$ be the uncountable set of disjoint pairs $\{\{d(0), d(1)\} : d \in D_1\}$. By applying Lemma 5.6, we may choose an integer c and an uncountable $D_2 \subset D_1$ such that for all $a, b \in D_2$ with $a(1) < b(0)$, we have that $osc(a(0), b(0)) + c - 1 \leq osc(a(0), b(1)) \leq osc(a(0), b(0)) + c$. By Lemma 5.5, the value of $c - 1$ is positive.

For each real r, we let $[r]_{2\pi}$ denote the value $r - 2\pi\ell$, where $2\pi\ell \leq r < 2\pi(\ell + 1)$, i.e., $e^{i\,r} = e^{i\,[r]_{2\pi}}$. Note that for $a < d$, both in D_2 and $k = osc(a(0), d(0)) + 1$, the value of $\rho(w_{d(0)}(a(0)), w_{d(1)}(a(0)))$ equals one of

$$\rho(z^k_{a(0)}, z^{k+c}_{a(0)}) = \rho(1, z^c_{a(0)}) = [c\, r_{a(0)}]_{2\pi} \text{ and}$$

$$\rho(z^k_{a(0)}, z^{k+c-1}_{a(0)}) = \rho(1, z^{c-1}_{a(0)}) = [(c-1)\, r_{a(0)}]_{2\pi}\ .$$

Now we choose three uncountable subsets of D_2. First of all, choose any pair $0 < s_1 < s_2$ so that

(1) s_1 and s_2 are complete accumulation points of $(s_1, s_2) \cap \{r_{d(0)} : d \in D_2\}$,
(2) and there is an ℓ with $\pi\ell < cs_1 < cs_2 < \pi(\ell + 1)$,

and let $s = \frac{s_1+s_2}{2}$. Note that for $s_1 < r_1 < r_2 < s_2$, $[cr_2]_{2\pi} - [cr_1]_{2\pi} = c(r_2 - r_1)$.

Choose $0 < \epsilon < \frac{s_2-s_1}{5}$ so that $D_4 = \{b \in D_2 : r_{b(0)} \in (s_1 + 2\epsilon, s_2 - 2\epsilon)\}$ is uncountable. Also let $D_3 = \{a \in D_2 : r_{a(0)} \in (s_1, s_1 + \epsilon)\}$, and $D_5 = \{b \in D_2 : r_{b(0)} \in (s_2 - \epsilon, s_2)\}$. We note that each of D_3, D_4, D_5 is uncountable.

For any $a \in D_i$, $a < b \in D_j$ $(3 \leq i \neq j \leq 5)$, $b < d \in D_2$, and $c' \in \{c-1, c\}$, we have that

$$|\,[c'\,r_{a(0)}]_{2\pi} - [c'\,r_{b(0)}]_{2\pi}| = c'|r_{a(0)} - r_{b(0)}| > \epsilon\ .$$

This implies that if $a_3 < a_4 < a_5 < d$ with $a_i \in D_i$ and $d \in D_2$, then there is some choice $\{a, b\} \subset \{a_3, a_4, a_5\}$ so that

$$|\rho(w_{d(0)}(a(0)), w_{d(1)}(a(0))) - \rho(w_{d(0)}(b(0)), w_{d(1)}(b(0)))| > \epsilon\ .$$

We prove that the uncountable set $\{(z_{d(0)}, z_{d(1)}) : d \in D_2\}$ has no complete accumulation point in $\mathcal{L}[X] \times \mathcal{L}[Y]$. Fix any pair $(u, v) \in X \times Y$ and choose $a \in D_3$ so that $\max\{u, v\} < a(0)$; then choose b, e so that $a < b \in D_4$, and $b < e \in D_5$. Use the neighborhood $U(w_u; \{a(0), b(0), e(0)\}, \epsilon/4) \times U(w_v; \{a(0), b(0), e(0)\}, \epsilon/4)$ for (w_u, w_v) in $\mathcal{L}[X] \times \mathcal{L}[Y]$. Let $r = \rho(z_u, z_v)$ and recall that $w_u(\xi) = 1$ and $w_v(\xi) = 1$ for all $a(0) \leq \xi \in \omega_1$. Now suppose that $(w_{d(0)}, w_{d(1)})$ is in this neighborhood for some $d \in D$. It follows that for each $\xi \in \{a(0), b(0), e(0)\}$, $\rho(w_{d(0)}(\xi), 1) < \epsilon/4$ and $\rho(w_{d(1)}(\xi), 1) < \epsilon/4$. This implies that $\rho(w_{d(0)}(\xi), w_{d(1)}(\xi)) \in (-\epsilon/2, \epsilon/2)$ for each $\xi \in \{a(0), b(0), e(0)\}$. This implies that d is not an element in D_2 above e since for such $d \in D_2$ there is a pair $\xi, \zeta \in \{a(0), b(0), e(0)\}$ such that $|\rho(w_{d(0)}(\xi), w_{d(1)}(\xi)) - \rho(w_{d(0)}(\zeta), w_{d(1)}(\zeta))| > \epsilon$. □

Now we prove Lemma 5.5.

Proof of Lemma 5.5: Let Λ'' be the set of $\lambda \in \Lambda'$ such that $\Lambda' \cap \lambda$ is cofinal in λ. To prove the Lemma it will suffice to define uncountable subsets X and Y of ω_1 and to then prove that for all $x \in X$ and $y, y' \in Y$ with $y < x < y'$,

$$|\,Osc(y, x) \cap \omega| > 1 \text{ and } |\,Osc(x, y') \cap \omega| > 1\ .$$

In fact, to accomplish this we will be more prescriptive in our definitions and by the end of this proof we will have proven

Fact 5.8. *For all $x \in X$ and $y, y' \in Y$ with $y < x < y'$,*

$$\{30, 200\} \subset Osc(y, x) \text{ and } \{10, 90\} \subset Osc(x, y').$$

Of course the specific choice of $\{10, 30, 90, 200\}$ is quite arbitrary but the numbers do have to be some distance apart. Our next steps are to recall the underlying definitions from [8] related to the Osc function and to then construct a specific C-sequence that will work. Our choice of the uncountable sets X and Y is also critical and are chosen so as to allow us to carefully control the values of $\mathrm{Osc}(\delta, z) \cap \omega$ for $\delta \in \Lambda'$ and $z \in (X \cup Y) \setminus \delta$.

The first definition we will need is that of the lower trace.

Definition 5.9 ([8]). *If $\alpha \leq \beta < \omega_1$, then $L(\alpha, \beta)$ is defined recursively by*

$$L(\alpha, \alpha) = \emptyset,$$
$$L(\alpha, \beta) = (L(\alpha, \min(C_\beta \setminus \alpha)) \cup \{\max(C_\beta \cap \alpha)\}) \setminus \max(C_\beta \cap \alpha).$$

We will see below that $Osc(x,y) \subset L(x,y)$, and so we will certainly need $L(x,y) \cap \omega$ to be non-empty (in fact, quite large). Since $\max(C_\beta \cap \alpha)$ is likely to be greater than ω, in order for us to be ensuring that $Osc(x,y) \cap \omega$ is non-empty for special pairs x, y, we will need to be *adding* elements to $Osc(x,y)$ as per the recursive nature of the definition. The following is Fact 1 from [8].

Fact 5.10. *If $\alpha \leq \beta \leq \gamma$ and $\max(L(\beta,\gamma)) < \min(L(\alpha,\beta))$, then*
$$L(\alpha,\gamma) = L(\alpha,\beta) \cup L(\beta,\gamma).$$

The next definition that we need is the following recursively defined function ϱ_1 of Todorčević.

Definition 5.11 ([14])**.** *If $\alpha \leq \beta$, then $\varrho_1(\alpha,\beta)$ is defined recursively by*
$$\varrho_1(\alpha,\alpha) = 0,$$
$$\varrho_1(\alpha,\beta) = \max\left(|C_\beta \cap \alpha|, \varrho_1(\alpha, \min(C_\beta \setminus \alpha))\right).$$

Following [8], we let $e_\beta : \beta \to \omega$ be the function defined by $e_\beta(\xi) = \varrho_1(\xi,\beta)$. Again we loosely note that in order to ensure that for some integer k and $\omega < x < y \in \omega_1$, $\rho_1(k,x) < \rho_1(k,y)$, we will somehow need to have different values for $|C_\beta \cap k|$ for appropriate β arising in Definition 5.11. And now we have the definition of Osc and can see why we will need to have $\rho_1(k,x) < \rho_1(k,y)$ at critical integers k.

Definition 5.12. *For $\alpha < \beta \in \omega_1$, $Osc(\alpha,\beta)$ is the set of $\xi \in L(\alpha,\beta) \setminus \min(L(\alpha,\beta))$ such that $e_\alpha(\xi^-) \leq e_\beta(\xi^-)$ and $e_\alpha(\xi) > e_\beta(\xi)$, where ξ^- is the maximum of $L(\alpha,\beta) \cap \xi$.*

The value of $osc(\alpha,\beta)$ is equal to the cardinality of $Osc(\alpha,\beta)$.

We are ready to make our choices of X and Y:
$$X = \{\delta + \omega \cdot 8 : \delta \in \Lambda''\} \quad \text{and} \quad Y = \{\delta + \omega \cdot 9 : \delta \in \Lambda' \setminus \Lambda''\}\ .$$
For each $\min(\Lambda') \leq \alpha \in \omega_1$, let δ_α be the supremum of $\Lambda' \cap \alpha$ (thus $\delta_\alpha \in \Lambda'$), and let $T = \{\alpha : \min(\Lambda') < \alpha \text{ and } \delta_\alpha < \alpha \leq \delta_\alpha + \omega \cdot 9\}$.

We now give the definition of our C-sequence. We hope that the discussion above about our goal in ensuring that $L(x,y) \cap \omega$ and $L(y,x) \cap \omega$ are substantial, as well as occasionally needing large values of $|C_\beta \cap k|$ for integers k, partially serves to motivate the definition. In particular, we make special coherent choices for $C_{\delta+\omega\cdot k}$ for $\delta \in \Lambda'$ and $k \leq 9$. We also need $\rho(\delta, \delta + \omega \cdot 8)$ and $\rho(\delta, \delta + \omega \cdot 9)$ to be substantially different and we will accomplish this by ensuring that the recursive definition of $L(\delta, \delta + \omega \cdot 8)$ proceeds through $\{\delta + \omega \cdot 6, \delta + \omega \cdot 4, \delta + \omega \cdot 2\}$ and that of $L(\delta, \delta + \omega \cdot 9)$ through $\{\delta + \omega \cdot 7, \delta + \omega \cdot 5, \delta + \omega \cdot 3, \delta + \omega\}$. At each stage an integer from $C_{\delta+\omega\cdot k}$ will be added to $L(\delta, \delta + \omega \cdot 8)$ but only if it is larger than those added at earlier steps (see Definition 5.9). We use the integer 1000 in cases when we do not want small integers added to $L(\alpha,\beta)$. We will use the following sets $F_1, F_2, \ldots, F_9$ in presenting our definition of our C-sequence:

$F_1 = [5,10] \cup [20,40] \cup \{90\} \cup [100,200]$	$F_2 = [2,10] \cup \{30\} \cup [40,90] \cup \{200\}$
$F_3 = [5,10] \cup [20,40] \cup \{90\}$	$F_4 = [2,10] \cup \{30\} \cup [40,90]$
$F_5 = [5,10] \cup [20,40]$	$F_6 = [2,10] \cup \{30\}$
$F_7 = [5,10]$	$F_8 = \emptyset$
$F_9 = \emptyset$	

Recall that $T = \{\alpha : \min(\Lambda') < \alpha \text{ and } \delta_\alpha < \alpha \leq \delta_\alpha + \omega \cdot 9\}$.

Definition 5.13. *For each ordinal $\alpha < \omega_1$, we define C_α as follows:*

(1) $C_0 = \{0\}$ *and if* $\omega < \alpha = \beta + 1$, *then* $C_\alpha = \{0, 1000, \beta\}$,
(2) *if* $0 < \alpha < \omega$, *then* $C_\alpha = \{0, \alpha - 1\}$,
(3) *if* $\alpha \in \Lambda \cap \min(\Lambda')$ *or if* $\alpha \in \Lambda'$, *then* C_α *is any suitable cofinal subset of* α *such that* $C_\alpha \cap [0, 1000] = \{0, 1000\}$,
(4) *if* $\alpha \in T$ *and* $\alpha = \delta_\alpha + \omega \cdot k$, *let* $j = \max(0, k-2)$ *and* $C_\alpha = \{0\} \cup F_k \cup \{\delta_\alpha + \omega \cdot j\} \cup (\delta_\alpha + \omega \cdot (k-1), \delta_\alpha + \omega \cdot k)$.
(5) *if* $\delta_\alpha < \alpha \in \Lambda \setminus T$, *and if* $\beta < \alpha \leq \beta + \omega$ *for some* $\beta \in \Lambda$, *then* $C_\alpha = \{0, 1000, \delta_\alpha\} \cup (\beta, \alpha)$.

Let us note that $1000 \in C_\beta$ for all $\omega \leq \beta \notin T$ and prove the following:

Fact 5.14. *For all* $\beta \notin T$ *and* $0 < k < \min(\beta, 1000)$, $\varrho_1(k, \beta) = 1$.

Proof of Fact 5.14: For each $0 < n < \omega$, $C_n = \{0, n-1\}$ and so, by induction on $n > k$, $\rho_1(k, n) = 1$. If $\omega \leq \beta \notin T$, then $C_\beta \cap k = \{0\}$ and $1000 = \min(C_\beta \setminus k)$. Therefore, $\rho_1(k, \beta) = 1$

It will be helpful to notice that $\max(C_\beta \cap \alpha)$ is the minimum element of $L(\alpha, \beta)$.

Fact 5.15. *If* $\delta \in \Lambda'$ *and* $\alpha = \delta + \omega \cdot 8 \in X$ *then* $L(\delta_\alpha, \alpha) = \{0, 30, 90, 200\}$.

Proof of Fact 5.15: For each $0 \leq i \leq 4$, let $\beta_i = \delta + \omega \cdot (2i)$. For each $0 \leq i < 4$, $\min(C_{\beta_{i+1}} \setminus \delta) = \beta_i$ and $\max(C_{\beta_{i+1}} \cap \delta) = \min(L(\delta, \beta_{i+1})) > \max(C_{\beta_i} \cap \delta)$. This implies that $\max(C_{\beta_{i+1}} \cap \delta) \cup L(\delta, \beta_i) \subset L(\delta, \beta_{i+1})$. Now $L(\delta, \beta_1) = \{200\}$, $L(\delta, \beta_2) = \{90, 200\}$, $L(\delta, \beta_3) = \{30, 90, 200\}$, $L(\delta, \beta_4) = \{0, 30, 90, 200\}$.

By a similar argument, which we skip, we also have:

Fact 5.16. *If* $\delta \in \Lambda'$ *and* $\alpha = \delta + \omega \cdot 9$ *then* $L(\delta_\alpha, \alpha) = \{0, 10, 40, 90, 200\}$.

Using Definition 5.11 and following the ideas of the proofs of Fact 5.15 and Fact 5.16 we have the following. We leave the simple checking to the reader.

Fact 5.17. *Let* $\delta \in \Lambda'$, *and, for* $k = 0, \ldots, 9$, *let* $\beta_k = \delta + \omega \cdot k$. *Then, for each* $j < 201$, $e_{\beta_8}(j) = \varrho_1(j, \beta_8) = |C_{\beta_2} \cap j| = |F_2 \cap j|$ *and* $e_{\beta_9}(j) = \varrho_1(j, \beta_9) = |F_1 \cap j|$.

Again using that for all $1000 < \xi < \delta \in \Lambda'$, 1000 is in C_δ, we have that $1000 \leq \min(L(\xi, \delta))$ and we record this next fact.

Fact 5.18. *If* $\omega \leq \xi < \delta \in \Lambda'$, *then*

(1) $L(\delta, \delta + \omega \cdot 9) = L(\xi, \delta + \omega \cdot 9) \cap 1000$, *and*
(2) $L(\delta, \delta + \omega \cdot 8) = L(\xi, \delta + \omega \cdot 8) \cap 1000$.

Now we are ready to complete the proof of Lemma 5.5 by proving Fact 5.8. Let $x \in X$ and $y, y' \in Y$ with $y < x < y'$. By definition, $x = \delta_x + \omega \cdot 8$, $y = \delta_y + \omega \cdot 9$, $y' = \delta_{y'} + \omega \cdot 9$ and $\delta_y < \delta_x < \delta_{y'}$. We must prove that $\{30, 200\} \subset Osc(y, x)$ and

$\{10, 90\} \subset Osc(x, y')$.
By Facts 5.15 and 5.16 we have that

$$L(\delta_x, x) = \{0, 30, 90, 200\} \quad \text{and} \quad L(\delta_y, y) = L(\delta_{y'}, y') = \{0, 10, 40, 90, 200\}\ .$$

It then follows from Fact 5.18 that

$$L(y, x) = \{0, 30, 90, 200\} \quad \text{and} \quad L(x, y') = \{0, 10, 40, 90, 200\}\ .$$

By Facts 5.17 and the fact that

$$C_{\delta_x+\omega\cdot 2} \cap 1000 = \{0, 30, 200\} \cup [2, 10] \cup [40, 90],$$

we have the following values:

$$e_x(0) = 0, e_x(10) = 9, e_x(30) = 10, e_x(40) = 11, e_x(90) = 62, e_x(200) = 63.$$

Similarly

$$C_{\delta_y+\omega} \cap 1000 = C_{\delta_{y'}+\omega} \cap 1000 = \{0, 90\} \cup [5, 10] \cup [20, 40] \cup [100, 200],$$

and we have the following values for e_y (and $e_{y'}$):

$$e_y(0) = 0, e_y(10) = 6, e_y(30) = 18, e_y(40) = 28, e_y(90) = 29, e_y(200) = 129\ .$$

Now we verify that $\{30, 200\} \subset Osc(y, x)$. First let $\xi = 30 \in L(y, x) = \{0, 30, 90, 200\}$ and note that $\xi^- = 0$. Since $e_y(\xi^-) = e_x(\xi^-)$ and $e_y(30) = 18 > e_x(30) = 10$, we have $30 \in Osc(y, x)$. Similarly, with $\xi = 200$, we have $\xi^- = 90$, $e_y(90) = 29 < e_x(90) = 62$, and $e_y(200) = 129 > e_x(200) = 63$.

Next we verify that $\{10, 90\} \subset Osc(x, y')$. With $\xi = 10 \in L(x, y') = \{0, 10, 40, 90, 200\}$, we have $\xi^- = 0$ and $e_x(0) = e_{y'}(0)$ while $e_x(10) = 9 > e_{y'}(10) = 6$. Next let $\xi = 90$ and $\xi^- = 40$. We again have $e_x(40) = 11 \le e_{y'}(40) = 28$ and $e_x(90) = 62 > e_{y'}(90) = 29$.

This completes the proof of Lemma 5.5. □

Proof of Lemma 5.6: Let $\mathcal{A}$ be an uncountable set of pairwise disjoint two element subsets of ω_1. For $a \in \mathcal{A}$ we let $a(0) = \min(a)$ and $a(1) = \max(a)$. For each limit ordinal δ, choose any $a_\delta \in \mathcal{A}$ so that $\delta \le \min(a_\delta)$. Let $c_\delta = |\, Osc(a_\delta(0), a_\delta(1)) \cap \delta|$.

We will need this next Fact 3 from [8] (proven in [14]).

Fact 5.19. *For each $\alpha \le \beta \in \omega_1$, the set of $\xi < \alpha$ such that $e_\alpha(\xi) \neq e_\beta(\xi)$ is finite. Also, for each $\alpha \in \omega_1$, the set $\{e_\beta \restriction \alpha : \alpha \le \beta \in \omega_1\}$ is countable.*

Let $\{t_\xi : \xi \in \omega_1\}$ be an enumeration of the entire family $\{e_\beta \restriction \alpha : \alpha \le \beta < \omega_1\}$. By standard arguments, we may choose a cub $C \subset \omega_1$ such that for all $\delta \in C$,

$$\{t_\xi : \xi < \delta\} = \{e_\beta \restriction \alpha : \alpha < \delta, \text{ and } \alpha \le \beta < \omega_1\}\ .$$

For each $\delta \in C$, there is a $\gamma_\delta \in C_\delta$ such that

(1) $L(\delta, a_\delta(0)) \cup L(\delta, a_\delta(1)) \subset \gamma_\delta$,
(2) each of $e_{a_\delta(0)} \restriction [\gamma_\delta, \delta)$ and $e_{a_\delta(1)} \restriction [\gamma_\delta, \delta)$ is equal to $e_\delta \restriction [\gamma_\delta, \delta)$.

For all $\gamma_\delta < \xi < \delta \in C$, $\max(C_\delta \cap \xi)$ is the minimum element of $L(\xi, \delta)$, and so we have that $\gamma_\delta \leq \min(L(\xi, \delta))$. By Fact 5.10, for $\gamma_\delta < \xi < \delta \in C$, and $\delta \leq \beta$, if $\max(L(\delta, \beta)) < \gamma_\delta$, then $L(\xi, \beta) = L(\xi, \delta) \cup L(\delta, \beta)$.

By the pressing down lemma, there is a $\gamma \in \omega_1$, functions $s_0, s_1 \in \omega^\gamma$, a pair of finite sets $L_0, L_1 \subset \gamma$, an integer $\bar{c}$, and a stationary set $S \subset C$ satisfying that for all $\delta \in S$,

(1) $\gamma_\delta = \gamma$, $L(\delta, a_\delta(0)) = L_0$, $L(\delta, a_\delta(1)) = L_1$,
(2) $e_{a_\delta(0)} \restriction \gamma = s_0$, $e_{a_\delta(1)} \restriction \gamma = s_1$,
(3) $e_{a_\delta(0)} \restriction [\gamma, \delta) = e_{a_\delta(0)} \restriction [\gamma, \delta) = e_\delta \restriction [\gamma, \delta)$,
(4) $c_\delta = \bar{c}$.

Now let $\eta < \delta$ both be elements of S and we calculate each of $osc(a_\eta(0), a_\delta(0))$ and $osc(a_\eta(0), a_\delta(1))$. Let $\zeta = \min(L(a_\eta(0), \delta))$ and note that $\gamma = \gamma_\delta \leq \zeta$. To determine the cardinality of $Osc(a_\eta(0), a_\delta(0))$, we consider each of

$$Osc(a_\eta(0), a_\delta(0)) \cap \gamma \quad \text{and} \quad Osc(a_\eta(0), a_\delta(0)) \setminus \gamma.$$

The set $Osc(a_\eta(0), a_\delta(0)) \cap L(\delta, a_\delta(0))$ is empty because $L(a_\eta(0), \delta) \subset \gamma_\delta < a_\eta(0)$ and $e_{a_\eta(0)} \restriction \gamma = e_{a_\delta(0)} \restriction \gamma$. Also, $\zeta^- < \gamma$ and so $e_{a_\eta(0)}(\zeta^-) = e_{a_\delta(0)}(\zeta^-)$. Let $k = 1$ if $e_{a_\eta(0)}(\zeta) > e_\delta(\zeta)$ and otherwise let $k = 0$. Since $L(a_\eta(0), a_\delta(0)) \setminus \gamma$ equals $L(a_\eta(0), \delta)$ and $e_{a_\delta}(0) \restriction [\gamma, \delta) = e_\delta \restriction [\gamma, \delta)$, we have that

$$Osc(a_\eta(0), \delta) \subset Osc(a_\eta(0), a_\delta(0)) \ \setminus \gamma \subset Osc(a_\eta(0), \delta) \cup \{\zeta\}.$$

Similarly, we evaluate each of $Osc(a_\eta(0), a_\delta(1)) \cap \gamma$ and $Osc(a_\eta(0), a_\delta(1)) \setminus \gamma$. We first consider $Osc(a_\eta(0), a_\delta(1)) \setminus \gamma$. Since $e_{a_\delta(1)} \restriction [\gamma, \delta)$ is also equal to $e_\delta \restriction [\gamma, \delta)$, and $L(a_\eta(0), a_\delta(1)) \setminus \gamma$ is also equal to $L(a_\eta(0), \delta)$, we have that

$$Osc(a_\eta(0), \delta) \subset Osc(a_\eta(0), a_\delta(1)) \ \setminus \gamma \subset Osc(a_\eta(0), \delta) \cup \{\zeta\}.$$

So the only possible difference between

$$Osc(a_\eta(0), a_\delta(1)) \ \setminus \gamma \quad \text{and} \quad Osc(a_\eta(0), a_\delta(0)) \ \setminus \gamma$$

is the singleton ζ. If $k = 0$, then $\zeta \notin Osc(a_\eta(0), a_\delta(1)) \setminus \gamma$. However, if $k = 1$, then $\zeta \in Osc(a_\eta(0), a_\delta(0)) \setminus \gamma$ but ζ may or may not be in $Osc(a_\eta(0), a_\delta(1)) \setminus \gamma$.

Next, we note that $Osc(a_\eta(0), a_\delta(1)) \cap \gamma$ equals $Osc(a_\eta(0), a_\eta(1)) \cap \gamma$, and that this latter set is the same for all $\eta \in S$. This proves that

$$Osc(a_\eta(0), a_\delta(0)) \setminus \{\zeta\} \subset Osc(a_\eta(0), a_\delta(1)) \subset Osc(a_\eta(0), a_\delta(0)) \ .$$

Let the value of c stated in Lemma 5.6 be the cardinality of $Osc(a_\eta(0), a_\eta(1)) \cap \gamma$, and this completes the proof. ☐

6. Remarks by the second author and by both authors of this article

While the second author was preparing reviews of [13] and [15], he sent to several members of the Carolina Seminar some related product questions and observations, including a counterexample to the method of proof of Theorem 3.12 in [15]. Within a few days of each such communication, the first author responded with proofs or indications of proofs answering those questions and disproving Theorem 3.12. At the Seminar's meeting the following month (on February 22, 2020), the first author presented a portion of his answers, and before doing so, he informed the second author that he would like for the two of them to write this article. This was agreed upon, and then some weeks later the first author succeeded in proving (and writing up) the results in Section 5.

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Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223
Email address: adow@uncc.edu

Department of Mathematics, University of South Carolina, Columbia, SC 29208
Email address: stephenson@math.sc.edu