A PROBLEM BY E. LANDIS AND GENERIC BEHAVIOR OF NON-GENERIC SETS

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Abstract. We consider question motivated by an old problem of E. Landis in the study of the complete set of zeros of a C^∞ functions and all its derivatives.

1. Introduction

In the mid 1960's at the Moscow State University (MSU), Dr. E. Landis, one of the leading specialists in the qualitative theory of partial differential equations, published an elegant problem in a popular journal *Mathematical Education*.

(L) Landis: Let $f(x) \in C^\infty(\mathbb{R})$ and assume that for each $x \in \mathbb{R}$ there is some non-negative integer n such that $f^{(n)}(x) = 0$. Prove that $f(x)$ is a polynomial.

A brief solution to this problem appeared in the next volume of the same journal. The *Mathematical Education* journal was the Soviet analogue of College Mathematics Journal of the MAA, but due to technical difficulties it existed less than a year and it is now effectively inaccessible even within Russia. Fortunately after the publication of the problem (L) it has become part of the research mathematical folklore of the MSU. Our information about the problem (L) is based completely on the recollections of the fourth author but without benefit of the original publication.

Recently, *Applicable Analysis* dedicated a special volume to the memory of E. Landis (Vol 71, Numbers 1-4, 1999, Landis Special Issue) but this volume also does not mention the problem.

The goal of this paper is the discussion of (L) in a broader setting.

Definition 1. If $f(x) \in C^\infty(\mathbb{R})$ then a point x is a *generic* point for f if for each non-negative integer n, $f^{(n)}(x) \neq 0$. The set of generic points of f will be denoted as $G(f)$.

Definition 2. A point $x \in \mathbb{R}$ is a *non-generic* point for f if there is a non-negative integer $n = n(x)$, such that $f^{(n)}(x) = 0$. The set of non-generic points for f is denoted $NG(f)$.

It is obvious that $NG(f) = \mathbb{R} \setminus G(f)$ and that $G(f)$ belongs to the Baire class G_δ. Similarly, $NG(f)$ is an F_σ. Problem (L) can be formulated as: if $G(f)$ is empty, then f is a polynomial.

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Besides problem (L), we will also discuss some properties of the set $G(f)$. We will show that it cannot be an arbitrary G_δ set. In particular, it will be seen that if $G(f)$ is not empty then it has the full cardinality $\mathfrak{c} = |\mathbb{R}|$.

However, the main focus of the present paper is on the structure of the "typical" $G(f)$, where "typical" will be understood either in the sense of Baire category (in certain complete metric spaces of C^∞ functions) or in the sense of measure (probability).

2. Analytic Examples

This section contains a "herbarium" of particular (analytic) functions f and their corresponding non-generic sets $NG(f)$. We will use $\mathrm{sh}\,(x)$ and $\mathrm{ch}\,(x)$ to denote the standard hyperbolic trig functions $\mathrm{sh}\,(x) = \dfrac{e^x - e^{-x}}{2}$, $\mathrm{ch}\,(x) = \dfrac{e^x + e^{-x}}{2}$, and $\mathrm{th}\,(x) = \dfrac{\mathrm{sh}\,(x)}{\mathrm{ch}\,(x)}.\mathrm{th}\,(x) = \frac{\mathrm{sh}\,(x)}{\mathrm{ch}\,(x)}$.

Example 1. $f(x) = P_n(x) = a_n x^n + \cdots + a_1 x + a_0$, yields $NG(f) = \mathbb{R}$.

Example 2. $f(x) = e^x$, yields $NG(f) = \emptyset$.

Example 3. $f(x) = \mathrm{ch}\,(x)$ yields $NG(f) = \{0\}$.

Example 4. $f(x) = \cos x + \varepsilon \mathrm{ch}\,(x)$ results in a finite set for $NG(f)$ of arbitrarily large finite cardinality, which is determined by a suitably small value for $\varepsilon > 0$.

Example 5. $f(x) = \sin x$ of course has an infinite discrete set for $NG(f)$. In this case $NG(f)$ has no limit points.

Example 6. $f(x) = x \sin x + \varepsilon \mathrm{ch}\, x \quad (\varepsilon > 0)$ produces an example with $NG(f)$ an infinite set with infinitely many accumulation points (specifically, points $n\pi/2$, $n \in \mathbb{Z}$); these limit points are not included in $NG(f)$.

Example 7. $f(x) = x \sin x$ gives an infinite set for $NG(f)$ with an infinite discrete set of accumulation points; in this example, some limit points are included in $NG(f)$.

To see this, observe that $f^{(1)}(x) = \sin x + x \cos x$, $f^{(2)}(x) = 2 \cos x - x \sin x$, $f^{(3)}(x) = -3 \sin x - x \cos x$, $f^{(4)}(x) = -4 \cos x + x \sin x$, etc. Thus if $\sin x = 0$ but $x \neq 0$, then $x \cos x \neq 0$, and for each large enough odd n there will be a value of x_n near x (but different from x), such that $\pm n \sin x_n + x_n \cos x_n$ is equal to 0. It follows that each zero of $\sin x$ (except for 0) is a limit of points in $NG(f)$. Moreover, these points belong to $NG(f)$. Similarly, the zeros of $\cos x$ are limit points of $NG(f)$, but they are not members of $NG(f)$.

Example 8. Let $\varepsilon_k : k = 0, 1, \ldots$ be chosen so that $0 < \varepsilon_k < 1$ for each k. Assume also that ε_k monotonically converges to 0. Set $f(x) = \mathrm{sh}\, x - \displaystyle\sum_{k=0}^{\infty} \varepsilon_k \frac{x^{2k}}{(2k)!}$. Let us show that for each $n \geq 0$, $f^{(2n)}(x) = 0$ has a unique positive solution x_{2n} and it satisfies $\lim_{n\to\infty}(x_{2n}/\varepsilon_{2n}) = 1$, while $f^{(2n+1)}(x)$ has no roots. Therefore, we can arrange that $NG(f)$ will be a sequence $\{x_0, x_2, \ldots, x_{2n}, \ldots\}$ and the sequence can decay as fast (or as slow) as we wish.

Since $f^{(2n)}$ and $f^{(2n+1)}$ have the same form as f and f' respectively, it is sufficient to study solutions of the two equations $f(x) = 0$ and $f'(x) = 0$.

First we note that

$$f'(x) = \operatorname{ch} x - \sum_{k=1}^{\infty} \varepsilon_k \frac{x^{2k-1}}{(2k-1)!} \geq \operatorname{ch} x - |\operatorname{sh} x| = e^{-|x|} > 0\,.$$

Secondly, we have

$$\operatorname{sh} x - \varepsilon_0 \operatorname{ch} x \leq f(x) \leq \operatorname{sh} x - \varepsilon_0.$$

Here all the three functions strictly increase; in addition, each of the functions $\operatorname{sh} x - \varepsilon_0 \operatorname{ch} x$ and $\operatorname{sh} x - \varepsilon_0$ has a (unique) zero. Hence $f(x)$ has a unique zero ($x(\varepsilon_0)$, say) located between them:

$$\operatorname{th}^{-1}\varepsilon_0 \geq x(\varepsilon_0) \geq \operatorname{sh}^{-1}\varepsilon_0.$$

It also follows that $x(\varepsilon_0)/\varepsilon_0 \to 1$ as $\varepsilon_0 \to 0+$.

Now we look at examples where the set $NG(f)$ is a dense countable subset of $\mathbb{R}$.

Example 9. Let $f(x) = \dfrac{1}{1+x^2}$ and note that $f(x) = \dfrac{1}{2i}\left(\dfrac{1}{x-i} - \dfrac{1}{x+i}\right)$.

It follows that

$$f^{(n)}(x) = \frac{1}{2i}(-1)^n \left[\frac{n!}{(x-i)^n} - \frac{n!}{(x+i)^n}\right].$$

Thus $f^{(n)}(x) = 0$ implies that $\left(\dfrac{x+i}{x-i}\right)^n = 1$, or $\dfrac{x+i}{x-i} = e^{2\pi i k/n}$ for some $k \in \{1, 2, \ldots, n-1\}$. If $x_{k,n}$ is the solution to $\dfrac{x+i}{x-i} = e^{2\pi i k/n}$, then a routine calculation shows that $x_{k,n}$ is equal to $\cot(k\pi/n)$.

This shows that the set $NG(f)$ is equal to the set $\{\cot(k\pi/n) : 1 \leq k \leq n-1,\ n \geq 2\}$; hence it is dense in $\mathbb{R}$.

Example 10. Let $f(x) = e^{-x^2}$. It is well known that $f^{(n)}(x)$ is equal to $e^{-x^2} H_n(x)$ where $H_n(x),\ n = 1, 2, \ldots,$ are the Hermite polynomials. The distribution of their zeros is very well studied. It is known, in particular, that for each fixed interval Δ and $n \to \infty$, the roots $\{x_{n,i} : i = 1, 2, \ldots, n\}$ are distributed on Δ asymptotically uniformly with a step $O(n^{-1/2})$. For more details on the distribution of the roots (their density after rescaling, etc.) the reader can consult Szego's book [10].

3. Non-analytic examples

In the examples above, the functions $f(x)$ were analytic in a complex region around their domain and therefore these functions could not have an uncountable non-generic set. To see this, note that if $NG(f)$ is uncountable, then there would be some n such that $f^{(n)}(x)$ will be equal to 0 on an uncountable set. However, every uncountable subset of $\mathbb{R}$ will have a limit point, while the set of zeros of an analytic function cannot have a limit point in its domain [7, 10.18]. In this section we examine more complex constructions of C^∞ functions with a view towards the density of $NG(f)$ and the countability of $NG(f)$.

Our fundamental example is based on Weierstrass' construction of a nowhere differentiable continuous function.

The significance of this example will become clear later when we study generic properties of $NG(f)$ in function spaces.

Example 11. Consider the following Fourier series:

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \sin n_k x$$

with

$$\alpha_k = \frac{1}{n_k^{k-\frac{1}{2}}} .$$

Then

$$f^{(k)}(x) = n_1^{k-\frac{1}{2}} \sin\left(n_1 x + \frac{k\pi}{2}\right) + n_2^{k-\frac{3}{2}} \sin\left(n_2 x + \frac{k\pi}{2}\right) + \dots \tag{3.1}$$
$$+ n_{k-1}^{\frac{3}{2}} \sin\left(n_{k-1} x + \frac{k\pi}{2}\right) + n_k^{\frac{1}{2}} \sin\left(n_k x + \frac{k\pi}{2}\right) + n_{k+1}^{-\frac{1}{2}} \sin\left(n_{k+1} x + \frac{k\pi}{2}\right) + \dots$$

Set $n_k = 2^{4^k}$ for each $k \geq 1$, so that the series is *super lacunar*. It is easy to see that for this selection of parameters, the term $n_k^{\frac{1}{2}} \sin(n_k x + \frac{k\pi}{2})$ dominates all other terms of (3.1).

More precisely, for an appropriate $\delta > 0$ (namely, $\delta = 1/8$),

$$f^{(k)}(x) = 2^{\frac{1}{2}4^k} \left(\sin\left(n_k x + \frac{k\pi}{2}\right) + O\left(2^{-\delta 4^k}\right)\right). \tag{3.2}$$

The estimate (3.2) implies that $NG(f)$ is a dense set: for each k, the distance from any point x to the nearest root of the equation $f^{(k)}(x) = 0$ is $O(n_k^{-1})$.

It is fairly difficult to determine if the set $NG(f)$ is countable. Using a simple randomization of the parameters (see Proposition 4 below), we can more easily get definite answers. At this point, we do not know if for some parameters, $NG(f)$ will be uncountable (while f is C^∞).

We will use Sard's theorem. This is a result about the measure of the set of critical values of a differentiable function f that maps a manifold to a manifold. We will use it in the simplest case of $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}$. A point $y \in \mathbb{R}$ is a *critical value* of f if there is a point $x \in U$ such that $f'(x) = 0$ and $f(x) = y$. Sard's theorem, in this case, simply states that *for a continuously differentiable function f, the set of critical values has measure zero* (see [9] for more information).

Lemma 3. *Let f, $g \in C^\infty(\mathbb{R})$. Then*
(a) for almost all real numbers c, the function $(f - cg)(x)$ has only simple zeros in the open set $U = \{x \in \mathbb{R} : g(x) \neq 0\}$;
(b) if a C^∞ function has only simple zeros, there are at most countably many;
(c) if g has only simple zeros, then so does $f - cg$ for almost all $c \in \mathbb{R}$;
(d) if g has at most countably many zeros, then so does $(f - cg)(x)$ for almost all $c \in \mathbb{R}$.

Proof. (a) Both f and g are in $C^\infty(U)$, as is their ratio f/g. Suppose a root $a \in U$ of $f - cg$ is not simple. Then $f(a) = cg(a)$ and $f'(a) = cg'(a)$; this implies that

$(f/g)(a) = c$ and $(f/g)'(a) = 0$, so that c is a critical value of f/g. By Sard's theorem, such values of c form a zero measure set.

(b) Obvious.

(c) By (a), for almost all c the function $f - cg$ can have a multiple zero only in the set $\mathbb{R} \setminus U := \{x \in \mathbb{R} : g(x) = 0\}$. But for any $x \in \mathbb{R} \setminus U$ there exists at most one value of c for which $f - cg$ can have a multiple root at x – this immediately follows from the fact that $g'(x) \neq 0$ by assumption. Since g has at most countably many zeros, the statement is proven.

(d) Follows from (a) and (b). □

We can apply Lemma 3 to the above function f.

Proposition 4. *Define a function*

$$\psi_h(x) = h \sin(n_1 x) + \sum_{k>1} \alpha_k \sin(n_k x),$$

where n_k ($k \geq 1$) and α_k ($k \geq 2$) are defined as above. Then for almost all h, $NG(\psi_h)$ is countable and dense, and all zeros of all the derivatives $\psi_h^{(k)}$ are simple.

Proof. Let $f(x)$ denote the function $\sum_{k>1} \alpha_k \sin(n_k x)$ and let $g(x)$ be $\sin(n_1 x)$. By Lemma 3(c), for almost all $h \in \mathbb{R}$, $f + hg$ has only simple zeros. Similarly, for almost all h and each k, $(f^{(k)} + hg^{(k)})(x)$ has only simple zeros. (We use the fact that g and all its derivatives have only simple zeros.) In particular, for almost all h, $NG(\psi_h)$ is countable. That it is dense for all h, follows from the above argument based on the estimate (3.2). □

It is interesting to also consider adding the condition that the different order derivatives of f do not share roots. This can easily be accomplished in Proposition 4 by introducing a randomization of each of the parameters α_k, but we do not know if the single parameter h can be so chosen. Proposition 4 can also be derived from the following fundamental result by E. Bulinskaja related to the random processes of the class C^1. We present the statement in the form from [2, 4.5]:

Proposition 5. *Let $g(x, \omega)$, $x \in [\alpha, \beta]$, ω in a probability space $(\Omega, \mathcal{F}, P)$, be random variables (r.v.) for which $(d/dx)\, g(x, \omega)$ is P-a.s. continuous. Assume also that for any fixed $x \in [\alpha, \beta]$, the distribution of the r.v. $g(x, \cdot)$ has density $P_x(a) = (d/da)P\{g(x, \cdot) < a\}$ which is uniformly bounded: for all $x \in [\alpha, \beta]$, $a \in \mathbb{R}$, $P_x(a) \leq c_0 < \infty$. Then for any fixed $u \in \mathbb{R}$, P-a.s. we have:*

1. *the set $\{x \in [\alpha, \beta] : g(x, \cdot) = u\}$ is finite;*
2. *all roots of the equation $g(x, \cdot) = u$ are simple; i.e. the system $g(x, \cdot) = u$, $g'(x, \cdot) = 0$ has no solutions.*

Example 12. Let $f(\cdot)$ be the same as in the previous example and let $\psi(\cdot) \in C_0^\infty$ be compactly supported and strictly positive and analytic at inner points in the support. An example of such ψ is

$$\psi(x) = \begin{cases} 0, & x \notin (0,1) \\ \exp(-\frac{1}{x} - \frac{1}{1-x}), & x \in (0,1) \end{cases} \tag{3.3}$$

Notice that $\psi^{(k)}(0)$ and $\psi^{(k)}(1)$ are 0 for all $k \geq 0$.

Put $\psi_\varepsilon(x) = \psi(x)(1 + \varepsilon f(x))$, where $\varepsilon \neq 0$. Then the non-generic set $NG(\psi_\varepsilon)$ contains $\mathbb{R} \setminus \operatorname{supp} \psi$ and on $\operatorname{supp} \psi$ it is dense. In addition, for almost every $\varepsilon > 0$ the set $NG(\psi_\varepsilon) \cap \operatorname{supp} \psi$ is countable.

The proof of the first statement is based on the equality

$$\psi_\varepsilon^{(k)}(x) = \psi^{(k)}(x) + \varepsilon(\psi(x)f^{(k)}(x) + k\psi'(x)f^{(k-1)}(x) + \ldots + \psi^{(k)}(x)f(x)).$$

Here, as $k \to \infty$, the first term dominates the sum for any $\varepsilon > 0$ in any fixed closed interval Δ contained in the interior of the support of ψ (due to (3.2)). The second statement follows from Lemma 3(d) (we use the obvious fact that the set $NG(\psi) \cap \operatorname{supp} \psi$ is countable).

Example 13. If we take linear combinations of the antiderivative of functions such as ψ from example 12, we can find, for example, C^∞ functions, g, with compact support and constantly 1 on any given interval Δ. If f is any analytic function, then the function $f \cdot g$ is such that $NG(f \cdot g) \cap \Delta$ is equal to $NG(f) \cap \Delta$.

Using such functions we can glue together different local examples as constructed above to get quite a variety of behavior for $NG(f)$. For instance, for any partition $\{\Delta_i : i = 1, \ldots n\}$ of $\mathbb{R}$ into semi-open intervals, one can construct a function $f(x) \in C^\infty(\mathbb{R})$ such that for each odd integer i, $f \upharpoonright \Delta_i$ is any specified analytic function and for each even integer i, $NG(f) \cap \Delta_i$ is a dense countable subset of Δ_i.

Example 14. Let Δ_n $(n = 1, 2, \ldots)$ enumerate the Cantor middle third intervals (e.g. $\Delta_1 = (\frac{1}{3}, \frac{2}{3})$). For each n, also let

$$\psi_n(x) = \begin{cases} 0 & x \notin \Delta_n \\ \exp(-\frac{1}{x-a_n} - \frac{1}{b_n - x}) & x \in (a_n, b_n) = \Delta_n \end{cases}. \tag{3.4}$$

For each n, let A_n be large enough so that $|\psi_n^{(k)}(x)| < A_n$ for all $k \leq n$ and $x \in \Delta_n$. Then the function $g = \sum_n \frac{\psi_n}{2^n \cdot A_n}$ is in C^∞ and $NG(g)$ is uncountable and has measure 0. Multiplying g by $1 + \varepsilon f$, where f is the Weierstrass type function from example 11, we can further arrange a function $\tilde{g}$ such that $NG(\tilde{g})$ is still uncountable and has measure 0 and, in addition, is dense in $[0, 1]$.

4. The proof of (L) and corollaries

Lemma 6. *If $f(x) \in C^\infty(\mathbb{R})$ and $I = [a, b]$ is a closed bounded non-degenerate interval in $\mathbb{R}$ such that for all $x \in I$, there is an integer $n = n(x)$ such that $f^{(n)}(x) = 0$, then there is a non-empty subinterval $I_0 = (a_0, b_0) \subset I$ such that $f(x)$ is a polynomial on I_0.*

Proof. For each n let $\Gamma_n = \{x \in I : f^{(n)}(x) = 0\}$. By the assumption, the interval I is contained in the union of the Γ_n's. By the Baire Category theorem (see [7, 5.6]), there is an integer m such that the closure of Γ_m contains some open interval $I_0 = (a_0, b_0)$. Since $f^{(m)}$ is continuous on I_0, it is constantly 0 on I_0. It follows now that $f \upharpoonright I_0$ is a polynomial of degree at most m. □

Theorem 7. *If $f(x) \in C^\infty(\mathbb{R})$ is such that for all $x \in \mathbb{R}$, there is an integer $n = n(x)$ such that $f^{(n)}(x) = 0$, then f is a polynomial.*

Proof. By Lemma 6, it follows that every non-degenerate interval contains a subinterval on which f is a polynomial. Let U_n denote the interior of $\Gamma_n = \{x \in I : f^{(n)}(x) = 0\}$ and let F denote $\mathbb{R} \setminus \bigcup_n U_n$. Assume that $(a, b) \subset U_j$ and $(b, c) \subset U_k$, we first show that $(a, c) \subset U_{\min(j,k)}$. Note that $f^{(j)}(b) = f^{(k)}(b) = 0$. By symmetry, we may assume that $k > j$ is minimal such that $f^{(k)}$ is identically 0 on (b, c). It follows that $f^{(k-1)}$ is some non-zero constant on (b, c), contradicting that $f^{(k-1)}(b) = 0$. Similarly if (c, d) is an interval contained in $\bigcup_n U_n$, there is a minimal integer k such that $f^{(k)}$ is constantly 0 on (c, d) and if $k > 0$, then $f^{(k-1)}$ is a non-zero constant function on (c, d). It follows immediately then that F has no isolated points.

If F is empty we are done since then $\mathbb{R}$ is a maximal interval in $\bigcup_n U_n$. Since F is a closed subset of $\mathbb{R}$, it also satisfies the hypotheses of the Baire category theorem, hence there is some interval (a, b) such that $(a, b) \cap F$ is a non-empty subset of Γ_m for some integer m. We obtain a contradiction by showing that $(a, b) \subset U_m$. Since F has no isolated points and $f^{(m)}$ is identically 0 on $F \cap (a, b)$, it follows that $f^{(j)}$ is also 0 on $F \cap (a, b)$ for all $j \geq m$. Let (c, d) be a maximal subinterval of $(a, b) \setminus \Gamma_m$. By maximality either c or d is in the closure of Γ_m, let us assume it is c. Note that (c, d) is disjoint from F. Therefore there is a minimal $j > m \geq 0$ such that $f^{(j)}$ is constantly 0 on (c, d), hence $f^{(j-1)}$ is a non-zero constant on (c, d). We now have our contradiction since $f^{(j-1)}(c) = 0$. □

We may explicitly record the following corollary which is proven by simply replacing $\mathbb{R}$ in the above proof by the interval I.

Corollary 8. *If I is a non-degenerate open interval of $\mathbb{R}$ and $f(x) \in C^\infty(\mathbb{R})$ is not a polynomial on I, then $G(f) \cap I$ is not empty.*

We can now show that if $G(f)$ is not empty it is quite complex.

Corollary 9. *If $f \in C^\infty(\mathbb{R})$ and $a < b \leq c < d$ are reals such that $(a, b) \cup (c, d) \subset NG(f)$, then $c, d \in NG(f)$, and (b, c) is not contained in $G(f)$ unless $b = c$. Therefore, if the set $G(f)$ is closed, it is either empty or all of $\mathbb{R}$.*

Proof. By Corollary 8, there are integers j, k (minimal) so that $f^{(j)}$ is constant on (a, b) and $f^{(k)}$ is constant on (c, d). Clearly then $c, d \in NG(f)$. If $b < c$, and there is a some $x \in G(f) \cap (b, c)$, then by the mean-value theorem, there is a point $y \in (b, c)$ such that $f^{(1+j')}(y) = 0$ where $j' = \max(j, k)$. □

Corollary 10. *If $f \in C^\infty(\mathbb{R})$ is not a polynomial, then $G(f)$ has cardinality $\mathfrak{c}$.*

Proof. By Corollary 9, the set $G(f)$ can have no isolated points. In addition, if we set F to be the closure of $G(f)$ in $\mathbb{R}$, F will have no isolated points and will be a complete metric space. In addition, $G(f)$ is a dense G_δ subset of F. It is well-known that when a complete separable metric space has no isolated points each dense G_δ subsets has cardinality $\mathfrak{c}$ (in fact it is itself a complete metric space under a compatible metric). □

Theorem 11. *If $f \in C^\infty(\mathbb{R})$ then $NG(f)$ is a countable union of pairwise disjoint closed sets.*

Proof. For each n, let $F_n = \{x \in \mathbb{R} : f^{(n)}(x) = 0\}$. Since $f^{(n)}$ is continuous, it follows that F_n is a closed subset of $\mathbb{R}$. To prove the theorem it suffices to show that $F_{n+1} \setminus F_n$ can be written as a countable union of pairwise disjoint closed sets, which we do for $n = 0$. If an interval $(a, b) \subset F_1$, then either $[a, b] \subset F_0$ or $[a, b] \subset F_1 \setminus F_0$. Now let (a, b) be any maximal subinterval of $\mathbb{R} \setminus F_0$. Since no interval of the form $(a, a + \varepsilon)$ or $(b - \varepsilon, b)$ can be contained in F_1, we may choose $a < \cdots < a_{-n} < \cdots < a_{-1} < a_0 < a_1 < \cdots < a_n < \cdots < b$ such that $a_{-n} \to a$, $a_n \to b$ $(n \to \infty)$ and none of the a_n's are in F_1. For each integer n, $F_1 \cap (a_{n-1}, a_n)$ is a closed set and this collection of pairwise disjoint closed sets covers $F_1 \cap (a, b)$. This can be repeated for each of the countably many maximal intervals (a, b) contained in $\mathbb{R} \setminus F_0$ which completes the proof. $\square$

5. Generic in topology behavior of non-generic sets

In this section we continue the study of the nature of the set $NG(f)$ for a typical function f. Specifically we consider completely metrizable linear topological spaces of C^∞ functions and study the collection of functions $f \in X$ whose non-generic set is dense and countable. In fact, we are interested in an even stronger property defined below.

We consider the space $C^\infty(\mathbb{R})$ with the topology of local uniform convergence of all derivatives (notation emphasizing the topology is $C^\infty_{\mathrm{loc}}(\mathbb{R})$). This topology is generated by a countable set of seminorms

$$\|f\|_n := \max_{0 \le k \le n} \max_{-n-1 \le x \le n+1} |f^{(k)}(x)|, \quad n \ge 0,$$

or equivalently, by a metric

$$d(f, g) := \sum_{n=0}^{\infty} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}.$$

Throughout this section, X will denote a linear topological space of C^∞ functions. We will consider the following properties that X may or may not have:
(a) X is completely metrizable;
(b) the natural embedding of X into $C^\infty_{\mathrm{loc}}(\mathbb{R})$ is continuous;
(c) X contains the linear space $C^\infty_0(\mathbb{R})$ of all C^∞ functions that are compactly supported.

Definition 12. A linear topological space X of C^∞ functions that has properties (a), (b) and (c), will be called *rich*.

$C^\infty_{\mathrm{loc}}(\mathbb{R})$ and the Schwartz space $\mathcal{S}(\mathbb{R})$ of fast decaying functions (see, e.g., [6]) are examples of rich spaces. The spaces BR_σ and $BR_{\sigma-0}$ introduced later in this section have properties (a) and (b) but not (c).

Now we introduce notation that we will need below.

For $f \in C^\infty(\mathbb{R})$ and $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, we set

$$Z_k(f) := \{x \in \mathbb{R} \ : f^{(k)}(x) = 0\},$$

so that $NG(f) = \bigcup_{k=0}^\infty Z_k(f)$.

For a collection Y of C^∞ functions set

$$Q_0(Y) := \{f \in Y \ : \ NG(f) \text{ is dense in } \mathbb{R}\}; \tag{5.1}$$

$$J(Y) := \{f \in Y \ : \ Z_j(f) \cap Z_k(f) = \emptyset \ \text{ if } j \ne k\}; \tag{5.2}$$

$$Q(Y) := Q_0(Y) \cap J(Y). \tag{5.3}$$

Theorem 13. *Assume X is a rich space of C^∞ functions on $\mathbb{R}$. Then $Q(X)$ is a dense G_δ subset of X.*

The proof is based on two lemmas. We formulate them and show how the theorem follows, then prove the lemmas.

Lemma 14. *(i) Let $a, b \in \mathbb{R}$ $(a < b)$ and $j, k \in \mathbb{Z}_+$ $(j \neq k)$. If X satisfies (b), then the set*

$$J_{ab}^{jk}(X) := \{f \in X \ : \ Z_j(f) \cap Z_k(f) \cap [a, b] = \emptyset\}$$

is open.
(ii) If X satisfies (c), then $J_{ab}^{jk}(X)$ is dense.

Corollary 15. *If X is rich, then $J(X)$ is a dense G_δ subset of X.*

Proof. We have

$$J(X) = \bigcap_{j,k\colon 0 \le j < k} \ \bigcap_{m \ge 1} J_{-m,m}^{jk} ,$$

so the statement follows from the Baire theorem, according to which the intersection of a countable family of dense open sets in a complete metric space is dense. □

Lemma 16. *(i) Let $a, b \in \mathbb{R}$ $(a < b)$. If X satisfies (b), then the set*

$$D_{ab}(X) := \{f \in X \ : \ (\exists j \in \mathbb{Z}_+)\,(\exists c, d \in [a, b])\,(f^{(j)}(c) < 0 < f^{(j)}(d))\}$$

is open.
(ii) If X satisfies (c), then $D_{ab}(X)$ is dense.

Corollary 17. *If X is rich, then the set*
(5.4)

$$D(X) := \{f \in X \ : \ (\forall a < b)\,(\exists j \in \mathbb{Z}_+)\,(\exists c, d \in [a, b])\,(f^{(j)}(c) < 0 < f^{(j)}(d))\}$$

is a dense G_δ subset of X.

Proof. We have

$$D(X) = \bigcap_{a,b \in \mathbb{Q}\colon a<b} D_{ab}(X),$$

where $\mathbb{Q}$ denotes the set of all rationals, hence $D(X)$ is a G_δ set, which is dense by the Baire theorem. □

Note that

$$Q(X) \equiv J(X) \cap Q_0(X) = J(X) \cap D(X). \tag{5.5}$$

The latter set, according to Corollaries 15 and 17 and the Baire theorem, is a dense G_δ subset of X. Therefore, to prove the theorem it remains to prove the two lemmas.

Before proving them we will establish two auxiliary facts. Fix two numbers $a, b \in \mathbb{R}$ $(a < b)$ and an arbitrary compactly supported C^∞ function $\phi_{ab}(\cdot)$, such that $\phi_{ab}(x) = 1$ on the interval

$$\Delta = (a - \eta,\, b + \eta), \tag{5.6}$$

where η is an arbitrary number > 0.

Lemma 18. *Fix $k \in \mathbb{Z}_+$ and two numbers $r > 0, R \geq 0$. Set*

$$g_\lambda(x) := \phi_{ab}(x)\sin(\lambda x)/\lambda^{k-\frac{1}{2}}.$$

Then there exists $\Lambda > 0$, such that for all $\lambda \geq \Lambda$

$$\max_{0\leq j\leq k-1}\max_{x\in\mathbb{R}}|g_\lambda^{(j)}(x)| \leq r \tag{5.7}$$

and

$$\max_{a\leq x\leq b}(\theta g_\lambda^{(k)}(x)) \geq R \quad \text{for } \theta = \pm 1.$$

Proof. A straightforward application of the Leibnitz rule yields: if $j \leq k-1$, then there exists such a constant C that for all $\lambda \geq 1$

$$\max_{x\in\mathbb{R}}|g_\lambda^{(j)}(x)| \leq C\lambda^{j-k+\frac{1}{2}} \leq C\lambda^{-\frac{1}{2}},$$

while each of the two maxima $\max_{a\leq x\leq b}(\pm g_\lambda^{(k)}(x))$ equals $\lambda^{1/2}$ if $\lambda(b-a) \geq 2\pi$. □

Lemma 19. *Given a non-degenerate finite interval $[a,b] \subset \mathbb{R}$ and a sequence $B_k \to \infty$, there exists a compactly supported C^∞ function $g(\cdot)$, such that for all $k \in \mathbb{Z}_+$*

$$\max_{a\leq x\leq b} g^{(k)}(x) \geq B_k, \quad \min_{a\leq x\leq b} g^{(k)}(x) \leq -B_k. \tag{5.8}$$

Proof. Using notation of Lemma 18, set

$$g(x) := \sum_{k=0}^\infty g_{\lambda_k}(x) = \phi_{ab}(x)\sum_{k=0}^\infty \frac{\sin(\lambda_k x)}{\lambda_k^{k-\frac{1}{2}}}. \tag{5.9}$$

Select $\lambda_k > 0$ according to Lemma 18, where we put $r = r_k := 2^{-k}$ and $R = R_k$; the sequence R_k will be specified later.

By the choice of r_k we have

$$\sum_{k=j+1}^\infty \|g_{\lambda_k}^{(j)}(\cdot)\|_\infty \leq 2^{-j}, \tag{5.10}$$

so that $\sum_{k=0}^\infty \|g_{\lambda_k}^{(j)}(\cdot)\|_\infty < \infty$ for every $j \in \mathbb{Z}_+$. This guarantees that the series $\sum g_{\lambda_k}$ in (5.9) converges and its sum $g(\cdot)$ is in $C_0^\infty(\mathbb{R})$.

If x is in the interval (5.6), we have

$$g^{(j)}(x) = \sum_{k=0}^\infty \lambda_k^{j-k+\frac{1}{2}}\sin\left(\lambda_k x + j\frac{\pi}{2}\right).$$

Here $|\sum_{k=j+1}^\infty| \leq 1$ by (5.10), so the inequalities (5.8) will be ensured if we choose λ_j so large that $\lambda_j(b-a) \geq 2\pi$ and

$$\lambda_j^{1/2} \geq R_k := B_k + \sum_{k=0}^{j-1}\lambda_k^{j-k+\frac{1}{2}} + 1.$$

□

Proof of Lemma 16. Statement (i) is obvious, so we only need to verify statement (ii). Suppose $f \in X$ and let $B_k := k\max_{a\leq x\leq b}|f^{(k)}(x)|$. According to Lemma 19, there exists $g \in C_0^\infty(\mathbb{R})$ satisfying (5.8) for all $k \geq 0$. For $m \geq 1$ set

$$f_m := f + \frac{1}{m}g. \tag{5.11}$$

For any fixed m and all $k > m$ we have:

$$\max_{a\le x\le b} f_m^{(k)}(x) \ge \frac{1}{m}\max_{a\le x\le b} g^{(k)}(x) + \min_{a\le x\le b} f^{(k)}(x) \ge \frac{1}{m}B_k - \frac{1}{k}B_k > 0.$$

Similarly, $\min_{a\le x\le b} f_m^{(k)}(x) < 0$. Hence $f_m \in D_{ab}(X)$. On the other hand, (5.11) implies that $f_m \to f$ in X as $m \to \infty$. $\square$

Proof of Lemma 14. (i) The set $J_{ab}^{jk}(X)$ can be defined equivalently as

$$J_{ab}^{jk}(X) := \{f \in X \ : \ \min_{a\le x\le b}(|f^{(j)}(x)| + |f^{(k)}(x)|) > 0\},$$

which makes the statement obvious.

(ii) Set

$$g_\lambda(x) := \phi_{ab}(x)\sin(\lambda x + \alpha), \tag{5.12}$$

where $\lambda > 0$ and α are such that both $\sin(\lambda x + \alpha)$ and $\cos(\lambda x + \alpha)$ are nonzero on the interval $\overline{\Delta}$ – the closure of the open interval (5.6). Clearly, this is true (with the same α) not only for the initially chosen $\lambda = \lambda_0$ but also for all λ in a small enough open interval I centered at λ_0. We may assume that $0 \notin I$. Then for any choice of λ in I we have

$$g_\lambda^{(l)}(x) \ne 0 \text{ on } \Delta \text{ for all } l \in \mathbb{Z}_+. \tag{5.13}$$

Given $f \in X$, we are going to prove that *arbitrarily close (in X) to f there exists a function $h \in X$, such that*

$$Z_j(h) \cap Z_k(h) \cap [a,b] = \emptyset. \tag{5.14}$$

Define, for any $t \in \mathbb{R}$, a function

$$f_{t,\lambda}(x) := f(x) + t g_\lambda(x). \tag{5.15}$$

It follows from (5.13) and Lemma 3(a) that for a.e. $t \in \mathbb{R}$, all derivatives of $f_{t,\lambda}(\cdot)$ have only simple zeros (if any) in the interval Δ. Choosing such a t small enough (so that $f_{t,\lambda}$ is arbitrarily close to f) and substituting the resulting function $f_{t,\lambda}(\cdot)$ for $f(\cdot)$, we reduce our statement to its particular case where all zeros of $f^{(j)}$ and $f^{(k)}$ in the interval (5.6) are simple. To prove the statement in this case, we use the same family (5.15), but this time $\lambda \in I$ should be properly selected.

Proposition 20. *There exists an at most countable set $E \subset I$, such that for any $\lambda \in I \setminus E$ and all $t \ne 0$ with small enough $|t|$, the function (5.15) satisfies*

$$Z_j(f_{t,\lambda}) \cap Z_k(f_{t,\lambda}) \cap [a,b] = \emptyset.$$

Proof. Since all zeros of $f^{(j)}$ and $f^{(k)}$ in $[a,\, b]$ are simple, there are only finitely many. Let $y_1,\, y_2, \dots,\, y_m$ and $z_1,\, z_2, \dots,\, z_n$ be all the zeros of $f^{(j)}$ and $f^{(k)}$, respectively, in the interval $[a,\, b]$.

Consider a function $F(x,t) := f_{t,\lambda}^{(j)}(x) = f^{(j)}(x) + t g_\lambda^{(j)}(x),\ \ x, t \in \mathbb{R}$. When $t = 0$, the only zeros of $F(\cdot, t)$ in $[a, b]$ are y_l, and since they are simple, $(\partial F(x,t)/\partial x)(y_l,\, 0) = f^{(j+1)}(y_l) \ne 0$.

It follows easily from the Implicit Function Theorem that there exist $\delta > 0,\ \beta > 0$ and C^∞ functions $Y_l(t),\ l = 1, \dots, m$, on the interval $(-\delta,\, \delta)$ such that

(i) for each $t \in (-\delta,\, \delta)$, $Y_1(t), \dots, Y_m(t)$ are the only zeros of $F(\cdot, t) \equiv f_{t,\lambda}^{(j)}(\cdot)$ in $(a - \beta,\, b + \beta)$;

(ii) $Y_l(0) = y_l, \quad l = 1, \dots .m.$
Moreover, by the same theorem,

$$Y_l'(0) = -\frac{\partial F(x,t)/\partial t\,|_{x=y_l,t=0}}{\partial F(x,t)/\partial x\,|_{x=y_l,t=0}} = -\frac{g_\lambda^{(j)}(y_l)}{f^{(j+1)}(y_l)} = A\lambda^j \sin(\lambda y_l + \alpha + j\pi/2),$$

where A is a nonzero constant.

Similarly, for small $|t|$, all the zeros of $f_{t,\lambda}^{(k)}(\cdot)$ in some open interval containing $[a,b]$ are the values of n C^∞ functions $Z_q(t)$ such that $Z_q(0) = z_q$ $(q = 1, 2, \dots, n)$ and

$$Z_q'(0) = -\frac{g_\lambda^{(k)}(z_q)}{f^{(k+1)}(z_q)} = B\lambda^k \sin(\lambda z_q + \alpha + k\pi/2),$$

where $B \neq 0$.

Suppose $f^{(j)}$ and $f^{(k)}$ have r common zeros in $[a,\, b]$. Let $y_l = z_q$ be one of them. Then we have

$$\frac{Y_l'(0)}{Z_q'(0)} = C\lambda^{j-k}(\tan(y_l\lambda + \alpha))^H, \tag{5.14}$$

where C and H are constants, $C \neq 0$ and $H \in \{-1, 0, 1\}$. The right-hand side $s(\lambda)$ of (5) is real-analytic on I and, since $j - k \neq 0$, non-constant, so that the set $E_{lq} := \{\lambda \in I \,:\, s(\lambda) = 1\}$ is at most countable. For $\lambda \in I \setminus E_{lq}$ we have $Y_l'(0) \neq Z_q'(0)$, hence $Y_l(0) \neq Z_q(0)$ for small $t \neq 0$.

If we remove from I the union E of the r sets E_{lq}, then any remaining λ has the desired property: for all $t \neq 0$ with small enough $|t|$ the two finite sets $\{Y_l(t)\}_{l=1}^m$ and $\{Z_q(t)\}_{q=1}^n$ are disjoint. This completes the proof of the proposition and thereby that of Lemma 14. □

Theorem 13 is, therefore, also proven. □

Next we are interested in the Baire category properties of the sets $Q(X)$ and $Q_0(X)$ (defined by (5.3) and (5.1)) in the case of certain spaces X of real-analytic functions.

The Banach space B_σ is a classic function space of widespread interest. The functions are restrictions to $\mathbb{R}$ of entire functions of exponential type $\leq \sigma$ that are bounded on $\mathbb{R}$.

Definition 21. [5] Let $\sigma > 0$. An entire function $f(z)$ (i.e., a function $f : \mathbb{C} \to \mathbb{C}$ holomorphic on the whole complex plane $\mathbb{C}$) is called a *function of exponential type* $\leq \sigma$ if for any $\varepsilon > 0$ f satisfies inequality $|f(z)| < C_\varepsilon e^{(\sigma+\varepsilon)|z|}$ with some constant C_ε.

Let $C(\mathbb{R}, \mathbb{C})$ denote the collection of all complex-valued continuous functions on $\mathbb{R}$.

Definition 22. [1], [5] The *Bernstein space* B_σ $(\sigma > 0)$ is defined by

$$B_\sigma := \{f \in C(\mathbb{R}, \mathbb{C}) \,:\, \|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)| < \infty$$
$$\text{and } f \text{ has a holomorphic extension to } \mathbb{C} \text{ of exponential type} \leq \sigma\}.$$

Remark. It is well-known that B_σ endowed with the sup-norm is a complex Banach space [1].

A fundamental fact about B_σ is given by *Bernstein's inequality* (see [1], [5], [3]):

$$\text{If } f \in B_\sigma \text{ then for all } n \geq 0, \quad \|f^{(n)}\|_\infty \leq \sigma^n \|f\|_\infty. \tag{5.15}$$

Definition 23. [4, p. 592] $B_{\sigma-0} := \mathrm{cl}\,(\cup_{\tau\in(0,\sigma)} B_\tau)$, where cl denotes the closure in B_σ.

It may seem at first glance that $B_{\sigma-0}$ and B_σ must coincide, but it is not the case. In fact, the quotient space $B_\sigma/B_{\sigma-0}$ is infinite-dimensional, even non-separable (see [4, p. 592] for this result). Our results will further illustrate this distinction.

We will denote by BR_σ and $BR_{\sigma-0}$ the subcollection of real-valued members of B_σ and $B_{\sigma-0}$, respectively. Each of them endowed with the sup-norm is a real Banach space.

Our goal is to answer a natural question: is $Q(X)$ (or at least $Q_0(X)$) topologically generic (i.e., contains a dense G_δ subset of X) when X is BR_σ or $BR_{\sigma-0}$?

The answer is given by theorems 24 and 25.

Theorem 24. *The set $Q_0(BR_\sigma)$ is not dense in BR_σ.*

Note that $Q(BR_\sigma) \subset Q_0(BR_\sigma)$; therefore, the set $Q(BR_\sigma)$ is not dense in BR_σ either.

Proof. By scale transformation $y = \sigma x$ we may assume that $\sigma = 1$. The function $f_0(x) = \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ belongs to BR_1 and has maximal possible type 1. On the interval $\Delta_0 = (\pi/6, \pi/3)$, this function and all its derivatives (i.e. $\pm \sin x$, $\pm \cos x$) are large:

$$|\sin x| > \frac{1}{2} \quad , \quad |\cos x| > \frac{1}{2} \ .$$

Let us consider the ball of radius $\frac{1}{2}$ centered at f_0 in BR_1:

$$B_{\frac{1}{2}}(f_0) = \left\{ f \in BR_1 : \|f - f_0\|_\infty \leq \frac{1}{2} \right\} \ .$$

By Bernstein's inequality, we have that for each $n \geq 0$,

$$\|f^{(n)} - f_0^{(n)}\|_\infty \leq \frac{1}{2} \quad .$$

This implies that each function f in $B_{\frac{1}{2}}(f_0)$ and all its deriviatives are non-zero on Δ_0. Indeed, for $x \in \Delta_0$ we have

$$|f^{(n)}(x)| \geq |f_0^{(n)}(x)| - \|f^{(n)} - f_0^{(n)}\|_\infty > \frac{1}{2} - \frac{1}{2} \ = 0,$$

hence $B_{\frac{1}{2}}(f_0)$ is disjoint from $Q_0(BR_1)$. $\square$

Theorem 25. *The set $Q(BR_{\sigma-0})$ is a dense G_δ subset of $BR_{\sigma-0}$.*

The proof is based on two lemmas formulated similarly to Lemmas 14 and 16.

Lemma 26. *(i) Let $a, b \in \mathbb{R}$ $(a < b)$ and $j, k \in \mathbb{Z}_+$ $(j \neq k)$. The set*

$$J_{ab}^{jk}(B_{\sigma-0}) = \{f \in B_{\sigma-0} \ : \ Z_j(f) \cap Z_k(f) \cap [a, b] = \emptyset\}$$

is open.

(ii) The set $J_{ab}^{jk}(B_{\sigma-0})$ is dense.

Proof. Statement (i) is a particular case of Lemma 14(i). The proof of (ii) mimics that of Lemma 14(ii) while being different in two points. First, instead of the compactly supported function (5.12) (which is not in $B_{\sigma-0}$) we use the function $\sin(\lambda x + \alpha)$ itself. Secondly, in order that this function be in $B_{\sigma-0}$ we need to subject λ to an additional restriction $\lambda < \sigma$. □

Lemma 27. *(i) Let $a, b \in \mathbb{R}$ $(a < b)$. The set*

$$D_{ab}(B_{\sigma-0}) = \{f \in B_{\sigma-0} \ : \ (\exists j \in \mathbb{Z}_+)(\exists c, d \in [a,b])(f^{(j)}(c) < 0 < f^{(j)}(d)\}.$$

is open.
(ii) The set $D_{ab}(B_{\sigma-0})$ is dense.

Proof. Part (i) follows directly from Lemma 16(i), so it remains to prove (ii). We want to prove, therefore, that any $f \in B_{\sigma-0}$ can be approximated by elements of the set $D_{ab}(B_{\sigma-0})$. We may assume that $f \in B_\tau$, where $\tau < \sigma$. By a suitable scale transformation, we may also assume that $\tau < 1 < \sigma$. Let $c := (a+b)/2$ and $g(x) := \sin(x - c)$. Set $f_\varepsilon := f + \varepsilon g$, so that $f_\varepsilon \in B_{\sigma-0}$. By Bernstein's inequality,

$$\|f_\varepsilon^{(4k)} - \varepsilon g\|_\infty \le \tau^{4k} \|f\|_\infty \to 0 \text{ as } k \to \infty.$$

It follows that $f_\varepsilon^{(4k)}$ with large k changes sign in $[a, b]$. Hence we have $D_{ab}(B_{\sigma-0}) \ni f_\varepsilon \to f$ as $\varepsilon \to 0$. □

Proof of Theorem 25. Just as in the proof of Theorem 13, we conclude from Lemmas 26 and 27 and the Baire theorem that each of the sets $J(BR_{\sigma-0})$ and $D(BR_{\sigma-0})$ and consequently their intersection $Q(BR_{\sigma-0})$ is a dense G_δ set. □

6. Generic in measure behavior of non-generic sets

In this next example we find a function for which $NG(f)$ is a proper dense open set. We will later investigate varying the values of the measure of $G(f)$.

Example 15. There is a C^∞ compactly supported function f such that $G(f)$ is non-empty and has measure 0. That is, the function f has polynomial structure on intervals outside the uncountable set $G(f)$.

Proof. Let $Y_1, \ldots, Y_n, \ldots$ be a family of i.i.d.r.v with uniform distribution on $[-1, 1]$ and corresponding density function $\rho(y) = \frac{1}{2} I_{[-1,1]}(y)$. Its Fourier transform is

$$\int_{-1}^{1} \rho(y) e^{ity} dx = \frac{\sin t}{t} = \varphi(t) \ .$$

We note that $|\varphi(t)| \le \frac{c}{1+t}$.

Set $Z = \dfrac{Y_1}{3} + \dfrac{Y_2}{3^2} + \cdots + \dfrac{Y_n}{3^n} + \cdots$, hence

$$E\, e^{itz} = \Pi_{n=1}^{\infty} \left(\frac{3^n \sin \frac{t}{3^n}}{t} \right) = \varphi_Z(t) \ .$$

This function has estimation

$$|\varphi_Z(t)| \le \frac{c_n}{(1+|t|)^n} \quad , \quad (\forall n \ge 0) \quad .$$

This means that there exists a density

$$P_Z(x) \in C^\infty \quad .$$

Of course

$$|Z| \le \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$$

i.e. the support $P_Z(\cdot) = [-\frac{1}{2}, \frac{1}{2}]$, and the density $P_Z(\cdot)$ is strictly positive on $(-\frac{1}{2}, \frac{1}{2})$.

We must prove that it is polynomial outside the Cantor set consisting of the ternary points $\{\sum_{k=1}^{\infty} \frac{\varepsilon_k}{3^k} : \varepsilon_k = \pm 1\} = \Gamma$.

The proof is based on the following elementary lemma.

Lemma 28. *Let μ be measure supported on $[-\delta, \delta]$ and $f(x)$, $x \in \mathbb{R}$ be a continuous function, which coincides on some interval $[a, b]$ of length greater than 2δ, with a polynomial $P_n(x)$ of degree n. Then the convolution $f * \mu = \int_{\mathbb{R}} f(x-y)d\mu$ is a polynomial of the same degree n on the interval $[a+\delta, b-\delta]$.*

Proof. If $x \in (a+\delta, b-\delta)$, then

$$(f*\mu)(x) = \int_{-\delta}^{\delta} P_n(x-y)\mu(dy) = \int_{-\delta}^{\delta} \sum_{k=0}^{n} \frac{P_n^{(k)}(x)}{k!}(-y)^k \mu(dy) = Q_n(x) \qquad \square$$

Now we return to the function $P_Z(x)$ and study it using approximations (in C^∞):

$$P_Z(x) = \lim_{n\to\infty} P_{Z_n}(x) \quad , \quad Z_n = \frac{Y_1}{3} + \cdots + \frac{Y_n}{3^n} \ .$$

Let us note that $P_{Z_n}(x)$ is a polynomial of degree at most $(n-1)$ outside the finite set $\Gamma_n = \{x : x = \pm\frac{1}{3}, \pm\frac{1}{3^2} \pm \cdots \pm \frac{1}{3^n}\}$ and that $|\Gamma_n| = 2^n$.

For instance

$$P_{Z_1}(x) = \begin{cases} 3/2 & x \in [-\frac{1}{3}, \frac{1}{3}] \\ 0 & x \notin [-\frac{1}{3}, \frac{1}{3}] \end{cases}$$

$$P_{Z_2}(x) = \begin{cases} 0 & x \le -\frac{1}{3} - \frac{1}{9} \ \text{ or } \ x \ge \frac{1}{3} + \frac{1}{9} \\ \frac{27}{4}(x + \frac{1}{3} + \frac{1}{9}) & x \in [-\frac{1}{3} - \frac{1}{9}, -\frac{1}{3} + \frac{1}{9}] \\ \frac{3}{2} & x \in [-\frac{1}{3} + \frac{1}{9}, \frac{1}{3} - \frac{1}{9}] \\ 27(\frac{1}{3} + \frac{1}{9} - x) & x \in [\frac{1}{3} - \frac{1}{9}, \frac{1}{3} + \frac{1}{9}] \end{cases}$$

etc.

Let $\tilde{Z}_n = \frac{Y_{n+1}}{3^{n+1}} + \frac{Y_{n+2}}{3^{n+2}} + \ldots$, hence $|\tilde{Z}_n| \le \frac{1}{2\cdot 3^n} = \varepsilon_n$. But $P_Z(x) = P_{Z_n} * P_{\tilde{Z}_n}$ and the lemma shows that $P_Z(x)$ must be polynomial of degree at most $(n-1)$ on the intervals that are complementary to the ε_n-neighborhoods of the set Γ_n. Let $\Gamma_n^{\varepsilon_n}$ denote the union of these intervals. Of course the sum of the lengths of the intervals in $\Gamma_n^{\varepsilon_n}$, i.e. $|\Gamma_n^{\varepsilon_n}|$, will equal $2^n \cdot \frac{1}{2\cdot 3^n} = \frac{1}{2}(2/3)^n \to 0$.

Let us note also that the set of the limit points for the sequence $\{\Gamma_n : n = 1, 2, \ldots\}$ is a Cantor set of measure 0.

Therefore we have proven that $G(P_Z(\cdot)) \subset \Gamma$, hence the measure of $G(P_Z(\cdot))$ is 0. It is probably true that $\Gamma \setminus G(P_Z(\cdot))$ is countable but we were not able to verify this conjecture. $\square$

The function $P_Z(x)$ is self-similar. It satisfies the following functional equation:

$$P_Z(x) = 3\int_{-\frac{1}{3}}^{\frac{1}{3}} P_Z(3(x-y))dy \quad , \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

i.e. $P_Z'(x) = 3[P_Z(3x+1) - P_Z(3x-1)]$, which is equivalent to the relation

$$Z = \frac{Y_1}{3} + \frac{Z}{3} \quad .$$

Such kinds of equations are known as differential-functional equations and do have important probabilistic applications. A systematic study of the function $P_Z(\cdot)$, with applications to the interpolation theory, can be found in [8].

This first example shows that probabilistically the set $NG(f)$ is going to be a countable dense subset of $\mathbb{R}$. In the next section we investigate the topological properties of the collection of analytic f for which $NG(f)$ is a countable dense set.

Example 16. Let $W_\zeta(\omega)$, $\zeta \in \mathbb{R}$ be the Brownian motion (Wiener process), i.e. random Gaussian process with zero mean, $E\,W_\zeta \equiv 0$ and the correlator

$$B(\zeta,\zeta') = E\,W_\zeta W_{\zeta'} = \begin{cases} |\zeta| \wedge |\zeta'| & \zeta\cdot\zeta' \le 0 \\ 0 & \zeta\cdot\zeta' > 0 \end{cases}$$

In particular, $\operatorname{var} W_\zeta = B(\zeta,\zeta) = |\zeta|$. It means that W_ζ, $\zeta \ge 0$ and $W_{-\zeta}$, $\zeta > 0$ are two independent standard Wiener processes.

Let us introduce also $\widehat{W}_\zeta$, $\zeta \in \mathbb{R}$: the independent copy of W.

If $\varphi(\zeta)$ is a bounded continuous function decreasing faster than any degree of ζ,

$$\text{i.e. } |\varphi|(1+|\zeta|^n) \in L^1(\mathbb{R}) \ \text{ for any } \quad n \ge 1$$

then one can consider the following Gaussian homogeneous random process

$$f(x,\omega) = \int_{\mathbb{R}} \varphi(\zeta)\, \cos x\zeta\, dW_\zeta + \int_{\mathbb{R}} \varphi(\zeta)\, \sin x\zeta d\widehat{W}_\zeta$$

(Stochastic integrals, hence one can understand in the usual Riemannian sense). Of course, one can identify ω and realizations $(W_0, \widehat{W}_0)$.

As is easy to see, the process $f(x,\omega) \in C^\infty$ and for its derivatives one has

$$f^{(n)}(x,\omega) = \int_{\mathbb{R}} \varphi(\zeta)\zeta^n \cos(x + \frac{\pi n}{2}) d\,W_\zeta + \int_{\mathbb{R}} \varphi(\zeta)\zeta^n \sin(x+\frac{\pi n}{2}) d\,\widehat{W}_\zeta$$

It gives the following expressions for the correlations

$$E\, f^{(n)}(x,\cdot) \equiv 0 \quad , \quad n \ge 0$$

$$E\, f^{(n)}(x)\, f^{(n)}(y) = \int_{\mathbb{R}} \cos\zeta(x-y)\,\zeta^{2n}\varphi^2(\zeta) d\,\zeta = B_{2n}(x-y)$$

$$E\, f^{(n)}(x)\, f^{(m)}(y) = B_{m+n}(x-y) \tag{6.1}$$

$$= \textstyle\int_{\mathbb{R}} \pm\cos\zeta(x-y)\,\zeta^{m+n}\varphi^2(\zeta) d\,\zeta \quad , m+n \equiv 0 \mod 2 \tag{6.2}$$

$$= \textstyle\int_{\mathbb{R}} \pm\sin\zeta(x-y)\,\zeta^{m+n}\varphi^2(\zeta) d\,\zeta \quad , m+n \equiv 1 \mod 2 \tag{6.3}$$

We concentrate on the particular cases when

$$\varphi(\zeta) = \exp(-|\zeta|^\alpha/2) \quad , \alpha > 0$$

$$\varphi(\zeta) = I_{[\sigma,\sigma]}(\zeta) \quad , (\text{here } \alpha = \infty).$$

I.e. the spectral densities for $f(\cdot)$, $f^{(n)}(\cdot)$ are given by

$$\varphi^2(\zeta) = \exp(-|\zeta|^\alpha) \quad , \alpha > 0$$

$$\zeta^{2n}\varphi^2(\zeta) = \zeta^{2n}\exp(-|\zeta|^\alpha) \ , \quad \alpha = \infty.$$

Corresponding processes $f(x,\cdot)$, $f^{(n)}(x,\cdot)$ will be called $f_\alpha(x)$, $f_\alpha^{(n)}(x)$ to specify the dependence on the parameter α.

For $\alpha \geq 1$, one can extend $f(x,\cdot)$ into the complex plane $z = x+iy$. Namely, for complex z,

$$|\cos\zeta z| \leq e^{|\zeta||z|} \text{ and } |\sin\zeta z| \leq e^{|\zeta||z|}$$

and (by the Laplace method)

$$\sigma^2_{n,\alpha}(z) = E|f^{(n)}(z)| \leq \int_{\mathbb{R}^2} e^{|\zeta||z|}\zeta^{2n}e^{-|\zeta|^\alpha}d\zeta \simeq$$
$$\overset{\log}{\simeq} e^{c(\alpha)|z|^{\frac{\alpha}{\alpha-1}}} \quad , \quad c(\alpha) = \frac{\alpha-1}{\alpha^{\frac{\alpha-1}{\alpha}}} \, .$$

One can prove now that P-a.s. for $\alpha > 1$, $f(z,\omega)$ is an entire function of the order $\frac{\alpha}{\alpha-1}$ and of the type $c(\alpha)/2$. If $\alpha = \infty$, then the order is equal to 1 and the type is σ (calculations are very similar).

For $\alpha = 1$ the function $f(x,\omega)$ has analytic continuation in the strip $|Im(z)| < 1$. Finally for $\alpha < 1$, the functions $f(x,\cdot)$ are C^∞ on $\mathbb{R}$ but have no analytic continuation.

The variances of all derivatives can be calculated explicitly:

$$(6.4) \quad E[f^{(n)}(\cdot)]^2 = B_{2n}(0) = 2\int_0^\infty \zeta^{2n}e^{-\zeta^\alpha}d\zeta = \frac{2}{\alpha}\Gamma\left(\frac{2n+1}{\alpha}\right), \ \alpha < \infty$$
$$E[f^{(n)}(\cdot)]^2 = 2\int_0^\sigma \zeta^{2n}d\zeta = \frac{2\sigma^{2n+1}}{2n+1} \ , \alpha = \infty$$

Theorem 29. *For any $0 < \alpha \leq \infty$, the random function $f = f_\alpha(x,\omega)$ has a countable dense non-generic set $NG(f)$ (with probability 1, P-a.s.)*

We will prove the theorem for $\alpha < \infty$ as the case for $\alpha = \infty$ is much simpler. We will also provide some asymptotical formulas.

We consider, instead of $f^{(n)}(x)$, $n \geq 0$, the normalized versions. Set

$$\xi_n(x) = \frac{f^{(n)}(\alpha_n x)}{\beta_n} \quad , \quad \alpha_n = \sqrt{\frac{B_{2n}(0)}{B_{2n+2}(0)}} \sim \left(\frac{\alpha}{2n}\right)^{1/\alpha} = \frac{c(\alpha)}{n^{1/\alpha}}.$$

The process $\xi_n(\cdot)$ is a stationary one and its correlation function $b_{2n}(z)$ is the normalization of $B_{2n}(z)$. Namely

$$(6.5) \quad b_{2n}(z) = E\,\xi_n(x)\xi_n(x+z) = \frac{B_{2n}(\alpha_n z)}{B_{2n}(0)}$$
$$= 1 + \frac{B''_{2n}(0)\alpha_n^2}{2B_{2n}(0)} + \frac{B^{(4)}_{2n}(0)\alpha_n^4 z^4}{24B_{2n}(0)} + O(z^6)$$
$$= 1 - z^2 + \gamma_{4,n}\frac{z^4}{24} + O(z^6) \ ,$$
$$\text{where} \quad \gamma_{4,n} = \frac{B_{2n+4}(0)B^2_{2n}(0)}{B_{2n}(0)B_{2n+2}(0)} \to 1, \quad n \to \infty \, .$$

We used the obvious relations $B''_{2n}(0) = B_{2n+2}(0)$, $B^{(4)}_{2n}(0) = B_{2n+4}(0)$ and explicity formula 6.4 for $B_{2n}(0)$.

Formula 6.5 gives

$$E\xi_n(x)\xi_n'(x) \quad = \quad \frac{\partial b_{2n}}{\partial z}(0) \quad = \quad 0 \quad , E\xi_n'(0)\xi_n'(0) \quad = \quad -\frac{\partial^2 b_{2n}}{\partial z^2}(0) \quad = \quad 1 \quad . \tag{6.6}$$

For different n, m, the processes $\xi^{(n)}(x), \xi^{(m)}(x)$ are not stationary connected (due to different scalings α_n, α_m in the arguments of the derivatives). We have however

$$b_{k,m}(x_1, x_2) = E\xi_n(x_1)\xi_m(x_2) = \frac{Ef^{(n)}(\alpha_n x_1)f^{(m)}(\alpha_m x_2)}{\sqrt{B_{2n}(0)B_{2m}(0)}} =$$
$$\frac{B_{m+n}(\alpha_n x_1 - \alpha_m x_2)}{\sqrt{B_{2n}(0)B_{2m}(0)}}$$

and

$$|b_{n,m}(x_1, x_2)| \leq \frac{\int_{\mathbb{R}} |\zeta|^{m+n} e^{-|\zeta|^\alpha} d\zeta}{\sqrt{B_{2n}(0)B_{2m}(0)}} \leq \frac{\Gamma\left(\frac{m+n+1}{\alpha}\right)}{\sqrt{\Gamma\left(\frac{2n+1}{\alpha}\right)\Gamma\left(\frac{2m+1}{\alpha}\right)}}$$

The following estimation is very important for the coming asymptotical analysis. It shows that $\xi_n(\cdot)$ and $\xi_m(\cdot)$ are "almost independent" if $|n-m| >> \sqrt{m+n}$.

Lemma 30. *Assume that $m \leq n$, $m = n-\zeta$, and $\zeta = o(n^{2/3})$. Then asymptotically (for $n \to \infty$)*

$$h(\zeta) = \frac{\Gamma\left(\frac{2n-\zeta+1}{\alpha}\right)}{\sqrt{\Gamma\left(\frac{2n+1}{\alpha}\right)\Gamma\left(\frac{2n+1-2\zeta}{\alpha}\right)}} \sim exp\left(-\frac{\zeta^2}{2n\alpha}\right) \tag{6.7}$$

For fixed n, the function $h(\zeta)$ is decreasing.

The proof of this lemma can be based on the direct calculations using the Stirling formula

$$\Gamma(1+x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\frac{\theta}{12x}} \ .$$

The following observation shows that in reality 6.7 is equivalent to the local CLT for Bernoulli r.v (De Moivre-Laplace theorem). Assume temporarily that numbers $\frac{2m+1}{\alpha}, \frac{2n+1}{\alpha}$ are odd integers. Put

$$2X = \frac{2m+1}{\alpha} + 1 \ , \ 2Y = \frac{2n+1}{\alpha} + 1 \ , \ \frac{m+n+1}{\alpha} = X + Y + 1 \ ,$$

then

$$\frac{\Gamma\left(\frac{m+n+1}{\alpha}\right)}{\sqrt{\Gamma\left(\frac{2n+1}{\alpha}\right)\Gamma\left(\frac{2m+1}{\alpha}\right)}} = \frac{\Gamma(X+Y)}{\sqrt{\Gamma(2X)\Gamma(2Y)}} = \tag{6.8}$$

$$\sqrt{\frac{(2X+2Y)!}{(2X)!(2Y)!}\left(\frac{1}{2}\right)^{2X+2Y}} \Bigg/ \sqrt{\frac{(2X+2Y)!}{(X+Y)!(X+Y)!}\left(\frac{1}{2}\right)^{2X+2Y}} =$$

$$\sqrt{\frac{b(2X+2Y, 2X, \frac{1}{2})}{b(2X+2Y, X+Y, \frac{1}{2})}}$$

where $b(n,x,p) = \binom{n}{x} p^x q^{(n-x)}$, $(0 < q = 1-p < 1,\ 0 < p < 1)$ is the standard notation for the binomial probabilities. It is well known that

$$b(n,x,p) \sim \frac{1}{\sqrt{2\pi npq}} e^{-\frac{\zeta^2}{2npq}} \quad , \ \zeta = x - np = o(n^{2/q}) \ .$$

Application of this result to equation 6.8 gives the asymptotics 6.7. The monotonicity also follows from well known properties of binomial coefficients. But the proof of the De Moivre-Laplace theorem is based completely on the Stirling formula for $\Gamma(1+x)$, which is true for general (very large) x.

Let us return to the proof of the theorem 29.

Lemma 31. *Let $v_\alpha(n,\Delta) = \#\{x_i \in \Delta : \xi^{(n)}(x_i) = 0\}$, then*

$$E\, v_\alpha(n,\Delta) = \frac{|\Delta|}{\pi} \quad , \quad E \sum_{k=0}^{n} v_\alpha(k,\Delta) = (n+1)|\Delta| \tag{6.9}$$

Proof. It follows from the general theory of the smooth Gaussian random processes that for the roots of $\xi^{(n)}(\cdot)$ on fixed interval Δ one can use the following “symbolic” Kac-Rice formula (see [2, ch10,3]).

$$v_\alpha(n,\Delta) = \int_\Delta \delta_0(\xi^{(n)}(x))|\xi^{(n+1)}(x)|dx \tag{6.10}$$

The justification of this formula is based on the Bulinskaja theorem as used above. It yields

$$Ev_\alpha(n,\Delta) = \int_\Delta E\,\delta_0(\xi^{(n)}(x))|\xi^{(n+1)}(x)|dx = \tag{6.11}$$

$$= \int_\Delta dx \int_{\mathbb{R}^2} \delta_0(z_1)|z_2| P_{n,x}(z_1,z_2) dz_1\, dz_2$$

where $P_{n,x}(\cdot)$ is the joint distribution density for the r.v. $\xi^{(n)}(x), \xi^{(n+1)}(x)$. This (gaussian) density does not depend on x (process $\xi^{(n)}(\cdot)$ is the stationary one) and given by the formula

$$P_{n,x}(z_1,z_2) = \frac{1}{2\pi\sqrt{\det\Sigma}} \exp\left(-\frac{(\Sigma^1 z, z)}{2}\right) ,$$

where

$$\Sigma = \begin{bmatrix} E[\xi^{(n)}(x)]^2 & E\xi^{(n)}(x)\xi^{(n+1)}(x) \\ E\xi^{(n)}(x)\xi^{(n+1)}(x) & E\xi^{(n+1)}(x)\xi^{(n+1)}(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \ .$$

(This is due to 6.5. In fact $E[\xi^{(n)}(x)]^2 = b_{2n}(0) = 1$, $E[\xi^{(n)}(x)\xi^{(2n+1)}(x)] = -b'_n(0) = 0$, $E[\xi^{(n+1)}(x)]^2 = -\frac{\partial^2 b}{\partial z^2}(0) = 1$).

Finally

$$Ev_\alpha(n,\Delta) = |\Delta| \int_{\mathbb{R}} P_{0,n}(0,z_2)|z_2|dz_2 = 2\frac{|\Delta|}{2\pi} \int_0^\infty z_2 e^{-\frac{z_2^2}{2}} dz_2 = \frac{|\Delta|}{\pi} \ .$$

The second statement of the lemma is now obvious. ∎

Unfortunately the estimation of the first moment is not sufficient for the proof of theorem 29. We have to check that, for instance,

$$\operatorname{var}\left(\sum_{k=0}^{n} v_\alpha(k,\Delta)\right) = o\left(\left(\sum_{k=0}^{n} Ev_\alpha(k,\Delta)\right)^2\right) = o(n^2)\ .$$

In fact we will prove more:

Theorem 32. *For any fixed interval Δ and any $\delta > 0$,*

(1) $var(\sum_{k=0}^{n} v_\alpha(k,\Delta)) = o(n^{\frac{3}{2}+\delta})$,

(2) $\dfrac{\sum_{k=0}^{n} v_\alpha(k,\Delta)}{n} \xrightarrow{P} |\Delta|$

and for appropriate sequences $N_n \to \infty$, e.g. $N_n = n^{2+\delta_1}$, $\delta_1 > 0$,

$$\frac{\sum_{k=0}^{n} v_\alpha(n,\Delta)}{n} \to |\Delta| \quad (P\text{-}a.s.)\ .$$

Proof. Of course

$$\operatorname{var}\left(\sum_{k=0}^{n} v_\alpha(k,\Delta)\right) = \sum_{k=1}^{n} \operatorname{var} v_\alpha(k,\Delta) + 2\sum_{k<m} \operatorname{Cov}(v_\alpha(k,\Delta), v_\alpha(m,\Delta))\ .$$

Let us start from the estimation of $\operatorname{var} v_\alpha(k,\Delta)$. It is based on the general results of [2, 10.6,10.77]. First of all we will use the following (Kac-Rice type) formula for the second factorial moment of $v_\alpha(k,\Delta)$:

$$Ev_\alpha(k,\Delta)(v_\alpha(k,\Delta)-1) = 2\int_0^{|\Delta|} (|\Delta| - h)\psi_k(h)dh$$

where $\psi_k(h) = \int\int_{\mathbb{R}^2} |z_1||z_2| P_{k,k+1}(0,0,z_1,z_2)dz_1dz_2$ and $P_{k,k+1}(\cdot)$ is the joint gaussian distribution density for the vector $\xi_k(x)$, $\xi_k(x+h)$, $\xi_k(x)$, $\xi_k(x+h)$. It is given by the covariance matrix

$$\Lambda_k(h) = \left[\begin{array}{cc|cc} 1 & b_{2k}(h) & 0 & -b'_{2k}(h) \\ b_{2k}(h) & 1b'_{2k}(h) & 0 & \\ \hline 0 & b+2k'(h) & 1 & -b_{2k}(h) \\ -b'_{2k}(h) & 0 & -b_{2k}(h) & 1 \end{array}\right].$$

$\square$

Due to formula 6.5,

$$b_{2k}(h) = 1 - \frac{h^2}{2} + \frac{\gamma_{4,h}h^4}{2h} + O(h^6)$$

and the coefficient $\gamma_{4,h}$ is uniformly bounded in k. In fact, $\gamma_{4,h}$ converges to 1 as $k \to \infty$.

Let $b_{2k}(h) = 1 - \frac{h^2}{2} + \tau(h)$. Then for appropriate constants $c_i > 0$ ($i = 0,1$) uniformly in k,

$$\psi_k(h) \le \frac{c_0\theta'(h)}{h^2} \le c_1 h$$

and, as a result, for $|\Delta| << 1$, $Ev_\alpha(k,\Delta)(v_\alpha(k,\Delta)-1) = O(|\Delta|^3)$, i.e. $\operatorname{var}(v_\alpha(k,\Delta)) = Ev_\alpha(k,\Delta)+)(|\Delta|^2)$, and $Ev_\alpha(k,\Delta) = \frac{|\Delta|}{\pi}$.

This means that $\sum_{k=0}^{n} \operatorname{var} v_\alpha(k,\Delta) = O(n)$ and for $n \to \infty$ and $|\Delta| \to 0$,

$$\sum_{k=0}^{n} \operatorname{var} v_\alpha(k,\Delta) \sim \frac{n|\Delta|}{\pi} \ .$$

Let us estimate the covariances $\operatorname{Cov}(v_\alpha(k,\Delta), v_\alpha(m,\Delta))$. If $|k-m| \le (\max(k,m))^{\frac{1}{2}+\delta}$, i.e. k and m are "close" enough then

$$|\operatorname{Cov}(v_\alpha(k,\Delta), v_\alpha(m,\Delta))| \le \sqrt{\operatorname{var}(v_\alpha(k,\Delta))(\operatorname{var} v_\alpha(m,\Delta))} = O(1) \ .$$

Due to [2, 10.6], for $Ev_\alpha(k,\Delta)v_\alpha(m,\Delta)$ one can use the formula

$$\begin{aligned} (6.12)\quad & E\int_\Delta \int_\Delta \delta(\xi_k(x_1))\delta(\xi_m(x_2))|\xi_k'(x_1)|\ |\xi_m'(x_2)|dx_1dx_2 \\ &= \int\int_{\Delta\times\Delta} dx_1dx_2 \int\int_{\mathbb{R}^2} |z_1||z_2|P_{k,m,x_1,x_2}(0,z_1,0,z_2)dz_1dz_2 \end{aligned}$$

where $P_{k,m,x_1,x_2}(y_1,z_1,y_2,z_2)$ is the (gaussian) joint distribution density for the vector $(\xi_k(x_1), \xi_k'(x_1), \xi_m(x_2), \xi_m'(x_2))$ given by the covariance matrix $b_{k,m}(x_1,x_2) =$

$$= \left[\begin{array}{cc|cc} E\xi_k(x_1)\xi_k(x_1) & E\xi_k(x_1)\xi_k'(x_1) & E\xi_k(x_1)\xi_m(x_2) & E\xi_k(x_1)\xi_m'(x_2) \\ E\xi_k'(x_1)\xi_k(x_1) & E\xi_k'(x_1)\xi_k'(x_1) & E\xi_k'(x_1)\xi_m(x_2) & E\xi_k'(x_1)\xi_m'(x_2) \\ \hline E\xi_m(x_2)\xi_k(x_1) & E\xi_m(x_2)\xi_k'(x_1) & E\xi_m(x_2)\xi_m(x_2) & E\xi_m(x_2)\xi_m'(x_2) \\ E\xi_m'(x_2)\xi_k(x_1) & E\xi_m'(x_2)\xi_k'(x_1) & E\xi_m'(x_2)\xi_m(x_2) & E\xi_m'(x_2)\xi_m'(x_2) \end{array}\right]$$

$$= \left[\begin{array}{c|c} I & \varepsilon_{k,m}C \\ \hline \varepsilon_{k,m}C^* & I \end{array}\right] \ ,$$

where $\|C\| = 1$ (we can use here, for instance, the Hilbert-Schmidt norm : $\|C\|_{1+S} = \sqrt{\operatorname{Tr}(C\cdot C^*)}$) and, due to Lemma 6.7, $\varepsilon_{k,m} \le \exp\left(-\frac{|k-m|^2}{2m\alpha}\right)$, $k \le m$ and $|k-m| = O(m^{2/3})$.

Lemma 33. *Let* $B = \left[\begin{array}{c|c} I & \varepsilon C \\ \varepsilon C^* & I \end{array}\right]$ *be a* $2n\times 2n$ *matrix, let* $\|C\|_{HS} = 1$ *and let* ε *be a small parameter. Then the determinant of* B *is* $1+\varepsilon^2\|C\|_{HS}^2 + O(\varepsilon^3)$ *and*

$$B^{-1} = \left[\begin{array}{c|c} I+\varepsilon^2CC^* & -\varepsilon C \\ -\varepsilon C^* & I+\varepsilon^2C^*C \end{array}\right] + O(\varepsilon^3)$$

(of course, the remainder is a matrix with H-S norm which is $O(\varepsilon^3)$*).*

Proof. First, we have that

$$\ln\ B = \ln\left(I + \begin{bmatrix} 0 & \varepsilon C \\ \varepsilon C^* & 0 \end{bmatrix}\right) = \varepsilon\begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} + \frac{\varepsilon^2}{2}\begin{bmatrix} CC^* & 0 \\ 0 & C^*C \end{bmatrix} + O(\varepsilon^2)$$

$$\text{and}\ \ \det B = \exp\left(\operatorname{Tr}\ \ln B\right) = \frac{\varepsilon^2}{2}(\|C\|_{HS}^2 + \|C\|_{HS}^2) + O(\varepsilon^3) \ .$$

Secondly,

$$\begin{aligned} B^{-1} &= \left(I + \varepsilon\begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}\right)^{-1} = I - \varepsilon\begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} + \varepsilon^2\begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}^2 + O(\varepsilon^3) \\ &= \begin{bmatrix} I+\varepsilon^2CC^* & -\varepsilon C \\ \varepsilon C^* & I+\varepsilon^2C^*C \end{bmatrix} + O(\varepsilon^3) \ . \end{aligned}$$

□

Now we can easily complete the proof of the theorem 32.

Proof of Theorem 32. Set $\vec{x} = (0, z_1, 0, z_2)^*$ and

$$p^{(0)}(0, z_1, 0, z_2) = \frac{e^{-\frac{\vec{x}^2}{2}}}{(2\pi)^2} = \frac{1}{(2\pi)^2} e^{-\frac{z_1^2+z_2^2}{2}} .$$

Lemma 31 gives that

$$Ev_\alpha(k, \Delta) Ev_\alpha(m, \Delta) = \int_\Delta \int_\Delta dx_1 dx_2 \int \int_{\mathbb{R}^2} p^{(0)}(0, z_1, 0, z_2) |z_1||z_2| dz_1 dz_2$$

$$\text{i.e. } \operatorname{Cov}(v_\alpha(k, \Delta), v_\alpha(m, \Delta)) = \tag{6.13}$$

$$\int_\Delta \int_\Delta dx_1 dx_2 \int \int_{\mathbb{R}^2} |z_1||z_2| (P_{k,m,x_1,x_2}(\cdot) - p^{(0)}(\cdot)) dz_1 dz_2$$

$$= \int_\Delta \int_\Delta dx_1 dx_2 \int \int_{\mathbb{R}^2} |z_1||z_2| \frac{e^{-\frac{z_1^2+z_2^2}{2}}}{(2\pi)^2} \left(\frac{e^{\left(-(b_{k,m}^{-1} - I)\vec{x}, \vec{x}\right)/2}}{\sqrt{\det b_{k,m}(\cdot)}} - 1 \right) dz_1 dz_2$$

But $(((b_{k,m} - I)\vec{x}, \vec{x})) =$

$$\left(\begin{bmatrix} 0 & \varepsilon_{k,m} C \\ \varepsilon_{k,m} C^* & 0 \end{bmatrix} \vec{x}, \vec{x} \right) + \varepsilon_{k,m} \left(\begin{bmatrix} CC^* & 0 \\ 0 & C^* C \end{bmatrix} \vec{x}, \vec{x} \right) + O(\varepsilon_{k,m}^3) .$$

Divide $\int \int_{\mathbb{R}^2}$ into two parts: $\int \int_{|\vec{x}| \leq \frac{1}{\sqrt{\varepsilon}}} + \int \int_{|\vec{x}| > \frac{1}{\sqrt{\varepsilon}}}$ and use the formula $e^{\varepsilon f} - 1 = \varepsilon f + O(\varepsilon^2)$, if $|f| \leq 1$. Let us note that

$$\int \int_{|\vec{x}| \leq \frac{1}{\sqrt{\varepsilon}}} |z_1||z_2| \left(\begin{bmatrix} 0 & \varepsilon_{k,m} C \\ \varepsilon_{k,m} C^* & 0 \end{bmatrix} \vec{x}, \vec{x} \right) dz_1 dz_2 = 0 .$$

It gives that

$$\int \int_{|\vec{x}| \leq \frac{1}{\sqrt{\varepsilon}}} |z_1||z_2| \left(\begin{bmatrix} 0 & \varepsilon_{k,m} C \\ \varepsilon_{k,m} C^* & 0 \end{bmatrix} \vec{x}, \vec{x} \right) dz_1 dz_2 \leq c \cdot \varepsilon_{k,m}^2 .$$

Obviously

$$\int \int_{|\vec{x}| > \frac{1}{\sqrt{\varepsilon}}} |z_1||z_2| \left(\begin{bmatrix} 0 & \varepsilon_{k,m} C \\ \varepsilon_{k,m} C^* & 0 \end{bmatrix} \vec{x}, \vec{x} \right) dz_1 dz_2 = O(e^{-\frac{c_1}{\varepsilon_{k,m}}})$$

and we have proven that

$$\operatorname{Cov}(f_\alpha(k, \Delta), v_\alpha(m, \Delta)) = O(\varepsilon_{k,m}^2) = O\left(\exp\left(-\frac{|k-m|^2}{\alpha \max(k, m)} \right) \right)$$

This estimation implies the relation 1 in the Theorem 32. Other relations are trivial consequences of the Chebyshev's inequality and the Borel-Cantelli lemma. □

Some further development of the ideas in Theorem 32 gives the next theorem.

Theorem 34. *Let $N(k, \Delta) = \#\{x \in \Delta : f^{(k)}(x) = 0\}$. Then for $\alpha < \infty$,*

$$E\, N(k, \Delta) = \frac{|\Delta|}{\pi} \sqrt{\frac{B_{2k+2}(0)}{B_{2k}(0)}} \sim \frac{|\Delta|}{\pi} \left(\frac{2n}{\alpha} \right)^{1/\alpha} \quad \textit{and}$$

$$\text{(1)}\quad E \left(\sum_{k=0}^{n} N(k, \Delta) \right) n \overset{\sim}{\to} \infty \frac{|\Delta|}{\pi} \left(\frac{2}{\alpha} \right)^{1/\alpha} \frac{\alpha n^{1+\alpha} \alpha}{1+\alpha},$$

$$(2)\quad \frac{\sum_{k=0}^{n} N(k,\Delta)}{n^{\frac{1+\alpha}{\alpha}}} \xrightarrow{P} \frac{|\Delta|}{\pi}\left(\frac{2}{\alpha}\right)^{1/\alpha}\frac{\alpha}{1+\alpha} = |\Delta| c_\alpha$$

Proof. The proof of part 1 is the direct repetition of the Lemma 31. The proof of part 2 requires slightly longer calculations of the second moment, but essentially is also the same proof as that in Theorem 32. The central point is the "decorelation" Lemma 31 □

Corollary 35. *Let $f_\alpha(x)$ be an entire function of the exponential order α (i.e. $|f_\alpha(z)| \le c_0 \exp(c_1|z|^\alpha)$ as $|z| \to \infty$). Then for any $\varepsilon > 0$ and fixed interval Δ, one can find an entire function $f_{\alpha'}(x)$, $\alpha' > \alpha > 0$ such that $NG(f_\alpha + \varepsilon f_{\alpha'})$ is dense in Δ. In fact, one can take $f_{\alpha'}(x,\omega)$ is a stationary gaussian process with the spectral density $\psi^2(\zeta) = \exp(-2|\zeta|^{\alpha'})$ and prove the statement holds P-a.s.*

Corollary 36. *Let f be a C^∞ function supported on the compact interval $\Delta \subset \mathbb{R}$ and which is analytic on the interior of Δ. Then for $\alpha' < 1$ and $\tilde{f}_\varepsilon(x) = f(x)(1 + \varepsilon f_{\alpha'}(x,\omega))$, the function $\tilde{f}_\varepsilon$ almost surely has $NG(\tilde{f}_\varepsilon)$ intersecting Δ in a countable dense subset.*

For example if we use $\Delta = [0,1]$ and

$$f(x) = \begin{cases} 0 & x \notin (0,1) \\ e^{-\left(\frac{1}{x}+\frac{1}{1-x}\right)} = e^{-\frac{1}{x(1-x)}} & x \in (0,1) \end{cases}$$

we will also have that $NG(\tilde{f})$ contains $\mathbb{R} \setminus [0,1]$.

The proof is based on two facts. For any $\varepsilon > 0$, $\Delta' = (\alpha,\beta)$, $\Delta'_\varepsilon = (\alpha+\varepsilon, \beta-\varepsilon)$, the function $f(x)$ admits the estimation

$$|f^{(n)}(x)| \le n! A^n(\varepsilon) \quad , x \in \Delta'_\varepsilon \ .$$

At the same time (as we know), $|f^{(n)}_{\alpha'}(x)| = O\left((n!)^{1/\alpha'}\right)$ and for large n and fixed ε, the random term $\varepsilon f_{\alpha'}(x,\omega)$ will dominate.

Using functions from Corollary 36 and their integrals one can define for any two analytic functions, $f_1(x)$, $f_2(x)$ (including polynomials) and given interval $[\alpha,\beta]$ some C^∞ interpolation $F(x)$ such that

$$F(x) = \begin{cases} f_1(x) & x \le \alpha \\ f_2(x) & x \ge \beta \end{cases}$$

and arrange that $NG(F) \cap [\alpha,\beta]$ is a dense countable subset.

This yields the next example.

Example 17. For any partition of $\mathbb{R}$ into intervals

$$\Delta_0 = (\infty, x_1], \Delta_1 = (x_1, x_2], \ldots \Delta_{n-1} = (x_{n-1}, x_n], \Delta_n = (x_n, \infty)$$

and any assignment of "empty" or "contains" to each of the even intervals, there is a C^∞ function $F(x)$ such that the set $NG(F)$ contains all the even indexed intervals assigned as "contains" and is disjoint from all the even indexed intervals assigned as "empty". $NG(F)$ will meet each of the odd indexed intervals in a countable dense set.

In particular, one can construct a C^∞ function F which has polynomial structure outside a given finite set of intervals (with arbitrarily small measure) and such that $NG(F)$ is countable and dense on these intervals.

7. Open Problems

Question 1. Assume that D is a countable subset of $\mathbb{R}$; is there an analytic function f such that $NG(f) = D$?

Question 2. Assume that Γ_n $(n = 0, 1, 2, \ldots)$ is a sequence of pairwise disjoint countable subsets of $\mathbb{R}$, such that each Γ_n has no finite accumulation points. Also assume that the Γ_n's satisfy the intermittency condition

$$\text{if } x, y \in \Gamma_n \text{ and } x < y \text{ then there exists such } z \in \Gamma_{n+1} \text{ that } x < z < y. \tag{7.1}$$

Is there an analytic function f such that for each n, Γ_n is exactly the set of zeros of $f^{(n)}$?

Even the finite (polynomial) version of the previous question does not seem to have a known answer.

Question 3. Let Γ_n $(n = 0, 1, 2, \ldots, N)$ be disjoint finite subsets of $\mathbb{R}$ such that $|\Gamma_n| = N - n$. Assume they satisfy the intermittency condition (7.1) for $n = 0, 1, 2, \ldots, N-2$. Under what additional conditions is there a polynomial $P(x)$ such that for each $n = 0, 1, \ldots, N$, Γ_n is the set of zeros of $P^{(n)}(x)$? For example, if $\Gamma_0 = \{x, y\}$ and $\Gamma_1 = \{z\}$, then z has to be $(x + y)/2$.

Question 4. What is $NG(e^{-\frac{1}{x}})$? If $f(x) = e^{-\frac{1}{x}}$, and $f^{(n)}(x)$ is written as $e^{-\frac{1}{x}} \cdot P_n(x)/x^{2n}$, is there a natural recurrence relation between $P_{n-1}(x)$, $P_n(x)$ and $P_{n+1}(x)$?

Question 5. If F_n $(n = 0, 1, 2, \ldots)$ is an increasing family of closed subsets of $\mathbb{R}$ satisfying the intermittency condition

$$(\forall n)\, (\forall x < y \in F_n)\, (\exists z \in F_{n+1})\, (x < z < y)$$

is there a function $f \in C^\infty(\mathbb{R})$ such that $NG(f) = \bigcup_n F_n$?

Question 6. If $NG(f)$ is dense and has measure 0, and $f \in C^\infty$ is the sum of a convergent series of analytic functions, is $NG(f)$ necessarily countable?

Question 7. Is the set $Q_0(BR_\sigma)$ nowhere dense?

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