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# A NON-TRIVIAL COPY OF $\beta\mathbb{N}\setminus\mathbb{N}$

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ABSTRACT. There is a copy $K$ of the Stone-Cech remainder, $\beta\mathbb{N}\setminus\mathbb{N}=\mathbb{N}^*$, of the integers inside $\mathbb{N}^*$ that is not equal to $\overline{D}\setminus D$ for any countable discrete $D\subset\beta\mathbb{N}$. Such a copy of $\mathbb{N}^*$ is known as a non-trivial copy of $\mathbb{N}^*$. This answers a longstanding open problem of Eric van Douwen ([9, 219]).

## 1. Introduction

A subset $K$ of $\beta\mathbb{N}$ is a non-trivial copy of $\mathbb{N}^*$ if $K$ is homeomorphic to $\beta\mathbb{N}\setminus\mathbb{N}$ but is not equal to $\overline{D}\setminus D$ for any countable discrete $D\subset\beta\mathbb{N}$. It is of course a fundamental property of $\beta\mathbb{N}$ that every set of the form $\overline{D}\setminus D$, for countable discrete $D\subset\beta\mathbb{N}$, is a copy of $\mathbb{N}^*$. Non-trivial copies exist in abundance under CH, can be forced to exist in models of Martin's Axiom, and it has long been suspected that these would not exist under the proper forcing axiom, PFA [8, 3.14.2]. For example, it was proven by W. Just to follow from PFA that a nowhere dense P-set of $\mathbb{N}^*$ is not homeomorphic to $\mathbb{N}^*$. Analogous to the classification of autohomeomorphisms of $\mathbb{N}^*$, one could say that a non-trivial copy, $K$, of $\mathbb{N}^*$ was nowhere trivial if every relatively clopen subset of $K$ was also non-trivial. Our construction produces a nowhere trivial copy of $\mathbb{N}^*$. We discuss some variants in the final section.

A closed set $K\subset\mathbb{N}^*$ corresponds to the filter $\mathcal{F}_K\subset\mathcal{P}(\mathbb{N})$ consisting of all those subsets of $\mathbb{N}$ whose closure contains $K$. The task will be to construct such a *special* filter $\mathcal{F}$. This paper brings together a variety of methods that have been used to construct special filters in the study of Stone-Cech compactifications. This includes van Mill's notion of *nice* filter [11] in the construction of weak P-points and some tools from the construction of remote points [4]. We define the notion of a maximal nice filter on a $\sigma$-compact space. We strengthen the notion

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to that of a non-trivial maximal nice filter, and show the connection to the existence of non-trivial copies of $\mathbb{N}^*$. This method was anticipated in [6].

## 2. Aronszajn trees and a special filter

We will construct a special filter on $\mathcal{P}(\mathbb{N})$ as discussed in the introduction. If $X$ is a $\sigma$-compact space which can be embedded into $\mathbb{N}^*$, then a special filter on $X$ naturally induces a filter on $\mathcal{P}(\mathbb{N})$. The space we will use is obtained from the regular open algebra on $2^{\omega_1}$, $RO(2^{\omega_1})$. This is also a common method of constructing special points in $\beta\mathbb{N}$ (e.g. see [10, 7]).

**Definition 2.1.** For each $n \in \mathbb{N}$, let $X_n$ be a compact space. A *nice* filter $\mathcal{F}$ on $X = \bigcup_{n\in\mathbb{N}}\{n\} \times X_n$ is a free filter of closed sets satisfying that for each $F \in \mathcal{F}$, the set $\{n : F \cap (\{n\} \times X_n) = \emptyset\}$ is finite.

**Proposition 2.2.** *Let $X = \bigcup_{n\in\mathbb{N}}\{n\} \times X_n$ with each $X_n$ compact, and let $\pi$ denote the first coordinate projection map on $X$ to $\mathbb{N}$. If $\mathcal{F}$ is a nice filter on $X$, then $\pi^\beta[K_\mathcal{F}] = \mathbb{N}^*$, where $K_\mathcal{F} \subset \beta X$ is defined as $\bigcap\{\overline{F} : F \in \mathcal{F}\}$ and $\pi^\beta : \beta X \to \beta\mathbb{N}$.*

The regular open algebra, $RO(2^{\omega_1})$, on $2^{\omega_1}$, is a complete ccc Boolean algebra. The Stone space is often denoted as $E(2^{\omega_1})$. The points $\mathcal{U}$ of $E(2^{\omega_1})$ consist of ultrafilters of regular open subsets of $2^{\omega_1}$. There is a canonical map, $\varphi$, from $E(2^{\omega_1})$ onto $2^{\omega_1}$ sending $\mathcal{U}$ to the unique point $x \in \bigcap\{\overline{U} : U \in \mathcal{U}\}$, equivalently to the point whose neighborhood base of clopen sets is contained in $\mathcal{U}$. Additionally, if $U \in RO(2^{\omega_1})$, then $U^* = \{\mathcal{U} \in E(2^{\omega_1}) : U \in \mathcal{U}\}$ is a basic clopen subset of $E(2^{\omega_1})$. It follows that $\varphi[U^*] = \overline{U}$. Let us also note that for $U \in RO(2^{\omega_1})$, there is a $\delta = \delta_U \in \omega_1$ and a regular open set $U_\delta \subset 2^\delta$ such that $U = U_\delta \times 2^{\omega_1 \setminus \delta}$. Conversely there is an embedding $e_\delta : RO(2^\delta) \hookrightarrow RO(2^{\omega_1})$ (of course $e_\delta(U) = U \times 2^{\omega_1\setminus\delta}$) so that $RO(2^{\omega_1}) = \bigcup\{e_\delta[RO(2^\delta)] : \delta \in \omega_1\}$.

**Definition 2.3.** A set $K \subset \mathbb{N}^*$ is a weak P-set if for each countable $D \subset \mathbb{N}^* \setminus K$, the closure of $D$ is disjoint from $K$.

The following result is a special case of [14] (see also [12, 4.15] and [5, 3.5]).

**Lemma 2.4.** *There is an embedding of $\beta(\mathbb{N} \times E(2^{\omega_1}))$ as a weak P-set of $\mathbb{N}^*$.*

**Definition 2.5.** A nice filter $\mathcal{F}$ on $\bigcup_{n\in\mathbb{N}}\{n\} \times X_n$ will be said to be maximal, if for each sequence $\langle\langle A_0^n, A_1^n\rangle : n \in \mathbb{N}\rangle$ where, for each $n$, $\{A_0^n, A_1^n\}$ is a 2-element open cover of $X_n$, there is an $F \in \mathcal{F}$ such that for each $n$, $F \cap (\{n\} \times X_n)$ is contained in one of the elements of $\{\{n\} \times A^n_0, \{n\} \times A^n_1\}$. Equivalently, there is an $f \in 2^{\mathbb{N}}$ such that $F \cap \Big(\{n\} \times (X_n \setminus A^n_{f(n)})\Big)$ is empty for each $n \in \mathbb{N}$.

**Lemma 2.6.** *Suppose that $\mathcal{F}$ is a nice filter on $X = \bigcup_{n\in\mathbb{N}}\{n\} \times X_n$ and $K_{\mathcal{F}}$ is the corresponding subset of $\beta X$. The projection map $\pi^\beta$ is one-to-one on $K_{\mathcal{F}}$ if and only if $\mathcal{F}$ is a maximal nice filter.*

*Proof.* Assume that $\pi^\beta$ is one-to-one on $K = K_{\mathcal{F}}$ and let $\{A^n_0, A^n_1\}$ be an open cover of $X_n$ for each $n \in \mathbb{N}$. Notice that $H_0 = \bigcup_{n\in\mathbb{N}}\{n\}\times(X_n\setminus A^n_0)$ and $H_1 = \bigcup_{n\in\mathbb{N}}\{n\} \times (X_n \setminus A^n_1)$ have disjoint closures in $\beta X$. Since $\pi^\beta \restriction K$ is one-to-one, $\pi^\beta(H_0 \cap K)$ and $\pi^\beta(H_1 \cap K)$ are disjoint closed subsets of $\mathbb{N}^*$. There is a set $U_0 \subset \mathbb{N}$ such that $\pi^\beta(H_0 \cap K) \subset U_0^*$ and $\pi^\beta(H_1 \cap K) \subset (\mathbb{N} \setminus U_0)^*$. Thus, with $U_1$ denoting $\mathbb{N} \setminus U_0$, there is some $F \in \mathcal{F}$ which is disjoint from $\bigcup_{n\in U_0}\{n\}\times(X_n\setminus A^n_1)\cup\bigcup_{n\in U_1}\{n\}\times(X_n\setminus A^n_0)$. Therefore, if we define $f(n) = 1 - i$ if and only if $n \in U_i$, we have the needed function witnessing that $\mathcal{F}$ is maximal.

Conversely assume that $\mathcal{F}$ is maximal and assume that $x_0 \neq x_1 \in K$ satisfy that $\pi^\beta(x_0) = \pi^\beta(x_1) = \mathcal{U} \in \mathbb{N}^*$. Let $W_0, W_1$ be disjoint open neighborhoods in $\beta X$ of $x_0$ and $x_1$. For each $n \in \mathbb{N}$ and $e \in 2$, define $A^n_e$ so that $\{n\} \times A^n_e = (\{n\} \times X) \setminus \overline{W_e}$. By the maximal property, choose $F \in \mathcal{F}$ so that, for each $n \in \mathbb{N}$, there is an $e_n \in 2$ so that $F \cap (\{n\} \times X_n) \subset \{n\} \times A^n_{e_n}$. Choose $U \in \mathcal{U}$ so that there is an $e \in 2$ with $n_e = e$ for all $n \in U$. Notice that the closure of $\bigcup_{n\in U}\{n\} \times X_n$ is a neighborhood of $x_e$. However since we have that $F$ is disjoint from $W_e \cap \bigcup_{n\in U}\{n\} \times X_n$, we contradict that $x_e$ is in $K$. □

**Definition 2.7.** A nice filter $\mathcal{F}$ on $X = \bigcup_{n\in\mathbb{N}}\{n\} \times X_n$, will be called non-trivial (respectively nowhere trivial) if for each sequence $x_n \in X_n$ ($n \in \mathbb{N}$), there is an $F \in \mathcal{F}$ such that $\{(n, x_n) : n \in \mathbb{N}\} \setminus F$ is infinite (respectively $(n, x_n) \notin F$ for all $n \in \mathbb{N}$).

**Lemma 2.8.** *If $\beta(\mathbb{N}\times X)$, for a compact $X$, embeds into $\mathbb{N}^*$ as a weak P-set, and if $\mathcal{F}$ is a non-trivial (respectively nowhere trivial) maximal nice filter on $\mathbb{N} \times X$, then $K_{\mathcal{F}}$ is sent to a non-trivial (respectively nowhere trivial) copy of $\mathbb{N}^*$ under the embedding.*

*Proof.* Let $D \subset \beta\mathbb{N}$ be countable and discrete. Identify $\beta(\mathbb{N} \times X)$ with its image in $\beta\mathbb{N}$. We prove that $\overline{D} \setminus D$ is not equal to $K$. First note that $K$ is a nowhere dense subset of $\beta(\mathbb{N} \times X)$, hence it is nowhere dense in $\mathbb{N}^*$. Thus we may assume that $D$ is disjoint from $\mathbb{N}$. Also $D \setminus (\mathbb{N} \cup \beta(\mathbb{N} \times X))$ is a countable subset of $\mathbb{N}^*$ and so its closure will miss $\beta(\mathbb{N} \times X)$. Thus we have that $D$ is contained in $\beta(\mathbb{N} \times X)$. If $D \cap (\mathbb{N} \times X)$ is infinite, then there is an infinite subset $E$ of this intersection which is disjoint from some $F \in \mathcal{F}$. Since $\mathbb{N} \times X$ is normal, $E$ and $F$ have disjoint closures in $\beta(\mathbb{N} \times X)$, hence we have that the closure of $E$ is disjoint from $K$. Finally, we assume that $D$ is contained in the remainder of $\mathbb{N} \times X$. The image of $D$ by the projection map $\pi^\beta$ is a countable subset of $\mathbb{N}^*$, and so it is not dense. Again, it follows that $\overline{D}$ can not contain $K$ since $\pi^\beta[K] = \mathbb{N}^*$.

The proof for nowhere trivial is similar. □

Now we turn the construction of nowhere trivial maximal nice filters. We utilize the structure of an Aronszajn tree to ensure nowhere trivial. Let $T \subset 2^{<\omega_1}$ be an Aronszajn tree; specifically $T$ is downward closed, no maximal elements, and for all $\alpha < \beta \in \omega_1$, $T \cap 2^\alpha$ countable and for each $t \in T \cap 2^\alpha$, there is an extension of $t$ in $T \cap 2^\beta$. This next result is from [4].

**Lemma 2.9.** *There exists a family $\{t(n,\alpha) : n \in \mathbb{N}, \alpha \in \omega_1\} \subset T$ such that*

(1) *for each $\alpha \in \omega_1$, $\{t(n,\alpha) : n \in \mathbb{N}\} \subset T \cap 2^\alpha$,*
(2) *for $\beta < \alpha$, there is an $n \in \mathbb{N}$ so that for all $k \geq n$, $t(k,\beta) \subset t(k,\alpha)$.*

*Say that this is a nicely descending family.*

*Proof.* For each $n \in \mathbb{N}$, $t(0,n)$ is simply the empty function (root of $T$). Also, $t(1,n)$ is either element of $T \cap 2^1$ for all $n \in \mathbb{N}$. Let $\delta \in \omega_1$ and assume that $\{t(n,\alpha) : n \in \mathbb{N}, \alpha \in \delta\}$ have been chosen so that the conditions are satisfied for each $\beta < \alpha < \delta$. If $\delta = \alpha + 1$, then again $t(n,\delta)$ is simply either of the two extensions of $t(n,\alpha)$ for each $n \in \mathbb{N}$. If $\delta$ is a limit, let $\{\alpha_j : j \in \omega\}$ enumerate the ordinals below $\delta$. For each $j \in \omega$, let $\bar{\alpha}_j = \max\{\alpha_i : i \leq j\}$. Choose $n_j \in \mathbb{N}$ (strictly increasing with $j$) large enough so that for all $k \geq n_j$ and all $i \leq j$, $t(k,\alpha_i) \subset t(k,\bar{\alpha}_j)$. For each $j$ and $k \in [n_j, n_{j+1})$, choose $t(k,\delta) \in T \cap 2^\delta$ to be any extension of $t(k,\bar{\alpha}_j)$. It is immediate that for each $k \geq n_j$ and each $i \leq j$, $t(k,\alpha_i) \subset t(k,\delta)$. This completes the proof. □

This next lemma is not sufficient for our needs because $2^{\omega_1}$ does not embed into $\beta\mathbb{N}$ but it provides the fundamental idea.

**Lemma 2.10.** *For each $t \in 2^{<\omega_1}$, let $[t]$ denote the closed set $\{x \in 2^{\omega_1} : t \subset x\}$. Let $\{t(n,\alpha) : n \in \mathbb{N}, \alpha \in \omega_1\}$ be a nicely descending family as in Lemma 2.9. The filter $\mathcal{F}$ on $\mathbb{N} \times 2^{\omega_1}$ generated by the family*

$$\{\bigcup_{n\in\mathbb{N}} \{n\} \times [t(n,\alpha)] : \alpha \in \omega_1\}$$

*is a nowhere trivial maximal nice filter.*

*Proof.* Property (2) of a nicely descending family ensures that $\mathcal{F}$ is a nice filter. Actually the fact that it is a free filter follows from the fact that $T$ has no cofinal branches, but this would be simple to ensure just by enlarging the filter base. To show that $\mathcal{F}$ is nowhere trivial, we let $\{x_n : n \in \mathbb{N}\} \subset 2^{\omega_1}$. Since $T$ has no uncountable branches, there is a $\delta \in \omega_1$ such that, for each $n \in \mathbb{N}$, $x_n \restriction \delta \notin T$. Clearly $\bigcup_{n\in\mathbb{N}}\{n\} \times [t(n,\delta)]$ is a member of $\mathcal{F}$ which avoids $\{(n, x_n) : n \in \mathbb{N}\}$ as required.

Now we prove that $\mathcal{F}$ is maximal. Since $2^{\omega_1}$ is zero-dimensional and compact, it suffices to consider any sequence $\{A_0^n, A_1^n : n \in \mathbb{N}\}$ of complementary clopen sets. Choose $\delta \in \omega_1$ large enough so that each $A_0^n$ is in $e_\delta[RO(2^\delta)]$ as discussed above (i.e. in standard terminology, the support of the clopen sets are contained in $\delta$). Of course, for each $n \in \mathbb{N}$, $[t(n,\delta)]$ is contained in one of $\{A_0^n, A_1^n\}$. It is now trivial to see that $F = \bigcup_{n\in\mathbb{N}}\{n\} \times [t(n,\delta)]$ is the desired witness to maximality. □

We are now ready to prove our main result.

**Theorem 2.11.** *There is a non-trivial copy of* $\mathbb{N}^*$.

By Lemmas 2.4 and 2.8, it suffices to prove the following result which is of independent interest. We lift the construction from Lemma 2.10 to $\mathbb{N} \times E(2^{\omega_1})$.

**Lemma 2.12.** *There is a nowhere trivial maximal nice filter on* $\mathbb{N} \times E(2^{\omega_1})$.

*Proof.* Let $\{t(n,\alpha) : n \in \mathbb{N}, \alpha \in \omega_1\}$ be the nicely descending family constructed in Lemma 2.9. For each $n, \alpha$ we may regard the point $t(n,\alpha) \in 2^\alpha$ as generating the filter $\mathcal{T}(n,\alpha)$ of clopen subsets of $2^\alpha$ containing $t(n,\alpha)$. We will select by induction on $\alpha$, an ultrafilter $\mathcal{U}(n,\alpha)$ on $RO(2^\alpha)$ extending $\mathcal{T}(n,\alpha)$. For $\beta < \alpha$, let $e_{\beta,\alpha}$ be the natural embedding of $RO(2^\beta)$ into $RO(2^\alpha)$ given by $e_{\beta,\alpha}(U) = U \times 2^{\alpha\setminus\beta}$ for $U \in RO(2^\beta)$. Our inductive hypothesis is that for $\beta < \alpha$ there is an $n \in \mathbb{N}$, such that for all $k \geq n$, $\mathcal{U}(k,\alpha)$ extends the filter generated by $\mathcal{T}(k,\alpha) \cup e_{\beta,\alpha}[\mathcal{U}(k,\beta)]$. For $\beta < \omega$, the definition of $\mathcal{U}(n,\beta)$ is simply that it equals $\mathcal{T}(n,\beta)$. Now suppose that $\delta \in \omega_1$ and that we have constructed $\mathcal{U}(n,\alpha)$ for all $\alpha \in \delta$ and $n \in \mathbb{N}$. We proceed much as in the proof of Lemma 2.9. Let $\{\alpha_i : i \in \omega\}$ enumerate $\delta$ and, for each $j \in \omega$, let $\bar{\alpha}_j = \max\{\alpha_i : i \leq j\}$. By the induction hypothesis, there is a strictly increasing sequence $\{n_j : j \in \omega\} \subset \mathbb{N}$ so that for each $k \geq n_j$ and $i \leq j$, we have

(1) $t(k,\alpha_i) \subset t(k,\bar{\alpha}_j) \subset t(k,\delta)$, and
(2) $e_{\alpha_i,\bar{\alpha}_j}[\mathcal{U}(k,\alpha_i)] \subset \mathcal{U}(k,\bar{\alpha}_j)$

It follows then that, for each $k \in [n_j, n_{j+1})$ and each $i \le j$,

$$e_{\alpha_i,\delta}[\mathcal{U}(n,\alpha_i)] \cup \mathcal{T}(k,\delta) \subset e_{\bar\alpha_j,\delta}[\mathcal{U}(n,\bar\alpha_j)] \cup \mathcal{T}(k,\delta)$$

generates a regular filter on $RO(2^\delta)$. Indeed, if $W \in \mathcal{T}(k,\delta)$, then, since $t(k,\bar\alpha_j) \subset t(k,\delta)$, there is a $W' \in \mathcal{T}(k,\bar\alpha_j)$ such that $W'$ is the projection of $W$ into $2^{\bar\alpha_j}$. If $U \in \mathcal{U}(k,\bar\alpha_j)$, then $W' \cap U$ is not empty, and if follows that $W \cap e_{\bar\alpha_j,\delta}[U] = W \cap (U \times 2^{\delta\setminus\bar\alpha_j})$ is not empty. Choose any ultrafilter $\mathcal{U}(k,\delta)$ which extends $e_{\bar\alpha_j,\delta}[\mathcal{U}(k,\bar\alpha_j)] \cup \mathcal{T}(k,\delta)$. This completes the inductive construction.

We consider the nice filter $\mathcal{F}$ generated by the family

$$\{\bigcup_{n\in\mathbb{N}} \{n\} \times U_n^* : (\exists \alpha \in \omega_1)\ (\forall n \in \mathbb{N})\ U_n \in e_\alpha[\mathcal{U}(n,\alpha)]\}\ ,$$

recall that $U^*$ is a basic clopen subset of $E(2^{\omega_1})$ for $U \in RO(2^{\omega_1})$.

We omit the easy verification that $\mathcal{F}$ generates a nice filter. To see that $\mathcal{F}$ is nowhere trivial, fix any sequence $\{\mathcal{W}_n : n \in \mathbb{N}\} \subset E(2^{\omega_1})$. For each $n \in \mathbb{N}$, let $x_n = \varphi(\mathcal{W}_n)$ (i.e. the unique point whose clopen neighborhood base is contained in $\mathcal{W}_n$). Fix any $\delta \in \omega_1$ so that, for all $n \in \mathbb{N}$, $x_n \restriction \delta \notin T$. For each $n$, choose clopen $U_n \in \mathcal{T}(n,\delta)$ so that $x_n \restriction \delta \notin U_n$. It follows that $e_\delta[U_n] \notin \mathcal{W}_n$ for each $n \in \mathbb{N}$. Therefore, $\bigcup_{n\in\mathbb{N}}\{n\} \times U_n^*$ is the desired member of $\mathcal{F}$ which avoids the set $\{(n,\mathcal{W}_n) : n \in \mathbb{N}\}$.

Finally we check that $\mathcal{F}$ is maximal. Again, since $E(2^{\omega_1})$ is compact and zero-dimensional, it suffices to consider a sequence $\{(A_0^n)^*, (A_1^n)^* : n \in \mathbb{N}\}$ of basic clopen subsets of $E(2^{\omega_1})$. For each $n$, $A_0^n$ and $A_1^n$ are complementary regular open subsets of $2^{\omega_1}$. Choose $\delta \in \omega_1$ large enough so that each $A_e^n$ is in $e_\delta[RO(2^\delta)]$ and let $U_e^n \in RO(2^\delta)$ be chosen so that $e_\delta[U_e^n] = A_e^n$. Let $f \in 2^{\mathbb{N}}$ be chosen so that for each $n \in \mathbb{N}$, $U_{f(e)}^n$ is in the ultrafilter $\mathcal{U}(n,\delta)$. Clearly $\bigcup_{n\in\mathbb{N}}\{n\} \times (A_{f(n)}^n)^*$ is a member of $\mathcal{F}$. □

## 3. Maximal nice filters on $2^\omega$

In the previous section we established that $\mathbb{N} \times 2^{\omega_1}$ and $\mathbb{N} \times E(2^{\omega_1})$ carry nowhere trivial maximal nice filters. In [6] it was asked if $\mathbb{N} \times E(2^\omega)$ carries such a filter. It turns out that it does simply because $E(2^{\omega_1})$ can be embedded as a closed subspace of $E(2^\omega)$. In this section we consider the situation for metric spaces. We will establish that it is independent of Martin's Axiom plus $\mathfrak{c} = \omega_2$ whether $\mathbb{N} \times 2^\omega$ carries a non-trivial maximal nice filter. These results are closely related to the question of the existence of non-trivial autohomeomorphisms on $\mathbb{N}^*$.

*Question* 1. If $\mathbb{N}\times 2^\omega$ carries a nowhere trivial maximal nice filter, then does there exist a non-trivial autohomeomorphism on $\mathbb{N}^*$?

We skip the easy inductive proof of the next result, but just include it for context.

**Proposition 3.1** (CH). *If $\{X_n : n\in\mathbb{N}\}$ is a sequence of finite sets whose cardinalities diverge to infinity, then $\bigcup_{n\in\mathbb{N}}\{n\}\times X_n$ carries a nowhere trivial maximal nice filter.*

The connection to non-trivial autohomeomorphisms will be helpful. We will need some notions from [8].

**Definition 3.2.** An embedding $\varphi : \mathbb{N}^*\to\mathbb{N}^*$ is *trivial* on $A\subset\mathbb{N}$, if there is a function $\psi : A\to\beta\mathbb{N}$ such that $\psi^\beta\supset\varphi\restriction A^*$. An ideal $\mathcal{I}$ on a set $Y$ is said to be ccc over fin if every uncountable almost disjoint family of subsets of $Y$ has all but countably many members in $\mathcal{I}$. Analogously, a closed set $J\subset\mathbb{N}^*$ will be said to be ccc over fin if the dual ideal $\mathcal{I}$ of subsets of $\mathbb{N}$ whose closure misses $J$ is ccc over fin.

**Theorem 3.3.** [8, 3.8.1] *PFA implies that for every embedding $\varphi$ of $\mathbb{N}^*$ into $\mathbb{N}^*$, there is a partition $A\cup B$ of $\mathbb{N}$ such that $\varphi$ is trivial on $A$ and $\varphi[B^*]$ is ccc over fin.*

**Corollary 3.4** (PFA). *If $\{X_n : n\in\mathbb{N}\}$ is a family of finite sets, then every maximal nice filter on $\bigcup_{n\in\mathbb{N}}\{n\}\times X_n$ is trivial in that there is a sequence $\{x_n : n\in\mathbb{N}\}$ such that $\{(n,x_n) : n\in\mathbb{N}\}$ is a member of the filter.*

*Proof.* Since each $X_n$ is finite, the space $\beta X$ with $X=\bigcup_{n\in\mathbb{N}}\{n\}\times X_n$ is a copy of $\beta\mathbb{N}$. If $\mathcal{F}$ is a maximal nice filter on $X$, then $K=K_{\mathcal{F}}$ is such that the projection map $\pi^\beta$ is one-to-one on $K$. Let $\varphi$ be the inverse of the $\pi^\beta\restriction K$. By Theorem 3.3, there is a partition $A\cup B$ of $\mathbb{N}$ such that $\varphi$ is trivial on $A$ and $\varphi[B^*]$ is ccc over fin in $X^*$. It is immediate that this implies that $B$ is finite since for each infinite $D\subset B$, $(\pi^\beta)^{-1}(D^*)$ will meet $K$, meaning that $\varphi[B^*]$ could not be ccc over fin. Let $\psi : A\to\beta X$ be chosen so that $\psi^\beta\supset\varphi$. We note that we may assume that $\psi[A]\subset X$ since $\pi^\beta[\psi[A]\setminus X]$ is a nowhere dense subset of $\mathbb{N}^*$. It is easily checked that $\psi[A]$ will, mod finite, contain a set of the form $\{(n,x_n) : n\in\mathbb{N}\}$ as in the statement of the Corollary. ☐

In a paper [2, 2.5] studying the existence of non-trivial autohomeomorphisms in a forcing extension of a model of PFA, a generic filter on $\bigcup_{n\in\mathbb{N}}\{n\}\times 2^n$ is added satisfying the second statement of the next theorem.

**Theorem 3.5.** *It is consistent with MA + $\mathfrak{c} = \omega_2$ that there are no nowhere trivial maximal nice filters on any set of the form $\bigcup_{n\in\mathbb{N}}\{n\} \times X_n$ (with each $X_n$ finite), while there is a non-trivial maximal nice filter on $\bigcup_{n\in\mathbb{N}}\{n\} \times 2^n$.*

It is also shown in [2] that all embeddings of $\mathbb{N}^*$ into $\mathbb{N}^*$ are somewhere trivial, and so the first half of the statement holds in that model.

In a forthcoming paper, a proof of the following statement will be provided. The proof is similar to the construction of a nowhere trivial autohomeomorphism on $\mathbb{N}^*$ in a model of MA + $\mathfrak{c} = \omega_2$ given in [13].

**Proposition 3.6.** [1] *It is consistent with MA + $\mathfrak{c} = \omega_2$ that there is a nowhere trivial maximal nice filter on $\bigcup_{n\in\mathbb{N}}\{n\} \times 2^n$.*

Finally we strengthen Corollary 3.4 to the main result of this section. It modifies the method used in [3]. It uses the following key consequence of PFA established by Todorcevic (see [8, 2.2.7]).

**Proposition 3.7** (PFA)**.** *If $\{h_f : f \in \omega^\omega\}$ is a family of functions into $\mathbb{N}$ with the property that for each $f < g \in \omega^\omega$, $\mathrm{dom}(h_f) \subset \mathrm{dom}(h_g)$ and the set $\{x \in \mathrm{dom}(h_f) : h_f(x) \neq h_g(x)\}$ is finite, then there is a single function $h$ such that $h_f \subset^* h$ for all $f \in \omega^\omega$.*

**Theorem 3.8.** *PFA implies that $\mathbb{N} \times 2^\omega$ does not carry a non-trivial maximal nice filter.*

*Proof.* Let $\mathcal{F}$ be a maximal nice filter on $\mathbb{N} \times 2^\omega$. By identifying each $t \in 2^{<\omega}$ with the canonical clopen set $[t] \subset 2^\omega$, it is evident that, for each $f \in \omega^{\mathbb{N}}$, $\mathcal{F}$ induces a maximal nice filter, $\mathcal{F}_f$ on $\bigcup_{n\in\mathbb{N}}\{n\} \times 2^{f(n)}$. By Corollary 3.4, there is a sequence $\{t^f_n : n \in \mathbb{N}\}$ so that $t^f_n \in 2^{f(n)}$ (for $n \in \mathbb{N}$) and $\{(n, t^f_n) : n \in \mathbb{N}\}$ is a member of $\mathcal{F}_f$. Notice that, for $f < g$ both in $\omega^{\mathbb{N}}$, $t^f_n \subset t^g_n$ for all but finitely many $n \in \mathbb{N}$. Define $h_f : \bigcup_{n\in\mathbb{N}}\{n\} \times 2^{\leq f(n)} \to 2$ according to $h_f(t) = 1$ providing $t \subset t^f_n$. By Proposition 3.7, there is a single function $h : \mathbb{N} \times 2^{<\omega} \to 2$ which mod finite extends each $h_f$. It then follows that for all but finitely many $n \in \mathbb{N}$, $C_n = \{t \in 2^{<\omega} : h(n,t) = 1\}$ is a maximal chain. Let $\{x_n : n \in \mathbb{N}\} \subset 2^\omega$ be any sequence so that for each $n$ such that $C_n$ is a maximal chain, $x_n$ is the union of that chain. Let, for each $n \in \mathbb{N}$, $U_n$ be a clopen neighborhood of $x_n$. There is an $f \in \omega^{\mathbb{N}}$ such that $[x_n \restriction f(n)] \subset U_n$ for each $n$. Since $[x_n \restriction f(n)] = [t^f_n]$ for all but finitely many $n \in \mathbb{N}$, it follows that $\bigcup_{n\in\mathbb{N}}\{n\} \times U_n$ is in $\mathcal{F}$. Therefore $\mathcal{F}$ fails to be non-trivial as witnessed by $\{(n, x_n) : n \in \mathbb{N}\}$. □

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