A NON-PARTITIONABLE MAD FAMILY

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Abstract. It is consistent that there is a mad family which can not be partitioned into two nowhere mad families.

1. Introduction

In [Sim80], Simon showed that there is a pair of Frechet-Urysohn spaces whose product is not Frechet-Urysohn. The spaces he constructed were so-called Ψ-like spaces from almost disjoint families of subsets of ω. In particular he showed that there is a maximal almost disjoint (mad) family which could be suitably partitioned.

Definition 1. An almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ is *nowhere mad* if for each $X \subset \omega$, either X is almost contained in a finite union from $\mathcal{A}$, or there is an infinite $Y \subset X$ such that $Y \cap a$ is finite for each $a \in \mathcal{A}$. If, on the other hand, $\mathcal{A} \restriction X = \{a \cap X : a \in \mathcal{A}\}$ is infinite and for each infinite $Y \subset X$ there is an $a \in \mathcal{A}$ such that $a \cap Y$ is infinite, we would say that $\mathcal{A} \restriction X$ is a mad family on X.

Simon's key construction was to produce a mad family $\mathcal{A}$ which could be partitioned into two nowhere mad families. Let us say that such a family is partitionable. Simon actually proved a much stronger result.

Proposition 2. [Sim80] *For each mad family $\mathcal{A}$ on ω, there is an infinite $X \subset \omega$, such that $\mathcal{A} \restriction X$ is a partitionable mad family on X.*

The following idea is very well known.

Proposition 3. *If a mad family $\mathcal{A}$ satisfies that for each $X \subset \omega$, $\mathcal{A} \restriction X$ is either finite or has cardinality $\mathfrak{c}$, then $\mathcal{A}$ is partitionable.*

In this paper we show it is consistent to have a strongly non-partitionable mad family as follows.

Theorem 4. *It is consistent that there is a mad family $\mathcal{A}$ of cardinality $\mathfrak{c} > \omega_1$ such that for each $\mathcal{B} \subset \mathcal{A}$ of cardinality $\mathfrak{c}$, there is an X such that $\mathcal{B} \restriction X$ has cardinality ω_1 and is a mad family on X. Therefore, if $X \subset \omega$ and $\mathcal{A} \restriction X$ is partitionable then $\mathcal{A} \restriction X$ has cardinality ω_1.*

2. Main Lemma

We let S^1_2 denote the set of ordinals in ω_2 of cofinality ω_1. We define below a finite support iteration $\{P_\alpha, \dot{Q}_\alpha : \alpha \in \omega_2\}$ of σ-centered (hence ccc) posets. For each $\alpha \in S^1_2$, $\dot{A}_\alpha$ will be chosen (by a $\diamondsuit$-sequence on S^1_2) to be P_α-name of a cofinal subset of $\alpha \setminus S^1_2$, and for $\alpha \notin S^1_2$, $\dot{A}_\alpha$ will be the empty set (which is its own name). Each poset $\dot{Q}_\alpha$ will canonically define a name $\dot{a}_\alpha$ of a subset of ω.

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Definition 5. For each $\alpha < \omega_2$, define $\dot{Q}_\alpha$ a P_α-name of a poset as follows. We are given some P_α-name $\dot{A}_\alpha$ of a subset of $\alpha \setminus S^1_2$ and the family $\{\dot{a}_\beta : \beta < \alpha\}$ consisting of P_α-names of subsets of ω. The P_α-name $\dot{Q}_\alpha$ satisfies that

$$\Vdash_{P_\alpha} \dot{Q}_\alpha \text{ is an order on the set } 2^{<\omega} \times [\alpha \setminus \dot{A}_\alpha]^{<\omega} \text{ and } p = (S_p, F_p) \text{ for } p \in \dot{Q}_\alpha$$

where $p < q$ if $S_p \supset S_q$, $F_p \supset F_q$, and for all $\beta \in F_q$, $S_p(k) = 0$ for each $k \in \dot{a}_\beta \cap \mathrm{dom}(S_p) \setminus \mathrm{dom}(S_q)$. If $\alpha \in S^1_2$, then we will ensure that $\Vdash_{P_\alpha} \dot{A}_\alpha$ is cofinal in α. Otherwise $\Vdash_{P_\alpha} \dot{A}_\alpha$ is empty. The definition of $\dot{a}_\alpha$ is the set of $k \in \omega$ such that some $q \in \dot{Q}_\alpha$ in the generic filter satisfies $S_q(k) = 1$.

When selecting elements p of P_{ω_2}, we may assume that for each $\gamma \in \mathrm{dom}(p)$, there is an $S \in 2^{<\omega}$ and $F \in [\gamma]^{<\omega}$ such that $p \restriction \gamma \Vdash p(\gamma) = (S, F)$ (this is often referred to as *determined* conditions).

Lemma 6. *For each $\lambda \in S^1_2$, $1 \Vdash_{P_{\omega_2}}$ "the family $\{\dot{a}_\alpha \cap \dot{a}_\lambda : \alpha \in \dot{A}_\lambda\}$ is a mad family on $\dot{a}_\lambda$."*

Proof. Let $\lambda \in S^1_2$, $\dot{Y}$ be a P_{ω_2}-name of an infinite subset of ω and assume that $p_0 \Vdash \dot{Y} \subset \dot{a}_\lambda$. Towards a contradiction, assume that $p_0 \Vdash \dot{Y} \cap \dot{a}_\beta$ is finite for all $\beta \in \dot{A}_\lambda$. Since $\Vdash \dot{a}_\lambda \cap \dot{a}_\beta$ is finite for all $\beta \in \lambda \setminus \dot{A}_\lambda$, we have that p_0 forces that $\dot{Y} \cap \dot{a}_\beta$ is finite for all $\beta < \lambda$. Fix any countable elementary submodel M of $H(\theta)$ such that $p_0, \dot{Y}, \lambda$ and $\dot{A}_\lambda$ are in M. Let $\mu < \lambda$ be chosen so that $M \cap \lambda \subset \mu$ and fix any $\alpha \in (\mu, \lambda)$ such that there is a $p_1 < p_0 \restriction \lambda$, such that $p_1 \Vdash \alpha \in \dot{A}_\lambda$ (recall that $\dot{A}_\lambda$ is a P_λ-name).

Let p_2 be the meet of p_1 and p_0, hence $p_2 \restriction \lambda = p_1$ and $p_2 \restriction \gamma \Vdash p_2(\gamma) = p_0(\gamma)$ for $\gamma > \lambda$. Now let $p_3 < p_2$ be chosen so that there is some m_0 with $p_3 \Vdash \dot{Y} \cap \dot{a}_\alpha \subset m_0$. We may assume there is an m_0 so that for each $\gamma \in \mathrm{dom}(p_3)$, $p_3 \restriction \gamma \Vdash S_{p_3(\gamma)} \in 2^{m_0}$. Let G_α be any P_α-generic filter with $p_3 \restriction \alpha \in G_\alpha$. We may assume, by extending p_3, that for each $\gamma \in \mathrm{dom}(p_3)$, $F_{p_3(\gamma)} \subset \mathrm{dom}(p_3)$. In addition, we may assume that for $\gamma \in \mathrm{dom}(p_3) \setminus \lambda + 1$, $p_3 \restriction \gamma \Vdash \lambda \in F_{p_3(\gamma)}$.

In $V[G_\alpha]$, the terms $\dot{a}_\beta$ for $\beta \in F_{p_3(\alpha)}$ have all been evaluated and $\dot{Y}$ is forced to be almost disjoint from the union. Fix any $n > m_0$, not in this finite union, such that there is a $q \in P_{\omega_2} \cap M$ such that $q \restriction \alpha \in G_\alpha$, q is compatible with p_3, and $q \Vdash n \in \dot{Y}$. We may further assume that there is an $n < m_1 \in \omega$ such that for each $\gamma \in \mathrm{dom}(q) \setminus \alpha$, $q \restriction \gamma \Vdash S_\gamma = S_{q(\gamma)} \in 2^{m_1}$. Just as we did with p_3, we may also assume that for each $\gamma \in \mathrm{dom}(q)$, $F_{q(\gamma)} \subset \mathrm{dom}(q)$. Recall that since $q \in M$, we have that $\mathrm{dom}(q) \cap (\mu, \lambda)$ is empty. Since q is compatible with p_3, we have that for each $\gamma \in \mathrm{dom}(q) \cap \mathrm{dom}(p_3)$, and $m_0 \le k < m_1$ such that $S_\gamma(k) = 1$, and each $\beta \in F_{p_3(\gamma)}$, there is an extension r of $p_3 \restriction \gamma$ and $q \restriction \gamma$ so that $r \Vdash k \notin \dot{a}_\beta$.

We define $p_4 < p_3, q$ so that $p_4 \Vdash n \in \dot{a}_\alpha$. Let $p_4 \restriction \alpha$ be any member of G_α below both $p_3 \restriction \alpha$ and $q \restriction \alpha$ and so that $S_{p_4(\gamma)} \in 2^{<\omega} \setminus 2^{<m_1}$ for all $\gamma \in \mathrm{dom}(p_4) \cap \alpha$. Define $F_{p_4(\alpha)}$ to be $F_{p_3(\alpha)}$ and $S_{p_4(\alpha)} = S_{p_3(\alpha)}{}^\frown 0 \ldots 010 \ldots 0 \in 2^{m_1}$ where the 1 occurs at position n (hence $n \in \dot{a}_\alpha$). For $\gamma \in \mathrm{dom}(p_3) \setminus (\mathrm{dom}(q) \cup \alpha + 1)$, let $S_{p_4(\gamma)}$ be equal to $S_{p_3(\gamma)}{}^\frown 0 \cdots 0 \in 2^{m_1}$ and $F_{p_4(\gamma)}$ is equal to $F_{p_3(\gamma)}$. For $\gamma \in \mathrm{dom}(q)$, set $S_{p_4(\gamma)} = S_{q(\gamma)} \in 2^{m_1}$ and $F_{p_4(\gamma)} = F_{p_3(\gamma)} \cup F_{q(\gamma)}$. Note that $q \restriction \lambda \Vdash n \in \dot{a}_\lambda$ since $p_0 \Vdash \dot{Y} \subset \dot{a}_\lambda$ and so $\lambda \in \mathrm{dom}(q)$. Furthermore, $S_\gamma(n) = 0$ for $\gamma \in \mathrm{dom}(p_3) \cap \mathrm{dom}(q) \setminus (\lambda + 1)$ since $\lambda \in F_{p_3(\gamma)}$ and p_3, q are compatible. Since $p_3 \Vdash \alpha \in \dot{A}_\lambda$, it follows that $\alpha \notin F_{p_3(\lambda)}$.

It certainly follows that $p_4 \Vdash n \in \dot{a}_\alpha$. We check that $p_4 < p_3$. The previous paragraph shows that $p_4 \restriction \gamma \Vdash S_{p_4(\gamma)}(k) = 0$ for all $k \in \dot{a}_\alpha \setminus \mathrm{dom}(S_{p_3(\gamma)})$ for all $\gamma \in \mathrm{dom}(p_3)$ for which $\alpha \in F_{p_3(\gamma)}$. Now we consider other ordinals in the various $F_{p_3(\gamma)}$. It is also true, by construction, that for each $\gamma \in \mathrm{dom}(q) \cap \mathrm{dom}(p_3)$ and each $\beta \in F_{p_3(\gamma)} \setminus (\mathrm{dom}(q) \cup \{\alpha\})$, $p_4 \restriction \gamma \Vdash \dot{a}_\beta \cap [m_0, m_1)$ is empty. In addition, if $\beta \in F_{p_3(\gamma)} \cap \mathrm{dom}(q)$, then $q \restriction \gamma \Vdash S_\gamma(k) = 0$ for each $k \in [m_0, m_1) \cap \dot{a}_\beta$ since $\beta \in \mathrm{dom}(q)$ and p_3, q are compatible. Putting this all together, we have that for each $\gamma \in \mathrm{dom}(q) \setminus \alpha = \mathrm{dom}(q) \setminus \lambda$ and each $\beta \in F_{p_3(\gamma)}$ (which is empty if $\gamma \notin \mathrm{dom}(p_3)$),

$$p_4 \restriction \gamma \Vdash S_{p_4(\gamma)}(k) = S_\gamma(k) = 0 \text{ for each } k \in \dot{a}_\beta \setminus \mathrm{dom}(S_{p_3(\gamma)}).$$

For all $\gamma \in \mathrm{dom}(p_3) \setminus \mathrm{dom}(q) = \mathrm{dom}(p_4) \setminus \mathrm{dom}(q)$ and $k \in \mathrm{dom}(S_{p_4(\gamma)}) \setminus \mathrm{dom}(S_{p_3(\gamma)})$, the only case where $S_{p_4(\gamma)}(k) = 1$ is when $\gamma = \alpha$ and $k = n$. In this case, we chose $n \notin \bigcup\{a_\beta : \beta \in F_{p_3(\alpha)}\}$. This completes the proof that $p_4 < p_3$. Similarly, it follows very easily that $p_4 < q$ since for all $\gamma \in \mathrm{dom}(q)$, $S_{p_4(\gamma)} = S_{q(\gamma)}$.

Now that we have $p_4 < p_3, q$, we observe that $p_4 \Vdash n \in \dot{a}_\alpha \cap \dot{Y} \setminus m_0$, which is the contradiction we seek. $\square$

3. Proof of Main Theorem

In this section we apply Lemma **??** to prove Theorem **??**.

Definition 7. A sequence $\{A_\alpha : \alpha \in S^1_2\}$ is a $\diamondsuit$-sequence on S^1_2 if for each $\alpha \in S^1_2$, $A_\alpha \subset \alpha$, and for each $T \subset \omega_2$, the set $\{\lambda \in S^1_2 : T \cap \lambda = A_\lambda\}$ is stationary. The statement $\diamondsuit_{S^1_2}$ is the assertion that a $\diamondsuit$-sequence on S^1_2 exists.

It is well-known that $\diamondsuit_{S^1_2}$ is consistent and implies that $2^{\omega_1} = \omega_2$. The base set for our poset P_{ω_2} defined in Definition **??** is the family $\mathcal{P}$ of functions with domain a finite subset of ω_2 and range contained in $2^{<\omega} \times [\omega_2]^{<\omega}$. Therefore, we may think of subsets of $\mathcal{P} \times \omega_2$ as potential P_{ω_2}-names of subsets of ω_2. Recall that for each $\beta \in \omega_2$, $\check{\beta}$ is the canonical P_{ω_2}-name for β and each subset of $P_{\omega_2} \times \{\check{\beta} : \beta \in \omega_2\}$ is a P_{ω_2}-name of a subset of ω_2. We can abuse notation slightly and treat subsets of $P_{\omega_2} \times \omega_2$ as though they were such a name.

Let f be any 1-1 function from ω_2 onto $\mathcal{P} \times \omega_2$. For each $\alpha \in S^1_2$, fix any ω_1-sequence, C_α, of successor ordinals cofinal in α. For each $\alpha \in S^1_2$, we recursively define $\dot{A}_\alpha$ (and therefore P_α):

$$\dot{A}_\alpha = \begin{cases} f(A_\alpha) & \text{if } f(A_\alpha) \text{ is a } P_\alpha\text{-name and } 1 \Vdash f(A_\alpha) \text{ is cofinal in } \alpha \setminus S^1_2 \\ \check{C}_\alpha & \text{otherwise.} \end{cases}$$

Lemma 8. *The family $\{\dot{a}_\beta : \beta \in \omega_2 \setminus S^1_2\}$ is forced to be a mad family on ω.*

Proof. A routine density argument (which we leave to the reader) shows that for each P_{ω_2}-name, $\dot{Y}$, of an infinite subset of ω, and each $p \in P_{\omega_2}$, there is an $\alpha < \omega_2$ and a $q < p$ such that $q \Vdash \dot{Y} \cap \dot{a}_\alpha$ is infinite. If $\alpha \notin S^1_2$, we are done, while if $\alpha \in S^1_2$, we can apply Lemma **??**. $\square$

Proof of Theorem **??**. Assume $\diamondsuit_{S^1_2}$ and let $\dot{A}_\alpha$ and the iteration sequence $\{P_\alpha, \dot{Q}_\alpha : \alpha \in \omega_2\}$ be defined as above. Let G be a P_{ω_2}-generic filter and for each $\alpha \in \omega_2$, let $a_\alpha = \mathrm{val}_G(\dot{a}_\alpha)$. Our desired family is $\mathcal{A} = \{a_\beta : \beta \in \omega_2 \setminus S^1_2\}$. By Lemma **??**, $\mathcal{A}$ is a mad family on ω. Assume that $\mathcal{B} \subset \mathcal{A}$ has size $\omega_2 = \mathfrak{c}$. Let $I = \{\beta \in \omega_2 : a_\beta \in \mathcal{B}\}$ and let $\dot{I} \subset P_{\omega_2} \times \omega_2$ be a (pseudo) P_{ω_2}-name for I in the ground model. We now work in the ground model. We may (and do) assume that for each $\beta \in \omega_2$, the set of $p \in P_{\omega_2}$ such that $(p, \beta) \in \dot{I}$ is countable (since P_{ω_2} is ccc, there is such a name for I). In addition, we may assume that 1 forces that $\dot{I}$ is unbounded in ω_2 and disjoint from S^1_2.

Let $T = f^{-1}(\dot{I})$. It follows easily that there is a closed and unbounded $C \subset \omega_2$ such that for each $\gamma \in C$ and $\beta < \gamma$ and each $p \in P_{\omega_2}$ such that $(p, \beta) \in \dot{I}$, $\mathrm{dom}(p) \subset \gamma$ and $f(\beta) \in P_\gamma \times \gamma$. Moreover, since P_β has cardinality less than ω_2 for each $\beta < \omega_2$, we may assume that for each $\gamma \in C$ and each $\beta < \gamma$ and $p \in P_\beta$, there is $\zeta \in \gamma \cap T$ such that $f(\zeta) = (q, \xi)$ for some $\xi > \beta$ and $q < p$ (hence $1 \Vdash_{P_\gamma} f(T \cap \gamma) \cap (\beta, \gamma) = \dot{I} \cap (\beta, \gamma) \neq \emptyset$).

By $\diamondsuit_{S^1_2}$, there is a $\lambda \in C \cap S^1_2$ such that $T \cap \lambda = A_\lambda$. Since $\lambda \in C$, it follows that $f(A_\lambda) = f(T \cap \lambda)$ is a P_λ-name of a subset of $\lambda \setminus S^1_2$ which is forced to be cofinal in λ. It follows that $\dot{A}_\lambda$ is $f(A_\lambda)$ and that $1 \Vdash_{P_\lambda} \dot{I} \cap \lambda = \dot{A}_\lambda$. By Lemma **??**, we conclude that, in $V[G]$, $\{a_\beta \cap a_\lambda : \beta \in I \cap \lambda\}$ is a mad family on a_λ. Therefore we have shown that with $X = a_\lambda$, $\mathcal{B} \restriction X$ has cardinality ω_1 and is a mad family on X. $\square$

References

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