TWO TO ONE IMAGES AND PFA

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Abstract. We prove that all maps on N^* that are exactly two to one are trivial if PFA is assumed.

1. Introduction

A map $f : X \to K$ is precisely two to one if for each $k \in K$, there are exactly two points of X that map to k. For the remainder of the paper we are assuming that f is a precisely two to one mapping from N^* onto some (compact) space K. The question of whether there are non-trivial two to one maps on N^* is motivated by the papers of van Douwen [vD93] and R. Levy [Lev04]. In particular, Levy asks if every two to one image of N^* is homeomorphic to N^*. In fact there are several questions in [Lev04] that are consistently answered by the results in this paper. The behavior of two to one maps on N^* when CH is assumed is investigated in [DT04]. It is well known that van Douwen has shown in [vD93] that there is a compact separable space which is a $\leq$2 to one image of N^* and this pathology motivates the current study. R. Levy showed that if f is precisely two to one on N^* then K will have weight equal to $\mathfrak{c}$ and countable discrete subsets of K will have closure homeomorphic to βN.

In the two to one mapping context, it is natural to say that a mapping g from X to K is *trivial* if there are disjoint clopen subsets A, B of X such that $g[A] = g[B] = K$.

Proposition 1. *If f is locally one to one (every point has a neighborhood on which f is one to one), then N can be partitioned into $a \cup b$ such that $f[a^*] = f[b^*] = K$. Since f is two to one, f is then a homeomorphism on each of a^* and b^*.*

Proof. If each point of N^* has a neighborhood on which f is one to one, then there is a finite cover by such neighborhoods. Let $\mathcal{A}$ be a finite partition of N such that f is one to one on each $a^* \in \mathcal{A}$. Enumerate $\mathcal{A} = \{a_i : i \leq n\}$. We will use induction on n. Consider the compact set $B_0 = f^{-1}(f[a_0^*]) \setminus a_0^*$ and note that $f[B_0] = f[a_0^*]$. Since f is two to one, $f[B_0 \cap a_1^*]$ is disjoint from $f[\bigcup_{1<j} a_j^*]$. Therefore there is a $c_1 \subset a_1$ such that $B_0 \cap a_1^* \subset c_1^*$ and $f[c_1^*]$ also disjoint from $f[\bigcup_{1<j} a_j^*]$. Since f is precisely two to one, and is one to one on c_1^*, it follows that $f[c_1^*] \subset f[a_0^*]$. That is, we have shown that $B_0 \cap a_1^* = c_1^*$. The same argument applies for each $i > 0$ replacing 1, hence B_0 is equal to b_0^* for some infinite $b_0 \subset N \setminus a_0$. It follows that the restriction of f to the union of $\{(a_1 \setminus b_0)^*, (a_2 \setminus b_0)^*, \ldots, (a_n \setminus b_0)^*\}$ is precisely two to one and is one to one on each piece. $\square$

2000 *Mathematics Subject Classification.* Primary 54A25.
Key words and phrases. two to one maps, Stone-Cech, PFA.
This article was considerably improved by the careful referee's reports.
Supported by NSF grant DMS-0103985.

The statements of PFA, OCA, MA and MA(ω_1) can be found in [Tod89] and some familiarity will be assumed. Basic information about N^* can be found in [Wal74]. Of course it is well known (see [Vel93]) that OCA and MA implies that the mapping, $f^{-1} \circ f$ from a^* to b^* in the above proposition will actually be a trivial mapping. For a function f, we will use $f(\cdot)$ when the function is applied to a member of its domain and $f[\cdot]$ when we applying to a set of elements from the domain.

Definition 2. Let $\mathcal{J}$ be the collection of those sets $a \in [\omega]^\omega$ such that, on a^*, f is precisely two to one and locally one to one. Let $\mathcal{J}'$ denote the ideal generated by $\mathcal{J}$.

Proposition 3. *If a_0, a_1 are disjoint infinite subsets of N and f is one to one on each of a_0^* and a_1^*, then $f[a_0^*] \cap f[a_1^*]$ is clopen in K and is equal to $f[c^*]$ for some $c \subset a_0 \cup a_1$ in $\mathcal{J}$.*

Proof. Set $A = N \setminus (a_0 \cup a_1)$ and note that $f[A^*]$ is disjoint from $f[a_0^*] \cap f[a_1^*]$ by the two to one property of f. Thefore $K \setminus f[A^*]$ is open and is easily seen to be equal to the closed set $f[a_0^*] \cap f[a_1^*]$. Therefore there is an $c \subset N$ such that $c^* = f^{-1}(U)$ where U is the clopen set $f[a_0^*] \cap f[a_1^*]$. It is now routine to verify that c is as required since $f[c^*] = f[(a_0 \cap c)^*] = f[(a_1 \cap c)^*]$ and f is two to one. □

Proposition 4 (OCA + MA)**.** *For each $a \in \mathcal{J}$, there will be a permutation h_a on some cofinite subset of a such that h_a is never the identity, h_a^2 is the identity, and for each $b \subset a$, $f[b^*] = f[h_a[b]^*]$ and is clopen.*

Proof. By Proposition 1, a can be partitioned as $a_0 \cup a_1$ such that $f[a_0^*] = f[a_1^*]$. It follows easily that $f^{-1} \circ f$ restricts to a homeomorphism from a_0^* to a_1^*. By OCA and MA, this is a trivial homeomorphism and h_a is the witness together with its inverse (with possibly finitely many elements of a removed). For each $b \subset a$, each of $f[(b \cap a_0)^*] = f[(h(b \cap a_0)\ )^*]$ and $f[(b \cap a_1)^*] = f[(h(b \cap a_1)\ )^*]$ are clopen by Proposition 3. □

Naturally the task is to prove that $\mathcal{J}$ does not generate a proper ideal. The first step is to prove that $\mathcal{J}$ is not empty. We will proceed by first showing that if f is not locally one to one, then there is a point x such that for every countable family $\{A_n : n \in \omega\} \subset x$, there is an $A \in \mathcal{J}$ such that A is almost contained in each A_n.

There are two main results. The first is Lemma 17 which is critical to establishing that such an x exists. The second, Theorem 25, is to show that this leads to a contradiction.

We will certainly need the following results from [Far00]

Definition 5. [Far00, 3.3.2] An ideal $\mathcal{I} \subset \wp(N)$ is ccc over fin, if there is no uncountable family of almost disjoint subsets of N such that none are in $\mathcal{I}$.

The following are consequences of OCA and MA (and therefore of PFA).

Proposition 6 (OCA + MA)**.** [Far00, 3.8.2] *If $\Phi : \wp(N)/fin \to \wp(N)/fin$ is a homomorphism, then there is an $A \subset N$ and an $h : A \to N$, such that $\{a \subset N : \Phi(a) = h^{-1}(a)\}$ is ccc over fin.*

It will be useful to state the topological dual (a similar but slightly weaker formulation was given in [Far00, 3.5.5]). The kernel of Φ will form an ideal of subsets of N and so the closures of the complements will intersect to a closed set

$K \subset N^*$. Therefore Φ^{-1} will induce an isomorphism from a subalgebra of $\wp(N)/fin$ to the clopen subsets of K. A closed set $F \subset N^*$ will be said to be ccc over fin if there is no uncountable family of pairwise disjoint clopen subsets of N^* each meeting F. A closed set which is ccc over fin will be nowhere dense in N^*.

Proposition 7 (OCA + MA)**.** *If H is a continuous mapping from N^* onto a subset K of N^*, then there is an $A \subset N$ and a function $h : A \to N$ so that $H[A^*]$ is clopen, $H \restriction A^* = \beta h \restriction A^*$, and $H[(N \setminus A)^*]$ is ccc over fin.*

Corollary 8 (OCA + MA)**.** *If a nowhere dense set T of N^* is homeomorphic to N^*, then there does not exist an uncountable family of pairwise disjoint clopen subsets of N^* each of which meets T.*

2. Basic properties of f and K

We will have to show that K is nowhere ccc (a space is said to be nowhere ccc if no non-empty open subset is ccc). Fix any closed subset Z of N^* such that f restricted to Z is irreducible (meaning no proper closed subset of Z will map onto).

Lemma 9 (MA)**.** *The set $f[N^* \setminus Z]$ is dense in K.*

Proof. Let W be a non-empty open subset of K and assume, for a contradiction that $W \cap f[N^* \setminus Z]$ is empty. Therefore, $f^{-1}(W)$ is contained in Z. Let U be a non-empty clopen subset of Z such that $U \subset f^{-1}(W)$. Since $f \restriction Z$ is irreducible, $J_a = a \cap f^{-1}(f[Z - a])$ is nowhere dense in Z for each a clopen in Z. Also, $K \setminus f[Z \setminus U]$ is a non-empty open subset of K, hence there is a $b_0 \in [N]^\omega$ such that $f[b_0^*] \subset K \setminus f[Z \setminus U]$. Since we are assuming $f^{-1}(f[U]) \subset Z$, we have that $f^{-1}(f[b_0^*]) \subseteq U$. Let $\{b_\alpha : \alpha \in \mathfrak{c}\}$ enumerate $[b_0]^\omega$. For each $\alpha \in \mathfrak{c}$, $J_\alpha = J_{b_\alpha^*}$ is a nowhere dense subset of b_0^*. By Martin's Axiom, it is routine to inductively choose a sequence $\{d_\alpha : \alpha < \mathfrak{c}\} \subset [b_0]^\omega$, descending mod finite, so that $d_\alpha^* \cap J_\alpha = \emptyset$ for each $\alpha < \mathfrak{c}$. Therefore there is a point $x \in b_0^*$ such that $x \notin \bigcup\{J_\alpha : \alpha \in \mathfrak{c}\}$. Since f is precisely two to one, there is a point $x' \neq x$ such that $f(x') = f(x)$. Since $x' \neq x$, there is an $\alpha \in \mathfrak{c}$ such that $x \in b_\alpha^*$ and $x' \notin b_\alpha^*$. Since $x \notin J_\alpha$, it follows that $x' \notin Z \setminus b_\alpha^*$ contradicting that $f^{-1}(f[b_0^*]) \subset Z$. □

Lemma 10 (MA)**.** *For each $a \in [\omega]^\omega$, there is a $b \in [a]^\omega$ such that $f \restriction b^*$ is one to one.*

Proof. Since f is two to one, it will suffice to find a $b \in [a]^\omega$ so that $K = f[(N \setminus b)^*]$. If a^* is not contained in Z, let $b \subset a$ be such that $b^* \cap Z = \emptyset$. Since $(N \setminus b)^* \supset Z$ and $f[Z] = K$, it follows that $K = f[(N \setminus b)^*]$.

Otherwise we have that $a^* \subset Z$. In this case, $(N \setminus a)^* \supset N^* \setminus Z$. By Lemma 9, $f[(N \setminus a)^*]$ will contain a dense subset of K, and, being compact, will contain K. □

Lemma 11 (MA)**.** *If $f \restriction b^*$ is one to one, $b \in [N]^\omega$, then the interior of $f[b^*]$ is equal to the union of clopen sets of the form $f[c^*]$ for $c \in \mathcal{J}$ and $c^* \subset f^{-1}(f[b^*])$.*

Proof. Let b be as in the hypothesis of the Lemma and let y be in the interior of $f[b^*]$. Since f is one to one on b^*, there is an $x \in (N \setminus b)^*$ such that $f(x) = y$. By continuity, there is an a_1, disjoint from b such that $f[a_1^*]$ is contained in $f[b^*]$. If we let $a_0 = b$, then the Lemma now follows by Proposition 3. □

For $b \subset N$, let f_{+b} denote the mapping $f \restriction b^*$ and $f_{-b} = f \restriction (N \setminus b)^*$.

Lemma 12. *If f is one to one on b^*, then $g_b = f_{-b}^{-1} \circ f_{+b}$ is an embedding of b^* into $(N \setminus b)^*$. If $g_b[b^*]$ has some interior, say c^*, then $c \cup b$ contains a member of $\mathcal{J}$.*

Proof. It clearly follows from the two to one assumption on f, that g_b is an embedding. If $c \subset N \setminus b$ is such that c^* is contained in $g_b[b^*]$, then f is one to one on each of c^* and b^*. Proposition 3 implies that $c \cup b$ contains some member of $\mathcal{J}$ (in fact c will be in $\mathcal{J}'$). □

Proposition 13 (OCA + MA)**.** *If f is one to one on b^*, then b can be partitioned into two, b_0 and b_1, such that $f[b_0^*]$ is clopen and $f[b_1^*]$ is nowhere dense.*

Proof. Again let g_b denote the embedding of b^* into $(N \setminus b)^*$ given by $f_{-b}^{-1} \circ f_{+b}$. By Proposition 7, there is $c \subset N \setminus b$ such that $c^* \subset g_b[b^*]$ and $g_b[b^*] \setminus c^*$ is nowhere dense. There are $b_0 \subset b$ such that $g_b[b_0^*] = c^*$ and $b_1 = b \setminus b_0^*$ will satisfy $g_b[b_1^*]$ is nowhere dense. It follows that $f[b_0^*] = f[c^*]$ and $f[b_0^*] = K \setminus f[(N \setminus (b_0 \cup c)^*]$ is clopen. In addition, $f[b_1^*]$ is nowhere dense in K because $g_b[b_1^*]$ is equal to $f_{-b}^{-1}(f[b_1^*])$ and is nowhere dense in $(N \setminus b)^*$. □

Although we will not need this result until the next section, this still seems the most natural place to present it.

Proposition 14. *If $a \in \mathcal{J}$ then there is a $c \in [a]^\omega$ such that $c^* \subset Z$ and $f[c^*]$ is disjoint from $f[Z \setminus c^*]$.*

Proof. By Definition 2 and Proposition 1, a can be partitioned as $a_0 \cup a_1$ with $f[a_0^*] = f[a_1^*]$. Since f maps Z irreducibly onto K, $f[a^*]$ is covered by $f[a_0^* \cap Z]$ and $f[a_1^* \cap Z]$, so assume $a_0^* \cap Z$ is not empty. Let $W = K \setminus f[Z \setminus a_0^*] \subseteq f[a_0^* \cap Z]$ and recall that W is non-empty open since $f \restriction Z$ is irreducible. Choose any infinite $c \subset a_0$ such that $f[c^*] \subset W$. Since $f[a_1^*]$ contains $f[c^*]$ and $f[a_1^* \cap Z]$ is disjoint from $f[c^*]$, it follows that $f[c^*] \subseteq f[a_1^* \setminus Z] \subseteq f[a_0^* \cap Z]$. Since f is one to one on a_0^*, we have that $f[c^*]$ is disjoint from $f[(a_0 \setminus c)^*]$ and so $c^* \subset a_0^* \cap Z$. Finally, since $f[Z \setminus c^*] \subseteq f[Z \setminus a_0^*] \cup f[(a_0 \setminus c)^*]$, we also have that $f[c^*]$ is disjoint from $f[Z \setminus c^*]$. □

Lemma 15 (MA)**.** *The space K is nowhere ccc.*

Proof. Observe that if U, W are disjoint non-empty open subsets of Z, then $K \setminus f[Z \setminus U]$ and $K \setminus f[Z \setminus W]$ are disjoint non-empty open subsets of K. Therefore, it suffices to show that Z is nowhere ccc. Let $A \subset N$ and $A^* \cap Z \neq \emptyset$ and assume that $A^* \cap Z$ is ccc. Since $f[Z \setminus A^*]$ and $f[A^* \cap Z]$ meet in a nowhere dense subset of K, there is a clopen subset b_0 of $A^* \cap Z$ such that $f[b_0] \cap f[Z \setminus A^*]$ is empty. Since $Z \setminus A^*$ is compact, there is a clopen set B^* of N^* such that $Z \setminus A^* \subset B^*$ and $f[b_0] \cap f[B^*]$ is empty.

Let $\mathcal{B} = \{b_\alpha : \alpha \in \mathfrak{c}\}$ enumerate the collection of clopen subsets of b_0. We may view b_0 as the Stone space of the Boolean algebra $\{b_\alpha : \alpha \in \mathfrak{c}\}$. Fix an unbounded set $C \subset \mathfrak{c}$ so that for each $\lambda \in C$, $\{b_\alpha : \alpha \in \lambda\}$ is a subalgebra of $\mathcal{B}$. Further let $\{A_\alpha : \alpha \in \mathfrak{c}\}$ enumerate all the infinite subsets of N with the property that their closures are disjoint from Z.

For each clopen subset a of b_0, let $J_a = a \cap f^{-1}[f[Z \setminus a]]$. Since f is irreducible on Z, each J_a is nowhere dense in Z. Also let $Y_\alpha = Z \cap A^* \cap f^{-1}[f[A_\alpha^*]]$ for each $\alpha \in \mathfrak{c}$. Since f is one to one on A_α^*, each of $f[A_\alpha^*]$ and $Z \cap f^{-1}(f[A_\alpha^*])$ are nowhere ccc. Since we are assuming that $Z \cap A^*$ is ccc, it follows that Y_α is also nowhere dense in Z.

We inductively define a family, $\{a_\beta : \beta < \mathfrak{c}\}$ of clopen subsets of $A^* \cap Z$. Let $a_0 = b_0$. Our inductive hypotheses are that for each $\alpha < \mathfrak{c}$,

(1) $\{a_\beta : \beta < \alpha\}$ has the finite intersection property;
(2) a_β is contained in one of $\{b_\beta, b_0 \setminus b_\beta\}$;
(3) for each $\beta + 1 < \alpha$, $a_{\beta+1}$ is contained in $a_\beta \setminus (J_{a_\beta} \cup Y_\beta)$.

Suppose we have chosen the family $\{a_\beta : \beta < \alpha\}$. Let $Z_\alpha \subset b_0$ denote the closed set $\bigcap\{a_\beta : \beta < \alpha\}$. For each integer n, the selection of a_n is trivial, and since we are assuming that a_0 is ccc, we can assume that $\alpha \geq \omega$ and that Z_α is nowhere dense in Z. If α is a limit, then simply let a_α equal b_α if b_α meets Z_α, otherwise set $a_\alpha = b_0 \setminus b_\alpha$. Otherwise, α is a successor and there is a $\bar\beta$ such that $\alpha = \bar\beta + 1$. We must avoid $J_{a_{\bar\beta}} \cup Y_{\bar\beta}$. It suffices to show that Z_α is not contained in $J_{a_{\bar\beta}} \cup Y_{\bar\beta}$ because then we can select $a_\alpha \subset a_{\bar\beta}$ to meet Z_α, miss $J_{a_{\bar\beta}} \cup Y_{\bar\beta}$ and to be contained in one of $\{b_\alpha, b_0 \setminus b_\alpha\}$.

For each $\gamma < \alpha$, $J_{a_\gamma} \cup Y_\gamma \cup Z_\alpha$ is a nowhere dense subset of $A^* \cap Z$. Since $b_0 \subset A^* \cap Z$ is ccc, we may fix a countable family $\mathcal{U}_\gamma$ of clopen subsets of b_0 so that $\bigcup \mathcal{U}_\gamma$ is a dense open subset of $b_0 \setminus (J_{a_\gamma} \cup Y_\gamma \cup Z_\alpha)$. Since MA implies that $\mathfrak{c}$ is a regular cardinal, there is a $\lambda_\alpha \in C$ larger than α so that $\{a_\gamma\} \cup \mathcal{U}_\gamma \subset \{b_\zeta : \zeta < \lambda_\alpha\}$ for each $\gamma < \alpha$. Let $\mathcal{B}_\alpha$ be the Boolean subalgebra $\{b_\zeta : \zeta < \lambda_\alpha\}$. It follows that there is a map g_α from b_0 onto $S(\mathcal{B}_\alpha)$, the Stone space of $\mathcal{B}_\alpha$, such that $g_\alpha^{-1}(b_\zeta) = b_\zeta$ for each $\zeta < \lambda_\alpha$. Let $F_\alpha = g_\alpha[Z_\alpha]$. It follows that F_α is also equal to the intersection of the family $\{a_\gamma : \gamma < \alpha\}$, but in the different Stone space of course. Our assumptions have guaranteed that F_α is nowhere dense in $S(\mathcal{B}_\alpha)$.

Since $\mathcal{B}$ is ccc, so is $\mathcal{B}_\alpha$. In addition, $\mathcal{B}_\alpha$ is of cardinality less than $\mathfrak{c}$, thus it follows from MA that $\mathcal{B}_\alpha$ is σ-centered and $S(\mathcal{B}_\alpha)$ is separable. Recall that $\alpha = \bar\beta + 1$ and let U_α denote the dense open subset of $S(\mathcal{B}_\alpha)$ which is generated by the family $\mathcal{U}_{\bar\beta}$. By construction, $g_\alpha^{-1}(U_\alpha)$ is disjoint from Z_α. Let D be a countable dense subset of U_α. The neighborhood filter of F_α traces a filter on D which has a filter base of cardinality less than $\mathfrak{c}$. Since we are assuming Martin's Axiom, there is a countable set $\{x_n : n \in \omega\} \subset D$ which is mod finite contained in every member of that filter base. In other words, the sequence $\{x_n : n \in \omega\}$ converges to F_α. For each n, let $z_n \in Z \cap b_0$ be chosen so that $g_\alpha(z_n) = x_n$. Note that since $x_n \in U_\alpha$, we have ensured that $z_n \notin J_{a_{\bar\beta}} \cup Y_{\bar\beta} \cup Z_\alpha$ and all but finitely many are in $a_{\bar\beta}$. Therefore all the limit points of $\{z_n : n \in \omega\}$ are in Z_α. By passing to a subsequence, we can assume that $f(z_n) \neq f(z_m)$ for $n < m$.

For each n, let $z'_n \in N^*$ be distinct from z_n such that $f(z'_n) = f(z_n)$. Let $T = f[Z_\alpha]$ which is a nowhere dense subset of K. By construction, the image of $\overline{\{z_n : n \in \omega\}}$ is contained in $\{f(z_n) : n \in \omega\} \cup T$. Since K has no isolated points, and $\{f(z_n) : n \in \omega\}$ is a relatively closed subset of $Z \setminus T$, it is discrete and nowhere dense. Since f maps $\{z_n : n \in \omega\} \cup \{z'_n : n \in \omega\}$ onto the discrete set $\{f(z_n) : n \in \omega\}$ by a two to one map, $\{z_n : n \in \omega\} \cup \{z'_n : n \in \omega\}$ is a discrete subset of N^*. It follows that $\{z_n : n \in \omega\}$ and $\{z'_n : n \in \omega\}$ have disjoint closures and $f[\overline{\{z_n : n \in \omega\}}] = f[\overline{\{z'_n : n \in \omega\}}]$.

Let $x \in Z$ be a limit point of $\{z_n : n \in \omega\}$, hence $x \in Z_\alpha$. There is a limit point x' of $\{z'_n : n \in \omega\}$ such that $f(x) = f(x')$. Clearly $x \in a_{\bar\beta}$ and we claim that $x \notin J_{a_{\bar\beta}} \cup Y_{\bar\beta}$. To show this it is sufficient (and necessary) to show that x' is not in

$(Z \setminus a_{\bar\beta}) \cup A^*_{\bar\beta}$. For each n, $z_n \notin Y_{\bar\beta}$, hence $z'_n \notin A^*_{\bar\beta}$. Since $A^*_{\bar\beta}$ is clopen and x' is a limit of $\{z'_n : n \in \omega\}$ it follows that $x' \notin A^*_{\bar\beta}$. The collection $\{z'_n : n \in \omega\} \cap Z$ has all but finitely of its elements contained in $a_{\bar\beta}$, which is clopen in Z, hence none of the limit points are in $Z \setminus a_{\bar\beta}$. We will be finished if we show that the closure of $\{z'_n : n \in I\} = \{z'_n : n \in \omega\} \setminus Z$ is disjoint from Z. Since $\overline{\{z_n : n \in \omega\}}$ is a nowhere dense subset of b_0 and $f[b_0]$ is disjoint from $f[B^*]$, $f[\overline{\{z'_n : n \in \omega\}}]$ is a nowhere dense subset of $K \setminus f[B^*]$. It follows now that $Z \cap \overline{\{z'_n : n \in \omega\}}$ is a nowhere dense subset of $Z \setminus B^*$. Since $Z \setminus B^*$ is ccc, there is a collection $\{c_n : n \in \omega\}$ of clopen subsets of N^* such that each is disjoint from $\{z'_n : n \in \omega\}$ and the union contains a dense subset of $Z \setminus B^*$. For each $n \in I$, let d_n be a clopen subset of $N^* \setminus B^*$ such that $z'_n \in d_n$ and $d_n \cap (Z \cup \bigcup\{c_k : k \le n\})$ is empty. For each $n \in \omega$, shrink c_n by removing $\bigcup\{d_k : k < n\}$; note that this does not change $c_n \cap Z$. Then we have that $\bigcup_{n\in I} d_n$ and $B^* \cup \bigcup_n c_n$ are disjoint, and as is well-known, they have disjoint closures in N^*. Since the latter closure contains Z we have finished the proof. □

3. TREE-LIKE FAMILIES

An embedding of N^* into N^* is said to be trivial, if the embedding lifts to an embedding of βN into βN. It is an open problem to determine if there can be a non-trivial embedding of N^* into N^* under OCA and MA (see [HvM90, Problem 219] and [Far00, Question 3.14.2]). If there are none, then it is easy to show that that the set b_1 in Proposition 13 would be empty by using Levy's proof from [Lev04, 2.4] which shows that the preimages of closures of countable discrete sets are again closures of countable discrete sets. The main result of this section is used as an alternative approach.

The set of finite sequences $\{0,1\}^{<\omega}$ has a standard tree ordering by set inclusion. A family $\mathcal{A}$ of subsets of N is said to be *tree-like* if there is an embedding T of N into $\{0,1\}^{<\omega}$ such that for each $A \in \mathcal{A}$, $T[A]$ is contained in a single branch of $\{0,1\}^{<\omega}$, and distinct members of $\mathcal{A}$ are sent to distinct branches (see [Far00, 3.12.2]).

Proposition 16 (MA(ω_1))**.** [Vel93, 2.3] *Let $\mathcal{A}$ be an uncountable almost disjoint family of infinite subsets of N. Then there is an uncountable $\mathcal{B} \subset \mathcal{A}$ and for each $a \in \mathcal{B}$ a partitition $a = a_0 \cup a_1$ such that the family $\mathcal{B}_i = \{a_i : a \in \mathcal{B}\}$ is tree-like for each $i \in \{0,1\}$.*

Lemma 17 (OCA + MA)**.** *Suppose that $\{a_\alpha : \alpha \in \omega_1\}$ is a tree-like family of subsets of N with the property for all α there is a $b_\alpha \in [a_\alpha]^\omega$ such that $f(b^*_\alpha)$ is disjoint from $f[(N \setminus a_\alpha)^*]$. Then there is an α such that, with $b = b_\alpha$, $g_b[b^*]$ has interior.*

Proof. We may assume that f is one-to-one on b^*_α by Lemma 10. For each $c \subset a_\alpha$ define $F(c) \subset b_\alpha$ as follows. Since $f[(a_\alpha \setminus b_\alpha)^*]$ contains $f(b^*_\alpha)$ and f is precisely two to one, there will be a subset $F(c)$ of b_α such that

$$f[F(c)^*] = f(b^*_\alpha) \cap f[(c \setminus b_\alpha)^*] \ .$$

The definition of F on $\wp(a_\alpha)$ can also be expressed as $F(c)^*$ is the clopen subset of $b^* = b^*_\alpha$ which is equal to $g_b^{-1}[c^* \cap g_b(b^*)]$.

It is easily seen that F is a homomorphism from $\wp(a_\alpha)/fin$ onto $\wp(b_\alpha)/fin$. By Corollary 8, if we find some α and some uncountable family of pairwise disjoint clopen sets each of which meets $g_{b_\alpha}[b^*_\alpha]$, then this copy of N^* will not be nowhere dense. Equivalently, by Farah [Far00] (Proposition 6), if for some α the kernel of

$F \restriction \wp(a_\alpha)$ is not ccc over fin, then there is a $c \subset a_\alpha \setminus b_\alpha$ such that F is a trivial isomorphism from $\wp(c)$ to $\wp(b_\alpha)$. We proceed as in [Vel93].

Let $\mathcal{X}$ denote the set of all pairs $\langle c, d\rangle$ such that for some α, $d \subset c \subset a_\alpha \setminus b_\alpha$, and each of $F(d)$ and $F(c \setminus d)$ are not 0.

We define a set $K_0 \subset [\mathcal{X}]^2$ according to $\{\langle c, d\rangle, \langle \bar{c}, \bar{d}\rangle\} \in K_0$ providing

(1) $c \subset a_\alpha$ and $\bar{c} \subset a_{\bar{\alpha}}$ implies $\alpha \neq \bar{\alpha}$
(2) $c \cap F(\bar{c})$ and $\bar{c} \cap F(c)$ are empty;
(3) $c \cap \bar{d} = \bar{c} \cap d$;
(4) $F(c) \cap F(\bar{d})$ is not equal to $F(\bar{c}) \cap F(d)$.

The appropriate separable metric topology on $\mathcal{X}$ (given by considering it as embedded in $\wp(\omega)^4$ by the mapping sending $\langle c, d\rangle$ to $\langle c, d, F(c), F(d)\rangle$) will result in K_0 being an open subset of $[\mathcal{X}]^2$ (see [Vel93]).

Assume that $\mathcal{Y}$ is an uncountable subset of $\mathcal{X}$ and that $[\mathcal{Y}]^2 \subset K_0$. Let $I \subset \omega_1$ be the set of α such that there is $\langle c, d\rangle \in \mathcal{Y}$ such that $c \subset a_\alpha$. Also let $\langle c_\alpha, d_\alpha\rangle \in \mathcal{Y}$ be chosen for each $\alpha \in I$ so that $c_\alpha \subset a_\alpha$. Since $[\mathcal{Y}]^2 \subset K_0$ and $\mathcal{Y}$ is uncountable, it follows that I is uncountable.

Let $C = \bigcup\{c_\alpha : \alpha \in I\}$ and $D = \bigcup\{d_\alpha : \alpha \in I\}$. Let $\langle c, d\rangle$ and $\langle \bar{c}, \bar{d}\rangle$ be an arbitrary distinct pair from $\mathcal{Y}$. Note that $c \cap (F(c) \cup F(\bar{c}))$ is empty. It follows that C is disjoint from $\bigcup\{F(c) : (\exists d)\langle c, d\rangle \in \mathcal{Y}\}$. Also, $D \cap c$ will equal d for each $\langle c, d\rangle \in \mathcal{Y}$. Hence $(C \setminus D) \cap c = c \setminus d$ for each $\langle c, d\rangle \in \mathcal{Y}$.

Now consider the two families $\{F(d_\alpha) : \alpha \in I\}$ and $\{F(c_\alpha \setminus d_\alpha) : \alpha \in I\}$. Assume that $E \subset \omega$ and that $E \cap F(c_\alpha) =^* F(d_\alpha)$ for each $\alpha \in I$. Let $n \in \omega$ and $I' \in [I]^{\omega_1}$ such that $(E \cap F(c_\alpha))\Delta F(d_\alpha)$ is contained in n for all $\alpha \in I'$. Let $\alpha \neq \beta$ both be in I'. We may assume that $F(c_\alpha) \cap n = F(c_\beta) \cap n$, $F(d_\alpha) \cap n = F(d_\beta) \cap n$. Also, we may assume that $F(d_\alpha) \setminus F(c_\alpha)$ is contained in n for all $\alpha \in I'$.

Since $\{(c_\alpha, d_\alpha), (c_\beta, d_\beta)\} \in K_0$, there is some $j \in F(c_\alpha) \cap F(d_\beta)$ such that $j \notin F(c_\beta) \cap F(d_\alpha)$. Clearly j must be larger than n. Since $j \in F(d_\beta)$, it follows that $j \in F(c_\beta)$. Therefore j is in E. On the other hand, since j is in $E \cap F(c_\alpha)$, it must follow that $j \in F(d_\alpha)$, contradicting that $j \notin F(c_\beta) \cap F(d_\alpha)$.

It follows then that there is no such E. This means that $\bigcup\{F(d_\alpha)^* : \alpha \in I\}$ and $\bigcup\{F(c_\alpha \setminus d_\alpha)^* : \alpha \in I\}$ do not have disjoint closures in N^*. Fix any $x \in N^*$ which is in each of the closures. Notice that $x \notin C^*$ since C is disjoint from $F(c_\alpha)$ for all $\alpha \in I$.

Since $f[d_\alpha^*]$ is equal to $f[F(d_\alpha)^*]$ and $f[(c_\alpha \setminus d_\alpha)^*] = f[(F(c_\alpha \setminus d_\alpha))^*]$, it follows that $f(x)$ is in the image of the closure of $\bigcup_{\alpha \in I} d_\alpha^*$ and of $\bigcup_{\alpha \in I}(c_\alpha \setminus d_\alpha)^*$. Therefore $f(x)$ is in the image of D^* and of $(C \setminus D)^*$. However this contradicts that $f(x)$ only has two points mapping to it as we have found points in $D^*, (C \setminus D)^*$, and $(N \setminus C)^*$.

Therefore by OCA, $\mathcal{X}$ can be expressed as a countable union $\bigcup_n \mathcal{Y}_n$ such that $[\mathcal{Y}_n]^2 \cap K_0$ is empty for each n. For each n, there is a countable $Y_n \subset \mathcal{Y}_n$ such that for each integer m and each $\langle c, d\rangle \in \mathcal{Y}_n$, there is some $\langle \bar{c}, \bar{d}\rangle \in Y_n$ such that $c \cap m = \bar{c} \cap m$, $d \cap m = \bar{d} \cap m$, $F(c) \cap m = F(\bar{c}) \cap m$, and $F(d) \cap m = F(\bar{d}) \cap m$.

Fix any $\alpha \in \omega_1$ such that $\bar{c} \cap a_\alpha$ is finite for each $\langle \bar{c}, \bar{d}\rangle \in \bigcup_n Y_n$. Construct an increasing sequence $\{k_n : n \in \omega\}$ of integers so that for each n and each $i \leq n$ and each sequence c', d', a', b' of subsets of k_n, if there is a $\langle c, d\rangle \in \mathcal{Y}_i$ such that $c \cap k_n = c'$, $d \cap k_n = d'$, $F(c) \cap k_n = a'$, and $F(d) \cap k_n = b'$, then there is a pair $\langle c, d\rangle \in Y_i$ that also has this property, and in addition, $a_\alpha \cap a_{\bar{\alpha}} \subset k_{n+1}$ where $c \subset a_{\bar{\alpha}}$.

Define E_i to be $\bigcup\{a_\alpha \cap [k_{3n+i}, k_{3n+i+1}) : n \in \omega\}$ for $i \in 3$. There is an $i \in 3$ such that $F(E_i)$ is not finite. There is a $j \in 3$ such that $E_j \cap F(E_i)$ is not finite. Fix any $c \subset E_i$ such that $F(c)$ is infinite and is contained in E_j modulo finite. Let d_0, d_1, d_2 be a partition of c so that $F(d_0)$, $F(d_1)$, and $F(d_2)$ are each infinite. For simplicity we will assume that $\{i, j\} = \{1, 2\}$. Note that for all $x \subset d_1$, the pair $\langle c, d_0 \cup x\rangle$ is a member of $\mathcal{X}$ (since $c \setminus (d_0 \cup x)$ contains d_2) and we may assume that $(c \cup F(c)) \cap [k_{3m}, k_{3m+1})$ is empty for all n.

It can now easily be shown that $F \restriction \wp(d_1)$ is "σ-Borel" (see [Far00, p103]) which, in the case that F is an isomorphism, was shown to imply $F \restriction \wp(d_1)$ has a Borel representation ([Vel93, 2.2]) and would complete the proof by Proposition 7. However it is pointed out in [Far00, p103] that this is not sufficient in the case that F is only a homomorphism as it is here. The proof of [Far00, 3.12.1] certainly handles a very similar situation but is not directly applicable to ours. Therefore to finish the proof we will directly produce an uncountable family of clopen subsets of N^* each of which will meet $g_{b_\alpha}[b^*_\alpha]$.

We construct an increasing sequence $\{m_i : i \in \omega\} \subset \{k_{3\ell} : \ell \in \omega\}$ together with subsets $t_i \subset d_1 \cap [m_i, m_{i+1})$ and possibly infinite sets $\{J_i : i \in \omega\}$ by induction. We can let $m_0 = 0$ and $J_{-1} = \emptyset$. For each $J \subset N$, let $D(J) = d_0 \cup (d_1 \cap \bigcup\{[k_{3j}, k_{3j+3}) : j \in J\})$.

Given that m_i and J_{i-1} have been chosen we will construct the set t_i also by a finite induction and will then choose m_{i+1}. Fix an enumeration $\{(e_\ell, n_\ell, n'_\ell) : \ell < L\}$ of $\wp(m_i)\times i\times i$. We will construct $\{t(i,\ell) : \ell < L\}$ such that $t(i{+}1,\ell)\cap\max(t(i,\ell)) + 1 = t(i,\ell)$ and will let $t_i = \bigcup\{t(i,\ell) : \ell < L\}$. Our inductive hypotheses on J_i are that $F(D(N \setminus J_i)) \setminus F(d_0)$ is infinite and that if $j \in J_i \setminus J_{i-1}$, then $k_{3j} > m_i$.

As we define $t(i,\ell)$, we will also define $J(i,\ell)$ and will set $J_i = \bigcup\{J(i,\ell) : \ell < L\}$. For convenience let $t(i,-1) = \emptyset$ and $J(i,-1) = J_{i-1}$.

Suppose we have chosen $t(i,\ell-1)$ and $J(i,\ell-1)$ such that $F(D(N \setminus J(i,\ell-1))) \setminus F(d_0)$ is infinite. Let $n = n_\ell$ and $n' = n'_\ell$.

First we simply try to get into $\mathcal{Y}_n$. That is, choose, if possible, $J'(i,\ell) \supset J(i,\ell{-}1)$ such that $J'(i,\ell)\cap\max(t(i,\ell{-}1)) \subset J(i,\ell{-}1)$, $\langle c, D(J'(i,\ell))\rangle \in \mathcal{Y}_{n'}$, and $F(D(N \setminus J'(i,\ell))) \setminus F(d_0)$ is infinite. If there is no such $J'(i,\ell)$, then let $J'(i,\ell) = J(i,\ell-1)$.

Next we try to get into $\mathcal{Y}_n$ with infinite growth. That is, choose, if possible, an $x \subset d_1 \setminus \max(t(i,\ell-1))$ such that

$$F(e_\ell \cup t(i,\ell-1) \cup D(J'(i,\ell)) \cup x) \setminus F(D(J'(i,\ell-1)))$$

is infinite and $\langle c, d_0 \cup e_\ell \cup t(i,\ell-1) \cup D(J(i,\ell-1)) \cup x\rangle \in \mathcal{Y}_n$.

If such an x exists, call this case one, and choose some large enough $j' \in x$ such that there is a $k_{3\ell'} < j'$ such that $\ell' \notin J'(i,\ell)$ and

$$F(e_\ell \cup t(i,\ell-1) \cup D(J'(i,\ell)) \cup x) \setminus F(D(J'(i,\ell)))$$

contains some element of $F(c) \cap [m_i, k_{3\ell'})$. We define $J(i,\ell) = J'(i,\ell)$, and $t(i,\ell) = t(i,\ell-1) \cup (x \cap j+1)$. Note that ℓ' will not be in J_i.

On the other hand, if no such an x exists, then choose $J(\ell,i) \supset J'(\ell,i)$ such that each of

$$F(e_\ell \cup t(i,\ell-1) \cup D(J(i,\ell))) \setminus F(D(J'(i,\ell)))$$

and

$$F(D(N \setminus J(i,\ell)))$$

are infinite. Again choose an $\ell' \notin J'(i,\ell)$ so that $k_{3\ell'} > \max(t(i,\ell-1))$, and ensure that $J(i,\ell) \cap \ell'+1$ equals $J'(i,\ell)\cap\ell'$. Also ensure that $d_1 \cap [k_{3\ell'}, k_{3\ell'+3})$ is not empty and let $t(i,\ell) = t(i,\ell-1) \cup (d_1 \cap [k_{3\ell'}, k_{3\ell'+3}))$.

Let $m_{i+1} = k_{3n}$ be chosen so that $3n \notin J_i = \bigcup\{J(i,\ell) : \ell < L\}$ and $t_i = \bigcup\{t(i,\ell) : \ell < L\} \subset [m_i, m_{i+1})$.

Let $J_\omega = \bigcup_i J_i$ and note that by construction $D(J_\omega)\cap m_{i+1} = D(J_i)\cap m_{i+1}$. For each infinite $I \subset \omega$, set $e_I = \bigcup\{t_i : i \in I\} \subset d_1$. We have found our uncountable family since we will show that each clopen set e_I^* meets $g_{b_\alpha}[b_\alpha^*]$. To show this, it suffices to show that $F(e_I)$ is infinite. Since $F(e_I)$ contains $F(D(J_\omega)\cup e_I)\setminus F(D(J_\omega))$ mod finite, it suffices to show this latter set is infinite.

Since $\langle c, D(J_\omega)\cup e_I\rangle \in \mathcal{X}$, there is an n such that $\langle c, D(J_\omega)\cup e_I\rangle \in \mathcal{Y}_n$. There is also an n' such that $\langle c, D(J_\omega)\rangle \in \mathcal{Y}_{n'}$. Let $i \in I$ be any integer greater than both n and n'. We will show that $F(D(J_\omega)\cup e_I)\setminus F(D(J_\omega))$ is not contained in i.

Set $d = D(J_\omega)\cup e_I$ and fix ℓ such that at stage i in the construction of the m_i's, $e_\ell = d\cap m_i$ and $n_\ell = n$, and $n'_\ell = n'$. We consider the properties of $t(i,\ell)$. Let $Y = D(J'(i,\ell))$. Since $\langle c, D(J_\omega)\rangle$ is in $\mathcal{Y}_{n'}$, it follows that we were able to ensure that $\langle c, D(J'(i,\ell))\rangle = \langle c, Y\rangle$ is in $\mathcal{Y}_{n'}$.

Recall that there was an ℓ' such that $m_i < k_{3\ell'} < m_{i+1}$ and that the maximum of $t(i,\ell)$ was greater than $k_{3\ell'}$. If $J(i,\ell)\setminus J'(i,\ell)$ was infinite, then we know $F(e_\ell \cup t(i,\ell-1)\cup D(J(i,\ell)))\setminus F(Y)$ is infinite. Therefore $F(d)\setminus F(Y)$ is infinite because $d \supset e_\ell \cup t(i,\ell-1)\cup D(J(i,\ell))$. If we set $\bar{x} = d\setminus \max(t(i,\ell-1))$, then $\bar{x}$ would be a witness to the fact that we should have been in case one when defining $t(i,\ell)$. Therefore, in fact, $J(i,\ell)$ does equal $J'(i,\ell)$ and at stage ℓ we were able to find some x as in case one. In addition, $t(i,\ell)$ was defined as $t(i,\ell-1)\cup(x\cap j+1)$ for some $j \in x\setminus k_{3\ell'}$.

Since $e_\ell = d\cap m_i$ and $i \in I$, we have that

$$d \cap k_{3\ell'} = (e_\ell \cup t(i,\ell-1)\cup D(J'(i,\ell))\cup x)\cap k_{3\ell'}.$$

That is, if we let $d_x = e_\ell \cup t(i,\ell-1)\cup D(J'(i,\ell))\cup x$, then $\langle c, d_x\rangle \in \mathcal{Y}_n$ and $d_x \cap k_{3\ell'} = d\cap k_{3\ell'}$.

Now also $\langle c,d\rangle \in \mathcal{Y}_n$, so there is a pair $\langle \bar{c},\bar{d}\rangle \in Y_n$ such that $\bar{c}\cap k_{3\ell'} = c\cap k_{3\ell'}$, $\bar{d}\cap k_{3\ell'} = d\cap k_{3\ell'}$, $F(\bar{c})\cap k_{3\ell'} = F(c)\cap k_{3\ell'}$, and $F(\bar{d})\cap k_{3\ell'} = F(d)\cap k_{3\ell'}$. In addition, if we let $\bar{\alpha}\in\omega_1$ such that $\bar{c}\subset a_{\bar{\alpha}}$, we have that $a_{\bar{\alpha}}\cap a_\alpha \subset k_{3\ell'+1}$. Further, recall that $(c\cup F(c))\cap[k_{3\ell'}, k_{3\ell'+1})$ is empty. Observe also that $Y\cap k_{3\ell'}$ is equal to $D(J_\omega)\cap k_{3\ell'}$.

Each of the following are straightforward consequences:

(1) $c\subset a_\alpha$ and $\bar{c}\subset a_{\bar{\alpha}}$ and $\alpha\neq\bar{\alpha}$
(2) $c\cap F(\bar{c})$ and $\bar{c}\cap F(c)$ are empty: because $c\cap F(\bar{c}) \subset c\cap(F(\bar{c})\cap a_\alpha)\subset c\cap(F(\bar{c})\cap k_{3\ell'+1})\subset c\cap F(c)$ and similarly, $\bar{c}\cap F(c)\subset \bar{c}\cap F(c)\cap k_{3\ell'}\subset c\cap F(c)$
(3) $c\cap\bar{d} = \bar{c}\cap d$: because each of $c\cap\bar{d}$ and $\bar{c}\cap d$ are equal to $d\cap k_{3\ell'}$.

Therefore, the only reason that $\{\langle c,d\rangle, \langle\bar{c},\bar{d}\rangle\}$ is not in K_0 is that $F(c)\cap F(\bar{d})$ must be equal to $F(\bar{c})\cap F(d)$. The same is true with d_x in place of d, hence $F(c)\cap F(\bar{d})$ must be equal to $F(\bar{c})\cap F(d_x)$. By the choice of x, there is some integer $m'\in F(c)\cap[m_i,k_{3\ell'})\cap F(d_x)\setminus F(Y)$, and therefore $m'\in F(\bar{c})\cap F(\bar{d})$. This then means that $m'\in F(d)$.

By the same reasoning, there is a pair $\langle c',d'\rangle \in Y_{n'}$ such that $c'\cap k_{3\ell'} = c\cap k_{3\ell'}$, $d'\cap k_{3\ell'} = D(J_\omega)\cap k_{3\ell'}$, $F(c')\cap k_{3\ell'} = F(c)\cap k_{3\ell'}$, and $F(d')\cap k_{3\ell'} = F(D(J_\omega))\cap k_{3\ell'}$. In addition, if we let $\alpha'\in\omega_1$ such that $c'\subset a_{\alpha'}$, we have that $a_{\alpha'}\cap a_\alpha\subset k_{3\ell'+1}$.

Further, recall that $(c \cup F(c)) \cap [k_{3\ell'}, k_{3\ell'+1})$ is empty. Repeating the argument above with $\langle c, D(J_\omega)\rangle$ and $\langle c', d'\rangle$ in place of $\langle c, d\rangle$ and $\langle \bar{c}, \bar{d}\rangle$ respectively, we have that $F(D(J_\omega)) \cap k_{3\ell'}$ is equal to $F(d') \cap k_{3\ell'}$. Furthermore, $F(Y) \cap k_{3\ell'}$ will also equal $F(d') \cap k_{3\ell'}$ because $Y \cap k_{3\ell'} = D(J_\omega) \cap k_{3\ell'}$. Since $m' \in F(c) \cap F(c') \setminus F(Y)$, it follows that $m' \notin F(D(J_\omega))$.

This shows that $m' \in F(d) \setminus F(D(J_\omega))$ and completes the proof. □

Corollary 18 (OCA + MA). *For each $A \subset N$ such that $A^* \cap Z \neq \emptyset$, there is a $c \in \mathcal{J}'$ such that $c \subset A$ and $c^* \subset Z$.*

Proof. Let $A \subset N$ and assume that $A^* \cap Z \neq \emptyset$. Since f is irreducible on Z, $f[A^* \cap Z]$ has interior in K. Let W be an non-empty open subset of $f[A^* \cap Z]$.

Assume we find a set $b \subset N$ such that $f[b^*] \subset W$ and $f^{-1}(f[b^*])$ contains a^* for some $a \in \mathcal{J}$. By Proposition 14, there is $c \in [a]^\omega$ such that $c^* \subset Z$ and $f[c^*] \cap f[Z \setminus c^*]$ is empty. Since $f[c^*] \subset f[A^* \cap Z]$, it follows that $c^* \subset A^*$, or by removing finitely many integers, $c \subset A$ as required. The remainder of the proof is to show there is such a set b.

Assume there is some $b \subset N$ such that $b^* \cap Z$ is empty, and $f[b^*] \subset W$ has interior. Since $b^* \cap Z$ is empty and $f[Z] = K$, it follows that $f \restriction b^*$ is one to one. By Lemma 11, this set b has the property that $f^{-1}(f[b^*])$ contains a^* for some $a \in \mathcal{J}$.

Now, by Lemma 15, we may fix an uncountable family $\{U_\alpha : \alpha \in \omega_1\}$ of pairwise disjoint open subsets of W. By Lemma 9, we may choose, for each α, some infinite c_α such that $c_\alpha^* \cap Z$ is empty and $f[c_\alpha^*]$ is a subset of U_α. By the previous paragraph, we may assume that $f[c_\alpha^*]$ is nowhere dense in K for each $\alpha \in \omega_1$. For each α, fix an $a_\alpha \subset N \setminus c_\alpha$ such that $f[a_\alpha^*]$ is a subset of U_α and meets $f[c_\alpha^*]$. By Proposition 16 and with re-indexing, we may assume that $\{a_\alpha : \alpha \in \omega_1\}$ is tree-like.

For each α let $w_\alpha \in f[a_\alpha^*] \cap f[c_\alpha^*]$. Since f is two to one, $w_\alpha \notin f[(N \setminus (a_\alpha \cup c_\alpha))^*]$. Let $W_\alpha \subset U_\alpha$ be an open set neighborhood of w_α which is disjoint from $f[(N \setminus (a_\alpha \cup c_\alpha))^*]$. Since $f[c_\alpha^*]$ is nowhere dense, $W_\alpha \setminus f[c_\alpha^*]$ is non-empty so there is an infinite $b_\alpha \subset a_\alpha \cup c_\alpha$ such that $f[b_\alpha^*] \subset W_\alpha \setminus f[c_\alpha^*]$. Clearly b_α is also almost disjoint from c_α, so we may choose it to be contained in a_α. By Lemma 10, ensure that $f \restriction b_\alpha^*$ is one to one. We have that for each α, $f[b_\alpha^*] \cap f[(N \setminus a_\alpha)^*]$ is empty. Now apply Lemma 17 and let $b = b_\alpha$ be chosen so that $g_b[b^*]$ has interior. By Lemma 12, $b^* \cup g_b[b^*]$ contains a^* for some $a \in \mathcal{J}$. Since $f[b^*] = f[g_b[b^*]]$, it follows that $a^* \subset f^{-1}(f[b^*])$ as required. □

Lemma 19 (OCA + MA). *If f is not locally one to one, then there is a point x such that for every countable family $\{A_n : n \in \omega\} \subset x$, there is an $a \in \mathcal{J}$ such that a is almost contained in each A_n.*

Proof. Since f is not locally one to one, there is a pair $\{x, x'\}$ in N^* such that $f(x) = f(x')$ and neither is in a^* for any $a \in \mathcal{J}$. Since $f[Z] = K$, we may assume that $x \in Z$. This is our choice for the point x.

Assume that $\{A_n : n \in \omega\} \subset x$ and, by possibly shrinking, we may assume that $A_0 \notin x'$ and $A_{n+1} \subset A_n$ for each n. Let B be any member of x which is contained in A_0. We check that $f[B^*] \setminus f[(N \setminus A_0)^*]$ is non-empty. If $f[B^*]$ was contained in $f[(N \setminus A_0)^*]$, then $f \restriction B^*$ would be one to one. By Proposition 13, we may assume that either $f[B^*]$ is clopen or is nowhere dense. By Lemma 11, $f[B^*]$ cannot be clopen, since $f(x)$ is not in $f[a^*]$ for any $a \in \mathcal{J}$. However, since $x \in Z$, $f[B^* \cap Z]$ has interior and so cannot be nowhere dense.

Next we show that $f[Z \cap B^*] \setminus f[(N \setminus A_0)^*]$ is also non-empty. We have that $f(x) \notin f[(A_0 \setminus B)^*]$ since $x \in B^*$ and $x' \notin A_0^*$. Fix any $B_1 \in x$ with $B_1 \subset B$ such that $f[B_1^*] \cap f[(A_0 \setminus B)^*]$ is empty. By the previous paragraph, there is a $y \in B_1^*$ such that $f(y) \notin f[(N \setminus A_0)^*]$, and there is a $z \in Z$ such that $f(z) = f(y)$ and which is not in $f[(N \setminus A_0)^*] \cup f[(A_0 \setminus B)^*]$. We check that $z \in B^* \cap Z$. Clearly $z \notin (N \setminus A_0)^* \cup (A_0 \setminus B)^*$, hence z must be in B^*.

We recursively construct a sequence of infinite sets $B_n \in x$ and $b_n \subset B_n$ so that

(1) $B_{n+1} \subset A_{n+1} \cap B_n \setminus b_n$,
(2) $f[B_{n+1}^*] \cap (f[b_n^*] \cup f[(A_0 \setminus B_n)^*])$ is empty,
(3) $b_n \in \mathcal{J}'$,
(4) and $f[b_n^*] \cap f[(N \setminus A_0)^*]$ is empty.

Let $B_0 \subset A_0$ be any member of x. Since $f[Z \cap B_0^*] \setminus f[(N \setminus A_0)^*]$ is not empty, there is some $C_0 \in [B_0]^\omega$ such that $C_0^* \cap (Z \cap B_0^*)$ is not empty and $f[C_0^*] \cap f[(N \setminus A_0)^*]$ is empty. By Corollary 18, there is an infinite $b_0 \subset C_0$ such that $b_0 \in \mathcal{J}'$. Clearly $b_0 \notin x$, and since $x' \notin b_0^*$, $f(x)$ is not in $f[b_0^*]$. We can choose $B_1 \in x$ so that $B_1 \subset B_0 \cap A_1 \setminus b_0$ and so that $f[B_1^*]$ is disjoint from each of $f[b_0^*]$ and $f[(A_0 \setminus B_0)^*]$. The induction proceeds for each n in the same way.

By possibly shrinking each b_n we can assume that $f \restriction b_n^*$ is one to one, and we can choose c_n so that $b_n \cup c_n \in \mathcal{J}$. Since $f[b_n^*] \cap f[b_m^*]$ is empty for all $n < m$, and $f[b_n^*] = f[c_n^*]$ for all n, we can assume that $(b_n \cup c_n)$ is disjoint from $(b_m \cup c_m)$ for $n < m$. Recall that, by Lemma 4, $f[b^*]$ is clopen for any b which is contained in b_n for any n.

For each n, let $\{b(n, \alpha) : \alpha \in \omega_1\}$ be an almost disjoint family of infinite subsets of b_n. Inductively define an almost disjoint family $\{a_\alpha : \omega \le \alpha < \omega_1\}$ of subsets of $\bigcup_n b_n$ so that $a_\alpha \cap (b_n \cup c_n)$ is almost equal $b(n, \alpha)$ for each n. Apply Proposition 16, to find a partition a_α as $a_\alpha^0 \cup a_\alpha^1$ so that there is an uncountable $I \subset \omega_1 \setminus \omega$ such that each of the families $\{a_\alpha^0 : \alpha \in I\}$ and $\{a_\alpha^1 : \alpha \in I\}$ are tree-like. For each $\alpha \in I$, at least one of a_α^0 or a_α^1 will meet infinitely many of the b_n's. Therefore, by re-indexing, we can assume we have an uncountable tree-like family $\{a_\alpha : \omega \le \alpha < \omega_1\}$ such that each a_α meets infinitely many of the b_n's in an infinite set.

Claim 1. there is some $b \subset \bigcup_n b_n$ such that $f \restriction b^*$ is one to one, $f[b^*] \cap f[(N \setminus A_0)^*]$ is empty, $b \cap b_n$ is finite for each n, and $f[b^*]$ has interior.

To prove the Claim, assume first there is some α, $\omega \le \alpha < \omega_1$, such that $f[(N \setminus a_\alpha)^*]$ contains $f[a_\alpha^*]$. Therefore $f \restriction a_\alpha^*$ is one to one. By Lemma 13, there is a partition $d_0 \cup d_1$ of a_α so that $f[d_1^*]$ is nowhere dense and $f[d_0^*]$ is clopen. Again note that d_0 is contained in some member $\mathcal{J}$ by Lemma 11. By the assumption on the family $\{b_n : n \in \omega\}$, we must have that $d_1 \cap b_n$ is finite for all n. Let y be any point in $\overline{\bigcup_n (d_0 \cap b_n)^*} \setminus \bigcup_n (d_0 \cap b_n)^*$. Notice that $f[(d_0 \cap b_n)^*] \subset f[c_n^*]$ for each n, hence there is a point $y' \in \overline{\bigcup_n c_n^*} \setminus \bigcup_n c_n^*$ such that $f(y') = f(y)$. Since $\overline{\bigcup_n b_n^*} \subset (\bigcup_n b_n)^*$ and $\overline{\bigcup_n c_n^*} \subset (\bigcup_n c_n)^*$, $y \neq y'$. Also, $f[b_n^*] = f[c_n^*]$ is disjoint from $f[(N \setminus A_0)^*]$, hence each of $\overline{\bigcup_n b_n^*}$ and $\overline{\bigcup_n c_n^*}$ are contained in A_0^*. We have shown then that $f(y)$ is not in $f[(N \setminus A_0)^*]$ since f is two to one. Let $Y \in y$ be chosen so that $f[Y^*] \cap f[(N \setminus A_0)^*]$ is empty. Since $Y \cap (d_0 \cap b_n)$ is infinite for infinitely many n, there is an infinite $b \subset Y \cap d_0$ such that $b \cap b_n$ is finite for all n. Since $b \subset d_0$ and $d_0 \in \mathcal{J}'$, we have that $f[b^*]$ is clopen by Lemma 4. This proves the Claim in this case.

Now we assume that, for each $\alpha \geq \omega$, there is some infinite $b_\alpha \subset a_\alpha$ such that $f[b_\alpha^*]$ is disjoint from $f[(N \setminus a_\alpha)^*]$. Clearly $f[b_\alpha^*] \cap f[(N \setminus A_0)^*]$ is therefore empty. It also follows that b_α is almost disjoint from each b_n since $c_n \subset N \setminus a_\alpha$. By Lemma 17, there is some $\alpha \geq \omega$ such that $f[b_\alpha^*]$ has interior. This proves the claim.

We can now complete the proof of the Lemma. Let $b \subset \bigcup_n b_n$ be as in the Claim. By Lemma 11, there is some $a \in \mathcal{J}$ such that $a^* \subset f^{-1}(f[b^*])$. Clearly $f[a^*] \subset f[b^*]$, hence $f[a^*] \cap f[(N \setminus A_0)^*]$ is empty. It follows that $a^* \subset A_0^*$. Let $n > 0$, and notice that $f[a^*] \subset f[b^*] \subset f[\bigcup_{m>n} b_m^*] \subset f[B_{n+1}^*]$. Since $f[B_{n+1}^* \cap f[(A_0 \setminus B_n)^*]$ is empty, we have that $f[a^*]$ is disjoint from $f[(N \setminus A_n)^*]$. Therefore it follows that $a^* \subset A_n^*$. This completes the proof of the Lemma. $\square$

4. Locally one to one

In this section we prove Lemma 23 in a (slightly) more general setting than we have developed in the paper and prove the main theorem, Theorem 25, as a simple consequence. The approach is almost a routine application of OCA except it is made more complicated because we must first add a Cohen real.

Definition 20. A family $\mathcal{Y}$ is *σ-cofinal* in an ultrafilter x if for each countable family $\{A_n : n \in \omega\} \subset x$, there is an $a \in \mathcal{Y}$ such that $a^* \subset A_n^*$ for all n.

Definition 21. A family $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\}$ is a Hausdorff-Luzin family of pairs if for each $\alpha < \beta < \omega_1$, $c_\alpha \cap d_\alpha$ is empty and $(c_\alpha \cap d_\beta) \cup (c_\beta \cap d_\alpha)$ is not empty.

Proposition 22. *Suppose that $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\}$ is a Hausdorff-Luzin family of pairs of subsets of N, then $\bigcup\{c_\alpha^* : \alpha \in \omega_1\}$ and $\bigcup\{d_\alpha^* : \alpha \in \omega_1\}$ do not have disjoint closures in N^*.*

Proof. Assume, for a contradiction, that the two sets do have disjoint closures. It follows then, that there is a $Y \subset N$ such that $c_\alpha^* \subset Y^*$ and $Y^* \cap d_\alpha^* = \emptyset$ for all α. Furthermore, there is an integer m and an uncountable set $I \subset \omega_1$ such that $c_\alpha \setminus m \subset Y$ and $Y \cap d_\alpha \subset m$ for all $\alpha \in I$. Fix any $\alpha < \beta$, both in I, so that $c_\alpha \cap m = c_\beta \cap m$ and $d_\alpha \cap m = d_\beta \cap m$. Since $c_\alpha \cap d_\alpha$ is empty, it follows that $(c_\alpha \cap m) \cap (d_\beta \cap m)$ is empty. Also, $(c_\alpha \setminus m) \subset Y$ and $d_\beta \subset (N \setminus Y)$, hence $c_\alpha \cap d_\beta$ is empty. By the same reasoning it follows that $c_\beta \cap d_\alpha$ is empty, contradicting that the family of pairs was assumed to be Hausdorff-Luzin. $\square$

Lemma 23. *Assume that $\mathcal{I}$ is a family of infinite subsets of N and that for each $a \in \mathcal{I}$ there is a one to one function h_a from a cofinite subset of a onto some cofinite subset of a such that $h_a(n) \neq n$ for all $n \in a$. If $\mathcal{I}$ is σ-cofinal in x for some $x \in N^*$, then there is a proper poset P such that if G is P-generic, then in $V[G]$ there is a partition $\{C_0, C_1, C_2\}$ of N and an uncountable family $\{a_\alpha : \alpha \in \omega_1\} \subset \mathcal{I}$ such that $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\}$ forms a Hausdorff-Luzin family of pairs, where, for each $\alpha \in \omega_1$,*

$$c_\alpha = C_2 \cap h_{a_\alpha}(a_\alpha \cap C_0) \quad \textit{and} \quad d_\alpha = C_2 \cap h_{a_\alpha}(a_\alpha \cap C_1)$$

We defer the proof of Lemma 23 until after Theorem 25. The following corollary is a routine application of PFA proven by selecting the appropriate family of ω_1 many P-names and dense sets for a P-filter to meet. We use the convention that $\check{v}$ is the P-name for the ground model set v.

Corollary 24 (PFA). *Let $\mathcal{I}$ be a family of infinite subsets of N and for each $a \in \mathcal{I}$ let h_a be given which is a one to one function from a cofinite subset of a onto a* *cofinite subset of a such that $h_a(n) \neq n$ for all $n \in a$. If there is an $x \in N^*$ such that $\mathcal{I}$ is σ-cofinal in x, then there is a partition $\{C_0, C_1, C_2\}$ of N and an uncountable family $\{a_\alpha : \alpha \in \omega_1\} \subset \mathcal{I}$ such that $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\}$ forms a Hausdorff-Luzin family of pairs, where, for each $\alpha \in \omega_1$,*

$$c_\alpha = C_2 \cap h_{a_\alpha}(a_\alpha \cap C_0) \quad \textit{and} \quad d_\alpha = C_2 \cap h_{a_\alpha}(a_\alpha \cap C_1)$$

Proof. Let P be the proper poset given by Lemma 23. Let $\dot{C}_0, \dot{C}_1, \dot{C}_2$, and $\{\dot{a}_\alpha : \alpha \in \omega_1\}$ be the family of P-names so that for some $p_0 \in P$, p_0 forces

" $\{C_0, C_1, C_2\}$ is a partition of N, $\{a_\alpha : \alpha \in \omega_1\}$ is contained in $\check{\mathcal{I}}$ and $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\}$ forms a Hausdorff-Luzin family of pairs, where, for each $\alpha \in \omega_1$,

$$c_\alpha = C_2 \cap h_{a_\alpha}(a_\alpha \cap C_0) \ \text{ and } \ d_\alpha = C_2 \cap h_{a_\alpha}(a_\alpha \cap C_1)\text{" .}$$

By replacing P with the poset of all $p \in P$ such that $p \leq p_0$, we may assume that p_0 is the largest element of P. For each α, let $\dot{c}_\alpha$ and $\dot{d}_\alpha$ also denote the P-names for c_α and d_α respectively. For each $\alpha \in \omega_1$, let $D_\alpha = \{p \in P : (\exists a_\alpha \in \mathcal{I})\ p \Vdash \dot{a}_\alpha = \check{a}_\alpha\}$ For each $n \in N$, let $E_n = \{p \in P : (\exists i \in 3)(p \Vdash \check{n} \in \dot{C}_i)\}$. For each $\alpha < \beta < \omega_1$, let $D_{\alpha,\beta}$ be

$$\{p \in P : (\exists k \in N, i \in 2)(p \Vdash k \in \dot{C}_2 \cap \dot{a}_\alpha \cap a_\beta, h_{\dot{a}_\alpha}(k) \in \dot{C}_i, \text{ and } h_{\dot{a}_\beta}(k) \in \dot{C}_{1-i})\}$$

Note that for each $p \in D_{\alpha,\beta}$, there is a k such that $p \Vdash \check{k} \in (\dot{c}_\alpha \cap \dot{d}_\beta) \cup (\dot{d}_\alpha \cap \dot{c}_\beta)$.

It is routine to check that all of the above sets are dense in P, and that if G is a P-filter which meets each of them, then C_0, C_1, C_2 and $\{a_\alpha : \alpha \in \omega_1\}$ is our desired family where for each $i \in 3$, $C_i = \{n : (\exists p \in G \cap E_n)\ p \Vdash \check{n} \in \dot{C}_i\}$ and, for each $\alpha \in \omega_1$, a_α is the unique element of $\mathcal{I}$ such that there is a $p \in G \cap D_\alpha$ with $p \Vdash \check{a}_\alpha = \dot{a}_\alpha$. □

Theorem 25 (PFA)**.** *The function f is locally one to one.*

Proof. By Lemmas and 4 and 19, the hypotheses of Corollary 24 are satisfied by letting $\mathcal{I}$ be $\mathcal{J}$ from Definition 2 and the family of functions from Lemma 4. Let C_0, C_1, C_2 and $\{a_\alpha : \alpha \in \omega_1\}$, $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\}$ be as in Corollary 24. By Proposition 22, there is a point $w \in N^*$ such that

$$w \in \overline{\bigcup_{\alpha\in\omega_1} c^*_\alpha} \cap \overline{\bigcup_{\alpha\in\omega_1} d^*_\alpha} \subset C^*_2 .$$

Let $\alpha \in \omega_1$, and observe that by Proposition 4,

$$f[(a_\alpha \cap C_0)^*] = f[(h_{a_\alpha}(a_\alpha \cap C_0))^*] \ \text{ and } \ f[(a_\alpha \cap C_1)^*] = f[(h_{a_\alpha}(a_\alpha \cap C_1))^*] \ .$$

Therefore $f[C^*_0] \supset f[\bigcup_{\alpha\in\omega_1} c^*_\alpha]$ and $f[C^*_1] \supset f[\bigcup_{\alpha\in\omega_1} d^*_\alpha]$. It follows that $f(w)$ has a preimage in each of C^*_2, C^*_0 and C^*_1. This contradicts that f is two to one. □

Proof of Lemma 23. The remainder of the section is a proof of Lemma 23. Assume that $\mathcal{I}$ is a family of infinite subsets of N and that for each $a \in \mathcal{I}$ there is a one to one function h_a from a cofinite subset of a onto a cofinite subset of a such that $h_a(n) \neq n$ for all $n \in a$. Also assume that $\mathcal{I}$ is σ-cofinal in x for some $x \in N^*$. By choosing one representative in each equivalence class in $\mathcal{I}$ mod finite, we may assume that if a, a' are distinct members of $\mathcal{I}$, then a and a' are not equal mod finite.

Lemma 26. *If $\mathcal{I}$ is covered by a countable family $\{\mathcal{Z}_n : n \in \omega\}$, then $\mathcal{Z}_n$ is σ-cofinal in x for some n.*

Proof. For each n, let $\mathcal{A}_n$ be a countable subset of x such that if $\mathcal{Z}_n$ is not σ-cofinal in x, then $\mathcal{Z}_n$ has no member which is contained mod finite in each member of $\mathcal{A}_n$. Then $\mathcal{A} = \bigcup_{n\in\omega} \mathcal{A}_n$ is a countable subset of x and there is some $a \in \mathcal{I}$ which is contained mod finite in each member of $\mathcal{A}$. Since there is some n such that $a \in \mathcal{Z}_n$, it follows that $\mathcal{Z}_n$ is σ-cofinal in x. □

Proposition 27. *If H is a function from N to N and $H(n) \neq n$ for all n, then there is a set $X \in x$ such that $H[X] \cap X$ is empty.*

Proof. The three set lemma [Wal74, 6.25], implies there is a partition X_0, X_1, X_2 of N such that $H[X_i] \cap X_i$ is empty for each i. □

Lemma 28. *If $\mathcal{Y} \subset \mathcal{I}$ is σ-cofinal in x and $m \in \omega$, then there are $a, b \in \mathcal{Y}$ and $k > m$ such that $h_a^{-1}(k) \neq h_b^{-1}(k)$ and both are greater than m.*

Proof. Let X be any member of x such that $N \setminus X$ is infinite. Let Y denote the set of integers $k \in X$ so that k is in the domain and range of h_{a_k} for some $a_k \in \mathcal{Y}$. Since $\mathcal{Y}$ is σ-cofinal in x, Y meets every member of x in an infinite set. Since x is an ultrafilter, $Y \in x$. For each $k \in Y$, define $H(k) = h_{a_k}^{-1}(k)$. Note that $H(k) \neq k$ for all $k \in Y$. Extend H to $N \setminus Y$ arbitrarily to a permutation on N so that $H(n) \neq n$ for all n. By Proposition 27, there is an $X_1 \in x$ so that $H[X_1] \cap X_1$ is empty. Let $a \in \mathcal{Y}$ such that $a \subset^* Y \cap X_1$ and choose $k \in Y \cap X_1 \cap a \setminus (h_a[m] \cup h_a^{-1}[m] \cup H[m] \cup H^{-1}[m])$. Note that since $k \in Y \cap X_1$, $h_a^{-1}(k) \neq H(k) = h_{a_k}^{-1}(k)$. It follows that if we let $b = a_k$ then we have our desired pair $a, b \in \mathcal{Y}$. □

The standard countable poset for adding a Cohen real has many forcing equivalent forms. The form most useful for us is the set, ${}^{<\omega}3$, of all functions from some integer into $3 = \{0, 1, 2\}$. This poset is ordered by extension.

Corollary 29. *If $G \subset {}^{<\omega}3$ is a generic filter and if $\mathcal{I}$ is covered, in $V[G]$, by a countable family $\{\mathcal{Y}_n : n \in \omega\}$, then there are $p \in G$, $n \in \omega$, $a, b \in \mathcal{Y}_n$ and $k \in a \cap b$ such that $p(k) = 2$, $p(h_a^{-1}(k)) = 0$, and $p(h_b^{-1}(k)) = 1$.*

Proof. Let $\{\dot{\mathcal{Y}}_n : n \in \omega\}$ be ${}^{<\omega}3$-names and assume that some p is any member of G which forces that $\mathcal{I}$ is covered by $\bigcup_{n\in\omega} \dot{\mathcal{Y}}_n$. Let $p' < p$ be arbitrary. For each $q < p'$ and integer n, let $\mathcal{Y}_{q,n}$ be the set of $a \in \mathcal{I}$ such that $q \Vdash \check{a} \in \dot{\mathcal{Y}}_n$. Since $p' < p$, we have that $\bigcup_{q<p'} \bigcup_{n\in\omega} \mathcal{Y}_{q,n}$ covers $\mathcal{I}$. By Lemma 26, there is a $q < p'$ and an n such that $\mathcal{Y}_{q,n}$ is σ-cofinal in x. Let m be large enough so that the domain of q is contained in m. By Lemma 28, there is a pair $a, b \in \mathcal{Y}_{q,n}$ and $k \in a \cap b \setminus m$ such that $h_a^{-1}(k) \neq h_b^{-1}(k)$ and both are greater than m. We can extend q to a condition $q' \in {}^{<\omega}3$ so that $q'(k) = 2$, $q'(h_a^{-1}(k)) = 1$ and $q'(h_b^{-1}(k)) = 2$. Since p' was an arbitary element below p, the set of conditions with this property of q' is dense below p', hence there will be such a $q' \in G$. □

For the remainder of the proof we work in the extension $V[G]$ for a generic filter $G \subset {}^{<\omega}3$. We have our desired partition $\{C_0, C_1, C_2\}$ of N given by $g = \bigcup G$. That is, g is a function from N onto 3, and we let $C_i = g^{-1}(i)$ for $i \in 3$. We will now show that our uncountable family $\{a_\alpha : \alpha < \omega_1\}$ can be found in a subsequent proper forcing extension.

For each $a \in \mathcal{I}$, let $c_a = C_2 \cap h_a(a \cap C_0)$ and $d_a = C_2 \cap h_a(a \cap C_1)$. Since h_a is one to one and $C_0 \cap C_1$ is empty, we have that $c_a \cap d_a$ is empty for all $a \in \mathcal{I}$. We leave it as an exercise to verify that, since $\mathcal{I} \in V$ and distinct members of $\mathcal{I}$

have infinite symmetric difference, c_a and $c_{a'}$ also have infinite symmetric difference for $a \neq a' \in \mathcal{I}$. We define a separable metric topology on the set $\mathcal{I}$ by declaring $[a;m] = \{b \in \mathcal{I} : c_a \cap m = c_b \cap m$ and $d_a \cap m = d_b \cap m\}$ to be open for each $a \in \mathcal{I}$ and $m \in \omega$.

Let $K_0 \subset \mathcal{I}$ be the set of pairs $(a,b) \in \mathcal{I}$ such that $(c_a \cap d_b) \cup (d_a \cap c_b)$ is not empty. To see that K_0 is an open subset of $\mathcal{I}^2$ assume that $(a,\bar{a})$ is K_0. Let k be any integer in $(c_a \cap d_{\bar{a}}) \cup (d_a \cap c_{\bar{a}})$ and let $m > k$. Let $(b,\bar{b})$ be an element of $[a;m] \times [\bar{a};m]$. Clearly $k \in (c_b \cap d_{\bar{b}}) \cup (d_b \cap c_{\bar{b}})$ which shows that $[a;m] \times [\bar{a};m] \subset K_0$. Our desired Hausdorff-Luzin family of pairs is $\{(c_a, d_a) : a \in Y\}$ for any uncountable subset $Y \subset \mathcal{I}$ such that $Y^2 \setminus \Delta_Y \subset K_0$, where Δ_Y denotes the diagonal in Y^2.

We are nearly ready for a standard application of OCA, however we must recall that we are no longer in a model in which OCA holds since we have added a Cohen real. Instead, we use the following result which comes from the well-known proof that PFA implies OCA.

Proposition 30. [Do91, 6.2] *If X is a separable metric space and K_0 is an open subset of X^2, then either X is covered by a countable collection $\{X_n : n \in \omega\}$ such that $X_n^2 \setminus \Delta_X$ is disjoint from K_0 for all n, or there is a proper poset P which introduces an uncountable set $Y \subset X$ such that $Y^2 \setminus \Delta_X \subset K_0$.*

By Proposition 30, we will finish the proof of Lemma 23, if we show that $\mathcal{I}$ cannot be covered by a countable union, $\bigcup_{n\in\omega} \mathcal{Y}_n$, of sets such that $\mathcal{Y}_n^2 \setminus \Delta_{\mathcal{I}}$ is disjoint from K_0 for each n. By Corollary 29, there is an n, $p \in G$, and $a, b \in \mathcal{Y}_n$ and $k \in a \cap b$ so that $p(k) = 2$, $p(h_a^{-1}(k)) = 0$, and $p(h_b^{-1}(k)) = 1$. Let $i = h_a^{-1}(k)$ and $j = h_b^{-1}(k)$. It follows that $i \in a \cap C_0$ and $j \in b \cap C_1$. Therefore $k \in c_a \cap d_b$. Since this means that $(a,b) \in K_0 \cap \mathcal{Y}_n^2 \setminus \Delta_{\mathcal{I}}$ this finishes the proof. □

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