SPACES WHOSE PSEUDOCOMPACT SUBSPACES ARE CLOSED SUBSETS

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Abstract. Every first countable pseudocompact Tychonoff space X has the property that every pseudocompact subspace of X is a closed subset of X (denoted herein by "FCC"). We study the property FCC and several closely related ones, and focus on the behavior of extension and other spaces which have one or more of these properties. Characterization, embedding and product theorems are obtained, and some examples are given which provide results such as the following. There exists a separable Moore space which has no regular, FCC extension space. There exists a compact Hausdorff Fréchet space which is not FCC. There exists a compact Hausdorff Fréchet space X such that X, but not X^2, is FCC.

1. Introduction and terminology.

For topological spaces X and Y, $C(X,Y)$ will denote the family of continuous functions from X into Y, $C(X)$ will denote $C(X,\mathbb{R})$, and $C^*(X)$ will denote the family of bounded functions in $C(X)$. A space X is called *pseudocompact* provided that $C(X) = C^*(X)$. This definition was first given for Tychonoff spaces, i.e., completely regular T_1-spaces, by E. Hewitt [10].

For terms not defined here, see [5], [6] or [15]. Except where noted otherwise, no separation axioms are assumed.

Some properties of interest that are closely related to pseudocompactness are listed in Theorem 1.1.

Theorem 1.1. *Let X be a space. Then each statement below implies the next one, and all of properties* (B_1)–(B_6) *are equivalent.*

(A) *The space X is pseudocompact and completely regular.*

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(B_1) *Every locally finite family of open sets of X is finite.*
(B_2) *Every pairwise disjoint locally finite family of open sets of X is finite.*
(B_3) *Every sequence of nonempty open subsets of X has a cluster point in X.*
(B_4) *If $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a sequence of nonempty open subsets of X such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$, then $\mathcal{U}$ has a cluster point in X.*
(B_5) *Every countable open filter base on X has an adherent point.*
(B_6) *Every countable open cover of X has a finite subcollection whose union is dense in X.*
(C) *X is pseudocompact.*

We recall that the *adherence* of a filter base $\mathcal{F}$ on a space X is the intersection of the closures of the members of $\mathcal{F}$, and by a *cluster point* of a sequence $\{U_n : n \in \mathbb{N}\}$ of subsets of a space X is meant a point $p \in X$ such that for every neighborhood V of p, $V \cap U_n \neq \emptyset$ for infinitely many integers n. A sequence denoted $\{U_n : n \in \mathbb{N}\}$ will be referred to as a *pairwise disjoint sequence* provided that $U_i \cap U_j = \emptyset$ whenever $i \neq j$.

Proofs or references to proofs of the different results in Theorem 1.1 can be found in [1], [7], [8], [15] or [26]. These properties have been found useful by a number of authors, especially (B_2), which has been referred to in [26] as *feebly compact* and attributed to S. Mardešić and P. Papić, and (B_1), which was called *lightly compact* in [1]. I. Glicksberg [8] noted that every pseudocompact completely regular space satisfies (B_2), and every space satisfying (B_2) is pseudocompact.

One immediate corollary to Theorem 1.1 that will be used below is the following.

Corollary 1.2. [8] *Let X be a topological space.*

(a) *The union of finitely many feebly compact subspaces of X is feebly compact.*
(b) *If X is feebly compact and U is any open subset of X, then $\overline{U}$ is a feebly compact subspace of X.*
(c) *If D is a feebly compact subspace of X and $D \subseteq G \subseteq \overline{D}$, then G is feebly compact.*

Definition 1.3. We shall call a topological space X *feebly compact closed* (*"FCC"*) provided that X is feebly compact and every feebly compact subspace of X is a closed subset of X.

Definition 1.4. A space X will be called *sequentially feebly compact* provided that for every sequence $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of X there exist a point $p \in X$ and a strictly increasing sequence $\{n_i : i \in \mathbb{N}\}$ in $\mathbb{N}$ such that for every neighborhood V of p, $V \cap U_{n_i} \neq \emptyset$ for all but finitely many $i \in \mathbb{N}$.

2 The properties FCC and sequentially feebly compact.

The property FCC has been studied previously (but not named or labeled) by several authors. It was proved in [23] that every first countable feebly compact Hausdorff space, and hence every first countable pseudocompact Tychonoff space, is FCC. Then a proof was given in [14] that if a feebly compact space X is E_1, i.e., if every point x of X is an intersection of countably many closed neighborhoods of x, then X is FCC. The concept has been used in the study of maximal feeble compactness. By a *maximal feebly* *compact space* is meant a feebly compact space $(X, \mathcal{T})$ such that for every feebly compact topology $\mathcal{U}$ on X, if $\mathcal{T} \subseteq \mathcal{U}$ then $\mathcal{U} = \mathcal{T}$. Using the result of D. Cameron [3], that an FCC, *submaximal space* (i.e., a space in which every dense set is open) is maximal feebly compact, and a result of A.B. Raha [17], the authors proved in [16] that a topological space is maximal feebly compact space if and only if it is FCC and submaximal. Using the latter, a number of examples of maximal feebly compact spaces are given in [16], e.g., the well-known Isbell-Mrówka space Ψ described in [6, 5I]and in the proof below of Theorem 2.12. The property FCC was also considered in the article [9], where the relationship between countably compact regular spaces which are FCC and those which are Fréchet was studied.

The next lemma provides conditions each of which implies or is implied by, or under suitable restrictions is equivalent to, the property FCC. Let us recall that a space X is called *semiregular* provided that that the *regular open sets* (i.e., sets having the form $\operatorname{int}(\operatorname{cl}(A))$, where A is an open subset of X) form a base for the topology on X.

Lemma 2.1. *Let X be a topological space, and consider the conditions below.*

(F_1) *Every feebly compact subspace of X is a closed subset of X.*

(F_2) *For every feebly compact subspace S of X, dense subset D of $\overline{S}$, and point $p \in \overline{S} \setminus D$, there exists a pairwise disjoint sequence $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ of nonempty open subsets of D such that for every neighborhood V of p in $\overline{S}$, $\overline{V} \supseteq K_n$ for all but finitely many $n \in \mathbb{N}$.*

(F_3) *Every feebly compact subspace of X with dense interior is a closed subset of X.*

(F_4) *For every open subset S of X and point $p \in \overline{S} \setminus S$, there exists a pairwise disjoint sequence $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ of nonempty open subsets of S such that $\mathcal{K}$ has no cluster point in $X \setminus \{p\}$, and for every neighborhood V of p in $\overline{S}$, $\overline{V} \supseteq K_n$ for all but finitely many $n \in \mathbb{N}$.*

Then the following hold.

(a) *Property* (F_1) *implies* (F_2) *and* (F_3), *and if X is a Hausdorff space then* (F_2) *implies* (F_1).

(b) *Property* (F_4) *implies* (F_3), *and if X is feebly compact then* (F_3) *implies* (F_4).

(c) *If $\overline{S}$ is semiregular then in each of the statements* (F_2) *and* (F_4), *the containment "$\overline{V} \supseteq K_n$" may be replaced by "$V \supseteq K_n$."*

(d) *If X is Fréchet, Hausdorff and scattered, then it has property* (F_1).

(e) *If X is Fréchet and Hausdorff and has a dense set of isolated points, then it has property* (F_3).

Proof. We prove (b). The proof of (a) is similar.

(F_4) implies (F_3). Suppose (F_3) is false. Then there exist an open subset S of X, a feebly compact subspace F of X with $S \subseteq F \subset \overline{S}$, and a point $p \in \overline{F} \setminus F$. It would follow that $\overline{S} = \overline{F}$ and thus $p \in \overline{S} \setminus S$. By Corollary 1.2 (c), the feeble compactness of F, and the relation $F \subseteq S \setminus \{p\} \subseteq \overline{F}$, the subspace $S \setminus \{p\}$ would be feebly compact. By Theorem 1.1, every sequence $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ of nonempty open subsets of X such that each $K_n \subset S$ would have a cluster point in $\overline{S} \setminus \{p\}$. Therefore, ($\mathrm{F}_4$) would not hold.

Suppose X is feebly compact and (F_3) holds. Let S and p be as in the hypothesis of (F_4). It follows from (F_3) and the characterizations in Theorem 1.1 that there exists a pairwise disjoint sequence $\mathcal{W}$ of nonempty open sets of the space $\overline{S} \setminus \{p\}$ such that $\mathcal{W}$ has no cluster point in $\overline{S} \setminus \{p\}$. Define $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$, where for each $n \in \mathbb{N}$, $K_n = W_n \cap S$. Since S is dense in $\overline{S} \setminus \{p\}$ and open in X, it follows from the properties of $\mathcal{W}$ that $\mathcal{K}$ is a pairwise disjoint sequence of nonempty open subsets of X, as well as of S, which has no cluster point in $X \setminus \{p\}$. By the feeble compactness of X, $\mathcal{K}$ must have a cluster point, so p is the unique cluster point of $\mathcal{K}$ in X. If there were an infinite subset J of $\mathbb{N}$ and a neighborhood V of p in $\overline{S}$ such that $K_j \setminus \overline{V} \neq \emptyset$ for every $j \in J$, then $\{K_j \setminus \overline{V} : j \in J\}$ would be an infinite locally finite family of open subsets of X, in contradiction of Theorem 1.1.

Statement (c) is obvious. Let us prove (d). The proof of (e) is similar. Suppose $Y \subseteq X$ is feebly compact and $p \in \overline{Y}$. Let I be the set of isolated points of the space Y. Then $\mathrm{cl}_Y I = Y$ since X is scattered, and thus $p \in \overline{I}$. As X is Fréchet, there is a sequence $\{x_n : n \in \mathbb{N}\}$ in I which converges to p. Then $\{\{x_n\} : n \in \mathbb{N}\}$ is a sequence of nonempty open sets of the feebly compact Hausdorff space Y which has only p as a cluster point. Hence $p \in Y$. Therefore, Y is a closed subset of X. $\square$

Theorem 2.2. *Let X be a topological space. Then the following hold.*

(a) *If X is a feebly compact space which is either (i) E_1, or (ii) compact Hausdorff and either hereditarily metacompact or hereditarily realcompact, or (iii) Fréchet, Hausdorff and scattered, then it is FCC.*
(b) *If X is FCC, then it is a feebly compact T_1-space and has the properties* (F_1)–(F_4).
(c) *If X is a countably compact FCC space, then it is (i)* (Y. Tanaka) *Fréchet and (ii) sequentially compact.*
(d) *If X is feebly compact and either (i) has property* (F_3) *or (ii) is a sequential space, then X is sequentially feebly compact. In particular, FCC implies sequentially feebly compact.*
(e) *If X is feebly compact, Fréchet and Hausdorff and has a dense set of isolated points, then it has properties* (F_3)–(F_4).
(f) *If X is sequentially feebly compact, then it is feebly compact.*

Proof. As noted above, part (i) of (a) is obtained in [14]. Since by results of E. Hewitt [10] and S. Watson [28], every realcompact and every metacompact pseudocompact Tychonoff space is compact, one obtains (ii) of (a). Statement (iii) follows from Lemma 2.1 (d).

Obviously (b) holds. In [9] a proof was given that a statement like (c) (i) holds for regular spaces, and the author of [9] attributed the result to Y. Tanaka. Here is a similar proof that does not require regularity of the space X: Suppose $A \subset X$ and $x \in \overline{A} \setminus A$. Since $\overline{A} \setminus \{x\}$ is not feebly compact and has A as a dense subset, there exists a pairwise disjoint sequence $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of A which has no cluster point in $\overline{A} \setminus \{x\}$. Choose $x_n \in U_n$ for each $n \in \mathbb{N}$. Then the set $C = \{x_n : n \in \mathbb{N}\}$ is a discrete subspace of the countably compact T_1-space $\overline{C} = C \cup \{x\}$, and consequently, the sequence $\{x_n\}$ in A must converge to x. The statement (c) (ii) follows from the easily verified fact that every countably compact T_1 Fréchet space is sequentially compact.

We prove (d). Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a sequence of nonempty open subsets of the space X. We wish to show that there exist a point $p \in X$ and a strictly increasing sequence $\{n_i : i \in \mathbb{N}\}$ in $\mathbb{N}$ such that for every neighborhood V of p, $V \cap U_{n_i} \neq \emptyset$ for all but finitely many $i \in \mathbb{N}$ (or equivalently, there exist a point $p \in X$ and an infinite subset J of $\mathbb{N}$ such that $V \cap U_j \neq \emptyset$ for all but finitely many $j \in J$).

Suppose first that the hypothesis of (d) (i) holds. Let us consider two cases.

Case 1: suppose there are an infinite subset J of $\mathbb{N}$ and a point $p \in X$ such that $p \in \overline{U_j}$ for every $j \in J$. Then p and J have the required properties.

Case 2: suppose that Case 1 does not hold. Since X is feebly compact, the sequence $\mathcal{U}$ has a cluster point p. There exists $k \in \mathbb{N}$ such that for every integer $n > k$, the point $p \notin \overline{U_n}$. Define $S = \bigcup_{n \geq k+1} U_n$, and for each $i \geq k+1$, let $S_i = \bigcup_{n=k+1}^{i} U_n$. Note that $p \in \overline{S} \setminus S$ and $\overline{S_i} \subseteq \overline{S} \setminus \{p\}$ for every $i \geq k+1$. As every feebly compact subspace of X with dense interior is closed, it follows from Lemma 2.1 that there exists a pairwise disjoint sequence $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ of nonempty open subsets of X such that each $K_n \subseteq S$, $\mathcal{K}$ has no cluster point in $X \setminus \{p\}$, and for every neighborhood V of p, $\overline{V} \supseteq K_n$ for all but finitely many $n \in \mathbb{N}$. Since each $\overline{S_i}$ is feebly compact (by Corollary 1.2), then for each $i \geq k+1$, $\overline{S_i} \cap K_n \neq \emptyset$ for at most finitely many $n \in \mathbb{N}$. By mathematical induction one can find strictly increasing sequences $\{m_i : i \in \mathbb{N}\}$ and $\{t_i : i \in \mathbb{N}\}$ in $\mathbb{N}$ such that for each $i \in \mathbb{N}$:

$$K_{m_i} \cap U_{k+t_i} \neq \emptyset;\text{ and if } i > 1, \text{ then } K_m \cap \overline{S_{t_{i-1}+k}} = \emptyset \text{ for every } m \geq m_i.$$

Define $n_i = k + t_i$ for each $i \in \mathbb{N}$. Then the sequence $\{n_i : i \in \mathbb{N}\}$ and the point p have the properties required in the definition of sequentially feebly compact.

Next, we assume the hypothesis of (d) (ii) holds. Consider again the two cases named above. The proof in Case 1 proceeds as above.

Assume Case 2 holds. Then as in Case 2 above, there are a cluster point q of $\mathcal{U}$ and and $k \in \mathbb{N}$ so that for every integer $n > k$, the point $q \notin \overline{U_n}$. Then the set $T = \bigcup_{n \geq k+1} \overline{U_n}$ is not a closed set since $q \in \overline{T} \setminus T$. Because X is a sequential space, it follows that there exists a sequence $\{x_n : n \in \mathbb{N}\}$ in T which converges to a point $p \in X \setminus T$. For each integer $n \geq k+1$, note that since $p \notin \overline{U_n}$ then $x_m \in \overline{U_n}$ for at most finitely many $m \in \mathbb{N}$. Thus, there are strictly increasing sequences $\{m_i : i \in \mathbb{N}\}$ and $\{t_i : i \in \mathbb{N}\}$ such that for each $i \in \mathbb{N}$, one has $x_{m_i} \in \mathrm{cl}(U_{k+t_i})$. Therefore, the sequence $\{n_i = k + t_i : i \in \mathbb{N}\}$ and point p satisfy the definition of sequentially feebly compact.

Statement (e) follows from Lemma 2.1, and statement (f) follows from the characterizations in Theorem 1.1 and the appropriate definitions. $\square$

The next result will be used in §5.

Corollary 2.3. *Let X be a feebly compact space which has property (F_3). Suppose $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a sequence of nonempty open subsets of X such that $\overline{U_m} \cap \overline{U_n} = \emptyset$ whenever $m \neq n$. Then there are a point p in X, an infinite subset J of $\mathbb{N}$, and a sequence of nonempty open sets $\mathcal{P} = \{P_n : n \in J\}$ such that $P_n \subseteq U_n$ for each $n \in J$, and for every neighborhood O of p, $\overline{O}$ contains P_n for all but finitely many $n \in J$.*

Proof. This follows from the proof of Case 2 of statement (d) (i) in Theorem 2.2. □

Here are some examples illustrating that these properties are distinct.

Example 2.4. Let X be the one-point compactification of some uncountable discrete space. Then X is a scattered, Fréchet, compact Hausdorff, and hence FCC, space (by Theorem 2.2 (a) (iii)) which is not first countable (or E_1).

Example 2.5. There exists a space X which is a countable, compact, maximal feebly compact, and hence FCC, space which is not Hausdorff: in [16, 2.12] a proof is given that a certain countable, non-Hausdorff, maximal compact space due to V.K. Balachandran is also maximal feebly compact.

Example 2.6. Let X be any feebly compact Hausdorff space which contains a non-isolated P-point p. Then X cannot have property (F_3): if $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ were any pairwise disjoint sequence of nonempty open subsets of $X \setminus \{p\}$ and one chose, for each $n \in \mathbb{N}$, a nonempty open set $V_n \subseteq U_n$ with $p \notin \overline{V_n}$, then the sequence $\{V_n : n \in \mathbb{N}\}$ would have a cluster point in $X \setminus \{p\}$, and hence $\mathcal{U}$ would also, i.e., $X \setminus \{p\}$ would be feebly compact. The next two spaces are of this type.

Example 2.7. Let $X = \omega_1 + 1$, the set of ordinals $\leq \omega_1$, with the order topology. The space X is a sequentially compact, and hence sequentially feebly compact, compact Hausdorff space which does not have property (F_3).

Example 2.8. Let T be the Tychonoff plank, $T = (\omega_1 + 1) \times (\omega_0 + 1) \setminus \{(\omega_1, \omega_0)\}$. Then T is a locally compact Hausdorff space that does not have property (F_3) and is not sequentially compact [5], [6]. Since T has a dense, sequentially compact subspace, namely $T \setminus \{(\omega_1, \alpha) : \alpha < \omega_0\}$, it follows easily that T is sequentially feebly compact.

Example 2.9. Let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$, where $\mathbb{N}$ has the discrete topology, and let X be any dense, pseudocompact subspace of $\beta\mathbb{N}$. Then X is a feebly compact Tychonoff space that is not sequentially feebly compact, and hence is not FCC: it is well-known that no nontrivial sequence in $\beta\mathbb{N}$ is convergent [6], and so for any infinite subset J of $\mathbb{N}$ and sequence $\mathcal{U} = \{\{j\} : j \in J\}$, there would exist no point $p \in X$ and infinite subset K of J with every neighborhood of p containing all but finitely many of the sets $\{\{k\} : k \in K\}$.

Example 2.10. Let $X = \mathbb{N} \cup \{-\infty, \infty\}$, where a subset T of X is defined to be open iff $T \subseteq \mathbb{N}$ or $X \setminus T$ is finite. Then X is a first countable, scattered, compact T_1-space satisfying property (F_2), but none of (F_1), (F_3) and (F_4).

The properties first countable, E_1, Fréchet and sequential are well-known to be closely related to one another. We shall give some examples illustrating further similarities and, in some cases, differences between these properties and the properties FCC and sequentially feebly compact. One such is the following familiar space.

Example 2.11. Let $(X, \mathcal{S})$ be [0,1], with its usual topology, and let $\mathcal{T}$ be the topology on X generated by $\mathcal{S}$ and the family of co-countable subsets of X. Then $(X, \mathcal{T})$ is an FCC space which is E_1, but not a sequential space. The latter follows from the fact that no infinite subset of $(X,\mathcal{T})$ is countably compact. It is also known that for every set $T \in \mathcal{T}$, $\mathrm{cl}_{\mathcal{T}} T = \mathrm{cl}_{\mathcal{S}} T$, and consequently *every* open filter base on $(X,\mathcal{T})$ has an adherent point. Thus $(X,\mathcal{T})$ has the stated properties.

Example 2.11 also illustrates the observation that for any FCC space $(X,\mathcal{S})$, if $\mathcal{T}$ is any feebly compact topology on X such that $\mathcal{S} \subseteq \mathcal{T}$, then $(X,\mathcal{T})$ is also FCC. More generally, see Theorem 3.2 (d) below.

A previously defined family of spaces related to FCC spaces was studied in [11], where M. Ismail and P. Nyikos called a space X *C-closed* if every countably compact subspace of X is a closed subset of X. They proved that (a) a sequential Hausdorff space is C-closed, and (b) a sequentially compact, C-closed Hausdorff space is a sequential space. In their statement (b), if one replaces "sequentially compact, C-closed Hausdorff" by "countably compact FCC," then (as noted above in 2.2 (c)), one can replace their conclusion by "Fréchet and sequentially compact." The next result shows that in (a), even for feebly compact symmetrizable spaces, one cannot replace "C-closed" by "FCC." (A space $(X,\mathcal{T})$ is called *symmetrizable* in the sense of A.V. Arhangel′skiĭ if there exists a symmetric d on X which induces $\mathcal{T}$, where by a *symmetric* on X one means a function $d : X \times X \to [0,\infty)$ which vanishes exactly on the diagonal and satisfies the symmetric property, $d(x,y) = d(y,x)$ for all x, $y \in X$.)

Before stating the result, let us first recall that an *almost disjoint ("AD") family* $\mathcal{P}$ on a set X is a collection $\mathcal{P} \subseteq [X]^\omega$ such that $P \cap P'$ is finite whenever P, P' are distinct members of $\mathcal{P}$. An AD family $\mathcal{M}$ on X (such that $\mathcal{M} \subseteq \mathcal{Q} \subseteq [X]^\omega$) is called a *maximal almost disjoint family ("MAD" family)* (respectively, *maximal almost disjoint subfamily of* $\mathcal{Q}$) provided that $\mathcal{M}$ is properly contained in no AD family on X (respectively, no AD subfamily of $\mathcal{Q}$).

Theorem 2.12. *There exists a symmetrizable (therefore, sequential), scattered, C-closed Hausdorff space $(X,\mathcal{T})$ which contains a non-isolated point p such that $X \setminus \{p\}$ is first countable, locally compact, feebly compact and zero-dimensional, and hence X is sequentially feebly compact and C-closed but not FCC.*

Proof. Let Ψ be the Isbell-Mrówka space described in [6, 5I]: Let $\mathcal{M}$ be an infinite MAD family on $\mathbb{N}$ and $\Psi = \mathbb{N} \cup \mathcal{M}$, where a subset U of Ψ is defined to be open provided that for any set $M \in \mathcal{M}$, if $M \in U$ then there is a finite subset F of M such that $\{M\} \cup M \setminus F \subseteq U$. The space Ψ is then a first countable pseudocompact locally compact Hausdorff space that is not countably compact [6]. List in a 1-1 manner as $\{M_n : n \in \mathbb{N}\}$ the members of an infinite subset $\mathcal{I}$ of $\mathcal{M}$, choose a point $p \notin \Psi$, and define $X = \Psi \cup \{p\}$. Next, define $d : X \times X \to [0,\infty)$ as follows: $d(p, M_n) = d(M_n, p) = 1/n$ for each $M_n \in \mathcal{I}$ and $d(p,y) = d(y,p) = 1$ for each $y \in \mathbb{N} \cup (\mathcal{M} \setminus \mathcal{I})$; for each $n \in \mathbb{N}$ and $y \in \Psi \setminus \{n\}$, $d(n,y) = d(y,n) = 1/n$ whenever $n \in y \in \mathcal{M}$, and $d(n,y) = d(y,n) = 1$ whenever either $y \in \mathcal{M}$ with $n \notin y$ or $y \in \mathbb{N} \setminus \{n\}$; and $d(x,x) = 0$ for all $x \in X$. Let $\mathcal{T}$ be the topology induced on X by d, i.e., define $\mathcal{T}$ to be the collection of all subsets T of X such that for each point $t \in T$ there exists $\epsilon > 0$ such that T contains the "ball" $\{x \in X : d(t,x) < \epsilon\}$.

It is straightforward to show that d is a symmetric for the space $(X,\mathcal{T})$, and $(X,\mathcal{T})$ has the stated properties. Furthermore, it is known and not difficult to prove that every symmetrizable space is sequential. □

Example 2.13. If one lets X be as in the proof above, but weakens the topology $\mathcal{T}$ on X by choosing the topology $\mathcal{S}$ for which $(X, \mathcal{S})$ is the one-point compactification of Ψ, then it was noted in [12] that Eric van Douwen and Peter Nyikos had noticed previously the resulting compact Hausdorff space $(X, \mathcal{S})$ was sequential but not Fréchet. Like $(X, \mathcal{T})$, the space $(X, \mathcal{S})$ is not FCC, and since $\Psi \in \mathcal{S}$, these spaces do not even have the property (F_3). Since every symmetrizable compact Hausdorff space is known to be metrizable, and every scattered Fréchet feebly compact space is FCC, then $\mathcal{S} \neq \mathcal{T}$ and $(X, \mathcal{T})$ is not Fréchet either. While $(X, \mathcal{T})$ fails to be countably compact, the space $(X, \mathcal{S})$ is known to be sequentially compact and C-closed.

It is natural to ask if the word "scattered" can be removed from the statement in Theorem 2.2 (a) (iii). In [9] a very nice proof was given that, assuming [MA], there exists a compact Hausdorff Fréchet space which is not FCC.

The next result, which does not require any special axioms beyond ZFC, shows that there is a compact Hausdorff Fréchet space X which is not FCC. Its proof is an elaboration on one due to Reznichenko that was outlined in 3.6 of [12]. The authors are grateful to Peter Nyikos for calling Reznichenko's space to our attention. In addition, we shall show that X can be used to construct a compact Hausdorff and Fréchet space $\mathbb{A}(X)$ which is not FCC, but which has property (F_3) and also has a dense set of isolated points.

Theorem 2.14. *There is a compact Hausdorff Fréchet space which is not FCC.*

Proof. Let κ denote the cardinality of continuum.

We first define a compact 0-dimensional Fréchet topology on $T = \kappa^{\leq\omega}$, i.e., T consists of the functions into κ which have domain either a nonnegative integer or the entire set of nonnegative integers.

For any $t \in \kappa^{<\omega}$ and $\alpha \in \kappa$, we will let $t\alpha$ denote the function obtained by extending the domain of t by one and setting the final value to α. For $n \in \omega$ and $t : n \to \kappa$, we occasionally denote t by $\langle t_0, \ldots, t_{n-1} \rangle$.

Recall that T forms a tree when ordered by simple inclusion, i.e., for $s, t \in T$, $s \subseteq t$ if $\mathrm{dom}(s) \leq \mathrm{dom}(t)$ and $s = t \upharpoonright \mathrm{dom}(s)$. Now T is endowed with the following topology. Simply for each $s \in \kappa^{<\omega}$, the set

$$[s] = \{t \in T : s \subseteq t\}$$

is defined to be clopen. Thus a neighborhood basis for $s \in \kappa^{<\omega}$ is the family

$$\{[s] \setminus \bigcup_{i<n} [s\alpha_i] : n \in \omega \text{ and } \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa\} .$$

Furthermore, for $f \in T \cap \kappa^\omega$, the family $\{[f \upharpoonright n] : n \in \omega\}$ forms a neighborhood base at f, and hence each such f is a point of countable character in T.

We leave as an exercise that T is compact, and thus for each $s \in \kappa^{<\omega}$, the clopen set $[s]$ is compact. One can note that $\{[s\alpha] : \alpha \in \kappa\}$, is a pairwise-disjoint family of clopen sets, and $[s]$ is the one-point compactification of $\bigcup\{[s\alpha] : \alpha \in \kappa\}$. It follows easily then that T is a Fréchet space.

Next, for each $n \in \omega$, let T_n denote the clopen set $[\langle n\rangle]$, i.e., all functions $t \in T$ such that $t(0) = n$. We will construct a compactification, X, of $\bigcup_n T_n$.

Our base space for X will be $\bigcup_n T_n \cup \kappa \cup \{\infty\}$. We will define a locally compact topology on $\bigcup_n T_n \cup \kappa$ in which $\bigcup_n T_n$ with the above topology is open and dense, and X will just be the one-point compactification. We will work to ensure that $X \setminus \{\infty\}$ is feebly compact, thus ensuring that X is not FCC.

For each $\alpha \in \kappa$, we will select a sequence $\langle t_n^\alpha : n \in \omega\rangle$ such that for each n, t_n^α is a member of $T_n \cap \kappa^{<\omega}$ and α is in its range. The neighborhood basis for α will be

$$\{U(\alpha, n) = \{\alpha\} \cup \bigcup_{n<k} [t_k^\alpha] : n \in \omega\},$$

and hence α will be the point at infinity in the one-point compactification of the union of the sequence of clopen sets $\{[t_k^\alpha] : k \in \omega\}$.

In order to ensure that the space is Hausdorff we will make sure that for $\beta < \alpha < \kappa$, there will be an n such that $U(\beta, n) \cap U(\alpha, n)$ is empty. This is equivalent to requiring that for each m larger than this n, $[t_m^\beta]$ and $[t_m^\alpha]$ are disjoint, i.e., t_m^β and t_m^α are incomparable members of T (or T_m).

The sequences are chosen by induction on κ. In order to ensure that $X \setminus \{\infty\}$ is feebly compact, it suffices for us to require that for every infinite set $I \subset \omega$ and every sequence $\{s_n : n \in I\}$ such that $s_n \in T_n \cap (\kappa^{<\omega})$ for $n \in I$, there is an $\alpha \in \kappa$ such that α is a cluster point of the sequence of clopen sets $\{[s_n] : n \in I\}$. To do so, let $\{\{s_n^\alpha : n \in I_\alpha\} : \alpha \in \kappa\}$ enumerate the family of all such sequences.

The selection of the sequence $\{t_n^0 : n \in \omega\}$ is handled the same as that for any α. That is, assume that $\alpha < \kappa$ and that for each $\beta < \alpha$ we have chosen the sequence $\{t_n^\beta : n \in \omega\}$ as described above (so that $t_n^\beta \in T_n$ and β is in the range of t_n^β). We therefore have defined, as above, a topology on the space $X_\alpha = \bigcup_n T_n \cup \{\beta : \beta < \alpha\}$ with the neighborhood base $\{U(\beta, n) : n \in \omega\}$ for each $\beta < \alpha$.

Fix any $\gamma < \kappa$ so large that for each $\beta < \alpha$ and $n \in \omega$, γ is not in the range of t_n^β. Observe that the sequence $\{[\langle n, \gamma\rangle] : n \in \omega\}$ is a discrete sequence of clopen sets in the space X_α. In fact, for each $\beta < \alpha$, $U(\beta, 0)$ is disjoint from each member of the sequence. If the sequence $\{[s_n^\alpha] : n \in I_\alpha\}$ already has a cluster point in the space X_α, we define t_n^α to be $\langle n, \gamma, \alpha\rangle$ for each n. Otherwise, the sequence $\{[s_n^\alpha] : n \in I_\alpha\}$ is also discrete (hence each $U(\beta, 0)$ meets only finitely many of these sets), and we define t_n^α to be $s_n^\alpha\,\alpha$ for each $n \in I_\alpha$ and set $t_n^\alpha = \langle n, \gamma, \alpha\rangle$ for $n \notin I_\alpha$. It follows easily that for each $\beta < \alpha$, there is an n such that $U(\beta, n) \cap U(\alpha, 0)$ is empty.

This completes our construction of the space. It should be clear that the space $X_\kappa = \bigcup\{X_\alpha : \alpha < \kappa\}$ is locally compact, Hausdorff and feebly compact. Furthermore, X_κ is easily seen to be Fréchet, for X_κ is first countable at each $\alpha \in \kappa$, and its open subspace $X_\kappa \setminus \kappa = \bigcup_n T_n$ is also a subspace of the Fréchet space T and hence is Fréchet. To finish the proof, we verify that the one-point compactification $X = X_\kappa \cup \{\infty\}$ of X_κ is Fréchet.

Assume that $Y \subseteq X_\kappa$ does not have compact closure in X_κ. We wish to show that there is a sequence $\{y_n : n \in \omega\} \subseteq Y$ which converges to ∞. Since $\{\alpha : \alpha \in \kappa\}$ is a closed discrete subset of X_κ, we may assume that Y is contained in $\bigcup_n T_n$. Two cases are considered.

Case 1: Suppose Y has only finitely many limit points in κ. Then one may intersect Y with a neighborhood of ∞ which does not have any of those points in its closure, and obtain a subset Y' of Y which still has ∞ in its closure but has no limit points in κ. Any sequence $\{y_n : n \in \omega\} \subseteq Y'$ such that $y_n \in Y' \setminus \bigcup_{k<n} T_k$ will have no cluster point in X_κ and hence will converge to ∞.

Case 2: Suppose there is an infinite countable set $A \subset \kappa$ which is contained in the closure of Y. Since X_κ is Fréchet and A is countable, there is a countable $Y' \subseteq Y$ whose closure contains A. It is easily seen that one can also assume $Y' \cap T_n$ is finite for each n. Since each member of Y is really a function with range contained in κ, we can let B denote the set of all $\beta \in \kappa$ such that β is in the range of some member y of Y'. If γ is any member of $\kappa \setminus B$, we claim that $U(\gamma, 0) \cap Y'$ is empty. Indeed, if $y \in Y' \cap U(\gamma, 0)$, then there is an n such that $y \in [t_n^\gamma]$, from which it follows that $t_n^\gamma \subseteq y$. However, γ is in the range of t_n^γ but not in the range of y, a contradiction. It follows now that the closure of Y' in X_κ is a countable non-compact set and therefore is not feebly compact. Thus its dense subspace Y' contains a sequence having no cluster point in X_κ, and that sequence must converge to ∞. $\square$

Before presenting the last example of this section, a lemma is needed.

Lemma 2.15. *If X is Fréchet and Hausdorff, then the Alexandroff double, $\mathbb{A}(X)$, is Fréchet and Hausdorff, and $\mathbb{A}(X)$ has property (F_3).*

Proof. It suffices to prove that the double of a Fréchet space is Fréchet (which is probably well-known), for by Lemma 2.1 (e) we already know that a Fréchet Hausdorff space with a dense set of isolated points has property (F_3), and it is well-known that the double of a Hausdorff space is Hausdorff. But let us check. Let $\mathbb{A}(X)$ be the usual $X \times \{0,1\}$ with $X \times \{0\}$ open and discrete, and neighborhood base for $(x,1)$ be the usual $(U \times \{0,1\}) \setminus \{(x,0)\}$ for open $U \subseteq X$ containing x. If $(x,1)$ is in the closure of $A \subseteq \mathbb{A}(X)$ then clearly there is a subsequence of A converging to $(x,1)$ if $(x,1)$ is in the closure of $A \cap (X \times \{1\})$ since this subspace is homeomorphic to X. But just as easily we see that if $A \subseteq X \times \{0\}$, then there is some $A' \subseteq X$ such that $A = A' \times \{0\}$ and x is a limit of A'. Any subsequence of A' which converges to x will yield a corresponding subsequence of A which converges to $(x,1)$. $\square$

Theorem 2.16. *There is a compact Hausdorff Fréchet space which is not FCC, but which has a dense set of isolated points and hence has property (F_3).*

Proof. Just take the space X as constructed in 2.14 and apply 2.15 on $\mathbb{A}(X)$. $\square$

3. Subspaces and images.

We consider next whether these properties are or can be inherited by subspaces, or preserved or reflected by continuous maps.

Some inheritance results hold that are analogous to the well-known ones concerning feeble, countable or sequential compactness. These are stated next.

Theorem 3.1. *Let X be a topological space.*

(a) *If X is FCC and A is a feebly compact subspace of X, then A is FCC.*
(b) *If X is FCC (respectively, sequentially feebly compact) and A is an open subset of X, then $\overline{A}$ is FCC (respectively, sequentially feebly compact).*
(c) *If X has a dense, sequentially feebly compact subspace, then X is sequentially feebly compact.*

The Tychonoff plank witnesses that sequential feeble compactness, as well as feeble compactness, is not closed hereditary. If every countable closed subset of a space X is pseudocompact (and the space X is T_1), then every countably infinite subset of X has a limit point (and X is countably compact).

Before stating some mapping theorems, we recall that a mapping $f : X \to Y$ is called *Z-closed* (*closed*) provided that for every zero-set F (closed subset F) of X, $f(F)$ is a closed subset of Y. In case a mapping $f : X \to Y$ satisfies $f(F)$ is a proper closed subset of Y for every proper closed subset F of X, then f is called *irreducible*. A closed mapping $f : X \to Y$ is called *perfect* (*quasi-perfect*) provided that for every point $y \in Y$, the fiber $f^{-1}(y)$ is compact (countably compact).

Some known theorems are these ([5], [20]): (a) if $f \in C(X, Y)$ and X is feebly compact (respectively, sequentially compact, countably compact) then so is $f(X)$; (b) if $f : X \to Y$ is a closed mapping, and Y and each $f^{-1}(y)$, $y \in Y$, are countably compact, then X is countably compact; and (c) if f is a Z-closed open mapping of a Tychonoff space X onto a Tychonoff space Y, and if Y and each $f^{-1}(y)$, $y \in Y$, are pseudocompact, then so is X. The following also hold.

Theorem 3.2. *Let X and Y be topological spaces and $f : X \to Y$ a mapping.*

(a) *If $f \in C(X, Y)$ and X is sequentially feebly compact, then $f(X)$ is sequentially feebly compact.*
(b) *If f is an open and Z-closed mapping of X onto Y, and Y and each fiber $f^{-1}(y)$, $y \in Y$, are pseudocompact, then X is pseudocompact.*
(c) *If f is an irreducible quasi-perfect mapping of X onto Y, and Y is feebly compact, then X is feebly compact.*
(d) *If f is a perfect mapping of X onto Y, and every feebly compact subspace of X is a closed subset of X, then every feebly compact subspace of Y is a closed subset of Y.*
(e) *If X is FCC and $f \in C(X, Y)$ is a perfect mapping of X onto Y, then Y is FCC.*
(f) *If X is feebly compact and semiregular, Y is FCC, and $f \in C(X, Y)$ is a bijection, then f is a homeomorphism.*

Proof. The proof of (a) is immediate. To verify (b), one can use a Urysohn's lemma-type of argument similar to the proof that establishes (b) for Tychonoff spaces.

(c). Suppose that the hypotheses in (c) hold. By Theorem 1.1, it suffices for us to prove that every countable open cover $\mathcal{U}$ of X has a finite subcollection whose union is dense in X. Let $\mathcal{U}$ be a countable open cover of X such that for every finite subcollection $\mathcal{G}$ of $\mathcal{U}$, $\bigcup \mathcal{G} \in \mathcal{U}$. Then, by the countable compactness of each fiber $f^{-1}(y)$, it follows that for each $y \in Y$, there exists $U \in \mathcal{U}$ such that $f^{-1}(y) \subseteq U$. Hence $\{Y \setminus f(X \setminus U) : U \in \mathcal{U}\}$ is a countable open cover of Y. By the feeble compactness of Y, there is a finite subcollection $\{U_i : i = 1, \ldots, n\}$ such that $Y = \bigcup_{i=1}^n \overline{Y \setminus f(X \setminus U_i)}$. Let $W = \bigcup_{i=1}^n f^{-1}(Y \setminus f(X \setminus U_i))$. Since f is closed, $f(\overline{W}) \supseteq \overline{f(W)}$. Because f is onto, $f(W) = \bigcup_{i=1}^n Y \setminus f(X \setminus U_i)$. Thus $f(\overline{W}) = \bigcup_{i=1}^n \overline{Y \setminus f(X \setminus U_i)} = Y$. As f is irreducible, the latter implies $X = \overline{W}$. Since each $f^{-1}(Y \setminus f(X \setminus U_i)) \subseteq U_i$, then $X = \bigcup_{i=1}^n \overline{U_i}$.

(d). Suppose the hypothesis of (d) holds and S is a feebly compact subspace of Y. Then $f|f^{-1}(S) : f^{-1}(S) \to S$ is a perfect surjection (e.g., see [15, 1.8 (f) (2)]). By Zorn's lemma [15, 6.5 (c)], there is a closed subset A of the space $f^{-1}(S)$ such that $f|A : A \to S$ is a perfect irreducible surjection. Thus by (c) above, A is feebly compact. Therefore, A is a closed subset of X, and since f is a closed mapping, then $S = f(A)$ must be a closed subset of Y.

(e). This is an immediate consequence of (d) and the fact that feeble compactness is preserved by continuous mappings.

(f). Suppose the hypothesis holds and F is any closed subset of the space X. By the semiregularity of X, the family $\mathcal{R}$ of regular closed subsets of X which contain F satisfies $F = \bigcap \mathcal{R}$. Each set $R \in \mathcal{R}$ is a feebly compact subspace of X by Corollary 1.2 (b). Thus the continuous image $f(R)$ of each such R under the mapping f is a feebly compact, hence closed, subset of the space Y. As f is one-to-one, $f(F) = \bigcap\{f(R) : R \in \mathcal{R}\}$, and hence $f(F)$ is a closed subset of Y. Therefore, f is a homeomorphism. $\square$

One interesting corollary to 3.2 (f) is the following.

Corollary 3.3. *Let $\mathcal{S}$ and $\mathcal{T}$ be topologies on a set X such that $(X, \mathcal{T})$ is a feebly compact semiregular space, $(X, \mathcal{S})$ is FCC, and $\mathcal{S} \subseteq \mathcal{T}$. Then $\mathcal{T} = \mathcal{S}$.*

The next example illustrates that the condition "irreducible" cannot be removed from the hypothesis of (c) in Theorem 3.2.

Example 3.4. Let Ψ be the Isbell-Mrówka space described in the proof of Theorem 2.12. Let $\mathbb{N}^-$ be the set of the negative integers, with the discrete topology, and let X be the discrete union of Ψ and $\mathbb{N}^-$. List in a 1-1 manner as $\{M_n : n \in \mathbb{N}\}$ the members of an infinite subset of $\mathcal{M}$, and define $f : X \to \Psi$ by the rule: $f(x) = x$ if $x \in \Psi$, and $f(x) = M_{-x}$ if $x \in \mathbb{N}^-$. Then X is not pseudocompact, Ψ is FCC, and $f : X \to \Psi$ is a closed, continuous map of X onto Ψ, each of whose fibers is finite.

Example 3.5. This example shows that if $f : X \to Y$ is a perfect continuous surjection and Y is FCC, then X need not be FCC. Moreover, it shows that if Y is a first countable compact Hausdorff (hence sequentially compact) space, and f is an irreducible, perfect continuous surjection, then X need not be sequentially feebly compact. Let $X = \beta\mathbb{N}$ and $Y = \mathbb{N}_\infty$ be the one-point compactification of $\mathbb{N}$, where $\mathbb{N}$ has the discrete topology. Let f be the Čech mapping in $C(\beta\mathbb{N}, \mathbb{N}_\infty)$ which extends the identity mapping on $\mathbb{N}$. Then Y has the stated properties, and as noted in Example 2.9, $\beta\mathbb{N}$ is not sequentially feebly compact. It follows from 6.11 of [6] that $f(\beta\mathbb{N} \setminus \mathbb{N}) = \{\infty\}$, and hence f is irreducible, as well as perfect.

4. Product spaces.

In [8] and [20] it was shown that a number of the properties considered there are well behaved in the formation of feebly compact product spaces, namely: sequentially compact; feebly compact and first countable; and feebly compact and locally compact. We shall show that the property sequentially feebly compact likewise is well behaved in the formation of feebly compact product spaces. As was done in [8] and [20] for the product theorem proofs presented in those articles, in several proof outlines below there will be no loss of generality for us to assume that the sets in Theorem 1.1 (B_3) are the standard basic open sets for the product topology.

Theorem 4.1. *The property sequentially feebly compact is productive.*

Proof. Let $X = \prod_{a\in A} X_a$ be a product of sequentially feebly compact spaces. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a sequence of nonempty basic open sets of X.

The set $R = \{a \in A : \mathrm{pr}_a(U_n) \neq X_a, \text{for some } n \in \mathbb{N}\}$ of restricted coordinates of the members of $\mathcal{U}$ is countable. Choose the smallest ordinal number λ such that $\lambda = |R|$. For notational simplicity, and with no loss of generality, we may assume that $R = \{\alpha : \alpha < \lambda\}$ and $R \neq \emptyset$. Since each factor space is sequentially feebly compact, we may use mathematical induction and choose points $\{p_\alpha \in X_\alpha : \alpha < \lambda\}$ and infinite subsets $\{I_\alpha \subseteq \mathbb{N} : \alpha < \lambda\}$ such that: whenever $\alpha < \gamma < \lambda$ then $I_\alpha \supseteq I_\gamma$; and for each $\alpha < \lambda$ and neighborhood V of p_α, $V \cap \mathrm{pr}_\alpha(U_n) \neq \emptyset$ for all but finitely many $n \in I_\alpha$. For each $a \in A \setminus R$, choose one point $p_a \in X_a$.

Next, we define an infinite set $I \subseteq \mathbb{N}$ as follows: if λ is finite, set $I = I_{\lambda-1}$; if $\lambda = \omega_0$, then select an infinite subset I of $\mathbb{N}$ such that for each $\alpha < \lambda$, $I \setminus I_\alpha$ is finite. In either case, let $\{n_i : i \in \mathbb{N}\}$ be a strictly increasing mapping of $\mathbb{N}$ onto I.

Then for every neighborhood V of p in X, $V \cap U_{n_i} \neq \emptyset$ for all but finitely many $i \in \mathbb{N}$. Therefore, X is sequentially feebly compact. $\square$

The next theorem extends an analogous theorem obtained by A.H. Stone for first countable spaces—see [20]. First a lemma is given.

Lemma 4.2. *Suppose that X is sequentially feebly compact and Y is feebly compact. Then $X \times Y$ is feebly compact.*

Proof. Let $\mathcal{U} = \{U_n \times W_n : n \in \mathbb{N}\}$ be a sequence of nonempty open sets in $X \times Y$. Since X is sequentially feebly compact, there exist $p \in X$ and a strictly increasing sequence $\{n_i : i \in \mathbb{N}\}$ in $\mathbb{N}$ such that for every neighborhood V of p, $V \cap U_{n_i} \neq \emptyset$ for all but finitely many $i \in \mathbb{N}$. Since Y is feebly compact, the sequence $\{W_{n_i} : i \in \mathbb{N}\}$ has a cluster point $q \in Y$. Then (p, q) is a cluster point of $\{U_{n_i} \times W_{n_i} : i \in \mathbb{N}\}$ and hence also of $\mathcal{U}$. $\square$

Theorem 4.3. *Every product of feebly compact spaces, all but one of which are sequentially feebly compact, is feebly compact.*

Since every FCC space is sequentially feebly compact, and every pseudocompact completely regular space is feebly compact, there are applications of the preceding product results to FCC spaces and pseudocompact completely regular spaces.

Because each factor of a product space is a continuous image of the product space and is homeomorphic to a subspace of the product space, the next result follows from Theorem 3.2 (a).

Theorem 4.4. *If a product space X is FCC (respectively, sequentially feebly compact), then so is every factor space of X.*

We consider next what can be said about products of FCC spaces. One obvious consequence of Theorem 2.2 (a) is the following.

Theorem 4.5. *Let $X = \prod_{a\in A} X_a$ be a product of FCC spaces.*

(a) *If A is countable and for each $a \in A$, X_a is an E_1-space, then X is FCC.*
(b) *If A is finite, X is Fréchet, and for each $a \in A$, X_a is Hausdorff and scattered, then X is FCC.*

A simple example, however, shows that in general $X = \prod_{a\in A} X_a$ is never FCC if $|A| \geq \aleph_1$.

Example 4.6. Let A be any set with $|A| \geq \aleph_1$, $D = \{0, 1\}$ have the discrete topology, and X be the product space D^A. Let C be the Corson Σ-subspace of X based at 0, i.e., let $C = \{x \in X : x_a = 0 \text{ for all but countably many } a \in A\}$. Then X is a product of first countable compact Hausdorff spaces. But it is straightforward to show (and known) that C is a countably compact, proper dense subspace of X. Therefore, X is not FCC.

Corollary 4.7. *Let $X = \prod_{a\in A} X_a$, where $|A| \geq \aleph_1$, and for each $a \in A$, $|X_a| \geq 2$. Then X is not FCC.*

Example 4.8. Let A, D and X be as in Example 4.6, but require that $|A| \geq 2^{\aleph_0}$. Then the product space X is sequentially feebly compact by Theorem 4.1. As noted above, it is not FCC. In [20] a proof was given that X fails to be sequentially compact.

While Example 4.6 does not answer the question as to whether or not the property FCC is countably productive, one can use theorems of V.I. Malykhin and a theorem of P. Simon (see [21]) to do so. It is shown below that there exist two FCC spaces whose product is not FCC, and furthermore each of those spaces can be chosen to be compact, Fréchet, scattered and Hausdorff. In order to develop this answer, some terminology and notation to be used are given next.

Let $\mathcal{P}$ be an AD family on $\mathbb{N}$. As in [21], we shall define $\mathcal{F}(\mathcal{P})$ to be the space previously introduced by S.P. Franklin, the set $\mathbb{N}\cup\mathcal{P}\cup\{\infty\}$, where $\infty \notin \mathbb{N}\cup\mathcal{P}$, topologized as follows: each $n \in \mathbb{N}$ is isolated; a basic open neighborhood of a point $P \in \mathcal{P}$ is $\{P\}\cup C$, where C is any cofinite subset of P; and $\mathcal{F}(\mathcal{P})$ is the one-point compactification of $\mathbb{N}\cup\mathcal{P}$. The family $\mathcal{P}$ will be called *nowhere infinitely MAD* provided that for every infinite subset X of $\mathbb{N}$, the set $\{X \cap P : X \cap P \text{ is infinite and } P \in \mathcal{P}\}$ fails to be an infinite MAD family on X. Note that a MAD family (say on $\mathbb{N}$) is always a MAD family when restricted to a cofinite subset of $\mathbb{N}$.

In [21] Simon attributed to Malykhin two results which can be stated as follows. The space $\mathcal{F}(\mathcal{P})$ is Fréchet iff $\mathcal{P}$ is nowhere infinitely MAD. If $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is AD on

$\mathbb{N}$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$, then the product space $\mathcal{F}(\mathcal{P}_1) \times \mathcal{F}(\mathcal{P}_2)$ is not Fréchet iff $\mathcal{P}$ is not nowhere infinitely MAD. Then Simon proved that there exists an infinite MAD family $\mathcal{P}$ on $\mathbb{N}$ having a partition $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$ such that for each i, $i = 0, 1$, $\mathcal{P}_i$ is nowhere infinitely MAD. We shall call such a partitioned family a *Simon MAD* family. Simon used his theorem to prove that the product of two Fréchet compact Hausdorff spaces need not be Fréchet. By Tanaka's theorem (Theorem 2.2 (c) above), such a product would then not be FCC. Moreover, one can prove the following.

Theorem 4.9. *Let $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$ be a Simon MAD family. Then the compact Hausdorff Fréchet spaces $\mathcal{F}(\mathcal{P}_i)$, $i = 0, 1$, are scattered and hence are FCC spaces, but their product $\mathcal{F}(\mathcal{P}_0) \times \mathcal{F}(\mathcal{P}_1)$ does not have property* (F_3).

Proof. Obviously any Franklin space is scattered, so each $\mathcal{F}(\mathcal{P}_i)$ is FCC by Theorem 2.2 (a) (iii). The proof that $\mathcal{F}(\mathcal{P}_0) \times \mathcal{F}(\mathcal{P}_1)$ does not have property (F_3) is similar to the proof that this space is not Fréchet. Let $D = \{(n, n) : n \in \mathbb{N}\}$. We show that (i) the point $p = (\infty_0, \infty_1)$ is in $\overline{D}$ and (ii) the set $F = \overline{D} \setminus \{p\}$ is feebly compact.

(i). Let $U_0 \times U_1$ be a basic open neighborhood of p. For each i, $i = 0, 1$, $\mathcal{F}(\mathcal{P}_i) \setminus U_i$ is compact, and so there exist a finite subset $\mathcal{N}_i$ of the closed discrete subset $\mathcal{P}_i$ of the space $\mathcal{F}(\mathcal{P}_i) \setminus \{\infty_i\}$ and finite subsets C_i and N_i of $\mathbb{N}$ such that $\mathcal{F}(\mathcal{P}_i) \setminus U_i \subseteq \mathcal{N}_i \cup ((\bigcup \mathcal{N}_i) \setminus N_i) \cup C_i$. Choose any set $P \in \mathcal{P} \setminus (\mathcal{N}_0 \cup \mathcal{N}_1)$, which we may do since $\mathcal{P}$ is infinite. Because $\mathcal{P}$ is AD, some cofinite subset C of P satisfies $C \cap (\mathcal{N}_i \cup ((\bigcup \mathcal{N}_i) \setminus N_i) \cup C_i) = \emptyset$ for $i = 0, 1$. Thus $C \subseteq U_0 \cap U_1$, and so for any $n \in C$, $(n, n) \in U_0 \times U_1$. Therefore $D \cap (U_0 \times U_1) \neq \emptyset$.

(ii). Since each point of D is isolated and D is dense in F, it suffices to prove that every infinite subset of D has a limit point in $(\mathcal{F}(\mathcal{P}_0) \times \mathcal{F}(\mathcal{P}_1)) \setminus \{p\}$. Let I be an infinite subset of D. Define $X = \{n \in \mathbb{N} : (n, n) \in I\}$. Since $\mathcal{P}$ is MAD, there exists $P \in \mathcal{P}$ such that $X \cap P$ is infinite. Then every neighborhood of the point P contains all but finitely many integers in $X \cap P$. Either $P \in \mathcal{P}_0$ or $P \in \mathcal{P}_1$. Suppose $P \in \mathcal{P}_0$. Since $\mathcal{F}(\mathcal{P}_1)$ is compact, the infinite set $X \cap P$ has a limit point $y \in \mathcal{F}(\mathcal{P}_1)$. Thus the point (P, y) is a limit point of I. (In fact, since $P \notin \mathcal{P}_1$, one can show that $y = \infty_1$, and any 1-1 listing of the members of $\{(n, n) \in I : n \in X \cap P\}$ defines a sequence in D which converges to (P, ∞_1).) $\square$

Corollary 4.10. *There exists a compact Hausdorff, scattered, Fréchet, and hence FCC, space X whose product with itself, X^2, does not have property* (F_3).

Proof. Let X be the discrete union of the spaces $\mathcal{F}(\mathcal{P}_0)$ and $\mathcal{F}(\mathcal{P}_1)$ in Theorem 4.9.

5. Extension spaces.

A space E is called an *extension space* of a space X if X is a dense subspace of E. We wish to examine necessary and sufficient conditions that a space X have an FCC or sequentially feebly compact extension space E, where E may be required to have other properties, such as complete regularity. Some embedding theorems will be given, and it will be shown that there exist Moore spaces, neither of which has a regular FCC extension space, and one of which is separable and has no regular sequentially feebly compact T_1-extension space. A feebly compact extension space E of a space X is sometimes called a *feeble compactification of X*.

The following links some of these concepts and maximal and minimal $\mathcal{P}$-spaces (for various properties $\mathcal{P}$) and shows that: as far as the underlying sets are concerned, an FCC extension space of a space X is a minimal feeble compactification of X; and from the point of view of topological properties, a semiregular FCC extension space E of a space X is minimal with respect to being an FCC extension space of X and is maximal with respect to being a semiregular feeble compactification of X.

Theorem 5.1. *Let $(X,\mathcal{U})$ be a space, and suppose $(E,\mathcal{S})$ and $(G,\mathcal{T})$ are feeble compactifications of $(X,\mathcal{U})$ such that $(E,\mathcal{S})$ is FCC, where $X \subseteq G \subseteq E$ and $\mathcal{S}|G \subseteq \mathcal{T}$. Then the following hold.*

(a) $G = E$.
(b) *If $(G,\mathcal{T})$ is semiregular then $\mathcal{T} = \mathcal{S}$.*

Proof. For statement (a), note that because $(G,\mathcal{S}|G)$ is a continuous image of $(G,\mathcal{T})$, the space $(G,\mathcal{S}|G)$ is feebly compact, and hence G is a closed subset of $(E,\mathcal{S})$. But $X \subseteq G$, and so G is a dense subset of $(E,\mathcal{S})$. Thus $G = E$. Statement (b) follows from (a) and Corollary 3.3.

Before stating the next result, some notation and terminology is needed. Recall that a filter base $\mathcal{F}$ on a space is called *free* iff the adherence of $\mathcal{F}$ is empty. Given a space X and a family $\mathcal{M}$ of free open filter bases on X, we shall denote by $X_{\mathcal{M}}$ the set $X \cup \mathcal{M}$, topologized as follows: a set $G \subseteq X_{\mathcal{M}}$ is defined to be open iff (i) $G \cap X$ is open in X, and (ii) if $\mathcal{F} \in G \cap \mathcal{M}$ then $G \cap X$ contains some member of $\mathcal{F}$. For a space Y, we shall denote by sY the *semiregularization* of Y, the semiregular space whose points are the same as those of Y, and whose topology has as a base the regular open subsets of Y (for a derivation of properties of sY, see [15]).

Theorem 5.2. *Let X be a topological space and $\mathcal{M}$ a maximal family of countable, free, open filter bases on X such that whenever $\mathcal{F}$ and $\mathcal{G}$ are distinct members of $\mathcal{M}$ then there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\mathcal{F} \cap \mathcal{G} = \emptyset$. Then the following hold.*

(a) *X is a dense open subspace of $X_{\mathcal{M}}$, $X_{\mathcal{M}}$ is a feeble compactification of X, and each point of $X_{\mathcal{M}} \setminus X$ is of countable character in $X_{\mathcal{M}}$ and is an E_1-point of $X_{\mathcal{M}}$, i.e., is an intersection of countably many of its closed neighborhoods.*
(b) *If $X \subseteq G \subseteq X_{\mathcal{M}}$ is feebly compact then $G = X_{\mathcal{M}}$.*
(c) *The space X is sequential (Fréchet, first countable, Hausdorff) iff $X_{\mathcal{M}}$ has the same property, and in case X is sequential (first countable and Hausdorff), then $X_{\mathcal{M}}$ is sequentially feebly compact (first countable and Hausdorff, and hence FCC).*
(d) *X is semiregular and first countable iff $sX_{\mathcal{M}}$ is a semiregular first countable extension space of X, and in this case, $sX_{\mathcal{M}}$ is FCC if X is Hausdorff.*
(e) *If X is scattered (has a dense set of isolated points), Fréchet and Hausdorff, then $X_{\mathcal{M}}$ is FCC (has property (F_3)).*

The proof of 5.2 is straightforward, and some of its statements are similar to known results: (a), (b) and (d) are extensions of results in [23]; the last statement in (c) follows from 2.2 (d) and the preceding statements in (c), each of which is a special case of a known result.

If one seeks conditions on a space X that it have an FCC extension space having other desirable properties, such as regular, completely regular, zero-dimensional or Moore, some results obtained previously that have applications to these questions are the next four theorems.

Theorem 5.3.

(a) ([24]) *Every locally feebly compact, first countable zero-dimensional T_1-space has a feebly compact, first countable zero-dimensional T_1-extension space.*

(b) ([18]) *Every locally compact Moore space which is zero-dimensional at each point of a countable dense subset has a locally compact, feebly compact Moore extension space.*

Theorem 5.4. ([27])

(a) *Every locally pseudocompact (locally compact), first countable Tychonoff space X has a pseudocompact, first countable Tychonoff extension space E such that E is locally compact if X is.*

(b) *Every metrizable space has a pseudocompact first countable Tychonoff extension space.*

Theorem 5.5. ([13]) *Let X be a separable, locally pseudocompact Tychonoff (locally compact) Moore space. Then X can be embedded densely in a pseudocompact Tychonoff (locally compact) Moore space.*

Theorem 5.6. ([22])

(a) *Every locally feebly compact regular T_1-space X can be embedded as an open dense subspace in a feebly compact regular T_1-space Y which is first countable at every point of $Y \setminus X$.*

(b) *Every separable, locally feebly compact (locally pseudocompact, Tychonoff) Moore space embeds as an open dense set in a feebly compact (pseudocompact Tychonoff) Moore space.*

Theorems 5.5 and 5.6 answered questions raised in [17], [18] and [24] (While made available to others, Theorem 5.5 and its proof have not been published yet by P. Nyikos.) Simon and Tironi's very nice Theorem 5.6 (a) implies that every first countable, locally feebly compact, regular T_1-space has a feebly compact, first countable, regular T_1, and hence FCC, extension space.

Remark 5.7. For converse properties, we make some observations. If a space X has an FCC extension space E, then X must have property (F_1), and if X is locally feebly compact then it is an open subset of E. If X is an open subspace of some feebly compact regular space (some space having property (F_3)), then X is locally feebly compact (has property (F_3)). As noted in [25] (see also [19]), every dense subspace of a feebly compact Moore space is separable and has a dense metrizable subspace.

An example due to T. Terada and J. Terasawa (which was a modification of an example due to E. van Douwen and T.C. Przymusiński) was given in [27] to prove that there is a first countable zero-dimensional Čech-complete T_1-space which has no first countable, feebly compact, regular T_1-extension space. In [25] the Terada-Terasawa example was modified and used to obtain the following result: There is a first countable, feebly compact zero-dimensional T_1-space which has no Urysohn, feebly compact, sequential extension space. (Recall that a space X is said to be *Urysohn* provided that every pair of distinct points of X can be separated by disjoint closed neighborhoods of those points.) We show next that this example can be used to establish the following.

Theorem 5.8. *There exists a zero-dimensional separable Moore space Y such that Y has no Urysohn, sequentially feebly compact extension space, and hence Y has no Urysohn, FCC extension space.*

Proof. We refer the reader to the proof on page 24 of [25]. Let Y be the space described there, and assume X is any sequentially feebly compact extension space of Y. One can replace the fourth–sixth sentences of the last paragraph on that page by the sentence: "Since X is sequentially feebly compact, the sequence $\mathcal{U} = \{\{(n,i)\} : i \in I_n\}$ has associated with it an infinite subset J_n of I_n and a cluster point x_n of $\mathcal{U}$ such that every neighborhood V of x_n contains all but finitely many of the sets in $\{\{(n,i)\} : i \in J_n\}$." Then one can use the rest of that proof to show that X cannot be a Urysohn space. □

In Theorem 4.1 of [2] Murray Bell showed that if $\mathbb{C}$ is the Cantor set, $F[\mathbb{C}]$ is the set of all finite subsets of $\mathbb{C}$, and $\mathcal{T}$ is the Pixley-Roy topology on $F[\mathbb{C}]$, then $(F[\mathbb{C}],\mathcal{T})$ has no first countable pseudocompact Tychonoff extension space. We show next how to modify his proof and strengthen his theorem. Due to the complexity of his proof and for the convenience of the reader, we give a self-contained extension of it, rather than just a fragmentary presentation of the changes needed. First let us recall that if $\mathcal{U}$ is the algebra of clopen subsets of $\mathbb{C}$, and for each $G \in F[\mathbb{C}]$ and $U \in \mathcal{U}$ with $G \subseteq U$ one defines $[G,U] = \{H \in F[\mathbb{C}] : G \subseteq H \subseteq U\}$, then $\mathcal{T}$ has as a base the family $\mathcal{C} = \{[G,U] : G \in F[\mathbb{C}],\ G \subseteq U \text{ and } U \in \mathcal{U}\}$, and each member of $\mathcal{C}$ is a clopen subset of $(F[\mathbb{C}],\mathcal{T})$. It is known ([4]) that $(F[\mathbb{C}],\mathcal{T})$ is a zero-dimensional, ccc, non-separable Moore space.

Theorem 5.9. *The Pixley-Roy space $(F[\mathbb{C}],\mathcal{T})$ has no regular T_1 feeble compactification that has property* (F_3)*, and hence $(F[\mathbb{C}],\mathcal{T})$ has no extension space that is FCC and regular.*

Proof. Assume that $(F[\mathbb{C}],\mathcal{T})$ is a dense subspace of an FCC, regular space X. We will construct by induction, a decreasing sequence of Cantor sets $\{K^n : n \in \mathbb{N}\}$ and a decreasing sequence of clopen sets $\{B^n : n \in \mathbb{N}\}$ with these properties for each $n \in \mathbb{N}$:

(1) $B^n \supseteq K^n$ and $\mathrm{diam}(B^n) < \dfrac{1}{n}$; and

(2) there are a point $s^n \in \bigcap\{\mathrm{cl}_X[\{p\},B^n] : p \in K^n\}$ and a sequence $\{G^n_k : k \in \mathbb{N}\}$ of points of $F[\mathbb{C}]$ converging to s^n with the property that $K^n \bigcap(\bigcup\{G^n_k : k \in \mathbb{N}\}) = \emptyset$.

Assuming the existence of $\{K^n, B^n, s^n, G^n_k : n,k \in \mathbb{N}\}$ with properties (1) and (2), a contradiction follows quickly. Let $p \in \bigcap\{K^n : n \in \mathbb{N}\}$ and $R = \mathrm{int}_X\mathrm{cl}_X([\{p\},B^1])$. Note that $R \cap F[\mathbb{C}] = [\{p\}, B^1]$ and $\{p\} \in R$. Since $\{\mathrm{cl}_X[\{p\}, B^n] : n \in \mathbb{N}\}$ is a neighborhood base for $\{p\}$ in X, there is some $n \in \mathbb{N}$ such that $\mathrm{cl}_X[\{p\}, B^n] \subseteq R$. By (2), $s^n \in R$ and there is some $k \in \mathbb{N}$ such that $G_k^n \in R$. But $p \in G_k^n$ since $G_k^n \in R \cap F[\mathbb{C}] = [\{p\}, B^1]$, a contradiction as $K^n \cap G_k^n = \emptyset$ by (2).

The inductive step goes as follows (the first step is similar). Choose a Cantor set $K \subseteq K^n$ and a clopen B^{n+1} with $K \subseteq B^{n+1}$ and the diameter of B^{n+1} less than $1/(n+1)$. Choose a sequence $\langle F_k : k < \omega \rangle$ of finite subsets of K that strictly increase up to a dense subset of K. Define $V_k = [F_k, B^{n+1}] \setminus [F_{k+1}, B^{n+1}]$ for each $k < \omega$. Since $\{V_k : k < \omega\}$ is a pairwise disjoint family of nonempty open subsets of the dense subset $F[\mathbb{C}]$ of the T_3-space X, there exists a sequence $\mathcal{U} = \{U_k : k < \omega\}$ of nonempty open subsets of X whose closures in X are pairwise disjoint, and which satisfy $U_k \cap F[\mathbb{C}] \subseteq V_k$ for each $k < \omega$. It follows from our Corollary 2.3 that there exist a point s^{n+1} of X, an infinite subset J of ω, and a sequence $\mathcal{P} = \{P_k : k \in J\}$ of nonempty open subsets of $F[\mathbb{C}]$ such that $P_k \subseteq U_k$ for each $k \in J$, and every neighborhood of s^{n+1} contains all but finitely many sets in $\mathcal{P}$. For each $k \in J$, choose $[G_k^{n+1}, W_k] \subseteq P_k$. Since the F_k's increase to a dense subset of K and each F_k is contained in the clopen set W_k, there exists an infinite $A \subseteq J$ such that $K \cap \bigcap_{k \in A} W_k$ contains a Cantor set K'. Since $\bigcup_{k \in A} G_k^{n+1}$ is countable there exists a Cantor set $K^{n+1} \subseteq K'$ with $K^{n+1} \cap \bigcup_{k \in A} G_k^{n+1} = \emptyset$. The sequence $\{G_k^{n+1} : k \in A\}$ converges to s^{n+1}. If O is any neighborhood of s^{n+1} then for some $k \in A$, $P_k \subseteq O$. Consider any $p \in K^{n+1}$. One has $p \in W_k$ and $G_k^{n+1} \subseteq B^{n+1}$. Hence $[\{p\}, B^{n+1}] \cap [G_k^{n+1}, W_k] \neq \emptyset$ and so $[\{p\}, B^{n+1}] \cap O \neq \emptyset$. Thus $s^{n+1} \in \mathrm{cl}_X([\{p\}, B^{n+1}])$. $\square$

We conclude by asking a question similar to one raised several years ago by M.V. Matveev. Let us call a feeble compactification E of a space X a *minimal feeble compactification of* X provided that for every point $p \in E \setminus X$, the space $E \setminus \{p\}$ fails to be feebly compact. In each of Theorems 5.2–5.7, the feeble compactifications obtained for the given space are minimal feeble compactifications.

Question. If $\mathcal{P}$ denotes one of the properties Urysohn, regular T_1, or Tychonoff, does every $\mathcal{P}$-space have a $\mathcal{P}$, minimal feeble compactification?

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