Set-Theoretic update on Topology

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1 Introduction

This is the third in the Recent Progress in General Topology series, and this is the author's third contribution with the assigned theme of recent progress of applications of set-theory to topology. To some (like me) in set-theoretic topology, this can just seem like being asked to write about applications of topology to topology. My view is that the focus should be on the new aspects of the set-theoretic methods and not simply a survey of results that have a strong set-theoretic flavor. This point of view was reinforced when I saw preliminary versions of some of the other contributors' articles because they, naturally enough, were filled with highly sophisticated set-theoretic results. It is also reasonable to feel that there is no benefit in trying to provide an updated explanation of what set-theory in topology is all about. Rather, as we have in our previous efforts, we make a personal selection of recent applications of set-theory with the hopes that many of the most up to date and innovative applications are well represented. The author thanks Istvan Juhasz for discussions in helping select the topics even though many of his suggestions were omitted due to my own lack of expertise. The topics selected include applications of the forcing axiom $\mathsf{PFA}(S)$ to such problems as Katětov's problem about hereditarily normal squares, the P-ideal dichotomy, the compact small diagonal problem, and we show that compact spaces of countable tightness are sequentially compact (a partial step related to the Moore-Mrowka problem). These results are, for the most part due to Todorcevic and Larson. We also give a forcing construction of a counterexample to the Katětov problem based on a CH example given by Gruenhage and Nyikos.

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There is a brief discussion of Efimov spaces (compact spaces containing neither $\beta\mathbb{N}$ nor a converging sequence) using Koszmider's notion of a T-algebra. There is an application of PFA to the structure of compact sequential spaces related to the question of the maximum possible sequential order, and a review of selectively separable spaces. The final section includes some discussion of properties of Justin Moore's L-space. Another word about the focus of the article. Since the article is about applications of set-theoretic methods, we have attempted to be quite complete in providing detailed proofs.

2 Katětov's problem and $\mathsf{PFA}(S)[S]$

Proposition 2.1 ([Kat48]). *If X^2 is hereditarily normal and compact, then X is hereditarily Lindelöf.*

For a set A in a space X, let $cap(A)$ denote the set of complete accumulation points of A, that is, the set of $x \in X$ with the property that each neighborhood of x meets A in a set with cardinality equal to that of A.

Proof. Assume otherwise, and choose open sets $\{U_\alpha : \alpha \in \omega_1\}$ and a point $x_\alpha \in U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$. Let $H = cap(\{x_\alpha\}_\alpha) \times \{x_n : n \in \omega\}$ and $K = \{x_\alpha : \alpha \in \omega_1\} \times cap(\{x_n : n \in \omega\})$. It is immediate that $\overline{H} \cap K$ and $H \cap \overline{K}$ are empty. Let U be an open subset of X^2 which contains H. There is an α such that $(x_\alpha, x_n) \in U$ for each n. Clearly then $\overline{U} \cap K$ is not empty.

Definition 2.2. A Souslin tree $S \subset \omega^{<\omega_1}$ is coherent if $s\Delta t = \{\xi : s(\xi) \neq t(\xi)\}$ is finite for all $s, t \in S$. The axiom $\mathsf{PFA}(S)$ is the statement that there is a coherent Souslin tree and for all proper posets $\mathbb{P}$ such that forcing with $\mathbb{P}$ preserves that S is Souslin, for each family $\mathfrak{D}$ of at most ω_1 dense subsets of $\mathbb{P}$ there is a $\mathfrak{D}$-generic filter on $\mathbb{P}$.

Following [LT10], the notation $\mathsf{PFA}(S)[S]$ denotes a generic extension by S of a model of PFA. In what follows we shall use the phrase "$\mathsf{PFA}(S)[S]$ implies ..." to express succinctly a certain statement holds in every extension by S of a model for $\mathsf{PFA}(S)$.

We sketch a proof of the following important result.

Theorem 2.3. $\mathsf{PFA}(S)[S]$ *implies that any compact space with hereditary normal square is metrizable.*

The following non-trivial result is the first step, but we omit the proof (but see Theorem 7.1).

Proposition 2.4 ([LT02]). $\mathsf{PFA}(S)[S]$ *implies that a compact hereditarily Lindelöf space is hereditarily separable.*

Definition 2.5. The P-ideal dichotomy, PID, is the statement that when $\mathcal{I}$ is a P-ideal on a set X, either X is a countable union of sets each of which is orthogonal to $\mathcal{I}$, or there is an uncountable subset of X whose countable subsets are all members of $\mathcal{I}$.

Here are three important consequences of this forcing axiom. To best illustrate the methods, we will prove (in detail) that the P-ideal dichotomy holds. We recall that OCA (a well known consequence of PFA with an evolving naming convention) is the statement that if X is an uncountable separable metric space, and K is an open subset of $[X]^2$ (with the natural topology), then there is a countable cover $\{X_n : n \in \omega\}$ of X and an uncountable $Y \subset X$, such that either $[Y]^2 \subset K$ or, for each n, $[X_n]^2 \cap K = \emptyset$.

Lemma 2.6 ([Tod]). $\mathsf{PFA}(S)[S]$ *implies the P-ideal dichotomy,* OCA*, and the equality* $\mathfrak{b} = \omega_2$*.*

Proof. We prove that $\mathsf{PFA}(S)$ implies that forcing with S gives a model in which PID holds. To do so, we consider $\dot{\mathcal{I}}$ which is an S-name of a P-ideal on a cardinal ν. Let $\dot{\mathcal{I}}^\perp$ denote the S-name of the collection of subsets Y of ν which satisfy that $[Y]^\omega \cap \dot{\mathcal{I}}$ is empty. We suppose that some condition (in fact the root of S) forces that if $\{\dot{Y}_n : n \in \omega\}$ are S-names of members of $\dot{\mathcal{I}}^\perp$, then there is an S-name $\dot{\xi}$ such that 1 forces that $\dot{\xi} \notin \dot{Y}_n$ for each n. Our task is to use $\mathsf{PFA}(S)$ to show that there is an S-name $\dot{Y}$ of an uncountable set all of whose countable subsets are members of $\dot{\mathcal{I}}$. In fact, we will produce a cub $C \subset \omega_1$ and a sequence of pairs of S-names $\{\dot{\xi}_\gamma, \dot{b}_\gamma : \gamma \in C\}$ and an S-name $\dot{\Gamma}$ for an uncountable subset of C so that it is forced that $\dot{b}_\gamma \in \dot{\mathcal{I}}$ for each $\gamma \in \dot{\Gamma}$, and $\{\dot{\xi}_\zeta : \zeta \in \dot{\Gamma} \cap \gamma\} \subset^* \dot{b}_\gamma$.

Since forcing with S adds no new countable sets, the names for $\dot{b}_\gamma$ can simply be a pair (s_γ, b_γ) where $s_\gamma \in S$, and $b_\gamma \in [\nu]^\omega$. It will be necessary that $s_\gamma \Vdash b_\gamma \in \dot{\mathcal{I}}$. For simplicity we ignore the need for the more formal (and correct) notation that this name should be $\{(s_\gamma, \check{b}_\gamma)\}$. However we will not be able to make such a reduction for $\dot{\xi}_\gamma$, but we will need that s_γ forces a value on $\dot{\xi}_\delta$ for all $\delta < \gamma$ in order to ensure the covering property required of b_γ. In fact the definition of ξ_γ will need to be independent of the choice of s_γ. To assist with this requirement, we fix a regular cardinal θ so that $2^\nu < \theta$ and, following [Tod], choose a well-ordering $<_\theta$ of $H(\theta)$.

Each $\gamma \in C$ will correspond to some M_γ which is a countable elementary submodel of $(H(\theta), <_\theta)$ such that $M_\gamma \cap \omega_1 = \gamma$. We will choose s_γ to be any member of S_γ (the minimal members of $S \setminus M_\gamma$). We leave as a simple exercise that there is a set $b_\gamma \in [\nu]^\omega$ (choose the $<_\theta$-least) such that $s_\gamma \Vdash b_\gamma \in \dot{\mathcal{I}}$ and b_γ mod finite contains all $a \in M_\gamma$ such that $s_\gamma \Vdash a \in \dot{I}$. In a similar fashion, consider the collection $\{\dot{Y}_n : n \in \omega\}$ of all S-names in M_γ. Each $t \in S_\gamma$ will decide the truth of the statement $\dot{Y}_n \in \dot{\mathcal{I}}^\perp$ for each n. There is a minimal $\beta_\gamma \in \omega_1 \setminus \gamma$ such that for each $s \in S_{\beta_\gamma}$ there is an ordinal $\xi_s \in \nu$ such that $s \Vdash \xi_s \notin \dot{Y}_n$ for all n such that $s \Vdash \dot{Y}_n \in \dot{\mathcal{I}}_n$. The set $\dot{\xi}_{M_\gamma} = \{(s, \check{\xi}_s) : s \in S_{\beta_\gamma}\}$ is an S-name such that

$$(\forall \dot{Y} \in M_\gamma) \quad 1 \Vdash \dot{Y} \in \dot{\mathcal{I}}^\perp \text{ implies } \dot{\xi}_{M_\gamma} \notin \dot{Y} \quad .$$

For each countable elementary submodel M of $(H(\theta), <_\theta)$, let $\dot{\xi}_M$ denote the $<_\theta$ least such name. When we choose M_γ, we will let $\dot{\xi}_\gamma$ be $\dot{\xi}_{M_\gamma}$. In addition, for each $s \in S_\gamma$, fix $b(M_\gamma, s)$ as discussed above so that $a \subset^* b(M_\gamma, s)$ for each $a \in M_\gamma$ such that $s \Vdash a \in \dot{\mathcal{I}}$. In this way, once we have selected M_γ and s_γ, the choices of ξ_γ and b_γ are automatic.

Now we construct a poset $\mathbb{P}$ which will create the cub C and the selection of S-names as described above. That is, we will show that $\mathsf{PFA}(S)$ can be applied to $\mathbb{P}$ and that a selection of ω_1-many dense sets and ω_1 many names with which to invoke the $\mathsf{PFA}(S)$ axiom will prove the theorem.

A condition $p \in \mathbb{P}$ will have the form $(F_p, \mathcal{M}_p, \mathcal{S}_p)$ where $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of $(H(\theta), <_\theta)$, $F_p = \{M \cap \omega_1 : M \in \mathcal{M}_p\} \in [\omega_1]^{<\omega}$, $\mathcal{S}_p = \{s^p_\delta : \delta \in F_p\}$ is such that $s_\delta \in S_\delta$ for each $\delta \in F_p$. Notice that for $M \in \mathcal{M}_p$ and $M \cap \omega_1 < \delta \in F_p$, s^p_δ forces a value on $\dot{\xi}_M$. We define $p < q$ providing

1. $\mathcal{M}_p \supset \mathcal{M}_q$,
2. $\mathcal{S}_p \supset \mathcal{S}_q$, and if
3. $M \in \mathcal{M}_p \setminus \mathcal{M}_q$, $M \cap \omega_1 = \delta < \gamma \in F_q$, and $s_\delta < s_\gamma$, then $s_\gamma \Vdash \dot{\xi}_\delta \in b_\gamma$.

A more elegant presentation of $\mathbb{P}$ would be to suppress any mention of F_p and to replace $\{\mathcal{M}_p, \mathcal{S}_p\}$ as the corresponding function sending $\mathcal{M}_p$ into S. In later proofs we will adopt that approach but feel it might obscure some of the intuition in this first introduction.

It is trivial to see that, for each $\gamma \in \omega_1$, the set of $p \in \mathbb{P}$ with $F_p \setminus \gamma \neq \emptyset$ is dense. Suppose that $G \subset \mathbb{P}$ is a filter that meets each of these dense sets and let $C = \bigcup\{F_p : p \in G\}$. There are ω_1 many more dense sets that can be chosen sufficient to ensure that C is a cub, but actually we do not require that C is a cub, only that it is uncountable. Also let $\{s_\gamma : \gamma \in C\}$ and $\{M_\gamma : \gamma \in C\}$ be the assignments where, for each $p \in G$ and $\gamma \in F_p$, $s_\gamma = s^p_\gamma$ and $M_\gamma \in \mathcal{M}_p$. We wish to notice here that the last condition in the definition of the ordering on $\mathbb{P}$ ensures that for each $\gamma \in C$, $s_\gamma \Vdash \{\dot{\xi}_\delta : \delta \in C \text{ and } s_\delta < s_\gamma\} \subset^* b_\gamma$. The required S-name $\dot{\Gamma}$ is simply $\{(s_\gamma, \check{\gamma}) : \gamma \in C\}$.

It remains only to show that $\mathsf{PFA}(S)$ can be applied to $\mathbb{P}$. Let κ be a sufficiently large regular cardinal so that $\mathbb{P} \in H(\kappa)$ and let $\mathbb{P} \in M \prec H(\kappa)$ be a countable elementary submodel. Let $\delta_0 = M \cap \omega_1$ and $M_0 = M \cap H(\theta)$. Choose any $s_0 \in S_{\delta_0}$. Let $p_0 \in \mathbb{P} \cap M$.

A somewhat, by now, standard proof to show that $\mathbb{P}$ is proper is to show that the extension $p = (F_{p_0} \cup \{\delta_0\}, \mathcal{M}_{p_0} \cup \{M_0\}, \mathcal{S}_{p_0} \cup \{s_{\delta_0}\})$ is $(M, \mathbb{P})$-generic. To show that $\mathbb{P}$ is proper and preserves that S is Souslin, we show that for each $s_0 \in S_{\delta_0}$ (s_0, p) is an M-generic condition for $S \times \mathbb{P}$ (see [Miy93]).

Let $D \subset S \times \mathbb{P}$ be a dense open set which is a member of M. Choose any extension $(s, q) \in S \times \mathbb{P}$ which is a member of D. Let $\{M_0, M_1, \ldots, M_{\ell-1}\}$ enumerate $\mathcal{M}_q \setminus M$ in increasing order. Similarly, let $\{\delta_0, \delta_1, \ldots, \delta_{\ell-1}\}$ enumerate $F_q \setminus M$. For each $i < \ell$, let b_{δ_i} denote $b(M_i, s^q_{\delta_i})$, which we recall will be the value of b_{δ_i}. Since D is open, we can assume that s is not in $M_{\ell-1}$ and even that it forces a value on $\dot{\xi}_{M_{\ell-1}}$. To postpone discussion of one of the complications, we will first assume that $s^q_{\delta_0} < s$.

Simple elementarity will ensure that there are members $(\bar{s}, \bar{q})$ of $D \cap M$ such that $\bar{s} < s$, and $F_{\bar{q}}$ end-extends $F_q \cap M$. The difficult step in the proof is to overcome the obstruction to $\bar{q}$ being compatible with q caused by the requirement that whenever $i < \ell$ and $s^{\bar{q}}_\gamma \in \mathcal{S}_{\bar{q}}$ is below s_{δ_i}, we require that s_{δ_i} forces that $\dot{\xi}_{\bar{M}} \in b_{\delta_i}$, where $\bar{M} \cap \omega_1 = \gamma$. This is quite subtle.

To get the flavor, let us consider the collection

$$\dot{X}_0 = \{(\bar{s}, \xi) : (\exists \bar{q} \in \mathbb{P}) \ \ (\bar{s}, \bar{q}) \in D, \ \ \mathcal{M}_{\bar{q}} \text{ end-extends } \ \mathcal{M}_q \cap M, \\ \bar{M} = \min(\mathcal{M}_{\bar{q}} \setminus (\mathcal{M}_q \cap M)) \text{ and } \bar{s} \Vdash \dot{\xi}_{\bar{M}} = \xi\} \ . \quad (1)$$

We denote this set as an S-name because we can view it as such. This is a set of pairs which is a member of $M \cap H(\theta)$, and so it is in M_0. Of course the pair $(s, \dot{\xi}_{M_0})$ corresponds to a member of $\dot{X}_0$ from which we conclude that $s \Vdash \dot{X}_0$ is not in $\dot{\mathcal{I}}^\perp$. Moreover we have that the statement $(\exists a \in [\nu]^\omega) s \Vdash a \in \dot{\mathcal{I}} \cap [\dot{X}_0]^\omega$ holds and we may assume that for each $\xi \in a$, there is an $s_\xi < s$, such that $(s_\xi, \xi) \in \dot{X}_0$. By elementarity (and using that S is ccc), there is an $s' \in S \cap M$ with $s' < s$ and an $a \in M$ such that $s' \Vdash a \in \dot{\mathcal{I}} \cap [\dot{X}_0]^\omega$ and again that for each $\xi \in a$, there is an $s_\xi < s'$ such that $(s_\xi, \xi) \in \dot{X}_0$. For each $j < \ell$ such that $s' < s_{\delta_j}$, we have that $a \subset^* b_{\delta_j}$. Therefore there is some $\xi_0 \in a \cap \bigcap\{b_{\delta_j} : j < \ell \text{ and } s' < s_{\delta_j}\}$. For any such ξ_0 we must select a corresponding $\bar{M}_0$ witnessing that $(s_{\xi_0}, \xi_0) \in \dot{X}_0$.

The next problem is that we want to be able to choose such a ξ_0 and to then continue to choose the next ξ_1 corresponding to the next smallest model in an eventual choice of $\mathcal{M}_{\bar{q}}$. This requires that a similar, but more complicated situation exists for the set

$$\dot{X}_{\langle \xi_0 \rangle} = \{(\bar{s}, \xi) : (\exists \bar{q} \in \mathbb{P}) \ \ (\bar{s}, \bar{q}) \in D, \ \ \mathcal{M}_{\bar{q}} \text{ end-extends } \ \{\bar{M}_0\} \cup (\mathcal{M}_q \cap M), \\ \bar{M} = \min(\mathcal{M}_{\bar{q}} \setminus \bar{M}_0) \text{ and } \bar{s} \Vdash \dot{\xi}_{\bar{M}} = \xi\} \ . \quad (2)$$

There is a situation in which we do not actually have to even worry about ensuring that ξ_1 is in any of the b_{δ_j}. First, we choose any $\gamma_0 \in M_0$ so that for each $i < \ell$, $s^q_{\delta_i} \restriction [\gamma_0, \delta_0) \subset s$; hence all the $s^q_{\delta_i}$ agree on the interval $[\gamma_0, \delta_0)$. We retroactively assume that $\gamma_0 \in \bar{M}_0$. Let $\bar{M}_0 \cap \omega_1$ be denoted as $\bar{\delta}_0$.

Suppose that there is some $\alpha \geq \delta^q_0$ such that $s(\alpha) \neq s^q_{\delta_1}(\alpha)$. The definition of $\dot{X}_{\langle \xi_0 \rangle}$ can then be modified to ensure that for each $(\bar{s}, \xi_{\bar{M}}) \in \dot{X}_{\langle \xi_0 \rangle}$ there is a $\bar{q} \in \mathbb{P}$ so that $\bar{s}$ and $s^{\bar{q}}_{\delta_{\bar{M}}}$ disagree at some α which is above $\bar{\delta}_0$. This will ensure that for any such $\delta_{\bar{M}}$ and any $j < \ell$, $s^{\bar{q}}_{\delta_{\bar{M}}} < s_{\delta_j}$ will fail; and so the difficult condition in the definition of $<_{\mathbb{P}}$ is vacuously satisfied.

On the other hand, if $s^q_{\delta_1}$ does agree with s on the interval $[\gamma_0, \delta_1)$ and $(\bar{s}, \xi) \in \dot{X}_{\langle \xi_0 \rangle}$ is such that $\bar{s} < s$ (as required) then we will have the requirement that ξ_1 needs to be in b_{δ_1}. To have any chance of this we will again need that s_{δ_1} forces that $\dot{X}_{\langle\xi_0\rangle}$ is not in $\dot{\mathcal{I}}^\perp$. This puts an extra demand on the choice of ξ_0 (and each subsequent choice) which is accomplished by starting again and examining q more closely. Since we will use it so often, let $\mathbb{P}_q$ denote the set of $\bar{q} \in \mathbb{P}$ such that $\mathbb{M}_{\bar{q}}$ end-extends $\mathcal{M}_q \cap M$ with ℓ elements, and for $\bar{q} \in \mathbb{P}_q$ let $\{M_0^{\bar{q}}, \ldots, M_{\ell-1}^{\bar{q}}\}$ list this extension in increasing order. We no longer keep the assumption that $s_{\delta_0} < s$, but we introduce some standard notation for this type of proof. For each $i < \ell$, let σ_i be the mapping on S which takes an element s' of S and changes its initial segment below γ_0 so as to agree with s_{δ_i}. Notice that $\sigma_i \in M_0$ for each $i < \ell$. Next, let L denote the set of $i < \ell$ such that $s_{\delta_i} < \sigma_i(s)$ (that is, s and s_{δ_i} have no disagreements above γ_0), and let $\{i_j : j < |L|\}$ enumerate L.

Define the collection

$$T = \{\{\xi_0, \xi_1, \ldots, \xi_{|L|-1}, \bar{s}\} \subset \omega_1 \cup S : (\exists \bar{q} \in \bar{P}_q) \quad (\bar{s}, \bar{q}) \in D, \\ \text{for each } \ j < |L|, \sigma_{i_j}(\bar{s}) \Vdash \xi_j = \dot{\xi}_{M^{\bar{q}}_{i_j}}\} \tag{3}$$

We showed above that the sequence $\{\xi^q_{i_0}, \ldots, \xi^q_{i_{|L|-1}}, s\}$ is in T (where $\sigma_{i_j}(s) \Vdash \dot{\xi}_{M_{i_j}} = \xi^q_{i_j}$). Following [Tod], let

$$\partial T = \{t = \{\xi_0, \xi_1, \ldots, \xi_{|L|-2}, \bar{s}\} : (\exists a \in [\nu]^\omega) \\ \sigma_{i_{|L|-1}}(s) \Vdash a \in \dot{\mathcal{I}} \text{ and } (\forall \xi \in a) t \cup \{\xi\} \in T\}. \tag{4}$$

We know that $\{\xi^q_{i_0}, \ldots, \xi^q_{i_{|L|-2}}, s\} \in \partial T$, and thus by elementarity, and the fact that D is open, there is a $\bar{s}_{|L|-1} \in M_{\ell-1}$, such that $\bar{s}_{|L|-1} < s$ and for all extensions $\bar{s}$ of $\bar{s}_{|L|-1}$, $\{\xi^q_{i_0}, \ldots, \xi^q_{i_{|L|-2}}, \bar{s}\} \in \partial T \cap M_{\ell-1}$. Then set

$$\partial^2 T = \{t = \{\xi_0, \ldots, \xi_{|L|-3}, \bar{s}\} : (\exists a \in [\nu]^\omega) \quad (\forall \xi \in a) \\ \sigma_{i_{|L|-2}}(\bar{s}) \Vdash a \in \dot{\mathcal{I}} \text{ and } t \cup \{\xi\} \in \partial T\}. \tag{5}$$

By induction, for $0 < k - 2 < |L|$ and $\partial^0 T = T$,

$$\partial^{k-1} T = \{t = \{\xi_0, \ldots, \xi_{\ell-k}, \bar{s}\} : (\exists a \in [\nu]^\omega) \quad (\forall \xi \in a) \\ \sigma_{|L|-k+1}(\bar{s}) \Vdash a \in \dot{\mathcal{I}} \text{ and } t \cup \{\xi\} \in \partial^{k-2} T\} \tag{6}$$

and we have that there is a $\bar{s}_{i_k} \in M_k$ such that $\bar{s}_{i_k} < s$ and, for all extensions $\bar{s}$ of $\bar{s}_{i_k}$, $\{\xi^q_{i_0}, \ldots, \xi^q_{i_{|L|-k}}, \bar{s}\} \in \partial^{k-1} T$. In particular, there is an $\bar{s}_0 \in M_0$ so that $\bar{s}_0 < s$ and, $\{\bar{s}_0\} \in \partial^{|L|} T$.

For each $j < |L|$, let $K_j = \{k \in L : s_{\delta_{i_j}} < \sigma_j(s)\}$. Also, for each $j < |L|$, let $b_j = \bigcap\{b_{\delta_k} : k \in K_j\}$. Working in M_0, we select a witness a_0 for the fact that $\{\bar{s}_0\} \in \partial^{|L|} T$, and note then that we may choose $\xi_0 \in a_0 \cap b_0$ so that $\{\xi_0, \bar{s}_0\} \in \partial^{|L|-1} T$. Continuing this recursion, we choose, for $j < |L|$,

$\xi_j \in b_j \cap M_0$ so that $\{\xi_0, \xi_1, \ldots, \xi_j, \bar{s}_0\} \in \partial^{|L|-j-1}T$. Having chosen $t = \{\xi_0, \ldots, \xi_{|L|-1}, \bar{s}_0\} \in T$, we choose $\bar{q} \in \mathbb{P}_q$ witnessing that $t \in T$. We omit the details that $\bar{q}$ is compatible with q, $(\bar{s}_0, \bar{q}) \in D$, and $(\bar{s}_0, \bar{q})$ is compatible with (s, q) as required.

Corollary 2.7. $\mathsf{PFA}(S)[S]$ *implies that a compact first countable space which is not hereditarily Lindelöf will contain an uncountable discrete set.*

Proof. Let $\{U_\alpha : \alpha \in \omega_1\}$ be a family of cozero sets so that for each α, there is an $x_\alpha \in X \setminus \bigcup_{\beta<\alpha} U_\beta$ and an open set V_α such that $x_\alpha \in V_\alpha \subset \overline{V_\alpha} \subset U_\alpha$. Define an ideal $\mathcal{I}$ on the set $\{x_\alpha : \alpha \in \omega_1\}$ according to $a \in \mathcal{I}$ if $a \cap V_\alpha$ is finite for all $\alpha \in \omega_1$. Since $\mathfrak{b} > \omega_1$, it is easy to check that $\mathcal{I}$ is a P-ideal. Assume that A is an uncountable subset of $\{x_\alpha : \alpha \in \omega_1\}$ and choose any complete accumulation point x of A. Choose any countable $a \subset A$ which converges to x. Notice that $x \notin \bigcup_{\alpha<\omega_1} U_\alpha$ since, for each α, $U_\alpha \cap A$ is countable. Therefore $x \notin \overline{V_\alpha}$ for all α, hence $a \in \mathcal{I}$. By the PID, it follows that there is an uncountable A such that $[A]^\omega \subset \mathcal{I}$. For each α such that $x_\alpha \in A$, we have that V_α is a neighborhood which meets A in a finite set, hence A is an uncountable discrete set.

Proof (Proof of Theorem 2.3). We assume that $\dot{X}$ is an S-name of a compact space with hereditarily normal square. We assume that the tree S is a subtree of $\omega^{<\omega_1}$ and that the base set for $\dot{X}^2$ is an ordinal. By Proposition 2.1, $\dot{X}$ is first countable and hereditarily Lindelöf. By Proposition 2.4, we may fix an S-name $\dot{D}$ of a countable dense subset of $\dot{X}^2$. Assume that $\dot{X}^2$ is forced by some s_0 to not be hereditarily Lindelöf (and so not metrizable). By Corollary 2.7, there is a set $\{\dot{x}_\alpha : \alpha \in \omega_1\}$ of S-names of points in $\dot{X}^2$ which is forced to be discrete. Also, for each $\alpha \in \omega_1$, we may fix a name $\dot{D}_\alpha$ of a subset of $\dot{D}$ which is forced to converge to $\dot{x}_\alpha$. Fix a cub subset C of ω_1 with the property that for each $\delta \in C$ and $\alpha < \delta$, each $s \in S_\delta$ above s_0 forces a value on $\dot{D}_\alpha$. We now define an S-name of a function $\dot{f} : C \to 2$: for each $\delta \in C$ and $\delta^+ = \min(C \setminus \delta + 1)$, each $s \in S_{\delta^+ +1}$ forces that $\dot{f}(\delta) = 0$ if and only if $s(\delta^+) = 0$.

Now we show that s_0 forces that $\dot{D} \cup \{\dot{x}_\delta : \delta \in C\}$ is not normal. Assume that $\dot{U}$ is an S-name of an open subset of $\dot{X}^2$ with the property that $\dot{x}_\delta \in \dot{U}$ for each δ such that $\dot{f}(\delta) = 0$. We prove that there is a $\delta \in C$ and a condition s above s_0 which forces that $\dot{U} \cap \dot{D}_\delta$ is infinite and $\dot{f}(\delta) \neq 0$. This is rather easy. Since forcing with S does not add any new countable sets, there is an s_1 above s_0 which forces a value E on $\dot{U} \cap \dot{D}$. Choose any $\delta \in C$ so that there is a proper extension s of s_1 in S_{δ^+}. Recall that s forces a value D_δ on $\dot{D}_\delta$. Let s^0 be the extension of s which as value 0 at δ^+, and let s^1 be an extension which has a value greater than 0. Since s^0 forces that $\dot{D}_\delta$ is mod finite contained in $\dot{U}$, we have that $D_\delta \cap E$ is infinite. Therefore, s^1 is the extension we seek, since it forces that $\dot{f}(\delta) \neq 0$ and $\dot{D}_\delta \cap \dot{U}$ is infinite.

Counterexamples to Katětov's problem were constructed in [GN93]. We modify the CH example from that paper in this next result. It uses a method of construction of Boolean algebras called being minimal generation.

Theorem 2.8. *If ω_1 Cohen reals are added, then there is a counterexample to Katětov's problem, namely, there is a compact non-metrizable space whose square is hereditarily normal.*

Luzin set?

Proof. We build an ω_1-increasing chain of (minimally generated) Boolean algebras by forcing. The rough idea is that we will have a Luzin family $\{x_\alpha : \alpha \in \omega_1\}$ of points in 2^ω. For each $\alpha \in \omega_1$, we will choose (by forcing, along with x_α itself) a regular open subset a^0_α of 2^ω (and let a^1_α denote the regular open complement). We let B_0 be the usual algebra of clopen subsets of 2^ω, and B_α will be the Boolean algebra generated, in the regular open algebra of 2^ω, by the collection $B_0 \cup \{a_\beta : \beta < \alpha\}$.

One inductive hypothesis is that, for each $\beta < \alpha$, the filter of clopen sets at x_β will generate an ultrafilter on B_β and will generate precisely two ultrafilters x^0_β, x^1_β in the algebra $B_{\beta+1}$ (with $\dot{a}_\beta \in x^0_\beta$). Additionally, for all $x \in 2^\omega \setminus \{x_\xi : \xi \leq \beta\}$, the filter of clopen neighborhoods of x will generate an ultrafilter in $B_{\beta+1}$, and for all $\xi < \beta$, the filters x^0_ξ, x^1_ξ from $B_{\xi+1}$ will continue to generate ultrafilters in $B_{\beta+1}$. It follows that there is a single ultrafilter, namely that generated by x_β, on B_β which does not generate an ultrafilter on $B_{\beta+1}$. In addition, this single ultrafilter generates just the two new ultrafilters. It can be shown that this is equivalent to the fact that there are no proper subalgebras of $B_{\beta+1}$ which properly extend B_β (see [Kop89]), i.e. $B_{\beta+1}$ is a minimal extension of B_β.

We will, recursively, define a finite support iteration sequence $\{\mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha \in \omega_1\}$. For each $\alpha < \omega_1$, $\dot{Q}_\alpha$ is the $\mathbb{P}_\alpha$-name of a countable atomless poset, hence the poset $\mathbb{P}_{\omega_1}$ will be isomorphic to adding ω_1 Cohen reals. In addition, we will define a $\mathbb{P}_{\alpha+1}$-name, $\dot{a}_\alpha$, of a regular open subset of 2^ω.

The definition of $\dot{Q}_\alpha$ is $\{(u_0, u_1) : u_0 \in B_0, u_1 \in B_\alpha, u_0 \cap u_1 = \emptyset\}$. We define $(u_0, u_1) < (v_0, v_1)$ if $u_0 \subset v_0$ and $u_1 \setminus v_0 = v_1$. If $q \in \dot{Q}_\alpha$, then let (q_0, q_1) denote the coordinates of q. The definition of $\dot{x}_\alpha$ will be that $q \Vdash \dot{x}_\alpha \in q_0$. The definition of $\dot{a}^0_\alpha$ will be that $q \Vdash a^0_\alpha \setminus q_0 = q_1$. That is, if $G_{\alpha+1}$ is $P_{\alpha+1}$-generic, then the definition of a_α is $\bigcup\{u_1 : (\exists p, u_0) \ \ (p, (u_0, u_1)) \in G_{\alpha+1}\}$.

For each $\alpha \leq \omega_1$, and $\beta \leq \alpha$, if G_α is $\mathbb{P}_\alpha$-generic, it should be clear that the Stone space of B_β can be regarded as a compact metric topology on $X_\beta = \{x^0_\xi, x^1_\xi : \xi < \beta\} \cup 2^\omega \setminus \{x_\xi : \xi < \beta\}$. Let X denote the Stone space of B_{ω_1} in $V[G_{\omega_1}]$. It is well known that if G is $\mathbb{P}_{\omega_1}$-generic, then the set $\{\dot{x}_\alpha : \alpha \in \omega_1\}$ is a Luzin set; it is interesting how this plays a role in ensuring that the square is hereditarily normal.

Now suppose that $\dot{H}$ and $\dot{K}$ are $\mathbb{P}$-names of subsets H, K of X^2 and assume that some $p \in \mathbb{P}$ forces that $\overline{H} \cap K = \emptyset = H \cap \overline{K}$, i.e., they are separated. Let M be a countable elementary submodel with $\dot{H}$, $\dot{K}$, and $\{\mathbb{P}, Q_\alpha : \alpha \in \omega_1\}$ all in M. Let $\delta = M \cap \omega_1$ and let $\dot{H}_M$ and $\dot{K}_M$ be the $\mathbb{P}_\delta$-names $\dot{H} \cap M$ and $\dot{K} \cap M$ respectively. In the final model, let f denote the canonical map from $X \times X$ onto $X_\delta \times X_\delta$ (of course this just means that points of the form (x^e_ξ, y) or (x^e_ξ, x^j_ζ) with $\xi, \zeta \geq \delta$ are sent to the points (x_ξ, y) or (x_ξ, x_ζ). We view f as being the identity map on $H_\delta = \mathrm{val}_G(\dot{H}_M)$ and $K_\delta = \mathrm{val}_G(\dot{K}_M)$. We show that, in X_δ, H_δ is dense in $f[H]$, K_δ is dense in $f[K]$, and, most surprisingly, $f[H]$ and $f[K]$ are separated in X_δ.

Let $x \in 2^\omega \setminus \{x_\alpha : \alpha \in \omega_1\}$ and $\alpha \leq \beta < \omega_1$. We consider the points $\mathbf{y} = (x, x^e_\beta)$ and $\mathbf{x} = (x^i_\alpha, x^e_\beta)$ and to determine if $f(\mathbf{x})$ or $f(\mathbf{y})$ is in $f[H] \setminus \overline{H_\delta}$, and similarly if $f(\mathbf{x})$ or $f(\mathbf{y})$ is in $f[K] \cap \overline{H_\delta}$. By symmetry this will be enough. The handling of the $\mathbf{x}$ case where $\alpha < \delta$ is really no different than the proof for $\mathbf{y}$ since $\{x^0_\alpha, x^1_\alpha\}$ are in M. Therefore we assume that $\delta \leq \alpha$.

Assume first that a condition $p \in \mathbb{P}_{\omega_1}$ forces that $\mathbf{x}$ is a member of $\dot{H}$. Let b_1 and b_2 be members of B_δ that are in the ultrafilters corresponding to x^i_α and $x^e_\beta \cap B_\delta$ respectively. We may assume that each of b_1 and b_2 are simply clopen subsets of 2^ω, hence they are members of M. Then by elementarity, we certainly have that $b_1 \times b_2$ meets H_δ. This proves that $f(\mathbf{x})$ is in $\overline{H_\delta}$.

Now we assume that $\mathbf{y}$ is in K, then there are basic clopen neighborhoods b_1 and b_2 of x and x_β respectively, and a forcing condition p, such that p forces that $\dot{H} \cap (b_1 \times (b_2 \cap a^e_\beta))$ is empty. By symmetry, we will assume that $x^e_\beta = x^0_\beta$. By shrinking b_2 and extending p, we may assume that $p(\beta)_0 = b_2$. We claim that p forces that $\dot{H} \cap b_1 \times b_2$ is empty. If this is not the case then, by elementarity, there is a value $\mathbf{x}' \in M$ and an extension q of $p \restriction \delta$ which forces that $\mathbf{x}'$ is in $\dot{H} \cap (b_1 \times b_2)$. Let v be a proper clopen subset of b_2 such that (by possibly extending q) $q \Vdash_{\mathbb{P}_\delta} \mathbf{x}' \in b_1 \times v$. Finally, extend the condition $p \cup q$ to $\bar{p}$ so that $\bar{p}(\beta)_1$ contains v. This is a contradiction since $\bar{p}$ forces that $\dot{H} \cap (b_1 \times (b_2 \cap a^0_\beta)$ contains $\mathbf{x}'$. Therefore we must have that p forces that $\dot{H} \cap M$ is disjoint from $b_1 \times b_2$, i.e. $f(\mathbf{y}) \notin \overline{H_\delta}$ as required.

Next assume that $p \in \mathbb{P}_{\omega_1}$ is such that $(b_1 \cap a^i_\alpha) \times (b_2 \cap a^e_\beta)$ is forced by p to be disjoint from $\dot{H}$, $b_1 = p(\alpha)_0$, $b_2 = p(\beta)_0$, and that $\mathbf{x} = (x^i_\alpha, x^e_\beta) \in \dot{K}$. If $\alpha < \beta$, the argument is very similar to the one given for $\mathbf{y}$. So we suppose that $\alpha = \beta$ and that $b_1 = b_2$. If, by extending p and shrinking b_1, we can arrange that p forces that $\dot{H}$ is disjoint from $b_1 \times b_1$, then we have that p forces that $f(\mathbf{x}) \notin \overline{H_\delta}$ as required. Otherwise we have, as above, that there is some $q \in M$ and $\mathbf{x}' \in (b_1 \times b_2) \cap M$ such that $q \Vdash \mathbf{x}' \in \dot{H}$. Again if $\mathbf{x}'$ is not on the diagonal of X_δ, then we can find disjoint v_1 and v_2 in B_δ, each contained in b_1 and an extension q of $p \restriction \delta$ forcing that $\mathbf{x}' \in (v_1 \times v_2) \cap H$. We can then define an extension $\bar{p}$ of p and q satisfying that $\bar{p}_1 \cap (v_1 \cup v_2)$ is any of $\{\emptyset, v_1, v_2, v_1 \cup v_2)\}$ as needed to ensure that $\mathbf{x}' \in \dot{H} \cap (b_1 \cap a^i_\beta) \times (b_1 \cap a^e_\beta)$ and the desired contradiction. Finally, we have the case that p forces that $(b_1 \times b_1)$ is disjoint from $\dot{H} \setminus \Delta_X$. Here is where we use the Luzin property to show that it is also the case that p forces that $b_1 \times b_1$ is disjoint from $\dot{H} \cap \Delta_X$. In fact, since $p \restriction M$ has the property that for all $a \in B_\delta$ with $a \subset b_1$, there is an extension (of p) which forces some element of $\dot{K}$ (namely $\mathbf{x}$) in $a \times a$, it follows that the closure of $\dot{K}$ contains $(b_1 \times b_1) \cap \Delta_X$. Since $\dot{K}$ and $\dot{H}$ are (forced to be) separated, we have that $\dot{H}$ is forced to be disjoint from $b_1 \times b_1$. This completes the proof that $f(\mathbf{x})$ is not in the closure of H_δ.

Now that we have shown that $f[H]$ and $f[K]$ are separated in the metric space X_δ^2, we have that H and K are separated by disjoint open sets in X^2. This shows that X^2 is hereditarily normal, and of course, we have that X is not metrizable since it has uncountable weight.

3 On compact spaces with small diagonal

Another potential metrization theorem has generated considerable interest of late (see Gruenhage's article this volume) concerns compact spaces with a *small diagonal*. A compact space has a small diagonal if for each collection $\{\{x_\alpha, y_\alpha\} : \alpha \in \omega_1\}$ of pairs from the space, there is an open set which meets uncountably many of the pairs in a single point. The ideas for this next result come from [DH12].

Theorem 3.1. $\mathsf{PFA}(S)[S]$ *implies that a compact sequentially compact space with a small diagonal is metrizable.*

Proof. Actually we will just use OCA. We have a copy of ω embedded in a compact space X such that the closure is not metrizable. Fix a well-ordering $\prec$ on X and let $\mathcal{A}$ denote the collection of disjoint ordered pairs (a, b) of subsets of ω such that a and b converge to points $x_a \prec x_b$. Define $\langle (a, b), (a', b') \rangle \in K_0$ providing $(a \cap b') \cup (a' \cap b)$ is not empty. Assume we have an uncountable collection $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$ which is K_0 homogeneous. Let $x_\alpha = x_{a_\alpha}$ and $y_\alpha = x_{b_\alpha}$ and assume there is an open set W such that $x_\alpha \in W$ and $y_\alpha \notin \overline{W}$ for all α in some uncountable set $I \subset \omega_1$. As usual, we may assume there is an m so that $a_\alpha \setminus W \subset m$ and $b_\alpha \cap W \subset m$ for all $\alpha \in I$. Now choose distinct $\alpha, \beta \in I$ so that $b_\alpha \cap m = b_\beta \cap m$, and we have a contradiction, since $a_\alpha \cap b_\beta \subset m$ and $a_\alpha \cap b_\beta \cap m = a_\beta \cap b_\beta \cap m = \emptyset$.

To complete the proof we assume that $\{\mathcal{A}_n : n \in \omega\}$ are subsets of $\mathcal{A}$ and that $[\mathcal{A}_n]^2 \cap K_0$ is empty. Choose a countable elementary submodel M of some $H(\theta)$ so that $\{X, \{\mathcal{A}_n : n \in \omega\}\} \in M$. Since $\overline{\omega}$ is not metrizable, there is a pair of points u, v satisfying that for each open set $U \in M$ such that $u \in U$ we also have that $v \in \overline{U}$.

For a set $Y \subset \omega$, let $Y^1 = Y$ and $Y^0 = \omega \setminus Y$. Let $\{Y_n : n \in \omega\}$ enumerate $M \cap [\omega]^\omega$ and recursively define a sequence $\{\ell_n : n \in \omega\}$ so that u is in the interior of the closure of $Y_0^{\ell_0} \cap \cdots \cap Y_n^{\ell_n}$ for each n. Of course it then follows that v is also in the closure of each such set. Let U and W be neighborhoods of u and v respectively which have disjoint closures. It follows then that each of $\{Y_n^{\ell_n} \cap U : n \in \omega\}$ and $\{Y_n^{\ell_n} \cap W : n \in \omega\}$ have the finite intersection property. Choose any infinite set $a \subset \omega$ such that $a \subset^* Y_n^{\ell_n} \cap U$ for all n, and an infinite set $b \subset \omega$ such that $b \subset^* Y_n^{\ell_n} \cap W$ for all n. By shrinking the sets (since X is sequentially compact) we may assume that a and b each converge and (by symmetry) that $x_a \prec x_b$. Assume now that there is some k so that $(a,b) \in \mathcal{A}_k$. Choose $n \in \omega$ so that $Y_n = \bigcup\{a' : \exists b'(a',b') \in \mathcal{A}_k\}$. Evidently, $a \subset Y_n$, hence $\ell_n = 1$. On the other hand, for each $(a',b') \in \mathcal{A}_k$, $\{(a',b'),(a,b)\} \notin K_0$ and so we have that $a' \cap b$ is empty. This contradicts that $b \subset^* Y_n^{\ell_n}$.

Compact space with a small diagonal can not contain a converging ω_1-sequence, and compact spaces of uncountable tightness contain such a sequence ([JS92]). This leads to the following useful conclusion.

Proposition 3.2. *Compact spaces with a small diagonal have countable tightness.*

Although Todorcevic has proven that $\mathsf{PFA}(S)[S]$ implies that compact spaces of countable tightness are sequential, this result has not yet been published. Therefore we prove a weaker result which suffices for the application to compact spaces with small diagonal.

Theorem 3.3 ([Tod, 8.2]). $\mathsf{PFA}(S)[S]$ *implies that compact spaces of countable tightness are sequentially compact.*

Proof. We assume that we have an S-name of a compactification K of the set ω, in which no infinite subset of ω converges. Let $\{\dot{f}_\alpha : \alpha \in \omega_2\}$ be an enumeration of all the (nice) S-names satisfying that 1 forces that $\dot{f}_\alpha$ is a function from ω into $[0,1]$ which has a continuous extension to K. For each $t \in S$, we will let F_t denote the set of $f \in [0,1]^\omega$ such that t forces that f has a continuous extension to all of K. Say that $a \subset \omega$ is split by F_t if there is an $f \in F_t$ such that the set $\{f(n) : n \in a\}$ does not converge.

We show there is a t and a $b \subset \omega$ such that each $a \subset b$ is split by F_t. Otherwise, working in $\mathsf{PFA}(S)$ (a model of $\mathfrak{p} = \mathfrak{c}$), fix any well-ordering of S in order type ω_1 and recursively choose a mod finite, length ω_1 chain of infinite subsets $\{a_t : t \in S\}$ of ω so that a_t is not split by F_t. Choose any a which is mod finite contained in a_t for all $t \in S$. It is easy to see that t forces that a is a converging sequence in K.

Now fix t and $b \subset \omega$ so that F_t splits every infinite subset of b. Fix the embedding of ω into $[0,1]^{F_t}$ and let X denote the closure of this embedded copy of ω. It follows that t forces that K maps continuously onto the closure of X. If we prove that X has uncountable tightness in the $\mathsf{PFA}(S)$ model, then there will be an ω_1 free sequence $\{x_\alpha : \alpha \in \omega_1\}$ with a unique complete accumulation point x. It is easily checked that this sequence remains a free sequence in the closure of X in the forcing extension by S (use basic open sets to witness the disjoint closures). It then would follow that K does not have countable tightness.

We must define a proper poset $\mathbb{P}$ which forces an uncountable free sequence in X (as in the Moore-Mrowka proof) and satisfies that S remains Souslin after forcing with $\mathbb{P}$.

Since ω is completely divergent in X and $\mathfrak{p} = \mathfrak{c}$, the cardinality of X is $2^{\mathfrak{c}}$. Let Y be any dense countably compact subset of X of cardinality at most $\mathfrak{c}$. We may also assume Y is elementary in the sense that if two countable subsets of Y have a common limit point in X, then they do so in Y as well. Choose any $z \in X \setminus Y$, let $\mathfrak{F}$ be a maximal filter of closed subsets of Y such that $z \in \overline{F}$ for all $F \in \mathfrak{F}$. We first note that if $H \in \mathfrak{F}^+$, then there is a countable $H_0 \subset H$ such that $\overline{H_0}$ contains a member of $\mathfrak{F}$. Indeed, if this were not the case, then inductively choose $h_\alpha \in F_\alpha \in \mathfrak{F}$ so that $\overline{\{h_\beta : \beta < \alpha\}} \cap F_\alpha$ is empty. Since we may assume that X is countably tight and by our elementarity assumption on Y, it follows that $\{h_\beta : \beta < \alpha\}$ and F_α have disjoint closure in X. Thus, we have constructed the desired free sequence.

Since X is forced to have countable tightness in the forcing extension by S, the above construction can be conducted in the forcing extension as well (and no new countable sets are added), hence we have the following claim.

Claim. If $\dot{H}$ is an S-name of a subset of Y and $t \in S$ forces that $\dot{H} \in \mathfrak{F}^+$, then there is a $\beta \in \omega_1$ such that for each $t < s \in S_\beta$, the closure of $\{y : s \Vdash y \in \dot{H}\}$ is in $\mathfrak{F}$.

For each $y \in Y$, choose open sets V_y, U_y of X so that $y \in V_y \subset \overline{V_y} \subset U_y$ and $z \notin \overline{U_y}$. It follows that $Y \setminus U_y \in \mathfrak{F}$. Let κ be a large enough regular cardinal so that $X, Y, \mathfrak{F}$ are in $H(\kappa)$. As usual, the conditions in the poset $\mathbb{P}$ consist of functions p where $\operatorname{dom} p = \mathcal{M}_p$ is a finite elementary $\in$-chain of countable elementary models of $H(\kappa)$ and the range is a subset of Y. For each $M \in \mathcal{M}_p$, the set $\{X, Y, \{V_y, U_y : y \in Y\}, \mathfrak{F}\} \in M$, and $p(M)$ is an element of $\bigcap(\mathfrak{F} \cap M)$.

For each $p \in \mathbb{P}$ and $M \in \mathcal{M}_p$, we define the neighborhood $W(p, M)$ of $p(M)$ as $\bigcap\{V_{p(Q)} : Q \in \mathcal{M}_p \text{ and } p(M) \in V_{p(Q)}\}$. We define $p < q$ if $q \subset p$, and for each $M \in \mathcal{M}_p \setminus \mathcal{M}_q$ such that $\mathcal{M}_q \setminus M \neq \emptyset$, we have that $p(M) \in W(q, Q)$ where Q is the minimal element of $\mathcal{M}_q \setminus M$.

A somewhat, by now, standard proof will show that $\mathbb{P}$ is proper. And in essence, to see that $\mathbb{P}$ preserves that S is ccc, we show that if $M \prec H(\theta)$ and $s \in S_\delta$ and $M_0 = M \cap H(\kappa) \in \operatorname{dom}(p)$ then (s, p) is an M-generic condition for $S \times \mathbb{P}$ (see [Miy93]). This will be easier than many of the proofs using $\mathsf{PFA}(S)[S]$ because the conditions in $\mathbb{P}$ do not depend on S.

Let $D \in M$ be a dense open subset of $S \times \mathbb{P}$ and choose any extension $(\bar{s}, \bar{p})$ in D extending (s, p). Let $\mathcal{M}_{\bar{p}} \setminus M = \{M_0, M_1, \ldots, M_{\ell-1}\}$ be enumerated in increasing order. Since D is open and dense it follows that $(t, \bar{p}) \in D$ for extensions t of $\bar{s}$. Thus we may assume that $\bar{s} \notin M_{\ell-1}$. Let $\bar{p}(M_i) = x_i$ for each $i < \ell$, $\mathbf{x} = (x_0, \ldots, x_{\ell-1})$, and define, for $t \in S$,

$$\begin{aligned} T_t = \{(y_0, y_1, \ldots, y_{\ell-1}) = (q(Q_0), q(Q_1), \ldots, q(Q_{\ell-1})) :& \\ (\exists (t, q) \in D)(\exists Q \in \mathcal{M}_q) p \cap M_0 = q \cap Q \text{ and}& \\ \mathcal{M}_q \setminus Q_0 = \{Q_0 \in Q_1 \in \cdots \in Q_{\ell-1}\}\}.& \quad (7) \end{aligned}$$

Each $\mathbf{y} \in T_t$ is simply a function with domain ℓ, and so we may easily define, for each $k < \ell$ and each $t \in S$, $T_{t,k} = \{\mathbf{y} \restriction k : \mathbf{y} \in T_t\}$. For $\mathbf{y} \in T_{t,\ell-1}$, we define the S-name $\dot{H}_{\mathbf{y}} = \{(s, \check{y}) : \mathbf{y}^\frown\langle y\rangle \in T_s\}$. Then define $T_{t,\ell-1}(\mathfrak{F}) = \{\mathbf{y} \in T_{t,\ell-1} : t \Vdash \dot{H}_{\mathbf{y}} \in \mathfrak{F}^+\}$. Next, by induction, for $\mathbf{y} \in T_{t,\ell-k-1}$, define $\dot{H}_{\mathbf{y}} = \{(s, \check{y}) : \mathbf{y}^\frown\langle y\rangle \in T_{s,\ell-k}(\mathfrak{F})\}$ and $T_{t,\ell-j-1}(\mathfrak{F}) = \{\mathbf{y} \in T_{t,\ell-j-1} : t \Vdash \dot{H}_{\mathbf{y}} \in \mathfrak{F}^+\}$.

Of course we have that $\mathbf{x} \in T_{\bar{s}}$ and we show by induction that $\mathbf{x} \restriction k$ is in $T_{\bar{s},k}(\mathfrak{F})$ for $k = \ell-1, \ell-2, \ldots, 0$. We have that $\mathbf{x} \restriction k \in M_k$ and, also in M_k, we can define the set $A = \{t \in S : \mathbf{x} \restriction k \notin T_{t,k}(\mathfrak{F})\}$. The set of minimal elements of A will be contained in M_k, so if $\bar{s} \in A$, then there is a predecessor $t \in M_k$ of $\bar{s}$ which is in A. However, there is then an $F \in \mathcal{F} \cap M_k$ which is forced by $\bar{s}$ to be disjoint from $\dot{H}_{\mathbf{x}\restriction k}$ (i.e. $\{y : \mathbf{x} \restriction k^\frown\langle y\rangle \in T_{t,\ell-j}(\mathcal{F})\}$), which contradicts that x_k is forced by $\bar{s}$ to be in each of those sets.

Now we have that there is a predecessor t_0 of $\bar{s}$ in M_0 such that $\emptyset \in T_{t_0,0}(\mathfrak{F})$. By Claim 1, and by possibly extending t_0, we may assume that the closure of $Y_0 = \{y \in M_0 : t_0 \Vdash y \in \dot{H}_\emptyset\}$ will be in $\mathcal{F} \cap M_0$. Set $W = W(\bar{p}, M_0)$ and choose y_0 in $W \cap Y_0$. Let $\mathbf{y}_0 = \langle y_0\rangle$ and apply Claim 1 again, to extend t_0 to $t_1 < \bar{s}$ so that $Y_1 = \{y \in M_0 : t_1 \Vdash y \in \dot{H}_{\mathbf{y}_0}\}$ meets W. Continuing in this way, we inductively construct $\{y_0, y_1, \ldots, y_{\ell-1}\} \subset W \cap M_0$ and $t_\ell < \bar{s}$ in M_0 so that that $(y_0, y_1, \ldots, y_{\ell-1}) \in T_{t_\ell}$. By elementarity, there is a $q \in \mathbb{P} \cap M_0$ so that $(t_\ell, q) \in D$ witnessing that $(y_0, y_1, \ldots, y_{\ell-1}) \in T_{t_\ell}$. It is now easy to verify that (t_ℓ, q) is compatible with $(\bar{s}, \bar{p})$ as required.

4 Efimov problem

An Efimov space (if there is one) is an infinite compact space containing no converging sequence and no copy of $\beta\mathbb{N}$. A Moore-Mrowka space is a compact space of countable tightness which is not sequential. Efimov spaces have been shown to exist in a number of models (e.g. any model of CH [Fed77]) while the existence of a Moore-Mrowka space is known to be independent of ZFC (Fedorchuk and Ostaszewski from $\diamondsuit$ [Fed75, Ost76] and Balogh proved they do not exist if PFA holds [Bal89]). An analysis of the constructions has led to the formulation of minimal Boolean algebras [Kop89] and to a refinement [Kos99] which will be called T-algebras.

Definition 4.1. A T-algebra is a Boolean algebra $B \subset \mathcal{P}(X)$ (for some set X) for which there is a tree $T \subset 2^{<\kappa}$ (for some cardinal κ) and a generating set $\{a_t : t \in T\}$ for B such that the following hold:

1. all non-maximal nodes of T are branching,
2. for each non-successor node $t \in T$, $a_t = X$
3. for each successor node $t \in T$, $a_{t^\dagger} = X - a_t$ where $t^\dagger \cap t$ is the predecessor of both t and $t^\dagger$, (for convenience, let $t^\dagger = t$ when t does not have an immediate predecessor)

4. for each $t \in T$, the collection $\{a_s : s < t\}$ generates a non-degenerate filter u_t, and for each $s < t$, $a_t - a_s$ is a member of the Boolean algebra $B_{\leq s}$ generated by $\{a_r : r \leq s\}$. Let B_s denote the Boolean algebra generated by $\{a_r : r < s\}$.

For any maximal branch b of T we also let u_b be the filter generated by $\{a_s : s \in b\}$ and B_b the Boolean algebra generated by $\{a_s : s \in b\}$.

Two quite surprising properties of T-algebras are the following.

Proposition 4.2. *Let B be a T-algebra with generating set $\{a_t : t \in T \subset 2^{<\kappa}\}$, and let b be a maximal branch of T, then*

1. u_b is an ultrafilter on B,
2. B_b is a superatomic Boolean (sub-)algebra and its Stone space is equal to $\{u_b\} \cup \{u_{t^\dagger} : t \in b \text{ a successor}\}$.

Corollary 4.3. *If $A \cup \{b\}$ is a set of maximal branches of T for a T-algebra B with generating family $\{a_t : t \in T\}$, then u_b is a limit of $\{u_x : x \in A\}$ in the Stone space $S(B)$ if, and only if, $u_b \cap B_b$ is a limit of the collection $\{u_x \cap B_b : x \in A\}$ in the Stone space $S(B_b)$. For each $x \in A$, $u_x \cap B_b$ is the ultrafilter $u_{t^\dagger}$ where t is the minimal element of $b \setminus x$.*

This next construction is quite similar to that used in the Katětov example (2.8), but the example there was not a T-algebra (we leave it as an exercise that it could not be a T-algebra). The analogous result using Cohen forcing was given in [PM10]. The construction, from the hypothesis $\mathfrak{b} = \mathfrak{c}$, of a T-algebra whose Stone space is an Efimov space was announced in [DS12].

Theorem 4.4. *If $\kappa = 2^\omega$ and G is $\mathcal{M}_\kappa$-generic, then $V[G]$ models that there is a T-algebra $B \subset \mathcal{P}(\omega)$ with generating family $\{a_t : t \in 2^{<\omega_1}\}$, such that $S(B)$ has countable tightness and no converging sequences. That is, $S(B)$ is a Moore-Mrowka Efimov space.*

Proof. It is well-known that the Random real poset factors readily. For each set $I \subset \kappa$, $\mathcal{M}_\kappa$ is (forcing) isomorphic to $\mathcal{M}_I * \mathcal{M}_{\kappa\setminus I}$. Let $\{\dot{t}_\xi : \xi \in \kappa\}$ be a listing of nice names $\dot{t}$ such that for each $\xi \in \kappa$

1. there is a countable set $I_\xi \subset \xi + \omega$ such that $\dot{t}_\xi$ is an $\mathcal{M}_{I_\xi}$-name,
2. there is a δ_ξ such that $1 \Vdash \dot{t} \in 2^{\delta_\xi}$,
3. for all $\zeta < \xi$, $1 \Vdash \dot{t}_\xi \neq \dot{t}_\zeta$,
4. $1 \Vdash \{\dot{t}_\xi \restriction \alpha : \alpha \in \mathrm{dom}(t_\xi)\} \subset \{\dot{t}_\gamma : \gamma \leq \xi\}$,
5. $1 \Vdash 2^{<\omega_1} = \{\dot{t}_\xi : \xi \in \kappa\}$

For each $\omega \leq \alpha \in \omega_1$, let g_α be a bijection from ω onto α. To start the induction, let $\{C_n : n \in \omega\}$ be any independent family of infinite subsets of ω, and for $t \in 2^{<\omega}$, define $a_{t^\frown 1} = C_{|t|} \cap \bigcap\{a_s : s \subseteq t\}$ and $a_{t^\frown 0}$ is the complement of $a_{t^\frown 1}$. Notice that $a_{t^\frown 0} \setminus a_s$ is empty for each $s \leq t$, and so $a_{t^\frown 1} \setminus a_s = \omega \setminus a_s$ for each $s \leq t$. The algebra generated by $\{a_t : t \in 2^{<\omega}\}$ is a T-algebra which is equal to the algebra generated by $\{C_n : n \in \omega\}$. Thus its Stone space is the Cantor set.

By induction on $\xi \in \kappa$, we define $\dot{a}_{\dot{t}_\xi^\frown 0}$ and $\dot{a}_{\dot{t}_\xi^\frown 1}$ as follows. Let M be a countable elementary submodel such that $\xi, \{\dot{t}_\zeta : \zeta \in \kappa\}$ and $\{\dot{a}_{\dot{t}_\zeta} : \zeta \in \xi\}$ are in M. For each $\zeta < \xi$, let J_ζ be a countable set such that $\dot{a}_{\dot{t}_\zeta}$ is an $\mathcal{M}_{J_\zeta}$-name. Choose β_ξ to be a minimal limit ordinal in κ such $[\beta_\xi, \beta_\xi + \omega)$ is disjoint from $M \cup \bigcup_{\zeta<\xi} J_\zeta$. Let $\dot{r}_{\beta_\xi}$ denote the random real added by $\mathcal{M}_{[\beta_\xi, \beta_\xi+\omega)}$. That is, for each $k \in \omega$, the basic clopen set $[(\beta_\xi + k, e)]$ forces that $\dot{r}_{\beta_\xi}(k) = e$.

We use $g_{\delta_\xi+1}$ to enumerate the predecessors of $\dot{t}_\xi$ (including $\dot{t}_\xi$ itself) which are at successor levels as $\{s_n : n \in \omega\}$. The family $\{\bigcap_{j<k} a_{s_j} \setminus a_{s_k} : k \in \omega\}$ is a partition of ω (where $\bigcap_{j<0} a_{s_j} = \omega$). We define $\dot{a}_{\dot{t}_\xi^\frown 0}$ as the union of the collection $\{\bigcap_{j<k} a_{s_j} \setminus a_{s_k} : \dot{r}_{\beta_\xi}(k) = 0\}$. Of course $\dot{a}_{\dot{t}_\xi^\frown 1}$ is the complement, and is also (forced to be) equal to the union of the collection $\{\bigcap_{j<k} a_{s_j} \setminus a_{s_k} : \dot{r}_{\beta_\xi}(k) = 1\}$. Notice that for $k \in \omega$, $a_{\dot{t}_\xi^\frown e} \setminus a_{s_k}$ is equal to the finite join of sets of the form $a \setminus a_{s_k}$ for $a \in B_{\dot{t}_\xi}$, and so, by induction on ξ, are in $B_{\leq s_k}$.

Claim. If $J \subset \kappa$ is disjoint from $[\beta_\xi, \beta_\xi + \omega)$, and if $\dot{Y}$ is a $\mathcal{M}_J$-name of an infinite subset of $\{s < \dot{t}_\xi : s \text{ is a successor}\}$, then it forced that either there is a $t < \dot{t}_\xi$ such that $a_t \notin u_{s^\dagger}$ for infinitely many $s \in \dot{Y}$, or for each $e = 0, 1$, $\{s \in \dot{Y} : \dot{a}_{\dot{t}_\xi^\frown e} \in u_{s^\dagger}\}$ is infinite.

Claim. If $\{\dot{b}_\gamma : \gamma \in \omega_1\}$ are $\mathcal{M}_\kappa$-names of maximal branches of T (i.e. each $\dot{b}_\gamma$ is forced to be a member of 2^{ω_1}) then there is a $\delta \in \omega_1$ such that the closure of $\{u_{\dot{b}_\gamma} : \gamma \in \delta\}$ contains $\dot{b}_\gamma$ for all γ.

It follows directly from Claim 1 that $S(B)$ has no converging sequences, and from Claim 2 that it is hereditarily separable.

We remark that it was shown in [DF07] that there is an Efimov space in extensions by random reals.

Proposition 4.5. *If random reals are added, then the Stone space of the ground model $\mathcal{P}(\mathbb{N})$ has no converging sequences. Therefore, if more than $\mathfrak{c}$ random reals are added, it is an Efimov space.*

Another wonderful example of a T-algebra based solution is

Proposition 4.6 ([JKS09]). *Let T be the tree consisting of those $t \in 2^{<\omega_2}$ such that if there is an $\alpha \in \mathrm{dom}(t)$ with $t(\alpha) = 1$, then $\mathrm{dom}(t) < \alpha + \omega$ (t is a finite extension of the first place it is non-zero). There is a proper forcing extension in which there is a T-algebra with generating set $\{a_t : t \in T\}$ such that the Stone space is a Moore-Mrowka space with a single point of uncountable character and the complement of that point is initially ω_1-compact.*

5 Sequential order

In a space X and for a set $A \subset X$, we recursively define the sequential order of elements in the sequential closure of A. We use the notation $A^{(\alpha)}$ to denote the set of all elements that have sequential order at most α. In particular $A^{(0)}$ is simply A, and $A^{(1)}$ is the set of all elements x of X for which there is a sequence from A converging to x (including constant sequences). For any limit ordinal α, $A^{(\alpha)}$ will simply equal $\bigcup_{\beta<\alpha} A^{(\beta)}$, and for successor $\alpha+1$, $A^{(\alpha+1)} = (A^{(\alpha)})^{(1)}$. For $n \in \omega$, a point has sequential order n if n is minimal such that $x \in A^{(n)}$. For $\alpha \geq \omega$, x will have sequential order α if α is minimal such that $x \in A^{(\alpha+1)} \setminus A^{(\alpha)}$. We denote this sequential order as $s.o.(x, A) = \alpha + 1$.

The fundamental open problem about sequential order is that we do not know how large the sequential order of compact sequential spaces can be. The known limits presently are: in ZFC there is an example with sequential order 2, CH implies there is an example with sequential order ω_1, in the Cohen model there is an example with sequential order ω_1, and $\mathfrak{b} = \mathfrak{c}$ implies there is an example with sequential order 4.

More progress has been made when restricted to a natural subclass of compact sequential spaces. Say that a compact space is CB-sequential (for Cantor-Bendixson) if the space is scattered, sequential, and the sequential order of each element with respect to the set of isolated points naturally coincides with the scattering level of the point. The crucial idea is the following new PFA result, which uses a Luzin family in its proof. An almost disjoint family $\mathcal{A} = \{a_\alpha : \alpha \in \omega_1\}$ of subsets of ω is called a Luzin family if for each $\alpha \in \omega_1$ and $n \in \omega$, the set $\{\beta < \alpha : a_\beta \cap a_\alpha \subset n\}$ is finite.

Theorem 5.1 (PFA). *Suppose that X is a compact sequential space containing a countable set which we identify with ω. If a point z is not in $\omega^{(1)}$, but it is in the closure of ω, then z is the unique complete accumulation point of some set of size ω_1 from $\omega^{(1)}$. Notice then that every point of $\overline{\omega}$ is in the radial closure of $\omega^{(1)}$. Thus we can say that the radial order of every compact sequential space is at most* 2.

Proof. We assume we have a compact space X with ω a dense subset of X and $z \notin \omega^{(1)}$. Let $\mathcal{W}$ be any ultrafilter on ω with the property that $z \in \overline{W}$ for all $W \in \mathcal{W}$. Assume that $\mathcal{A}$ is a Luzin family of converging subsequences of ω and, for $\alpha \in \omega_1$, let $x_\alpha \in \omega^{(1)}$ be the limit. Assume that every neighborhood of z contains uncountably many of the x_α's. It then follows that z is the unique complete accumulation point of $\{x_\alpha : \alpha \in \omega_1\}$. Indeed, if $z \in W$ and W is open, then $a_\alpha \subset^* W$ for uncountably many α. Since $\mathcal{A}$ is Luzin, $W \cap a_\alpha$ is infinite for all but countably many α, hence $\overline{W}$ contains all but countably many x_α.

We use PFA to produce $\mathcal{A}$. We use the method in [Dow11]. Let $\mathcal{A}$ denote any maximal almost disjoint family of infinite subsets of ω with the property that for each $a \in \mathcal{A}$, a converges to some point $x_a \in X$. For a finite set $t \subset \omega$, let $\mathcal{A}_t = \{a \in \mathcal{A} : t \subset a\}$. Say that a subfamily $\mathcal{C}$ of $\mathcal{A}$ is $\mathcal{W}$-large if for each countable $\tilde{W} \subset \mathcal{W}$, there is an $a \in \mathcal{C}$ such that $a \cap W$ is infinite for all $W \in \tilde{W}$.

Claim. If $\mathcal{C}$ is an almost disjoint family which is $\mathcal{W}$-large, then the set of ℓ such that $\mathcal{C}_{\{\ell\}}$ is $\mathcal{W}$-large is in $\mathcal{W}$.

Proof (of claim). Assume that $W = \{\ell : \mathcal{C}_{\{\ell\}}$ is not $\mathcal{W} -$ large$\}$ is in $\mathcal{W}$. For each $\ell \in W$, select countable $\mathcal{W}_\ell \subset \mathcal{W}$ witnessing that $\mathcal{C}_{\{\ell\}}$ is not $\mathcal{W}$-large. Using that $\mathcal{C}$ is $\mathcal{W}$-large, choose $a \in \mathcal{C}$ so that $a \cap W'$ is infinite for all $W' \in \bigcup_\ell \mathcal{W}_\ell \cup \{W\}$. Choose any $\ell \in a \cap W$, and observe that $a \in \mathcal{C}_{\{\ell\}}$ violating that $\mathcal{W}_\ell$ is supposed to witness that this collection is not $\mathcal{W}$-large.

We define a poset where $p \in \mathbb{P}$ if p is a finite function whose domain $\mathrm{dom}(p)$ is a finite $\in$-chain, $\mathcal{M}_p$, of countable elementary submodels of some suitable $H(\kappa)$, with each containing $\{z, X, \mathcal{W}, \mathcal{A}\}$. For each $M \in \mathcal{M}_p$ the value $p(M)$ is a member of $\mathcal{A}$ that meets each member of $\mathcal{W} \cap M$. We define $p < q$ if $p \supset q$ and for each $M \in \mathcal{M}_p \setminus \mathcal{M}_q$ and each $Q \in \mathcal{M}_q$ with $M \in Q$ the intersection $p(M) \cap p(Q)$ is not contained in $|q|$. The canonical name of the desired family $\mathcal{A}_G$ is simply the collection $\{p(M) : p \in G, M \in \mathcal{M}_p\}$. If $\mathbb{P}$ is proper, this requirement on extension will ensure that this collection in Luzin. We need an additional argument that there is a family of ω_1-dense sets that will ensure that z will be a complete accumulation point.

First we show that $\mathbb{P}$ is proper and that the existence of the $\mathcal{W}$-large families of the form $\mathcal{A}_t$ is critical for this. Let $\mathbb{P} \in H(\theta)$ and let $q, \mathbb{P}$ be elements of a countable $M \prec H(\theta)$. Choose any $a \in \mathcal{A}$ which meets each member of $\mathcal{W} \cap M$ in an infinite set. We prove that if $q \subset p$ and $M_0 = M \cap H(\kappa) \in \mathcal{M}_p$, then p is $(M, \mathbb{P})$-generic. As usual, let $\{M_0, \ldots, M_\ell\}$ enumerate $\mathcal{M}_p \setminus M_0$ in increasing order. Most of the argument is relatively standard in that we use elementarity to find a $\bar{p} \in M_0$ which reflects the relationship between $p \cap M_0$ and p. However a new idea is needed to ensure that $\bar{p}(Q_j) \cap p(M_i)$ is not contained in $|p|$ where $\mathcal{M}_{\bar{p}} \setminus \mathcal{M}_p = \{Q_0, \ldots, Q_\ell\}$ is also listed in increasing order.

Let $D \in M$ be dense in $\mathbb{P}$ and define a tree $T \subset \mathcal{A}^{\ell+1}$, where $\langle a_0, \ldots, a_\ell \rangle \in T$ providing there is a condition $r \in D$ such that $\mathcal{M}_r$ is an end extension of $p \cap M_0$, $\mathcal{M}_r \setminus \mathcal{M}_p \cap M_0 = \{Q_0, \ldots, Q_\ell\}$ is listed in increasing order, and $a_i = r(Q_i)$ for $0 \leq i \leq \ell$. Naturally $T \in M_0$. Notice that for $\mathbf{a} = \langle a_0, \ldots, a_\ell \rangle \in T$, and with r as given, the collection $\mathcal{C}_{\mathbf{a}\restriction\ell} = \{a : \mathbf{a} \restriction \ell^\frown\langle a \rangle \in T\}$ is $\mathcal{W}$-large. The reason this is so is that if it were not $\mathcal{W}$-large, then there would be no set in $\mathcal{C}_{\mathbf{a}\restriction\ell}$ which met each member of $\mathcal{W} \cap Q_\ell$. But of course, $r(Q_\ell)$ is such a set. By induction, for each $j \leq \ell$, $\mathcal{C}_{\mathbf{a}\restriction j}$ is also $\mathcal{W}$-large, where $\mathcal{C}_{\mathbf{a}\restriction j}$ is the set of $a \in \mathcal{A}$ for which there is a $\mathbf{a}' \in T$ extending $\mathbf{a} \restriction j^\frown\langle a \rangle$. In particular, $\mathcal{C}_\emptyset$ is $\mathcal{W}$-large and is an element of M_0. By Claim 1, for each $t \in [\omega]^{<\omega}$ and almost disjoint family $\mathcal{C} \in M_0$ such that $\mathcal{C}_t$ is $\mathcal{W}$-large, and each $0 \leq i \leq \ell$, there is a $t' \supset t$ such that $\mathcal{C}_{t'}$ is $\mathcal{W}$-large and $(t' \setminus t) \cap p(M_i)$ is not empty (recall that

$p(M_i)$ meets every member of $M_i \cap \mathcal{W}$). Therefore, by a recursion of length $\ell+1$, there is a finite set t_0 such that $(\mathcal{C}_\emptyset)_{t_0}$ is $\mathcal{W}$-large and $t_0 \cap p(M_i)$ has cardinality greater than $|p|$ for each $0 \le i \le \ell$. Choose any $a_0 \in (\mathcal{C}_\emptyset)_{t_0}$ which is in M_0. Recall that $\mathcal{C}_{\langle a_0\rangle}$ is $\mathcal{W}$-large. Thus we can recursively choose finite sets t_{i+1} so that $(\mathcal{C}_{\langle a_0,\ldots,a_i\rangle})_{t_{i+1}}$ is $\mathcal{W}$-large and $t_{i+1} \cap p(M_j)$ has size greater than $|p|$ for each j. Then choose $a_{i+1} \in (\mathcal{C}_{\langle a_0,\ldots,a_i\rangle})_{t_{i+1}}$ and continue.

Finally we have that there is some $r \in D \cap M_0$ which witnesses that $\langle a_0,\ldots,a_\ell\rangle \in T$, and we have ensured that $a_i \cap p(M_j)$ has cardinality greater than $|p|$ for each $i,j \le \ell$. It follows that r is compatible with p and completes the proof that $\mathbb{P}$ is proper.

Finally we prove that there is a set $\mathcal{D}$ of ω_1-many dense subsets of $\mathbb{P}$ such that each $\mathcal{D}$-generic filter G will result in z being a complete accumulation point of the family $\{x_a : a \in \mathcal{A}_G\}$. The new idea is to remember that PFA implies that every compact sequential space has a dense set of points of countable character [Dow88, 6.3] (but this does not follow from Martin's Axiom [Kos99]). Fix any $p \in \mathbb{P}$ and $M \in \mathcal{M}_p$. Fix an increasing cofinal chain $\{\mathcal{W}_\ell : \ell \in \omega\}$ of the countable subsets of $\mathcal{W}$ which are members of M. Also let $\{t_j : j \in \omega\}$ enumerate those $t \in [\omega]^{<\omega}$ such that $\mathcal{A}_t$ is $\mathcal{W}$-large. For each $j \le \ell \in \omega$, let $\mathcal{A}_{j,\ell}$ denote the (still $\mathcal{W}$-large collection of) members of $\mathcal{A}_{t_j}$ which meet every member of $\mathcal{W}_\ell$. Let $Y_{j,\ell} = \{x_a : a \in \mathcal{A}_{j,\ell}\}$. Since $\mathcal{A}_{j,\ell}$ is $\mathcal{W}$-large, it follows that z is a limit of $Y_{j,\ell}$. Therefore, since $z \in M$ and X has countable tightness, z is a limit point of $Y_{j,\ell} \cap M$ for all $j,\ell \in \omega$. Define the closed set $K(p,M)$ to be the intersection of the family $\{\overline{Y_{j,\ell}} : j,\ell \in \omega\}$.

Let $y \in K(p,M)$ be any point that has a countable relative local base in $K(p,M)$. Let $\{U_n : n \in \omega\}$ be open subsets of X such that the family $\{U_n \cap K(p,M) : n \in \omega\}$ is a base for y in $K(p,M)$. For each j,ℓ,n

$$D(y,j,\ell,n) = \{r \in \mathbb{P} : (\exists M' \in \mathcal{M}_r \cap M) \;\; r(M') = a \in \mathcal{A}_{j,\ell} \text{ and } x_a \in U_n\}.$$

We show that $D(y,j,\ell,n)$ is predense below p. To see this, suppose that $\bar{p} < p$. As in the proof that $\bar{p}$ is an $(M,\mathbb{P})$-generic condition there is a finite set $t \supset t_j$ which meets $\bar{p}(Q)$ in size at least $|\bar{p}|$ for each $Q \in \mathcal{M}_{\bar{p}} \setminus M$, and so that ($\mathcal{A}_t$ is still $\mathcal{W}$-large. There is a j' so that $t = t_{j'}$. Let $\bar{Q} \prec H(\kappa)$ be a countable member of M satisfying that $\mathcal{M}_{\bar{p}} \cap M \in \bar{Q}$, and choose ℓ so that $\bar{Q} \cap \mathcal{W} \subset \mathcal{W}_\ell$. Since y is in the closure of $Y_{j',\ell} \cap M$, there is an $a \in \mathcal{A}_t$ which meets every member of $\bar{Q} \cap \mathcal{W}$, and so that $x_a \in U_n$. It follows that $r = \bar{p} \cup \{(\bar{Q}, a)\}$ (i.e. $r(\bar{Q}) = a$) is a member of $\mathbb{P}$ which extends $\bar{p}$ and is in $D(y,j,\ell,n)$.

Let $p \in G$ be a filter on $\mathbb{P}$ which meets each $D(y,j,\ell,n)$. For each j,ℓ, let $Y_G = \{x_a : (\exists r \in G) \;\; (\exists Q \in \mathcal{M}_r) r(Q) = a\}$. Clearly the family $\{U_n \cap Y_G \cap Y_{j,\ell} \cap M : j,\ell,n \in \omega\}$ has the finite intersection property. It follows then that Y_G has limit points in $K(p,M)$ which are in U_n for each n, i.e. that y is a limit of Y_G. Finally, to find such a family for z, simply choose a countable set of $y \in K(p,M)$, each of countable relative character, which accumulates to z. It should now be clear that there is a family of ω_1 many dense subsets of $\mathbb{P}$ sufficient to ensure that z is a complete accumulation point of Y_G.

Corollary 5.2. PFA *implies there is no CB-sequential space of sequential order greater than* ω.

Proof. Assume that X is such a space. Let X_0 be the dense set of isolated points. Since the sequential order is assumed to be greater than ω, there must be an infinite set $\{w_n : n \in \omega\}$ of points with sequential order of each w_n equaling ω with respect to X_0. For each n, choose a sequence $\{z_{n,m} : n+1 < m \in \omega\}$ which converges to w_n and such that $s.o.(z_{n,m}, X_0)$ is between m and ω. Since X is CB-sequential, it follows that $s.o.(z_{n,m}, X_n) \geq m - n$ for each n, m. Apply Theorem 5.1, and select an uncountable set $\{x_{n,m,\alpha} : \alpha \in \omega_1\} \subset X_{n+1}$ so that $z_{n,m}$ is the unique complete accumulation point of this set.

Choose any countable elementary submodel M_0 which contains all the sets defined in the previous paragraph. The poset $\mathcal{P}(\omega)/fin$ is a countably closed, hence proper, poset. Choose any infinite set I_0 which is $(M_0, \mathcal{P}(\omega)/fin)$-generic. Let I_0, M_0 be elements of another countable elementary submodel M_1. Continue choosing an elementary $\in$-chain of models M_α and $I_\alpha \in M_{\alpha+1}$ which is $(M_\alpha, \mathcal{P}(\omega)/fin)$-generic so that $I_\alpha \subset^* I_\beta$ for all $\beta < \alpha$. For limit α, the chain $\{I_\beta : \beta \in \alpha\}$ is already an $(M_\alpha, \mathcal{P}(\omega)/fin)$-generic filterbase where $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.

Let $M = \bigcup_{\alpha\in\omega_1} M_\alpha$, and choose any $I \subset \omega$ which is mod finite contained in each I_α. Also, choose any $f \in \omega^\omega$ so that for all $h \in \omega^\omega \cap M$, we have that $h <^* f$. By possibly shrinking I, we may assume that the sequence $\{z_{n,f(n)} : n \in I\}$ converges to some point w. Of course we have that $w \in X_\omega$. For each n, choose clopen $W_n \in M$ such that $\{w_n\} = W_n \cap X_\omega$. Also choose clopen W so that $\{w\} = W \cap X_\omega$. It is routine to check that for each $n \in I$ we may assume that $z_{n,f(n)} \in W \setminus \bigcup_{k<n} W_k$, and there is an $\alpha_n \in \omega_1$ such that $x_{n,f(n),\alpha} \in W$ for all $\alpha > \alpha_n$. Choose any $\alpha \in \omega_1$ which is larger than each such α_n. Now we work briefly in $M_{\alpha+1}$ since $\{x_{n,m,\alpha} : n, m \in \omega\}$ is an element of $M_{\alpha+1}$. For each n, there is a maximal antichain $\mathcal{A}_n \in M_{\alpha+1}$ of subsets a of ω satisfying that $\{x_{n,m,\alpha} : m \in a\}$ converges. It follows that, for each n, there is an $a_n \in \mathcal{A}_n$ such that $I_{\alpha+1} \subset^* a_n$. The sequence $\{a_n : n \in \omega\}$ is in $M_{\alpha+2}$. For each n, let y_n denote the limit of the sequence $\{x_{n,m,\alpha} : m \in a_n\}$. Note that $y_n \in W_n \cap X_{n+2}$. By the same reasoning, there is a $z \in M$ such that the sequence $\{y_n : n \in I_{\alpha+3}\}$ converges to z and $z \in X_\omega$. There is clopen $W_z \in M$ such that $W_z \cap X_\omega = \{z\}$, and we may assume that $y_n \in W_z$ for all $n \in I_{\alpha+3}$. There is an $h \in M \cap \omega^\omega$ such that $\{x_{n,m,\alpha} : h(n) < m\} \subset W_z$ for all $n \in I_{\alpha+3}$. By removing a finite set from I, we then have that $\{x_{n,f(n),\alpha} : n \in I\}$ is contained in W_z. Notice also that, since $w_n \notin W_z$ for each n, there is another function $h_1 \in M \cap \omega^\omega$ satisfying that $z_{n,m} \notin W_z$ for all $m > h_1(n)$ for all n. Of course $h_1 <^* f$ and so now we have that $w \neq z$, and $\{x_{n,f(n),\alpha} : n \in I\} \subset W_z \cap W_w$. This is a contradiction since this infinite set must have limit point in X_ω while the closed set $W_z \cap W_w$ does not meet X_ω.

6 Selective separability

The notion of Selectively Separable, or M-separable, is an interesting selection principle which significantly strengthens the notion of a space being separable. It was formulated by Scheepers in [Sch99].

Definition 6.1. A space S is M-separable if for each sequence $\{D_n : n \in \omega\}$ of dense subsets of S, there is a selection $\{E_n : n \in \omega\}$ of finite sets with dense union satisfying that $E_n \subset D_n$ for all n.

It is strongly motivated by the connection to the C_p-theory of function spaces. It is immediate that each dense subset of an M-separable space is separable.

Proposition 6.2. *If X is a σ-compact space, then every separable subspace of $C_p(X)$ is M-separable.*

In fact, the more general result holds for spaces X which have the property that each their finite powers in Menger (hence the name M-separable). A space X is Menger if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence of finite subcollections $\mathcal{W}_n \subset \mathcal{U}_n$ satisfying that $\bigcup_n \mathcal{W}_n$ is itself a cover. Of course $C_p(X)$ is not separable unless X has countable weight. We have the following very interesting result based on an earlier result of Arhangel'skii (see [BBMT08, 2.9]).

Proposition 6.3. *For a separable metric space X, $C_p(X)$ is M-separable, if and only if, X^n has the Menger property for each $n \in \omega$.*

There are countable dense subsets of $2^{\mathfrak{c}}$ which are M-separable (in $2^{2^{\mathbb{N}}}$), and there are those that are not M-separable (in $2^{\mathbb{N}^\omega}$).

A variant of M-separable was introduced in [BD11] which can be called strategically M-separable (or SS+). A space S is strategically M-separable if Player II has a winning strategy in the following game. The game lasts for ω moves, at stage n, player I choose a dense subset D_n of S and player II selects a finite subset E_n of D_n. Player II wins the play of the game if the collection $\{E_n : n \in \omega\}$ has dense union. This next result, which comes from [GS11] and [BD12] respectively, shows a surprising distinction between the two properties. In particular there is an M-separable countable space which is not strategically M-separable.

Theorem 6.4. *The property of being M-separable is finitely additively, but there is a countable space which in not strategically M-separable but which is the union of two dense strategically M-separable subspaces.*

In proving the previous result, a useful equivalence to being M-separable was introduced.

Proposition 6.5. *A separable space S is M-separable if for each $x \in S$ and each descending sequence $\{D_n : n \in \omega\}$ of dense sets, there is a selection $\{E_n : n \in \omega\}$ of finite sets with $E_n \subset D_n$, such that x is in the closure of the union.*

Proof. Assume that S has the property as in the statement and let $\{D_n : n \in \omega\}$ be any sequence of dense subsets of S. Let $\{x_k : k \in \omega\}$ enumerate a dense subset of S. For each n let $\tilde{D}_n = \bigcup\{D_k : n \leq k\}$. Of course the sequence $\{\tilde{D}_n : n \in \omega\}$ is a descending sequence of dense sets. For each $k \leq n \in \omega$, let $E(k, n)$ be a finite subset of D_n such that x_k is in the closure of $\bigcup_{k \leq n} E(k, n)$. For each $n \in \omega$, let $\tilde{E}_n = \bigcup\{E(k, n) : k \leq n\}$. Since $x_k \in \bigcup_{k \leq n} \tilde{E}_n$ for each k, it follows that the sequence $\{\tilde{E}_n : n \in \omega\}$ has dense union. Similarly, the sequence $\{D_n \cap \bigcup_{k \leq n} \tilde{E}_k : n \in \omega\}$ has the same union and witnesses that S is M-separable.

Another unexpected but useful result about M-separable is that every separable Frechet space is M-separable [BD11]. The most interesting question about M-separable spaces is whether or not the property is (consistently) finitely productive. Interestingly it is independent if there are two countable Frechet spaces whose product is not M-separable.

Proposition 6.6 ([Bab09]). CH *implies there are metric spaces X and Y such that $(X \cup Y)^2$ is not Menger, but X^n and Y^n are Menger for all $n \in \omega$. Therefore, there are countable dense subsets A and B of $C_p(X)$ and $C_p(Y)$ that are M-separable, while the space $A \times B \subset C_p(X \cup Y)$ is not.*

Proposition 6.7 ([GS11, BD11]). *Martin's Axiom for countable posets implies there are countable M-separable spaces whose product is not M-separable.*

Proposition 6.8 ([BD11]). CH *implies there are two countable Frechet spaces whose product is not M-separable.*

We end the discussion of M-separable by establishing the following application of OCA which improves the result in [BD12].

Theorem 6.9. OCA *implies that the product of two countable Frechet spaces is again M-separable.*

Proof. Assume that A and B are countable Frechet spaces. Fix any $x \in A$ and $y \in B$ and descending sequence $\{D_n : n \in \omega\}$ of dense subsets of $A \times B$. We prove that $A \times B$ is M-separable by verifying the condition in Proposition 6.5. We leave as an exercise the case when either x or y is in the closure of a set of isolated points. Thus we may assume that there are no isolated points and therefore that for each $x' \in X$ and $y' \in Y$, there is an n such that D_n is disjoint from $\{x'\} \times Y$ and $X \times \{y'\}$. By passing to a subsequence, we may assume that $D_n \setminus D_{n+1}$ is infinite for each n, and let $\{d(n,i) : i \in \omega\}$ be an enumeration of this set.

Assume that $a \subset A$ converges to x and $b \subset B$ converges to y. By our assumption on D_n, it follow that if $d_n \in D_n \cap (a \times b)$ for each $n \in \omega$, then $\{d_n : n \in \omega\}$ converges to (x,y). Therefore we may now also assume that for each such a, b, there is an n such that $D_n \cap (a \times b)$ is empty.

Let $\mathcal{A}$ be the set of all infinite sequences converging to x, let $\mathcal{B}$ be the set of all infinite sequences converging to y, and let $\mathcal{F} \subset \omega^\omega$ be the family of all strictly increasing functions. In preparation for applying OCA, consider the family $\mathcal{X} = \{(a,b,f) : (\forall n)(\forall i < f(n)) \ \ d(n,i) \notin a \times b\}$. There is a separable metric topology on $\mathcal{X}$ where for each point $(x', y', (n,j)) \in A \times B \times \omega^2$, the set $\{(a,b,f) \in \mathcal{X} : x' \in a, y' \in b, \text{ and } f(n) = j\}$ is clopen. Define $K_0 \subset [\mathcal{X}]^2$ by putting $\{(a,b,f), (a',b',f')\} \in K_0$ providing there is an (n,i) such that $i < \min(f(n), f'(n))$ and $d(n,i)$ is in one of $a \times b'$ or $a' \times b$.

Assume that $\{(a_\alpha, b_\alpha, f_\alpha) : \alpha \in \omega_1\}$ is an uncountable subset of $\mathcal{X}$ with the property that all pairs are in K_0. Since $\mathfrak{b} > \omega_1$, we may pass to an uncountable subset so that there is a function $f \in \omega^\omega$ such that $f_\alpha < f$ for all α. For each n, set $E_n = \{d(n,i) : i < f(n)\}$. We show that (x,y) is in the closure of $\bigcup_n E_n$. Let U and W be open sets in A and B containing x and y respectively. Again pass to an uncountable family and choose any m so that there are fixed finite sets F_A, F_B satisfying that $a_\alpha \setminus U = F_A$ and $b_\alpha \setminus W = F_B$ for all α. Choose m so that D_m is disjoint from $F_A \times B$ and $A \times F_B$. Next, by further thinning, we can assume that $(a_\alpha \times b_\alpha) \cap \{d(n,i) : n < m \text{ and } i < f(n)\}$ is the same for all α. Choose distinct α, γ and pick $d(n,i)$ with $i < \min\{f_\alpha(n), f_\gamma(n)\}$ witnessing that $\{(a_\alpha, b_\alpha, f_\alpha), (a_\gamma, b_\gamma, f_\gamma)\} \in K_0$. By symmetry, assume that $d(n,i) \in a_\alpha \times b_\gamma$. It follows that $n \geq m$ and $d(n,i) \in (a_\alpha \setminus F_A) \times (b_\gamma \setminus F_B)$; which implies that $U \times W$ meets E_n.

Therefore, by OCA, we are finished if we can show that $\mathcal{X}$ can not be expressed as a countable union of sets $\mathcal{Y}$ with the property that $[\mathcal{Y}]^2$ is disjoint from K_0. We follow the approach in [BD12], but also note there is a similar argument in [Tod03, Theorem 2]. Fix a sequence $\{x_n : n \in \omega\}$ converging to x and a sequence $\{y_m : m \in \omega\}$ converging to y. Let $\mathcal{W}$ be any ultrafilter on $\omega \times \omega$ with the property that for each $W \in \mathcal{W}$, the set $\{n : \{m : (n,m) \in W\} \text{ is infinite}\}$ is infinite. Assume that $\mathcal{X}$ is equal to $\mathcal{X}_n$ $(n \in \omega)$. For each $n, m \in \omega$ and $t \in \omega^m$, define

$$\mathcal{X}_{n,m,t} = \{(a,b) : (\forall k \in \omega)(\exists f)(a,b,f) \in \mathcal{X}_n, t \subset f, \text{ and } f(m) > k\} \ .$$

We leave the reader to check that for all $(a,b,f) \in \mathcal{X}$, there are n, m, t such that $(a,b) \in \mathcal{X}_{n,m,t}$ (every unbounded set of f will diverge to infinity at some m).

Let $\{(n_k, m_k, t_k) : k \in \omega\}$ be an enumeration of $\omega \times \omega \times \omega^{<\omega}$. Let $A_0 = A$ and $B_0 = B$. By induction on k, we choose $A_{k+1} \subset A_k$ and $B_{k+1} \subset B_k$

so that we retain the property that for each $\ell \in \omega$, $\{(n,m) : (x_n, y_m) \in D_\ell \cap (A_k \times B_k)\}$ is a member of $\mathcal{W}$. The choice of A_{k+1}, B_{k+1} must also satisfy that either, for all $(a,b) \in \mathcal{X}_{n_k,m_k,t_k}$, $a \cap A_{k+1}$ is finite or, for all $(a,b) \in \mathcal{X}_{n_k,m_k}$, $b \cap B_{k+1}$ is finite. Having chosen A_k, B_k, choose define $I_k = \bigcup\{a \setminus \ell : (\exists b)(a,b) \in \mathcal{X}_{n_k,m_k,t_k}$ and $d(n,i) \notin ((a \setminus \ell) \times B)$ for all $n < m_k, i < t(n)\}$ and $J_k = \bigcup\{b \setminus \ell : (\exists a)(a,b) \in \mathcal{X}_{n_k,m_k,t_k}$ and $d(n,i) \notin (A \times (b \setminus \ell))$ for all $n < m_k, i < t(n)\}$. We check that $I_k \times J_k$ is disjoint from D_{m_k}. Suppose otherwise, and that $d(n,i) \in I_k \times J_k$ for some $n \geq m_k$. There must be (a,b) and (a',b') in $\mathcal{X}_{n_k,m_k,t_k}$ so that $d(n,i) \in a \times b'$. Therefore, there are f, f' so that $i < f(n), f'(n)$ and $(a,b,f), (a',b',f')$ are in $\mathcal{X}_{n_k}$. This however shows that $\{(a,b,f),(a',b',f')\} \in K_0$.

It now follows immediately that one of $(A_k \setminus I_K) \times B_k$ or $A_k \times (B_k \setminus J_k)$ will have meet each D_m is a set which has $\mathcal{W}$-large closure. This completes the induction.

Choose a sequence $\{(j_k, \ell_k) : k \in \omega\}$ so that, for each k, $k \leq j_k \leq \ell_k$ and (x_{j_k}, y_{ℓ_k}) is in the closure of $D_k \cap (A_k \times B_K)$. Now we will use that A, B are Frechet. For each k, choose a sequence $\{a(k,i) : i \in \omega\}$ converging to x_{j_k} such that $D_k \cap (\{a(k,i)\} \times B_k) \neq \emptyset$ for each i. Similarly choose $\{b(k,i) : i \in \omega\}$ converging to y_{ℓ_k} such that $D_k \cap (A_k \times \{b(k,i)\})$ is not empty for each i. It follows that x is in the closure of $\bigcup_k \{a(k,i) : i \in \omega\}$, and so there is an $a \subset \bigcup_k \{a(k,i) : i \in \omega\}$ which is in $\mathcal{A}$. Also $a \cap \{a(k,i) : i \in \omega\}$ is finite for each k, hence $a \subset^* A_k$ for all k. Similarly there is a $b \subset^* B_k$ for all k such that $b \in \mathcal{B}$. Choose f and D_m so that $d(n,i) \notin a \times b$ for all $n \geq m$ and $i < f(n)$. Remove a finite subset from each of a and b so that $(a,b,f) \in \mathcal{X}$. This shows that $(a,b,f) \in \mathcal{X} \setminus \mathcal{X}_n$ for all n.

7 Minimal walks and L-spaces

Three major results are reviewed in this final section. The first was mentioned and used in the section on Katětov's problem, which shows the consistency of the non-existence of a first-countable L-space. Of course an L-space is a regular space which is hereditarily Lindelöf but not hereditarily separable. Justin Moore showed that there is an L-space [Moo08] and the L-space he constructs (defined below) has character ω_1. The method of minimal walks ([Tod07]) is used in the construction of the L-space and is of great importance in a number of results.

Theorem 7.1 ([Sze80]). $\mathsf{MA}(\omega_1)$ *implies there is no first-countable L-space.*

In fact the principle $\mathcal{K}_2(rec)$ suffices, see [LT02, 5.1].

Proof. Suppose that X is a first-countable space which is hereditarily Lindelöf. We assume, for a contradiction, that X is not separable. Choose a sequence $\{x_\alpha : \alpha \in \omega_1\} \subset X$ with the property that x_α is not in the closure of $\{x_\beta : \beta < \alpha\}$. For each α, choose an open neighborhood U_α of x_α with the property that $\overline{U_\alpha}$ is disjoint from $\overline{\{x_\beta : \beta < \alpha\}}$. For convenience, identify x_α with the singleton α.

We define a poset $\mathbb{P}$ that is designed to force an uncountable discrete subset of $\{x_\alpha : \alpha \in \omega_1\}$ as witnessed by the neighborhood assignment we just made. Thus, $p \in \mathbb{P}$ if $p \in [\omega_1]^{<\omega}$ and for $\alpha < \beta$ in p, $\beta \notin W_\alpha$. The union of any uncountable subset of $\mathbb{P}$ of pairwise compatible elements would be a discrete subset of X. Since that would contradict the assumption that X is hereditarily Lindelöf, and we are assuming $\mathsf{MA}(\omega_1)$, it follows that $\mathbb{P}$ is not ccc.

Let $\{p_\xi : \xi \in \omega_1\} \subset \mathbb{P}$ be an antichain. By a standard Δ-system argument we may assume that that $\max p_\xi < \min p_\eta$ whenever $\xi < \eta$ and that there is an $\ell \in \omega$ such that $|p_\eta| = \ell$ for all η. For each $\xi \in \omega_1$, let $W_\xi = \bigcup\{U_\alpha : \alpha \in p_\xi\}$. We have that for $\xi < \eta$, $W_\xi \cap p_\eta$ is not empty since this is what makes $p_\xi \perp p_\eta$. The next step in the proof is very clever. Choose $\ell' \le \ell$ minimal so that there are uncountable $A, B \subset \omega_1$ and $\{p'_\eta \in [p_\eta]^{\ell'} : \eta \in B\}$ such that for all $\xi \in A$, the set $\{\eta \in B : W_\xi \cap p'_\eta =\}$ is countable.

very clever, but what is $W_\xi \cap P'_\eta$ supposed to be? Empty?

Of course we fix such an ℓ' and the requisite sequences. The property we now need is that if $D \subset Y = \bigcup\{p'_\eta : \eta \in B\}$ is uncountable, then the set $\{\xi \in A : |W_\xi \cap D| < \omega_1\}$ is countable. Otherwise, we could shrink A and show that for some uncountable $B' \subset B$, the family $\{p'_\eta \setminus D : \eta \in B'\}$ contradicts the minimality of ℓ'. Since Y is hereditarily Lindelöf, we may assume that it has no countable open subsets. Fix any $y \in Y$ and let $\{O_n : n \in \omega\}$ be a neighborhood base with $\overline{O_{n+1}} \subset O_n$ for each $n \in \omega$. For all but countably many $\xi \in A$, there is an n such that $O_n \cap W_\xi$ is empty. There is an n and an uncountable $A' \subset A$ such that O_n is disjoint from W_ξ for all $\xi \in A'$. Let $B' = \{\eta : O_n \cap p'_\eta \neq \emptyset\}$. But now we have that $\{W_\xi : \xi \in A'\}$ and $\{p'_\eta \setminus O_n : \eta \in B'\}$ contradicts the minimality of ℓ'.

An important fundamental tool is the notion of a minimal walk which is such a powerful tool that a person could write an entire book about it ([Tod07]). For each limit ordinal $\alpha \in \omega_1$, let $C_\alpha \ni 0$ be an increasing ω-sequence cofinal in α. For each $\alpha = \beta + 1$ in ω_1, let $C_\alpha = \{0, \beta\}$. For $\alpha < \beta < \omega_1$, the walk from β to α is defined recursively by letting $\beta_1 = \min(C_\beta \setminus \alpha)$ be the first step (hence $\beta_1 = \alpha$ if $\beta = \alpha + 1$) and continuing with the walk from β_1 to α. The first consequence we recall is that it very naturally gives rise to a coherent sequence.

Definition 7.2. A coherent sequence of finite-to-one functions is a sequence $\langle e_\beta : \beta \in \omega_1 \rangle$ such that

1. for each $\beta \in \omega_1$, e_β is a finite-to-one function from β into ω,
2. if $\beta < \gamma \in \omega_1$, then $e_\beta(\alpha) = e_\gamma(\alpha)$ for all but finitely many $\alpha \in \beta$.

To define such a sequence we first define $\rho_1(\alpha, \beta)$ for $\alpha < \beta$ by recursion as follows:

$$\rho_1(\alpha,\beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\}.$$

Suppose that β is a limit and let $\langle \beta_k K < \omega\rangle$ be the increasing enumeration of C_β. Note that $\rho_1(\alpha,\beta) \geq |C_\beta \cap \alpha| \geq k$ whenever $\alpha \geq \beta_k$. Thus, for a fixed k, we have $\{\alpha < \beta : \rho_1(\alpha,\beta) < k\} \subseteq \bigcup_{j \leq k}\{\alpha < \beta_j : \rho_1(\alpha,\beta_j) < k\}$ and so, by induction, it follows that the former set is finite.

If $\beta = \gamma + k$ for some limit γ and integer k then one readily verifies that $\rho_1(\gamma + j, \beta) = 1$ for $j < k$ and $\rho_1(\alpha,\beta) = \rho_1(\alpha,\gamma)$ when $\alpha < \gamma$.

Thus the functions $e_\beta : \beta \to \omega$ defined by $e_\beta(\alpha) = \rho_1(\alpha,\beta)$ are all finite-to-one.

We verify the second condition by induction on γ. Let $k = |C_\gamma \cap \beta|$ and $\gamma_1 = \min C_\gamma \setminus \beta$. As $|C_\gamma \cap \alpha| \leq k$ for $\alpha < \beta$ we must have $\rho_1(\alpha,\gamma) = \rho_1(\alpha,\gamma_1)$ whenever $\rho_1(\alpha,\beta) > k$. Thus the α for which $\rho_1(\alpha,\beta) \neq \rho_1(\alpha,\gamma)$ are among those for which $\rho_1(\alpha,\gamma) \leq k$ or $\rho_1(\alpha,\beta) \neq \rho_1(\alpha,\gamma_1)$ and there are only finitely many of these.

As explained in [Tod07] the function ρ_1 and variants thereof can be used to code many combinatorial structures on ω_1; one can write down an Aronszajn tree in terms of the e_β as follows:

$$T = \bigcup_{\beta<\omega_1} \{t \in 2^\beta : t = {*}e_\beta\}.$$

Below we shall see how to build a *Countryman line* from ρ_1, that is, an uncountable linear order whose square is the union of countably many chains.

Instrumental in Moore's construction of an L-space is the *lower trace* of the walk.

Definition 7.3. The lower trace L is a function $L : [\omega_1]^2 \to [\omega_1]^{<\omega}$ such that for any $0 < \alpha < \beta < \gamma < \omega_1$

1. $L(\alpha,\beta)$ is a nonempty subset of α,
2. if $\max L(\beta,\gamma) < \min L(\alpha,\beta)$, then $L(\alpha,\gamma) = L(\alpha,\beta) \cup L(\beta,\gamma)$,
3. if β is a limit ordinal then for each $\xi < \beta$, there is a $\zeta < \beta$ so that $\xi < \min L(\alpha,\beta)$ for all $\zeta < \alpha < \beta$.

The definition of $L(\alpha,\beta)$ is

$$\{\max(C_\beta \cap \alpha)\} \cup (L(\alpha, \min(C_\beta \setminus \alpha)) \setminus \max(C_\beta \cap \alpha))$$

(i.e. follow the minimal walk). Property (1) and (3) of 7.3 are immediate given that $0 \in C_\beta$ and C_β is cofinal in β. To check that property (2) holds, we use another presentation of $L(\alpha,\beta)$. Let $\{\beta_i^\alpha : i < \ell\}$ denote the minimal walk in descending order from β to α, hence $\beta_0^\alpha = \beta$ and $\beta_\ell^\alpha = \alpha$, and $\beta_{i+1}^\alpha = \min(C_{\beta_i^\alpha} \setminus \alpha)$. Next, let $\lambda_i = \max(\bigcup\{C_{\beta_j^\alpha} \cap \alpha : j \leq i\})$ and we will have that $\{\lambda_0, \ldots, \lambda_\ell\}$ enumerates $L(\alpha,\beta)$ (with possible repetitions) in increasing order. Now suppose that $\max L(\beta,\gamma) < \min L(\alpha,\beta)$. Let $\{\gamma_i^\alpha : i < m\}$ denote the minimal walk from γ to α. Also, let $\{\lambda_j^\beta : j < \ell\}$ denote $L(\beta,\gamma)$. By induction on $j < \ell$, we prove that $\{\gamma_i^\alpha : i < \ell\}$ is actually equal to the minimal walk from γ to β. Of course γ_0^α is γ, and consider $0 < i < \ell$. By the induction hypothesis, $\alpha \geq \lambda_i^\beta \geq \max(C_{\gamma_{i-1}^\alpha} \cap \beta)$, and so, $\min(C_{\gamma_{i-1}^\alpha} \setminus \alpha) \geq \beta$. This ensures that $\min(C_{\gamma_{i-1}^\alpha} \setminus \beta) = \gamma_i^\alpha$ as required. From this we can conclude that the first ℓ members in the monotone enumeration of $L(\alpha, \gamma)$ is exactly the same as $L(\beta, \gamma)$ and that at step ℓ of the walk from γ to α we are at the ordinal β. Of course the walk will continue as simply the walk from β to α. The fact that $\max L(\beta, \gamma) < \min L(\alpha, \beta)$, ensures also that each of the members of $L(\beta, \gamma)$ are added in turn to $L(\alpha, \gamma)$ (and not "cut out" by the max operation) and so we have that $L(\alpha, \gamma) = L(\beta, \gamma) \cup L(\alpha, \beta)$.

We now turn to the definition of Moore's L-space.

You must mean $F \setminus \{\min F\}$

Definition 7.4. For $F \in [\omega_1]^{<\omega}$ and $s, t \in \omega^F$, then

$$\mathrm{Osc}(s,t) = \{\xi \in F\setminus\{\min F\} : s(\xi) > t(\xi) \text{ and } s(\max(F\cap\xi)) \leq t(\max(F\cap\xi)) \, .$$

In practice (and in context) ξ^- will be used to denote $\max(F \cap \xi)$ for $\xi \in F \setminus \{\min F\}$.

Define, for $\alpha < \beta \in \omega_1$,

$$\mathrm{osc}(\alpha, \beta) = \left|\mathrm{Osc}\big(e_\alpha \restriction L(\alpha, \beta), e_\beta \restriction L(\alpha, \beta)\big)\right|.$$

Also, for convenience, let $\mathrm{Osc}(\alpha, \beta)$ abbreviate $\mathrm{Osc}\big(e_\alpha \restriction L(\alpha, \beta), e_\beta \restriction L(\alpha, \beta)\big)$.

Definition 7.5. Fix a rationally independent sequence ζ_α $(\alpha \in \omega_1)$ of elements of $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. For each $\beta \in \omega_1$, define $w_\beta \in \mathbb{T}^{\omega_1}$ by

$$w_\beta = \begin{cases} \zeta_\alpha^{\mathrm{osc}(\alpha,\beta)+\frac{1}{2}} & \alpha < \beta \\ 1 & \beta \leq \alpha \in \omega_1 \end{cases}$$

Let $\mathcal{L}$ be the subspace of $\mathbb{T}^{\omega_1}$ consisting of the set $\{w_\beta : \beta \in \omega_1\}$.

It is evident that $\mathcal{L}$ is not separable. Well beyond the scope of this article is the following celebrated result.

Theorem 7.6 ([Moo06]). *$\mathcal{L}$ is hereditarily Lindelöf.*

We will now report on the result from [Moo08] the interesting fact about the square of $\mathcal{L}$.

Which of these do you prefer?

We will now report the following interesting fact about the square of $\mathcal{L}$, from [Moo08].

Theorem 7.7. *$\mathcal{L}$ has a co-countable subspace $\mathcal{X}$ (which is an L-space) whose square has a σ-discrete dense subset.*

Before giving the proof we go over the proof of an earlier result that has the same flavor.

Proposition 7.8 ([Tod07]). *$C(\rho_1) = (\omega_1, <_{\rho_1})$ is a Countryman line. That is, the square is a countable union of chains, where*

$$\alpha <_{\rho_1} \beta \text{ if } \begin{cases} \rho_1(.,\alpha) \subset \rho_1(.,\beta) & \text{or} \\ (\exists \xi)\rho_1(\xi,\alpha) < \rho_1(\xi,\beta) & \text{and } (\forall \delta < \xi)\rho_1(\delta,\alpha) = \rho_1(\delta,\beta) \end{cases}$$

We copy the lexicographic order of the set $\{e_\beta : \beta \in \omega_1\}$ to ω_1.

Proof. $D_{\alpha,\beta}$ will be the set $\{\alpha\}$ together with $\{\xi < \alpha : \rho_1(\xi,\alpha) \neq \rho_1(\xi,\beta)\}$. Also, $\Delta(\alpha,\beta)$ will equal $\min D_{\alpha,\beta}$.

$n_{\alpha,\beta}$ is the maximum value $\{\rho_1(\xi,\alpha) : \xi \in D_{\alpha,\beta} \setminus \alpha\} \cup \{\rho_1(\xi,\beta) : \xi \in D_{\alpha,\beta}\}$; thus if $\rho_1(\xi,\alpha) \neq \rho_1(\xi,\beta)$, then both values are at most $n_{\alpha,\beta}$.

$\alpha \in F_{\alpha,\beta}$ will be the finite set of $\eta \leq \alpha$ such that one of $\rho_1(\eta,\alpha)$ or $\rho_1(\eta,\beta)$ is at most $n_{\alpha,\beta}$.

Consider $\alpha < \beta$, $\gamma < \delta$ and $\alpha <_{\rho_1} \beta$, further that $n = n_{\alpha,\beta} = n_{\gamma,\delta}$, $e_\alpha \restriction F_{\alpha,\beta} \approx e_\gamma \restriction F_{\gamma,\delta}$, $e_\beta \restriction F_{\alpha,\beta} \approx e_\delta \restriction F_{\gamma,\delta}$, then we conclude that $\gamma <_{\rho_1} \delta$. (which gives us the countably many chains). Let $\xi = \min(\Delta(\alpha,\gamma), \Delta(\beta,\delta))$ and notice that $F_{\alpha,\beta} \cap \xi = F_{\gamma,\delta} \cap \xi$. Because of the isomorphism assumption, we may therefore assume that $\xi \notin F_{\alpha,\beta} \cap F_{\gamma,\delta}$.

If $\Delta(\alpha,\gamma) \neq \Delta(\beta,\delta)$, then we must have that $\xi \in F_{\alpha,\beta} \cap F_{\gamma,\delta}$ (e.g. $\xi = \Delta(\beta,\delta) < \Delta(\alpha,\gamma)$ will ensure that $\rho_1(\xi,\alpha) = \rho_1(\xi,\gamma)$ and so one of $\rho_1(\xi,\beta)$ and $\rho_1(\xi,\delta)$ differ, and have value at most n, which then ensures that all have value at most n).

Therefore we now have that $\xi = \Delta(\alpha,\gamma) = \Delta(\beta,\delta)$.

Now, if $\xi \notin F_{\alpha,\beta} \cup F_{\gamma,\delta}$, then $\rho_1(\xi,\beta) = \rho_1(\xi,\alpha) < \rho_1(\xi,\gamma) = \rho_1(\xi,\delta)$ which gives $\beta <_{\rho_1} \delta$ as required. If $\xi \in F_{\alpha,\beta} \setminus F_{\gamma,\delta}$ then $\rho_1(\xi,\beta) \leq n < \rho_1(\xi,\delta)$ as required. Finally, if $\xi \in F_{\gamma,\delta} \setminus F_{\alpha,\beta}$, then $\rho_1(\xi,\gamma) \leq n < \rho_1(\xi,\alpha)$ which contradicts that $\alpha <_{\rho_1} \gamma$.

We return to Justin. $\mathcal{X}$ is $\mathcal{L}$ minus the union of all countable open sets. Define a dense set D in $\mathcal{X}^2$ by a simple induction. Let $U_\delta \times V_\delta$ ($\delta \in \omega_1$) be an enumeration of a base for $\mathcal{X}^2$. Choose any point $d_\delta = (w_{\beta_\delta}, w_{\gamma_\delta}) \in U_\delta \times V_\delta$ such that $\gamma_\eta < \beta_\delta < \gamma_\delta$ for all $\eta < \delta$. Even though it is not needed for the proof, we can also choose β_δ so that w_{β_δ} is not in the closure of the set $\{w_{\beta_\eta} : \eta < \delta\}$. To Justin? Or to $\mathcal{L}$?

Similar to the Countryman proof: we identify countably many isomorphism types so that each type ensures that the subset of D with that type is discrete. For each δ we associate a rational ε_δ, and integers n_δ, M_δ and k_δ identified below.

Lemma 7.9. *There is a finite set $F_\delta \subset \beta_\delta$ such that $L(\alpha,\beta_\delta) \setminus F_\delta = L(\alpha,\gamma_\delta) \setminus F_\delta$ whenever $\alpha < \beta_\delta$.*

Proof. We use property 3 of definition 7.3. Let $\eta_0 = \beta_\delta$ and recursively define $\eta_{i+1} < \eta_i$, where $\eta_{i+1} < \xi \leq \eta_i$ implies that $L(\xi,\beta_\delta) \cup L(\xi,\gamma_\delta) \subset \min L(\xi,\eta_i)$. Of course we stop when $\eta_\ell = 0$ and define $F_\delta = \bigcup\{L(\eta_i,\beta_\delta) \cup L(\eta_i,\gamma_\delta) : i < \ell\}$.

To verify this works, consider any $0 < \alpha \leq \beta_\delta$ and fix $i < \ell$ so that $\eta_{i+1} < \alpha \leq \eta_i$. Of course $L(\eta_i, \beta_\delta) \setminus F_\delta = L(\eta_i, \beta_\delta) \setminus F_\delta$ since both are empty, so we assume $\xi < \eta_i$. By property 2, each of $L(\alpha, \beta_\delta) \setminus F_\delta$ and $L(\alpha, \gamma_\delta) \setminus F_\delta$ are equal to $L(\alpha, \eta_i) \setminus F_\delta$.

Lemma 7.10. *There is an integer M_δ such that $|\operatorname{osc}(\xi, \beta_\delta) - \operatorname{osc}(\xi, \gamma_\delta)| < M_\delta$ for all $\xi < \beta_\delta$.*

Proof. recall that $\operatorname{Osc}(\alpha, \beta) = \{\xi \in L(\alpha, \beta) : e_\alpha(\xi^-) \leq e_\beta(\xi^-)$ and $e_\alpha(\xi) > e_\beta(\xi)\}$, so $\xi \in \operatorname{Osc}(\alpha, \beta_\delta) \setminus \operatorname{Osc}(\alpha, \gamma_\delta)$ would of course mean that $e_\alpha(\xi) \neq e_{\beta_\delta}(\xi)$ and $\xi \in L(\alpha, \beta_\delta) \Delta L(\alpha, \gamma_\delta)$.

Lemma 7.11. *There is a rational $\varepsilon_\delta > 0$, such that for each integer i with $|i| < M_\delta$, and each $z \in \mathbb{T}$, with $|z - \zeta_{\beta_\delta}| < \varepsilon_\delta$*

$$\big||z^i - 1| - |z^{n_\delta + \frac{1}{2}} - 1|\big| \geq \varepsilon_\delta$$

Proof. The value ζ_{β_δ} was from a set chosen so that no two rational powers coincide. Therefore there is an $\varepsilon > 0$ satisfying that $\big||z^i - 1| - |z^{n_\delta + \frac{1}{2}} - 1|\big| \geq \varepsilon$ for $z = \zeta_{\beta_\delta}$ and $|i| < M_\delta$. By continuity then, there is a $\delta > 0$ such that for all z with $|z - \zeta_{\beta_\delta}| < \delta$ we also have that $\big||z^i - 1| - |z^{n_\delta + \frac{1}{2}} - 1|\big| \geq \frac{\varepsilon}{2}$. Setting ε_δ to be smaller than each of δ and $\frac{\varepsilon}{2}$ then satisfies the lemma.

Finally, we make the choice of the integer $k_\delta = k$ simply so that $k \cdot \varepsilon_\delta < \zeta_{\beta_\delta} < (k+1) \cdot \varepsilon_\delta$. This ensures that if $k_\delta = k_\eta$ and $\varepsilon_\delta = \varepsilon_\eta$, then $|\zeta_{\beta_\delta} - \zeta_{\beta_\eta}| < \varepsilon_\delta$.

Getting informal are we?

Well, we are not going to say that it is obvious, but we now have that for each rational $\varepsilon > 0$ and integers M, n and k the set

$$D_{\varepsilon, M, n, k} = \{d_\delta = (w_{\beta_\delta}, w_{\gamma_\delta}) : \delta < \omega_1 \text{ and } (\varepsilon_\delta, M_\delta, n_\delta, k_\delta) = (\varepsilon, M, n, k)\}$$

is discrete.

The definition of the neighborhood of d_δ is

$$U_\delta = \big\{(x, y) \in \mathcal{X}^2 : \big||x(\beta_\delta) - y(\beta_\delta)| - |\zeta_{\beta_\delta}^{n+\frac{1}{2}} - 1|\big| < \varepsilon\big\}.$$

It is obvious that U_δ is open, but it is not obvious that d_δ belongs to U_δ. To see this note that

$$|w_{\beta_\delta}(\beta_\delta) - w_{\gamma_\delta}(\beta_\delta)| = |1 - \zeta_{\beta_\delta}^{\operatorname{osc}(\beta_\delta, \gamma_\delta)}| = |1 - \zeta_{\beta_\delta}^{n+\frac{1}{2}}|$$

Since we arranged that d_δ is not in the closure of the set $\{d_\eta : \eta < \delta\}$ simply by virtue of the first coordinate, it suffices to check that if d_δ and d_η are in $D_{\varepsilon, M, n, k}$, with $\delta < \eta < \omega_1$, then $d_\eta \notin U_\delta$. To see this note that

$$\begin{aligned} |w_{\beta_\eta}(\beta_\delta) - w_{\gamma_\eta}(\beta_\delta)| &= |\zeta_{\beta_\delta}^{\operatorname{osc}(\beta_\delta, \beta_\eta)} - \zeta_{\beta_\delta}^{\operatorname{osc}(\beta_\delta, \gamma_\eta)}| \\ &= |\zeta_{\beta_\delta}^{-\operatorname{osc}(\beta_\delta, \gamma_\eta)}| \cdot |\zeta_{\beta_\delta}^{\operatorname{osc}(\beta_\delta, \beta_\eta) - \operatorname{osc}(\beta_\delta, \gamma_\eta)}| = 1 \cdot |\zeta_{\beta_\delta}^i - 1| \end{aligned} \tag{8}$$

for some $|i| < M_\eta = M$. Of course since $M_\delta = M$, we have by Lemma 7.11, that

$$\big||\zeta^i_{\beta_\delta} - 1| - |\zeta^{n+\frac{1}{2}}_{\beta_\delta} - 1|\big| \geq \varepsilon$$

completing the verification that $d_\eta \notin U_\delta$.

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