# COSMIC DIMENSIONS

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Abstract. Martin's Axiom for $\sigma$-centered partial orders implies that there is a cosmic space with non-coinciding dimensions.

## Introduction

A fundamental result in dimension theory states that the three basic dimension functions, dim, ind and Ind, coincide on the class of separable metrizable spaces. Examples abound to show that this does not hold in general outside this class. In [1] Arkhangel′skiĭ asked whether the dimension functions coincide on the class of cosmic spaces. These are the regular **co**ntinuous images of **s**eparable **m**etric spaces and they are characterized by the conjunction of regularity and having a countable network. A *network* for a topological space is a collection of (arbitrary) subsets such that every open set is the union of some subfamily of that collection. In [6] Vedenisoff proved that ind and Ind coincide on the class of perfectly normal Lindelöf spaces, see also [3, Section 2.4]. As the cosmic spaces belong to this class Arkhangel′skiĭ's question boils down to whether dim = ind for cosmic spaces.

In [2] Delistathis and Watson constructed, assuming the Continuum Hypothesis, a cosmic space $X$ with $\dim X = 1$ and $\operatorname{ind} X \geqslant 2$; this gave a consistent negative answer to Arkhangel′skiĭ's question.

The purpose of this paper is to show that the example can also be constructed under the assumption of Martin's Axiom for $\sigma$-centered partial orders. The overall strategy is that of [2]: we refine the Euclidean topology of a one-dimensional subset $X$ of the plane to get a topology $\tau$ with a countable network, such that $\dim(X, \tau) = 1$ and in which the boundary of every non-dense open set is (at least) one-dimensional, so that $\operatorname{ind}(X, \tau) \geqslant 2$. The latter is achieved by ensuring that every such boundary contains a topological copy of the unit interval or else a copy of the Cantor set whose subspace topology is homeomorphic to Kuratowski's graph topology, as defined in [4].

The principal difference between our approach and that of [2] lies in the details of the constructions. In [2] the topology is introduced by way of resolutions; however, some of the arguments given in the paper need emending because, for example, Kuratowski's function does not have the properties asserted and used in Lemmas 2.2 and 2.3 of [2] respectively. We avoid this and use Kuratowksi's fuction explicitly in the defintion of new local bases at certain points and thus make the definition of the topology more straightforward. Also, the use of partial orders leads to a cleaner and more perspicuous construction of the Cantor sets.

We begin with a careful analysis of Kuratowski's function in Sections 1 and 2. We then show how to transplant the graph topology to an arbitrary Cantor set in the plane. The remainder of the paper is devoted to a recursive construction of the

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necessary Cantor sets and finishes with a verification of the properties of the new topology. An outline of the full construction can be found in Section 4.

## 1. Kuratowski's function

In this section we give a detailed description of Kuratowski's function ([4], see also [3, Exercise 1.2.E]) and the resulting topology on the Cantor set. We do this to make our note self-contained and because the construction makes explicit use of this description.

Let $C$ be the Cantor set, represented as the topological product $2^{\mathbb{N}}$, and for $x \in C$ write $\operatorname{supp} x = \{i : x(i) = 1\}$. We let $D$ be the set of $x$ for which $\operatorname{supp} x$ is finite, partitioned into the sets $D_k = \{x : |\operatorname{supp} x| = k\}$; put $k_x = |\operatorname{supp} x|$ and $N_x = \max \operatorname{supp} x$ for $x \in D$. Note that $D_0 = \{\mathbf{0}\}$, where $\mathbf{0}$ is the point with all coordinates 0 (so, $N_{\mathbf{0}} = \operatorname{dom} c_{\mathbf{0}} = \emptyset$). Let $E = C \setminus D$, the set of $x$ for which $\operatorname{supp} x$ is infinite.

For $x \in C$ let $c_x$ be the counting function of $\operatorname{supp} x$, so $\operatorname{dom} c_x = \{1, \ldots, k_x\}$ if $x \in D$ and $\operatorname{dom} c_x = \mathbb{N}$ if $x \in E$.

Now define

$$f(x) = \sum_{j \in \operatorname{dom} c_x} (-1)^{c_x(j)} 2^{-j}$$

Thus we use the parity of $c_x(j)$ to decide whether to add or subtract $2^{-j}$. By convention an empty sum has the value 0, so $f(\mathbf{0}) = 0$.

Notation: if $x \in C$ and $n \in \mathbb{N}$ then $x \restriction n$ denotes the restriction of $x$ to the set $\{1, 2, \ldots, n\}$. Also, $[x \restriction n]$ denotes the $n$-th basic open set around $x$: $[x \restriction n] = \{y : y \restriction n = x \restriction n\}$.

For $x \in D$ we write $V_x = [x \restriction N_x]$. Using the $V_x$ it is readily seen that the sets $D_k$ are relatively discrete: simply observe that $V_x \cap \bigcup_{i \leqslant k_x} D_i = \{x\}$. In fact, for a fixed $k$ the family $\mathcal{D}_k = \{V_x : x \in D_k\}$ is pairwise disjoint. For later use we put $D_x = \{y \in D_{k_x+1} : y \restriction N_x = x \restriction N_x\}$ and we observe that $V_x = \{x\} \cup \bigcup \{V_y : y \in D_x\}$.

### 1.1. Continuity.

We begin by identifying the points of continuity of $f$.

**Proposition 1.1.** *The function $f$ is continuous at every point of $E$.*

*Proof.* Let $x \in E$ and let $\varepsilon > 0$ be given. Choose $N$ so large that $2^{-N} < \varepsilon/2$ and let $M = c_x(N)$. If $x \restriction M = y \restriction M$ then $c_x(j) = c_y(j)$ for $j \leqslant N$, so that

$$f(x) - f(y) = \sum_{j>N} (-1)^{c_x(j)} 2^{-j} - \sum_{j>N} (-1)^{c_y(j)} 2^{-j}.$$

The absolute value of the right-hand side is not larger than $2 \sum_{j>N} 2^{-j} = 2 \cdot 2^{-N}$, which is less than $\varepsilon$. This shows that $f$ is continuous at $x$. □

### 1.2. Distribution of values.

The function $f$ is definitely not continuous at the points of $D$. This will become clear from the following discussion on the distribution of the values of $f$.

**Proposition 1.2.** *Let $t \in [-1, 1]$. The preimage $f^{\leftarrow}(t)$ is uncountable, crowded and its intersection with $E$ is closed in $E$.*

*Proof.* The last claim follows immediately from Proposition 1.1, so we concentrate on the other two claims.

We associate a sequence $\langle i_k : k \in \mathbb{N} \rangle$ of zeros and ones with $t$. Recursively let $i_k = 0$ if $t \geqslant \sum_{j<k} (-1)^{i_j} 2^{-j}$ and $i_k = 1$ otherwise (if $k = 1$ the sum is zero by our convention on empty sums). One shows by induction that $|t - \sum_{j=1}^{k} (-1)^{i_j} 2^{-j}| \leqslant 2^{-k}$, so that $t = \sum_{j=1}^{\infty} (-1)^{i_j} 2^{-j}$. For every strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ we define $x = x_{t,\phi} \in C$ to be zero everywhere except at $2\phi(j)$ when $i_j = 0$ and at

$2\phi(j+1)$ when $i_j = 1$. The parity of $c_x(j)$ is everywhere the same as that of $i_j$; this implies that $f(x) = t$. Clearly then the set $\{x_{t,\phi} : \phi \text{ strictly increasing}\}$ is an uncountable subset of $f^{\leftarrow}(t)$.

To see that $f^{\leftarrow}(t)$ is crowded assume $f(x) = t$. In case $x \in D$ define $y_n$ for each $n$ as follows: $y_n \restriction N_x = x \restriction N_x$ and above $N_x$ we let $y_n$ be zero, except at the coordinates $2N_x + 2n$, $2N_x + 2n + 1$, $2N_x + 2n + 3$, $2N_x + 2n + 5$, ... The sequence $\langle y_n \rangle_n$ converges to $x$ and for all $n$ we have

$$f(y_n) = f(x) + \bigl(2^{-(k_x+1)} - \sum_{j > k_x+1} 2^{-j}\bigr) = f(x)$$

In case $x \in E$ define $y_n$ for all $n$ as follows: $y_n \restriction n = x \restriction n$ and above $n$ put $y_n(n+1) = y_n(n+2) = 0$ and $y_n(n+k) = x(n+k-2)$ for $k \geqslant 3$. Again $y_n \to x$ and $f(y_n) = f(x)$ for all $n$. $\square$

**Proposition 1.3.** *Let $x \in D$ and $k = k_x$. Then $x$ is an accumulation point of $f^{\leftarrow}(t)$ if and only if $f(x) - 2^{-k} \leqslant t \leqslant f(x) + 2^{-k}$.*

*Proof.* To prove 'only if' note that $f(x) - 2^{-k} \leqslant f(y) \leqslant f(x) + 2^{-k}$ whenever $y \restriction N_x = x \restriction N_x$.

To prove 'if' let $t$ be as in the statement of the proposition. We localize the proof of Proposition 1.2: define a sequence $\langle i_j : j \in \mathbb{N} \rangle$ by starting with $i_j = c_x(j) \bmod 2$ for $j \leqslant k$ and proceeding as above for $j > k$. In the end $t = \sum_{j=1}^{\infty} (-1)^{i_j} 2^{-j}$. Then define a sequence $\langle y_n \rangle_n$ much as in the previous proposition. First ensure $y_n \restriction N_x = x \restriction N_x$, next $y_n(i) = 0$ for $N_x < i < 2(N_x + k + n)$ and to finish let $y_n(2(N_x + n + j) + i_j) = 1$ and $y_n(2(N_x + n + j) + 1 - i_j) = 0$ for $j > k$. In the end $y_n \to x$ and $f(y_n) = t$ for all $n$. $\square$

1.3. **The dimension of the graph.** We identify $f$ with its graph in $C \times [-1, 1]$ and we write $\mathbb{I} = [-1, 1]$. For $x \in D$ we let $I_x = [f(x) - 2^{-k_x}, f(x) + 2^{-k_x}]$. The discussion in the previous subsection can be summarized by saying that the closure of $f$ in $C \times \mathbb{I}$ is equal to the set $K = f \cup \bigcup_{x \in D} \bigl(\{x\} \times I_x\bigr)$.

**Proposition 1.4.** $\operatorname{ind} f \leqslant 1$.

*Proof.* This is clear, by the subset theorem, as $\operatorname{ind}(C \times \mathbb{I}) = 1$. $\square$

**Proposition 1.5.** *If $x \in E$ then $\operatorname{ind}_{\langle x, f(x) \rangle} f = 0$.*

*Proof.* For each $n$ the set $[x \restriction n] = \{y : y \restriction n = x \restriction n\}$ is a neighbourhood of $x$ and the family of all such sets is a local base at $x$ in $C$. Because $f$ is continuous at $x$ the family of intersections $\bigl([x \restriction n] \times \mathbb{I}\bigr) \cap f$ is a local base at $\bigl\langle x, f(x) \bigr\rangle$ in $f$. These sets are clopen. $\square$

**Proposition 1.6.** *If $x \in D$ then $\operatorname{ind}_{\langle x, f(x) \rangle} f = 1$.*

*Proof.* Work in the compact set $K$. Put $t = f(x) + 2^{-k_x}$ and let $F = (C \times \{t\}) \cap K$; observe that, by Proposition 1.3, $\langle x, t \rangle \in F$.

If $\operatorname{ind}_{\langle x, f(x) \rangle} f$ were zero then we could find a partition $L$ in $K$ between $\bigl\langle x, f(x) \bigr\rangle$ and $F$ such that $L \cap f = \emptyset$. As $L$ is compact, so is its projection $\pi_C[L]$; this projection is a subset of $D$, hence countable and therefore scattered.

We reach a contradiction by showing that $\pi_C[L]$ contains a crowded subset. To this end we define a sequence $\langle x_n \rangle_n$ in $D_{k_x+1}$ by stipulating that $\operatorname{supp} x_n = \operatorname{supp} x \cup \{2(N_x + n)\}$. Then $x_n \to x$ and for all $n$ we have $f(x_n) + 2^{-(k_x+1)} = t$ and $f(x_n) - 2^{-(k_x+1)} = f(x)$. It follows that $L$ is a partition in $K$ between $\bigl\langle x_n, f(x) \bigr\rangle$ and $\langle x_n, t \rangle$ for almost all $n$, which means that $L$ must intersect the interval $\{x_n\} \times [f(x), t]$ for all those $n$, so that $x$ is an accumulation point of $\pi_C[L]$. This may repeated for each point $x_n$ above as $L$ is also a partition between $\bigl\langle x_n, f(x_n) \bigr\rangle$ and one of $\langle x_n, f(x_n) \pm 2^{-(k_x+1)}\rangle$. Then repeat this for the terms of the sequences converging to the $x_n$ and so on to get the crowded subset. $\square$

## 2. Basic neighbourhoods in the topology $\tau_f$

It will be useful to have a description of the topology of $f$ in terms of $C$ alone, that is, we consider the (separable metric) topology $\tau_f$ on $C$ given by $\tau_f = \{O_f : O \text{ open in } C \times \mathbb{I}\}$, where $O_f = \{x : \langle x, f(x)\rangle \in O\}$.

2.1. **The points of $E$.** The proof of Proposition 1.5 shows that at the points of $E$ the topology $\tau_f$ is the same as the usual topology. To be precise: if $x \in E$ then the family $\{[x \restriction n] : n \in \mathbb{N}\}$ is a local base for $\tau_f$ at $x$.

2.2. **The points of $D$.** For ease of notation we describe the neighbourhoods of $\mathbf{0}$ (the point with all coordinates 0). For $n \in \mathbb{N}$ the $n$-the basic neighbourhood of $\langle \mathbf{0}, f(\mathbf{0})\rangle$ in $C \times \mathbb{I}$ is $B_n = [\mathbf{0} \restriction n] \times (-2^{-n}, 2^{-n})$. We let $U_n = \{x : \langle x, f(x)\rangle \in B_n\}$. Clearly $U_n \subseteq [\mathbf{0} \restriction n]$ but the question is how much of $[\mathbf{0} \restriction n]$ actually is in $U_n$. Until further notice we leave $n$ fixed.

**Lemma 2.1.** *If $x \in D_k$ and $k \geqslant 1$ then $|f(x)| \geqslant 2^{-k}$.*

*Proof.* If $c_x(1)$ is even then $f(x) \geqslant 2^{-1} - 2^{-2} - \cdots - 2^{-k} = 2^{-k}$ and if $c_x(1)$ is odd then $f(x) \leqslant -2^{-1} + 2^{-2} + \cdots + 2^{-k} = -2^{-k}$. $\square$

This lemma shows that $D_k \cap U_n = \emptyset$ whenever $k \leqslant n$, so that in the internal description we must subtract $\bigcup_{k=1}^n D_k$ from $[\mathbf{0} \restriction n]$.

We know that the set $D_{n+1}$ is relatively discrete and that the family $\mathcal{D}_{n+1} = \{V_x : x \in D_{n+1}\}$ is a pairwise disjoint family of clopen sets; note that $\bigcup \mathcal{D}_{n+1} = C \setminus \bigcup_{k=0}^n D_k$. For each $x \in C$ we let $p_x \in \{-1,1\}^{\operatorname{dom} c_x}$ be its *pattern*, i.e., $p_x(i) = (-1)^{c_x(i)}$ for $i \in \operatorname{dom} c_x$.

**Lemma 2.2.** *If $x \in D_{n+1}$ and $p_x(k) = p_x(1)$ for some $k \geqslant 2$ then $V_x \cap U_n = \emptyset$.*

*Proof.* Assume $p_x(1) = p_x(k) = 1$. Then $f(x) \geqslant 2^{-1} - 2^{-2} - \cdots - 2^{-n} + 2^{-(n+1)} = 2^{-n} + 2^{-(n+1)}$, and so $f(y) \geqslant 2^{-n}$ whenever $y \restriction N_x = x \restriction N_x$.

The same argument applies when $p_x(1) = p_x(k) = -1$, but with all signs opposite of course. $\square$

For want of a better term let us say that $x \in D$ satisfies (†) if its pattern satisfies $p_x(k) = -p_x(1)$ for all $k \geqslant 2$.

It follows that $[\mathbf{0} \restriction n] \setminus \bigcup_{k=1}^n D_k$ is the union of two disjoint closed sets: $C_n = \bigcup\{V_x : x \in D_{n+1},\ x \restriction n = \mathbf{0} \restriction n \text{ and } x \textit{ does} \text{ satisfy condition (†)}\}$ and $F_n = \bigcup\{V_x : x \in D_{n+1},\ x \restriction n = \mathbf{0} \restriction n \text{ and } x \textit{ does not} \text{ satisfy condition (†)}\}$.

We must certainly also subtract $F_n$ from $[\mathbf{0} \restriction n] \setminus \bigcup_{k=1}^n D_k$. Finally then, from what remains, which is $\{\mathbf{0}\} \cup C_n$, we must still delete the points at which $f$ takes on the values $\pm 2^{-n}$.

The internal description of these point is as follows: if $y \in C_n$ then $f(y) = \pm 2^{-n}$ iff $y \in E$ and there is $x \in D_{n+1}$ that satisfies (†) such that $y \in V_x$ (so that its pattern extends that of $x$) and such that $p_y(k) = p_x(1)$ for $k > n+1$

*The other points of $D$.* Basic neighbourhoods of the other points of $D$ are copies of the $U_n$, obtained by shifting and scaling, that is,

$$U_n(x) = \{(x \restriction N_x) * y : y \in U_n\},$$

where $*$ denotes concatenation.

## 3. Making one Cantor set

We intend to copy the topology $\tau_f$ to many Cantor sets in the plane, or rather, we intend to construct many Cantor sets and copy $\tau_f$ to each of them. Here we describe how we will go about constructing just one Cantor set $K$, together with a homeomorphism $h: C \to K$, and how to refine the topology of the plane so that all points but those of $h[D]$ retain their usual neighbourhoods and so that at the points of $h[D]$ the dimension of $K$ will be 1.

All we need to make a Cantor set are two maps $\sigma : D \to \mathbb{R}^2$ and $\ell : D \to \omega$. Using these we define $W(d) = B(\sigma(d), 2^{-\ell(d)})$ for each $d \in D$. We want the following conditions fulfilled:

(1) the sequence $\langle \sigma(e) : e \in D_d \rangle$ converges to $\sigma(d)$, for all $d$;
(2) $\operatorname{cl} W(e) \subseteq W(d) \setminus \{d\}$ whenever $e \in D_d$;
(3) $\{\operatorname{cl} W(d) : d \in D_n\}$ is pairwise disjoint for all $n$.

The following formula then defines a Cantor set:

$$K = \bigcap_{n=0}^{\infty} \operatorname{cl}\bigl(\bigcup \{W(d) : d \in D_n\}\bigr). \tag{‡}$$

One readily checks that $\{\sigma(d) : d \in D\}$ is a dense subset and that the map $\sigma$ extends to a homeomorphism $h : C \to K$ with the property that $h[V_d] = K \cap W(d)$ for all $d \in D$.

Copying the Kuratowski function from $C$ to $K$ is an easy matter: we let $f_K = f \circ h^{-1}$. To copy the topology $\tau_f$ to $K$ *and* to preserve as much as possible of the Euclidean topology we use the sets $W(d)$ once more.

The goal is of course to assign neighbourhoods $V_n(d)$ to $h(d)$ that satisfy $V_n(d) \cap K = h[U_n(d)]$ for $d \in D$ and $n \in \mathbb{N}$, where the $U_n(d)$ are as defined in the previous section.

We deal with $h(\mathbf{0})$ first and we mimic the construction of the $U_n(\mathbf{0})$ from Section 2. First we compute for each $n$ the distance $\varepsilon_n$ between $H_n = h\bigl[[\mathbf{0} \restriction n]\bigr]$ and $K \setminus H_n$ and we set $O_n = B(H_n, \varepsilon/3)$. Then $O_n$ and $\operatorname{cl}_e O_n$ have the same intersection with $K$, namely $H_n$. Next we subtract from $O_n$ what corresponds to $\bigcup_{k=1}^n D_k \cup F_n \cup f^{\leftarrow}\bigl[\{2^{-n}, -2^{-n}\}\bigr]$. The first and third parts are easy: we subtract $\bigcup_{k=1}^n h[D_k] \cup f_K^{\leftarrow}\bigl[\{2^{-n}, -2^{-n}\}\bigr]$. For the second part we must subtract $G_n = \bigcup_{k=1}^n \bigcup \{\operatorname{cl}_e W(d) : d \in D_{k+1},\ d \restriction k = \mathbf{0} \restriction k$ and $d$ *does not* satisfy condition (†)$\bigr\}$. Thus the $n$th basic neighbourhood of $h(\mathbf{0})$ in the new topology is $V_n = O_n \setminus \bigl(\bigcup_{k=1}^n h[D_k] \cup G_n \cup f_K^{\leftarrow}\bigl[\{2^{-n}, -2^{-n}\}\bigr]\bigr)$.

The other points of $h[D]$ are dealt with similarly, by shifting and 'scaling' this process. To make $V_n(d)$ for $d \in D$ let $H_n(d) = h\bigl[[d \restriction (k_d + n)]\bigr]$ and determine its distance $\varepsilon_n(d)$ to its complement $K \setminus H_n(d)$. Then put $O_n(d) = B\bigl(H_n(d), \varepsilon_n(d)/3\bigr)$ and subtract from it the union of $\bigcup_{k=1}^n h[D_{k_d+k},\ f_K^{\leftarrow}\bigl[\{f(d) + 2^{-k_d-n}, f(d) - 2^{-k_d-n}\}\bigr]$ and $\bigcup_{k=1}^n \bigcup \{\operatorname{cl}_e W(e) : e \in D_{k_d+k+1},\ e \restriction (k_d + k) = (d * \mathbf{0}) \restriction (k_d + k)$ and $e \restriction [k_d + 1, N_e]$ *does not* satisfy condition (†)$\bigr\}$.

## 4. The plan

In this section we outline how we will construct a cosmic topology $\tau$ on a subset $X$ of the plane that satisfies $\dim(X, \tau) = 1$ and $\operatorname{ind}(X, \tau) \geqslant 2$.

We let $\mathcal{Q}$ denote the family of all non-trivial line segments in the plane with rational end points. Our subset $X$ will be $\mathbb{R}^2 \setminus A$, where $A = \{\langle p + \sqrt{2}, q \rangle : p, q \in \mathbb{Q}\}$. Note that $A$ is countable, dense and disjoint from $\bigcup \mathcal{Q}$. Also note that, with respect to the Euclidean topology $\tau_e$, one has $\operatorname{ind}(X, \tau_e) = 1$: on the one hand basic rectangles with end points in $A$ have zero-dimensional boundaries (in $X$), so that $\operatorname{ind}(X, \tau_e) \leqslant 1$, and on the other hand, because $X$ is connected we have $\operatorname{ind}(X, \tau_e) \geqslant 1$.

We will construct $\tau$ in such a way that its restrictions to $X \setminus \bigcup \mathcal{Q}$ and each element of $\mathcal{Q}$ will be the same as the restrictions of $\tau_e$; this ensures that $(X, \tau)$ has a countable network: take a countable base $\mathcal{B}$ for the Euclidean topology of $X \setminus \bigcup \mathcal{Q}$, then $\mathcal{Q} \cup \mathcal{B}$ is a network for $(X, \tau)$. Also, the $\tau_e$-interior of every open set in $(X, \tau)$ will be nonempty so that $\bigcup \mathcal{Q}$ and $X \setminus \bigcup \mathcal{Q}$ will be dense with respect to $\tau$.

It what follows cl will be the closure operator with respect to $\tau$ and $\mathrm{cl}_e$ will be the Euclidean closure operator.

**The topology.** We let $\{(U_\alpha, V_\alpha) : \alpha < \mathfrak{c}\}$ numerate all pairs of disjoint open sets in the plane whose union is dense and for each $\alpha$ we put $S_\alpha = \mathrm{cl}_e U_\alpha \cap \mathrm{cl}_e V_\alpha$. We shall construct for each $\alpha$ a Cantor set $K_\alpha$ in $X \cap S_\alpha$, unless there is a $Q_\alpha \in \mathcal{Q}$ that is contained in $S_\alpha$. The construction of the $K_\alpha$ will be as described in Section 3, so that we will be able to extend $\tau_e$ to a topology $\tau_\alpha$ whose restriction to $K_\alpha$ is a copy of the topology $\tau_f$. For notational convenience we let $I$ be the set of $\alpha$s for which we have to construct $K_\alpha$ and for $\alpha \in \mathfrak{c} \setminus I$ we set $\tau_\alpha = \tau_e$. As an aside we mention that $\mathfrak{c} \setminus I$ is definitely not empty: if the boundary of $U_\alpha$ is a polygon with rational vertices then $\alpha \notin I$.

Thus we may (and will) define, for any subset $J$ of $\mathfrak{c}$ a topology $\tau_J$: the topology generated by the subbase $\bigcup_{\alpha \in J} \tau_\alpha$. The new topology $\tau$ will $\tau_{\mathfrak{c}}$.

There will be certain requirements to be met (the first was mentioned already):

(1) The restriction of $\tau$ to $X \setminus \bigcup \mathcal{Q}$ and each $Q \in \mathcal{Q}$ must be the same as that of the Euclidean topology;
(2) Different topologies must not interfere: the restriction of $\tau$ to $K_\alpha$ should be the same as that of $\tau_\alpha$;
(3) For each $\alpha$, depending on the case that we are in, the set $K_\alpha$ or $Q_\alpha$ must be part of the $\tau$-boundary of $U_\alpha$.

If these requirements are met then the topology $\tau$ will be as required. We have already indicated that (1) implies that it has a countable network.

**The Inductive dimensions.** To see that $\operatorname{ind}(X, \tau) \geqslant 2$ we take an element $O$ of $\tau$ and show that its boundary is at least one-dimensional. There will be an $\alpha$ such that $\mathrm{cl}_e O = \mathrm{cl}_e U_\alpha$: there is $O' \in \tau_e$ such that $O \cap \bigcup \mathcal{Q} = O' \cap \bigcup \mathcal{Q}$ and we can take $\alpha$ such that $U_\alpha = \operatorname{int} \mathrm{cl}_e O'$ and $V_\alpha = \mathbb{R}^2 \setminus \mathrm{cl}_e U$. In case $\alpha \in I$ the combination of (2) and (3) shows that $\operatorname{ind} \operatorname{Fr} O \geqslant \operatorname{ind} K_\alpha = 1$ and in case $\alpha \notin I$ we use (1) and (3) to deduce that $\operatorname{ind} \operatorname{Fr} O \geqslant \operatorname{ind} Q_\alpha = 1$.

**The covering dimension.** As $\operatorname{ind}(X, \tau) \geqslant 2$ it is immediate that $\dim(X, \tau) \geqslant 1$. To see that $\dim(X, \tau) \leqslant 1$ we consider a finite open cover $\mathcal{O}$. Because $(X, \tau)$ is hereditarily Lindelöf we find that each element of $\mathcal{O}$ is the union of countably many basic open sets. This in turn implies that there is a countable set $J$ such that $\mathcal{O} \subseteq \tau_J$. This topology is separable and metrizable and it will suffice to show that $\dim(X, \tau_J) \leqslant 1$.

If $J$ is finite then we may apply the countable closed sum theorem: $O = X \setminus \bigcup_{\alpha \in J} K_\alpha$ is open, hence an $F_\sigma$-set, say $O = \bigcup_{i=1}^\infty F_i$. Each $F_i$ is (at most) one-dimensional as is each $K_\alpha$ and hence so is $X$, as the union of countably many one-dimensional closed subspaces.

If $J$ is infinite we numerate it as $\{\alpha_n : n \in \mathbb{N}\}$ and set $J_n = \{\alpha_i : i \leqslant n\}$. Then $(X, \tau_J)$ is the inverse limit of the sequence $\langle (X, \tau_{J_n}) : n \in \mathbb{N} \rangle$, where each bonding map $i_n : (X, \tau_{J_{n+1}}) \to (X, \tau_{J_n})$ is the identity. By Nagami's theorem ([5], see also [3, Theorem 1.13.4]) it follows that $\dim(X, \tau_J) \leqslant 1$.

## 5. The execution

The construction will be by recursion on $\alpha < \mathfrak{c}$. At stage $\alpha$, if no $Q_\alpha$ can be found, we take our cue from Section 3 and construct maps $\sigma_\alpha : D \to S_\alpha$ and $\ell_\alpha : D \to \omega$, in order to use the associated balls $W_\alpha(d) = B(\sigma_\alpha(d), 2^{-\ell_\alpha(d)})$ in formula (‡) to make the Cantor set $K_\alpha$. We also get a homeomorphism $h_\alpha : C \to K_\alpha$ as an extension of $d_\alpha$ and use this to copy Kuratowski's function to $K_\alpha$: we set $f_\alpha = f \circ h_\alpha^{-1}$. We then use the procedure from the end of Section 3 to construct the topology $\tau_\alpha$.

5.1. **The partial order.** We construct $\sigma_\alpha$ and $\ell_\alpha$ by an application of Martin's Axiom to a partial order that we describe in this subsection. To save on notation we suppress $\alpha$ for the time being. Thus, $S = S_\alpha$, $\sigma = \sigma_\alpha$, etc.

To begin we observe that $\bigcup \mathcal{Q} \cap S$ is dense in $S$: if $x \in S$ and $\varepsilon > 0$ then there are points $a$ and $b$ with rational coordinates in $B(x, \varepsilon)$ that belong to $U$ and $V$ respectively. The segment $Q = [a, b]$ belongs to $\mathcal{Q}$, is contained in $B(X, \varepsilon)$ and meets $S$. Actually, $Q \cap S$ is nowhere dense in $Q$ because no subinterval of $Q$ is contained in $S$ — this is where we use the assumption that no element of $\mathcal{Q}$ is contained in $S$. There is therefore even a point $y$ in $Q \cap S$ that belongs to $\operatorname{cl}(Q \cap U) \cap \operatorname{cl}(Q \cap V)$: orient $Q$ so that $a$ is its minimum, then $y = \inf(Q \cap V)$ is as required. It follows that the set $S'$ of those $y \in S$ for which there is $Q \in \mathcal{Q}$ such that $y \in \operatorname{cl}(Q \cap U) \cap \operatorname{cl}(Q \cap V)$ is dense in $S$. We fix a countable dense subset $T$ of $S'$. We also fix a numeration $\{a_n : n \in \mathbb{N}\}$ of $A$, the complement of our set $A$.

The elements $p$ of our partial order $\mathbb{P}$ have four components:

(1) a finite partial function $\sigma_p$ from $D$ to $T$,
(2) a finite partial function $\ell_p$ from $D$ to $\omega$,
(3) a finite subset $F_p$ of $\alpha \cap I$,
(4) a finite subset $\mathcal{Q}_p$ of $\mathcal{Q}$.

We require that $\operatorname{dom} \sigma_p = \operatorname{dom} \ell_p$ and we abbreviate this common domain as $\operatorname{dom} p$. It will be convenient to have $\operatorname{dom} p$ downward closed in $D$, by which we mean that if $e \in \operatorname{dom} p \cap D_d$ then $d \in \operatorname{dom} p$.

The intended interpretation of such a condition is that $\sigma_p$ and $\ell_p$ approximate the maps $\sigma$ and $\ell$ respectively; therefore we also write $W_p(d) = B(\sigma_p(d), 2^{-\ell_p(d)})$. The list of requirements in Section 3 must be translated into conditions that we can impose on $\sigma_p$ and $\ell_p$.

(1) $\|\sigma_p(e) - \sigma_p(d)\| < 2^{-N_e}$ whenever $d, e \in \operatorname{dom} p$ are such that $e \in D_d$, this will ensure that $\langle \sigma(e) : e \in D_d \rangle$ will converge to $\sigma(d)$;
(2) $\operatorname{cl}_e W_p(e) \subseteq W_p(d) \setminus \{d\}$ whenever $d, e \in \operatorname{dom} p$ are such that $e \in D_d$; and
(3) for every $n$ the family $\{\operatorname{cl}_e W_p(d) : d \in D_n \cap \operatorname{dom} p\}$ is pairwise disjoint.

The order on $\mathbb{P}$ will be defined to make $p$ force that for $\beta \in F_p$ and $Q \in \mathcal{Q}_p$ the intersection $\{\sigma(d) : d \in D\} \cap (K_\beta \cup Q)$ is contained in the range of $\sigma_p$, and even that when $d \notin \operatorname{dom} p$ the intersection $\operatorname{cl}_e W(d) \cap (K_\beta \cup Q)$ is empty. We also want $p$ to guarantee that $K \cap \{a_i : i \leqslant |\operatorname{dom} p|\} = \emptyset$.

Before we define the order, however, we must introduce an assumption on our recursion that makes our density arguments go through with relatively little effort; unfortunately it involves a bit of notation.

For $x \in \bigcup \mathcal{Q}$ set $I_x = \{\beta \in I : x \in \sigma_\beta[D]\}$. For each $\beta \in I_x$ let $d_\beta = \overleftarrow{\sigma_\beta}(x)$ and write $D_{x,\beta} = D_{d_\beta}$. If it so happens that $q \in \mathbb{P}$ and $x = \sigma_q(d)$ for some $d \in D$ and if $e \in D_d \setminus \operatorname{dom} q$ then we must be able to choose an extension $p$ of $q$ with $e \in \operatorname{dom} p$, without interfering too much with the sets $W_\beta(a)$, where $\beta \in I_x$ and $a \in D_{x,\beta}$. The following assumption enables us to do this (and we will be able to propagate it):

($*$) If $x \in \bigcup \mathcal{Q}$ then for every finite subset $F$ of $I_x \cap \alpha$ there is an $\varepsilon > 0$ such that the family $\mathcal{W}_{F,\varepsilon} = \{\mathrm{cl}_e W_\beta(a) : \beta \in F, a \in D_{x,\beta} \text{ and } \sigma(a) \in B(x,\varepsilon)\}$ is pairwise disjoint.

It is an elementary exercise to verify that in such a case the difference $B(x,\varepsilon) \setminus \bigcup \mathcal{W}_{F,\varepsilon}$ is connected.

We define $p \preccurlyeq q$ if

(1) $\sigma_p$ extends $\sigma_q$ and $\ell_p$ extends $\ell_q$,
(2) $F_p \supseteq F_q$ and $\mathcal{Q}_p \supseteq \mathcal{Q}_q$,
(3) if $d \in \operatorname{dom} p \setminus \operatorname{dom} q$ and $i \leqslant |\operatorname{dom} q|$ then $a_i \notin \mathrm{cl}_e W_p(d)$.
(4) if $d \in \operatorname{dom} p \setminus \operatorname{dom} q$ and $J \in \mathcal{Q}_q \cup \{K_\beta : \beta \in F_p\}$ then $\mathrm{cl}_e W_p(d)$ is disjoint from $J$.
(5) if $d \in \operatorname{dom} q$ and $x = \sigma_q(d)$ and if $e \in \operatorname{dom} p \setminus \operatorname{dom} q$ is such that $e \in D_d$ then $\mathrm{cl}_e W_p(e)$ is disjoint from $\mathrm{cl}_e W_\beta(a)$ whenever $\beta \in F_q \cap A_x$ and $a \in D_{x,\beta}$

It is clear that $p$ and $q$ are compatible whenever $\sigma_p = \sigma_q$ and $\ell_p = \ell_q$; as there are only countably many possible $\sigma$s and $\ell$s we find that $\mathbb{P}$ is a $\sigma$-centered partial order.

5.2. **Dense sets.** In order to apply Martin's Axiom we need, of course, a suitable family of dense sets.

*For $\beta < \alpha$ the set $\{p : \beta \in F_p\}$ is dense.* Given $p$ and $\beta$ extend $p$ by adding $\beta$ to $F_p$.

*For $Q \in \mathcal{Q}$ the set $\{p : Q \in \mathcal{Q}_p\}$ is dense.* Given $p$ and $Q$ extend $p$ by adding $Q$ to $\mathcal{Q}_p$.

*For $n \in \mathbb{N}$ the set $\{p : |\operatorname{dom} p| \geqslant n\}$ is dense.* This follows from the density of the sets below.

*For $e \in D$ the set $\{p : e \in \operatorname{dom} p\}$ is dense.* Here is where we use assumption ($*$). Since every $e \in D$ has only finitely many predecessors with respect to the relation "$D_d \ni q$" it will suffice to consider the case where $q \in \mathbb{P}$ and $e \in D_d \setminus \operatorname{dom} q$ for some $d \in \operatorname{dom} q$.

We extend $q$ to a condition $p$ by setting $F_p = F_q$, $\mathcal{Q}_p = \mathcal{Q}_q$, $\operatorname{dom} p = \{e\} \cup \operatorname{dom} q$ and by defining $d_p(e)$ and $\ell_p(e)$ as follows. Let $x = \sigma_q(d)$, put $n = k_e$ and consider $H = \bigcup\{\mathrm{cl}_e W_q(a) : a \in D_{n+1} \cap \operatorname{dom} p \cap D_d\}$.

Fix $\varepsilon_1 \leqslant 2^{-N_e}$ so that $B(x, 2\varepsilon_1)$ is disjoint from $H$, this is possible because of condition (2) in the defintion of the elements of $\mathbb{P}$. Observe that if we choose $\sigma_p(e)$ and $\ell_p(e)$ in such a way that $\mathrm{cl}_e W_p(e) \subseteq B(x,\varepsilon_1)$ then $p$ is an element of $\mathbb{P}$.

Next, using ($*$), find $\varepsilon_2 \leqslant \varepsilon_1$ that works for the finite set $F_q \cap I_x$. The set $W = \{x\} \cup \bigcup \mathcal{W}_{F,\varepsilon_2}$ is closed and does not separate the ball $B(x,\varepsilon_2)$, the set $S$ *does* separate this ball because the latter meets both $U$ and $V$. Therefore we can find a point $y$ in $S \cap B(x,\varepsilon_2) \setminus W$; we choose $\delta > 0$ so small that $\mathrm{cl}_e B(y,\delta) \subseteq B(x,\varepsilon_2) \setminus W$.

The set $S \cap B(y,\delta)$ separates $B(y,\delta)$, hence it is (at least) one-dimensional, The union of the $K_\beta$ (for $\beta \in F_q$) together with the $Q \cap S$ (for $Q \in \mathcal{Q}_q$) is zero-dimensional because each individual set is: each $K_\beta$ is a Cantor set and each $Q \cap S$ is nowhere dense in $Q$ and hence zero-dimensional. This means that, finally, we can choose $\sigma_p(e)$ in $T \cap B(y,\delta)$ but not in this union and then we take $\ell_p(e)$ so large that $\mathrm{cl}_e W_p(e)$ is a subset of $B(y,\delta)$ minus that union. Also, at this point we ensure that $a_i \notin \mathrm{cl}_e W_p(e)$ for $i \leqslant |\operatorname{dom} q|$: this is possible because $\sigma_p(e) \notin A$.

We have chosen $W_p(e)$ to meet requirements (3), (4) and (5) in the definition of $p \preccurlyeq q$.

5.3. **A generic filter.** Let $G$ be a filter on $\mathbb{P}$ that meets all of the above dense sets. Then $\sigma_\alpha = \bigcup\{\sigma_p : p \in G\}$ and $\ell_\alpha = \bigcup\{\ell_p : p \in G\}$ are the sought after maps. We define $W_\alpha$ and $K_\alpha$ as in Section 3.

5.3.1. *Assumption* $(*)$ *is propagated.* In verifying this we only have to worry about the points in $\sigma_\alpha[D]$ of course.

Therefore let $x \in \sigma_\alpha[D]$ and let $F$ be a finite subset of $I_x \cap \alpha$; we have to find an $\varepsilon$ for $F' = F \cup \{\alpha\}$. First fix $\varepsilon_1$ that works for $F$ itself. Next take $p \in G$ such that $d_\alpha \in \operatorname{dom} p$ and $F \subseteq F_p$. Using condition (5) in the definition of $\preccurlyeq$ and a density argument we find that $\operatorname{cl}_e W_\alpha(e)$ is disjoint from $\operatorname{cl}_e W_\beta(a)$ whenever $e \in D_{x,\alpha} \setminus \operatorname{dom} p$, $\beta \in F$ and $a \in D_{x,\beta}$. Now choose $\varepsilon$ smaller than $\varepsilon_1$ and all distances $\|x - \sigma_\alpha(e)\|$, where $e \in D_{x,\alpha} \cap \operatorname{dom} p$. Then $\mathcal{W}_{F',\varepsilon}$ is pairwise disjoint.

5.3.2. $K_\alpha$ *meets* $K_\beta$ $(\beta < \alpha)$ *in a finite set.* Let $\beta \in \alpha \cap I$ and take $p \in G$ such that $\beta \in F_p$. Choose $n$ such that $\operatorname{dom} p \subseteq \bigcup_{k \leqslant n} D_k$. By formula (‡) we know that $K_\alpha \subseteq \operatorname{cl}_e\bigl(\bigcup\{W_d : d \in D_{n+1}\}\bigr)$ the latter closure is equal to $\bigcup_{k\leqslant n} D_k \cup \bigcup\{W_d : d \in D_{n+1}\}$ and the intersection of *this* set with $K_\beta$ is contained in $\operatorname{dom} p$; this follows from condition 4 in the definition of $\preccurlyeq$.

5.3.3. $K_\alpha$ *meets each* $Q \in \mathcal{Q}$ *in a finite set.* The proof is identical to the previous one: take $p \in G$ with $Q \in \mathcal{Q}$.

## 6. The remaining properties of the topologies

We check conditions (1), (2) and (3) from Section 4.

A useful observation is that a typical new basic neighbourhood $B$, in the topology $\tau_\alpha$, at a point of $\sigma_\alpha[D]$ is of the form $O \setminus F$, where $F$ is such that $F \subseteq \operatorname{cl}_e F \subseteq F \cup \sigma_\alpha[D]$. To see this refer to the end of Section 3. In the case where $B = V_n$ the set $F$ is the union of

- $\bigcup_{k=1}^n h_\alpha[D_k]$, which is closed in $\tau_e$;
- a subfamily of $\{\operatorname{cl}_e W_\alpha(e) : e \in \bigcup_{k=1}^n D_{k+1}\}$, which accumulates at the points above, and
- $f_\alpha^{\leftarrow}\bigl[\{2^{-n}, -2^{-n}\}\bigr]$, which has its extra adherent points in $\sigma_\alpha[D]$ (see Proposition 1.2).

6.1. $X \setminus \bigcup \mathcal{Q}$ **retains its Euclidean topology.** This immediate from the description of the basic neighbourhoods given above as the intersection $F \cap X \setminus \bigcup \mathcal{Q}$ is closed in the Euclidean topology of $X \setminus \bigcup \mathcal{Q}$.

6.2. **Each** $Q \in \mathcal{Q}$ **retains its Euclidean topology.** We should show that $B \cap Q$ is open in the natural topology of $Q$. But this, again, is immediate: in the present case $K_\alpha \cap Q$ is finite, hence $F \cap Q$ is closed in $Q$.

6.3. $\tau_\alpha$ **and** $\tau_\beta$ **do not interfere.** The same argument as above applies: if $\beta \neq \alpha$ then $K_\alpha \cap K_\beta$ is finite. Therefore $B \cap K_\beta$ is open in the natural topology of $K_\beta$.

6.4. $Q_\alpha$ **is still in the boundary of** $U_\alpha$**.** For notation we refer to the discussion around assumption $(*)$. Let $x \in Q_\alpha$ and assume $I_x \neq \emptyset$ (if $I_x = \emptyset$ then $x$ has no new neighbourhoods and there is nothing to prove). A typical new basic neighbourhood of $x$ in the full topology $\tau$ contains a set of the form $O_\varepsilon = B(x,\varepsilon) \setminus \bigcup \mathcal{W}_{F,\varepsilon}$. By our construction, if $\varepsilon$ is taken small enough we have $\bigcup \mathcal{W}_{F,\varepsilon} \cap Q_\alpha = \emptyset$. This means that $O_\varepsilon$ is a Euclidean neighbourhood of many points of $Q_\alpha$ and hence that it meets both $U_\alpha$ and $V_\alpha$.

6.5. $K_\alpha$ **is still in the boundary of** $U_\alpha$**.** Let $x \in K_\alpha$ and assume $I_x \neq \emptyset$.

If $\alpha \notin I_x$ then the same argument as above will work: any $O_\varepsilon$ is a Euclidean neighbourhood of many points of $K_\alpha$.

If $\alpha \in I_x$ then we assume $\alpha \in F$ and observe that $O_\varepsilon$ is a Euclidean neighbourhood of many points of $S_\alpha \setminus \bigcup\{\operatorname{cl}_e W_\alpha(a) : a \in D_{x,\alpha}\}$.

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