EFIMOV SPACES AND THE SPLITTING NUMBER

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Abstract. An Efimov space is a compact space which contains neither a non-trivial converging sequence nor a copy of the Stone-Cech compactification of the integers. We give a new construction of a space which consistently results in an Efimov space. The required set-theoretic assumption is on the splitting number $\mathfrak{s}$.

Efimov's problem is to determine if every compact space with no converging sequences contains a copy of βN? A counterexample could be called an Efimov space. Fedorchuk [Fed76] showed that $\diamondsuit$ implies there is a compact hereditarily separable space with no converging sequences – certainly an Efimov space. This result was later generalized, again by Fedorchuk [Fed77], by formulating what was called a Partition Hypothesis and proving that the existence of an Efimov space is a consequence. The Partition Hypothesis is very much like the assumptions that $\mathfrak{s} = \omega_1$ and $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ (see also the review **MR54:13827** for additional information). We improve this result to more general values of $\mathfrak{s}$. A family $\mathcal{S} \subset [\omega]^\omega$ is a *splitting* family if for each infinite $a \subset \omega$ there is an $I \in \mathcal{S}$ such that each of $a \cap I$ and $a \setminus I$ are infinite. The cardinal $\mathfrak{s}$ is the least cardinality of a splitting family.

We will need a special set-theoretic hypothesis about the cardinal $\mathfrak{s}$. For a cardinal κ, the order theoretic structure $([\kappa]^\omega, \subset)$ may or may not have a cofinal subset of cardinality κ. We will assume that for $\kappa = \mathfrak{s}$ it does. It is a "large cardinal" hypothesis to assume that there is a cardinal κ with uncountable cofinality such that this cofinality is greater than κ. It holds in ZFC that the cofinality of $([\aleph_n]^\omega, \subset)$ is $\aleph_n$ for each $n \in \omega$.

The above examples of Efimov spaces are constructed by inverse limits. Another interesting feature of this problem is the role of *simple extensions*. These are referred to as *minimmal extensions* in the Boolean algebra setting. Fedorchuk's spaces were constructed with inverse limit systems consisting of simple extensions (defined in the next section). S. Koppelberg [Kop89] has proven that such an inverse limit of compact 0-dimensional spaces will never contain a copy of βN. We give a proof of this in the general case (mostly for the reader's interest) in the next section.

1. Simple extensions

In an inverse limit $\langle X_\alpha : \alpha \in \kappa \rangle$ with bonding maps $\langle f_{\alpha,\beta} : \beta < \alpha < \kappa \rangle$, say that $X_{\alpha+1}$ is a **simple extension** if $f^{-1}(x)$ is a single point for all $x \in X_\alpha$ with but one exception x_α. Consult Engelking's book [Eng89] for background on inverse limit systems. Throughout the paper we will be assuming that our inverse limit systems are continuous at limits stages. By this we mean that for each limit ordinal $\gamma \in \kappa$, X_γ is the limit space of the system $\langle X_\alpha : \alpha \in \gamma \rangle$ with the corresponding bonding maps. Observe that if U is an open subset of $X_{\alpha+1}$ and $x_\alpha \notin f_{\alpha+1,\alpha}(\overline{U})$, then $f_{\alpha+1,\alpha}(U)$ is open in X_α, and $\overline{f_{\alpha+1,\alpha}(U)} = f_{\alpha+1,\alpha}(\overline{U})$,

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Proposition 1 (S. Koppelberg)**.** *If X_κ is constructed by simple extensions, then X_κ does not map onto 2^{ω_1} (unless X_0 does).*

Question: Is it consistent that every such inverse limit contains a converging sequence? (under PFA?)

Proof. For each $\gamma < \kappa$, let $\mathcal{T}_{\gamma+1}$ denote the family of open subsets of $X_{\gamma+1}$. For $\gamma < \zeta$, let $\mathcal{T}_{\zeta,\gamma} = \{f^{-1}_{\zeta,\gamma}(U) : U \in \mathcal{T}_\gamma\}$. We can also assume that that $\mathcal{T}_{\zeta,\gamma} \subset \mathcal{T}_\zeta$ for each $\gamma < \zeta \leq \kappa$. Finally, for each limit ζ, we can assume that $\mathcal{T}_\zeta$ is equal to $\bigcup\{\mathcal{T}_{\zeta,\gamma} : \gamma < \zeta\}$ since this union does form a base the topology on X_ζ. Observe that if $U, V \in \mathcal{T}_{\zeta,\gamma}$ are disjoint, then so are $f_{\zeta,\gamma}(U)$ and $f_{\zeta,\gamma}(V)$. Furthermore, since our system consists of simple extensions, if $U, V \in \mathcal{T}_{\zeta,\gamma+1}$ have disjoint closures and $x_\gamma \notin \overline{f_{\zeta,\gamma}(U \cup V)}$, then $U, V \in \mathcal{T}_{\zeta,\gamma}$, and $f_{\zeta,\gamma}(U)$ and $f_{\zeta,\gamma}(V)$ also have disjoint closures.

Let $\lambda \leq \kappa$, be minimal such that X_λ maps onto I^{ω_1} and let g denote such a map. For each $\xi \in \omega_1$, let g_ξ denote the projection $\pi_\xi \circ g : X_\lambda \mapsto I$ (where $\pi_\xi : I^{\omega_1} \mapsto I$ is the usual projection). For each $\xi \in \omega_1$, choose basic open sets U_ξ, $\tilde{U}_\xi$, V_ξ, and $\tilde{V}_\xi$ in $\mathcal{T}_\lambda$ such that

$$g_\xi^{-1}(0) \subset U_\xi \subset g^{-1}([0,1/4]) \subset \tilde{U}_\xi \subset g^{-1}([0,3/8])$$

and

$$g_\xi^{-1}(1) \subset V_\xi \subset g^{-1}([3/4,1]) \subset \tilde{V}_\xi \subset g^{-1}([5/8,1]) \ .$$

A family of pairs of sets $\{\langle A_i, B_i\rangle : i \in S\}$ is *dyadic*, if for each disjoint pair of finite sets $S_0, S_1 \subset S$, the set $\bigcap\{A_i : i \in S_0\} \cap \bigcap\{B_i : i \in S_1\}$ is not empty. Sapirovskiĭ [Sa80] has shown that a compact space will map onto I^{ω_1} whenever it has an uncountable dyadic family consisting of closed sets. We will show that there is some $\gamma < \lambda$ such that X_γ contains such a dyadic family. Observe that if S_0 and S_1 are disjoint finite subsets of ω_1, and $W = \bigcap\{U_i : i \in S_0\} \cap \bigcap\{V_i : i \in S_1\}$, then the family $\{\langle U_\xi \cap W, V_\xi \cap W\rangle : \xi \in S\}$ is dyadic for any $S \subset \omega_1 \setminus (S_0 \cup S_1)$.

If $\lambda = \gamma + 1$ then let $\{x, y\} = f^{-1}_{\lambda,\gamma}(x_\gamma)$. If $x \notin \tilde{U}_0$, let $W_0 = U_0$, otherwise $x \notin \tilde{V}_0$ and set $W_0 = V_0$. Similarly, if $y \notin \tilde{U}_1$, let $W_1 = U_1$, otherwise $y \notin \tilde{V}_1$ and we set $W_1 = V_1$. For each $\xi \in \omega_1 \setminus \{0, 1\}$, set $U'_\xi = U_\xi \cap W_0 \cap W_1$ and $V'_\xi = V_\xi \cap W_0 \cap W_1$. Now, $\overline{U'_\xi}$ and $\overline{V'_\xi}$ are disjoint closed sets and each are disjoint from $\{x, y\}$. Therefore $x_\gamma \notin \overline{f_{\lambda,\gamma}(U'_\xi \cup V'_\xi)}$ for each $\xi \in \omega_1 \setminus \{0, 1\}$ and by the above, it follows that $f_{\lambda,\gamma}(U'_\xi)$ and $f_{\lambda,\gamma}(V'_\xi)$ have disjoint closures. Since the family $\{\langle U'_\xi, V'_\xi\rangle : \xi \in \omega_1 \setminus \{0, 1\}\}$ is dyadic, so is the family $\{\langle f_{\lambda,\gamma}(\overline{U'_\xi}), f_{\lambda,\gamma}(\overline{V'_\xi})\rangle : \xi \in \omega_1 \setminus \{0, 1\}\}$.

Therefore λ is a limit and $\mathcal{T}_\lambda = \bigcup_{\gamma<\lambda} \mathcal{T}_{\lambda,\gamma}$. Suppose there is a $\gamma < \lambda$ such that $\{U_\xi, \tilde{U}_\xi, V_\xi, \tilde{V}_\xi\} \subset \mathcal{T}_{\lambda,\gamma}$ for all $\xi \in S \in [\omega_1]^{\omega_1}$. For each $\xi \in S$, $f_{\lambda,\gamma}(\tilde{U}_\xi)$ and $f_{\lambda,\gamma}(\tilde{V}_\xi)$ are disjoint and contain, respectively, $f_{\lambda,\gamma}(\overline{U_\xi})$ and $f_{\lambda,\gamma}(\overline{V_\xi})$. Therefore, X_γ contains an uncountable dyadic family of closed sets.

Now we consider the case that λ has uncountable cofinality. There is some $\delta < \lambda$ such that there is a countably infinite set $S \subset \omega_1$ such that $\{U_\xi, \tilde{U}_\xi, V_\xi, \tilde{V}_\xi\} \subset \mathcal{T}_{\lambda,\delta}$ for each $\xi \in S$.

For finite sets $S_0, S_1 \subset S$, let $W(S_0, S_1)$ denote the set $\bigcap\{U_\xi : \xi \in S_0\} \cap \bigcap\{V_\xi : \xi \in S_1\}$. We now show that for each $\alpha \in \omega_1$, there are disjoint finite sets $S_0^\alpha, S_1^\alpha \subset S$ such that if we set $W_\alpha = W(S_0^\alpha, S_1^\alpha)$, then

$$\{U_\alpha \cap W_\alpha, V_\alpha \cap W_\alpha\} \subset \mathcal{T}_{\lambda,\delta} \ .$$

Fix any $\alpha \in \omega_1$ and let γ be minimal such that $\{U_\alpha, V_\alpha\} \subset \mathcal{T}_{\lambda,\gamma+1}$. If $\gamma + 1 \leq \delta$ we can let $S_0^\alpha = S_1^\alpha = \emptyset$. Otherwise, fix any $\xi \in S$. Then either $f_{\gamma,\delta}(x_\gamma) \notin f_{\lambda,\delta}(\tilde{U}_\xi)$ or $f_{\gamma,\delta}(x_\gamma) \notin f_{\lambda,\delta}(\tilde{V}_\xi)$. If the former and $W = W(\{\xi\}, \emptyset)$, or the latter and $W =$

$W(\emptyset,\{\xi\})$, then $\{U_\alpha\cap W, V_\alpha\cap W\}\in\mathcal{T}_{\lambda,\gamma}$ and $f_{\lambda,\gamma}(U_\alpha\cap W)$ and $f_{\lambda,\gamma}(V_\alpha\cap W)$ have disjoint closures. We obtain S_0^α and S_1^α by a finite induction.

There is an uncountable set $L\subset\omega_1\setminus S$ and finite sets S_0, S_1 such that $S_0^\alpha = S_0$ and $S_1^\alpha = S_1$ for all $\alpha\in L$.

It follows that the family

$$\{\langle f_{\lambda,\delta}(\overline{U_\alpha\cap W(S_0,S_1)}), f_{\lambda,\delta}(\overline{V_\alpha\cap W(S_0,S_1)}))\rangle : \alpha\in L\}$$

is a dyadic family in X_δ. □

2. An Efimov space

Theorem 2. *Assume that cof*$([\mathfrak{s}]^\omega)=\mathfrak{s}$ *and* $2^{\mathfrak{s}}<2^{\mathfrak{c}}$*, Then there is an Efimov space.*

We prove the theorem in a (converging) sequence of lemmas.

Definition 3. A collection $\mathcal{Z}$ is a z-partition of a space X, if $\mathcal{Z}\subset {}^\omega(\wp(X))$, for each $Z\in\mathcal{Z}$, $\{Z(n):n\in\omega\}$ is a pairwise disjoint collection of (possibly empty) compact open sets, $Z(\omega)=X\setminus\bigcup_n Z(n)$ is nowhere dense, and $\{Z(\omega):Z\in\mathcal{Z}\}$ is a partition of X.

A typical z-partition will have cardinality $\mathfrak{c}$. For example, if X is compact first countable with no isolated points then there are natural z-partitions $\mathcal{Z}$ such that $\{Z(\omega):Z\in\mathcal{Z}\}$ consists of the singletons of X.

Definition 4. A family $\mathcal{P}$ of z-partitions of X is *point-separating* if for each $x\neq y\in X$, there is a $\mathcal{Z}\in\mathcal{P}$ and a $Z\in\mathcal{Z}$ such that $x\in Z(\omega)$ and $y\notin Z(\omega)$. We will say that a z-partition $\mathcal{Z}'$ refines a z-partition $\mathcal{Z}$, if for each $Z'\in\mathcal{Z}'$ is a $Z\in\mathcal{Z}$ such that $Z'(\omega)\subset Z(\omega)$. We will say that the family $\mathcal{P}$ is σ-directed if for each countable $\mathcal{P}'\subset\mathcal{P}$, there is a $\mathcal{Z}'\in\mathcal{P}$ which refines each $\mathcal{Z}\in\mathcal{P}'$.

Lemma 5. *If $\mathcal{Z}$ is a z-partition of a compact space X and $I\subset\omega$, then there is a space $X(\mathcal{Z},I)$ and a continuous irreducible map $f_{\mathcal{Z},I}:X(\mathcal{Z},I)\mapsto X$ such that $|X(\mathcal{Z},I)|=|X|$ and for each $Z\in\mathcal{Z}$, $f^{-1}\left(\bigcup_{n\in I}Z(n)\right)$ and $f^{-1}\left(\bigcup_{n\in N\setminus I}Z(n)\right)$ have disjoint closures whose union covers $X(\mathcal{Z},I)$.*

Proof. We simply define $X(\mathcal{Z},I)$ as a subset of a product space, $X\times 2^{\mathcal{Z}}$. Let $I_0=I$ and $I_1=N\setminus I$.

$$\langle x,\varphi\rangle\in X(\mathcal{Z},I)\ \text{ if } (Z\in\mathcal{Z}\text{ and } x\notin Z(\omega))\text{ implies}\left(\varphi(Z)=e\ \text{ iff }\ x\in\overline{\bigcup_{n\in I_e}Z(n)}\right).$$

For each $Z\in\mathcal{Z}$, we can let $[\langle Z,0\rangle]$ and $[\langle Z,1\rangle]$ denote the canonical basic open subsets of $X\times 2^{\mathcal{Z}}$ consisting of all those $\langle x,\varphi\rangle$ such that $\varphi(Z)=0$ and $\varphi(Z)=1$ respectively. The family of finite intersections from $\{X(\mathcal{Z},I)\cap W\times 2^{\mathcal{Z}} : W$ open subset of $X\}$ together with $\{X(\mathcal{Z},I)\cap[\langle Z,e\rangle]:(Z,e)\in\mathcal{Z}\times\{0,1\}\}$ forms a base for the topology on $X(\mathcal{Z},I)$. Let $f_{\mathcal{Z},I}$ denote the projection mapping onto the first coordinate X. Each of $f_{\mathcal{Z},I}([\langle Z,0\rangle])$ and $f_{\mathcal{Z},I}([\langle Z,1\rangle])$ are the complementary regular closed subsets of X, $\overline{\bigcup_{n\in I}Z(n)}$ and $\overline{\bigcup_{n\in N\setminus I}Z(n)}$.

More generally finite intersections from $\{X(\mathcal{Z},I)\cap[\langle Z,e\rangle]:(Z,e)\in\mathcal{Z}\times\{0,1\}\}$ also map to regular closed sets. From this we can show that the mapping $f_{\mathcal{Z},I}$ is irreducible. If F is a proper closed subset of $X(\mathcal{Z},I)$, we must show that $f_{\mathcal{Z},I}[F]$ is a proper subset of X. The complement of F contains some open set $W\times 2^{\mathcal{Z}}$ intersected with some finite intersection from $\{X(\mathcal{Z},I)\cap[\langle Z,e\rangle]:(Z,e)\in\mathcal{Z}\times\{0,1\}\}$. The finite intersection will map to a regular closed subset of X which can be assumed to contain the open set W. It follows that W is disjoint from F.

For each $Z \in \mathcal{Z}$, $f^{-1}_{\mathcal{Z},I}(Z(\omega))$ maps onto $Z(\omega)$ by an at most two to one mapping. In fact, there are two points only if each of $[\langle Z, 0\rangle]$ and $[\langle Z, 1\rangle]$ contain one. Since $\{Z(\omega) : Z \in \mathcal{Z}\}$ is a partition of X, $|X(\mathcal{Z}, I)| = |X|$. It is routine to check that $X(\mathcal{Z}, I)$ is a closed subset of $X \times 2^{\mathcal{Z}}$ and so is compact. □

Lemma 6. *Let $\mathcal{P}$ be a family of z-partitions of a compact 0-dimensional space X. If $|\mathcal{P}| \leq \mathfrak{s}$, then there is a σ-directed family $\mathcal{P}'$ containing $\mathcal{P}$, such that $|\mathcal{P}'| \leq cof([\mathfrak{s}]^\omega, \subset)$.*

Proof. Let $\{\mathcal{Z}_\alpha : \alpha \in \mathfrak{s}\}$ be an enumeration of $\mathcal{P}$. Fix a family $\mathcal{A} \subset [\mathfrak{s}]^\omega$ which is cofinal. For each $A \in \mathcal{A}$, we define $\mathcal{Z}_A$ so that $\mathcal{Z}_A$ refines $\mathcal{Z}_\alpha$ for each $\alpha \in A$. In fact, for each $\zeta = \langle Z_\alpha\rangle_{\alpha\in A} \in \Pi\{\mathcal{Z}_\alpha : \alpha \in A\}$ let $Z_\zeta(\omega)$ equal $\bigcap\{Z_\alpha(\omega) : \alpha \in A\}$. If $Z_\zeta(\omega)$ is not empty, then since X is 0-dimensional, we can choose a partition $\{Z_\zeta(n) : n \in \omega\}$ of $X \setminus Z_\zeta(\omega)$ consisting of compact open subsets of X. Then $\mathcal{Z}_A$ consists of all the partitions $\{Z_\zeta : \zeta \in \Pi\{\mathcal{Z}_\alpha : \alpha \in A\}$ such that $Z_\zeta(\omega) \neq \emptyset\}$. Let $\{A_n : n \in \omega\} \subset \mathcal{A}$ and $A \in \mathcal{A}$ such that $\bigcup_n A_n \subset A$. The construction of $\mathcal{Z}_A$ ensures that $\mathcal{Z}_A$ refines $\mathcal{Z}_{A_n}$ and $\mathcal{Z}_\alpha$ for each $n \in \omega$ and $\alpha \in A$. Therefore $\mathcal{P}'$ is σ-directed. □

Lemma 7. *Assume $cof([\mathfrak{s}]^\omega, \subset) = \mathfrak{s}$. Let X be a compact 0-dimensional space and $\mathcal{P}$ a point-separating family of z-partitions of X with $|\mathcal{P}| \leq \mathfrak{s}$. There is a space $X(\mathcal{P})$ and an irreducible mapping f onto X such that $|X(\mathcal{P})| \leq |X| \cdot 2^{\mathfrak{s}}$ and if $S \subset X(\mathcal{P})$ is a converging sequence then $f[S]$ is finite. Furthermore, there is a point-separating family $\mathcal{P}'$ of z-partitions of $X(\mathcal{P})$ of cardinality $\mathfrak{s}$.*

Proof. We may assume that $\mathcal{P}$ is σ-directed. Let $\mathcal{S} \subset [\omega]^\omega$ be a splitting family with cardinality $\mathfrak{s}$. For each $(\mathcal{Z}, I) \in \mathcal{P} \times \mathcal{S}$, let $f_{\mathcal{Z},I}$ and $X(\mathcal{Z}, I)$ be as in Lemma 5.

We define $X(\mathcal{P})$ as a subspace of $X \times \Pi\{X(\mathcal{Z}, I) : (\mathcal{Z}, I) \in \mathcal{P} \times \mathcal{S}\}$. A pair $\langle x, g\rangle \in X(\mathcal{P})$ if for each $(\mathcal{Z}, I) \in \mathcal{P} \times \mathcal{S}$, $f_{\mathcal{Z},I}(g(\mathcal{Z}, I)) = x$.

Let us show that $X(\mathcal{P})$ is a closed subset. Suppose that $\langle x, g\rangle \notin X(\mathcal{P})$ and fix $(\mathcal{Z}, I) \in \mathcal{P} \times \mathcal{S}$ such that $f_{\mathcal{Z},I}(g(\mathcal{Z}, I)) = x' \neq x$. Let U be an open subset of X such that $x \in U$ and $x' \notin \overline{U}$. Let $W = f^{-1}_{\mathcal{Z},I}(X \setminus \overline{U})$ and notice that

$$U \times \Pi\{X(\mathcal{Z}', I') : (\mathcal{Z}, I) \neq (\mathcal{Z}', I') \in \mathcal{P} \times \mathcal{S}\} \times W$$

is a neighborhood of $\langle x, g\rangle$ which is disjoint from $X(\mathcal{P})$.

We let f denote the restriction to $X(\mathcal{P})$ of the first coordinate projection mapping. The proof that f is irreducible is similar to the proof that $f_{\mathcal{Z},I}$ is irreducible in Lemma 5. For each $(\mathcal{Z}, I) \in \mathcal{P}\times\mathcal{S}$, the $(\mathcal{Z}, I)$-coordinate projection mapping induces a natural mapping $\tilde{f}_{\mathcal{Z},I}$ from $X(\mathcal{P})$ onto $X(\mathcal{Z}, I)$ such that $f = f_{\mathcal{Z},I} \circ \tilde{f}_{\mathcal{Z},I}$.

Assume that $S \subset X(\mathcal{P})$ and that $f[S]$ is a countably infinite subset of X. Let $x \in X$ be the image of any limit point of S. For each $y \in f[S] \setminus \{x\}$, there is a $\mathcal{Z}_y \in \mathcal{P}$ and some $Z_y \in \mathcal{Z}_y$ such that $x \in Z_y(\omega)$ and $y \notin Z_y(\omega)$. Since $\mathcal{P}$ is σ-directed, there is a $\mathcal{Z} \in \mathcal{P}$ and a $Z \in \mathcal{Z}$ such that $x \in Z(\omega)$, and $f[S] \setminus Z(\omega)$ is infinite. If there is some n such that $Z(n) \cap f[S]$ is infinite, then $f[S]$ does not converge to x. Otherwise, there is an $I \in \mathcal{S}$ such that $f[S] \cap \bigcup_{n\in I} Z(n)$ and $f[S] \cap \bigcup_{n\in N\setminus I} Z(n)$ are each infinite. By Lemma 5, $f^{-1}_{\mathcal{Z},I}(\bigcup_{n\in I} Z(n))$ and $f^{-1}_{\mathcal{Z},I}(\bigcup_{n\in N\setminus I} Z(n))$ have disjoint closures in $X(\mathcal{Z}, I)$. Therefore $\tilde{f}_{\mathcal{Z},I}[S]$ does not converge in $X(\mathcal{Z}, I)$. It then follows that S does not converge in $X(\mathcal{P})$.

For each $(\mathcal{Z}, I) \in \mathcal{P} \times \mathcal{S}$, we define two z-partitions of $X(\mathcal{P})$. Again by Lemma 5, the closures of $f^{-1}_{\mathcal{Z},I}(\bigcup_{n\in I} Z(n))$ and $f^{-1}_{\mathcal{Z},I}(\bigcup_{n\in N\setminus I} Z(n))$ in $X(\mathcal{Z}, I)$ form a clopen partition of $X(\mathcal{Z}, I)$. Let $C(\mathcal{Z}, I, Z, 0)$ and $C(\mathcal{Z}, I, Z, 1)$ denote the preimage under $\tilde{f}_{\mathcal{Z},I}$ of these clopen sets.

We define partitions Z_e for $e \in \{0,1\}$:

$$Z_e(n) = \begin{cases} f^{-1}(Z(n)) \cup C(\mathcal{Z}, I, Z, 1-e) & n = 0 \\ f^{-1}(Z(n)) \cap C(\mathcal{Z}, I, Z, e) & 0 < n \leq \omega \ . \end{cases}$$

The collection $\mathcal{W}(\mathcal{Z}, I) = \{Z_e : e \in \{0,1\},\ Z \in \mathcal{Z}\}$ is a z-partition of $X(\mathcal{P})$. Suppose that x, y are distinct points of $X(\mathcal{P})$. If $f(x) \neq f(y)$, then there is a $\mathcal{Z} \in \mathcal{P}$ such that $f(x) \in Z(\omega)$ and $f(y) \notin Z(\omega)$ for some $Z \in \mathcal{Z}$. Fix any $I \in \mathcal{S}$ and choose $e \in \{0,1\}$ such that $x \in C(\mathcal{Z}, I, Z, e)$. It follows that $x \in Z_e(\omega)$ and $y \notin Z_e(\omega)$. If $f(x) = f(y) = \bar{x}$, then there are g, h such that $x = \langle \bar{x}, g\rangle$ and $y = \langle \bar{x}, h\rangle$. There must be $(\mathcal{Z}, I)$ such that $g(\mathcal{Z}, I) \neq h(\mathcal{Z}, I)$. There is a unique $Z \in \mathcal{Z}$ such that $\bar{x} \in Z(\omega)$ and $f^{-1}_{\mathcal{Z},I}(\bar{x})$ contains at most two points; in this case $\{g(\mathcal{Z}, I), h(\mathcal{Z}, I)\}$. One of these points is in $\overline{f^{-1}_{\mathcal{Z},I}[\bigcup_{n\in I} Z(n)]}$ and the other in $\overline{f^{-1}_{\mathcal{Z},I}[\bigcup_{n\in N\setminus I} Z(n)]}$. That is, we have found $(\mathcal{Z}, I)$ and $Z \in \mathcal{Z}$ such that $x \in C(\mathcal{Z}, I, Z, 0)$ and $y \in C(\mathcal{Z}, I, Z, 1)$ (or vice-versa). Moreover, $x \in Z_0(\omega)$ and $y \in Z_1(\omega)$.

This shows that the family $\mathcal{P}'$ is point-separating where $\mathcal{P}'$ consists of $\{\mathcal{W}(\mathcal{Z}, I) : (\mathcal{Z}, I) \in \mathcal{P} \times I\}$. □

Proof of Theorem 2. Let X_0 be the Cantor set 2^ω and let $\mathcal{Z}_0$ be any z-partition such that $\{Z(\omega) : Z \in \mathcal{Z}_0\}$ are the singletons. Set $\mathcal{P}_0 = \{\mathcal{Z}_0\}$. Suppose that $\lambda \in \omega_1$ and we have defined an inverse limit $\langle X_\alpha : \alpha \in \lambda\rangle$ with bonding maps $\langle f_{\alpha,\beta} : \beta < \alpha < \lambda\rangle$. Suppose further that for each $\alpha \in \lambda$ we have specified a point-separating family $\mathcal{P}_\alpha$ of z-partitions of X_α such that $|\mathcal{P}_\alpha| \leq \mathfrak{s}$. Finally, assume that if $\beta < \alpha < \lambda$ and $S \subset X_\alpha$ is a converging sequence, then $f_{\alpha,\beta}[S]$ is a finite subset of X_β. If λ is a limit then define X_λ to be the inverse limit of the system $\langle X_\alpha, \{f_{\alpha,\beta} : \beta < \alpha\} : \alpha \in \lambda\rangle$. There are canonical mappings $f_{\lambda,\alpha}$ from X_λ onto X_α for each $\alpha < \lambda$. For each $\alpha < \lambda$ and $\mathcal{Z} \in \mathcal{P}_\alpha$, let $f^{-1}_{\lambda,\alpha}(\mathcal{Z})$ denote the natural z-partition of X_λ,

$$\{\langle f^{-1}_{\lambda,\alpha}(Z(n)) : n \in \omega + 1\rangle : Z \in \mathcal{Z}\}.$$

It is routine to check that $\mathcal{P}_\lambda = \{f^{-1}_{\lambda,\alpha}(\mathcal{Z}) : \alpha \in \lambda,\ \mathcal{Z} \in \mathcal{P}_\alpha\}$ is a point-separating family of z-partitions of X_λ. It also trivially follows that for each converging sequence $S \subset X_\lambda$ and each $\alpha < \lambda$, $f_{\lambda,\alpha}[S]$ is finite because $f_{\lambda,\alpha+1}[S]$ will be a converging sequence in $X_{\alpha+1}$ and $f_{\lambda,\alpha} = f_{\lambda,\alpha+1} \circ f_{\alpha+1,\alpha}$.

If $\lambda = \alpha + 1$, then we obtain X_λ as $X_\alpha(\mathcal{P}_\alpha)$ by applying Lemma 7.

We check that $X = X_{\omega_1}$ is an Efimov space. Since the cardinality of X is constructed to be no larger than $2^{\mathfrak{s}}$ and we are assuming that $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$, X can not contain a copy of βN. Suppose that $S \subset X$ is an infinite set. Let $\alpha < \omega_1$ be chosen so that $f_{\omega_1,\alpha}[S]$ is also infinite. By the inductive hypothesis, it follows that $f_{\omega_1,\alpha+1}[S]$ is not a converging sequence. Therefore X contains no converging sequences. □

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