

PFA(S) AND COUNTABLE TIGHTNESS

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ABSTRACT. Todorćevic introduced the forcing axiom PFA(S) and established many consequences. We contribute to this project. In particular, we consider status under PFA(S) of two important consequences of PFA concerning spaces of countable tightness. In particular we prove that the existence of a Souslin tree does not imply the existence of a compact non-sequential space of countable tightness. We contrast this with M. E. Rudin's result that the existence of a Souslin tree does imply the existence of an S-space (and the later improvement by Dahrough to a compact S-space).

1. INTRODUCTION

In the paper [8], the authors solved Katětov's problem by introducing what they called the Souslin Axiom (SA_{ω_1}) which is the statement that there is a coherent Suslin tree S such that for all posets P with $P \times S$ ccc, and any collection $\mathcal{D} = \{D_\xi : \xi < \omega_1\}$ of dense open subsets of P , there is a \mathcal{D} -generic filter $G \subset P$. The (independence, and thus solution) to the Katětov problem was established by then passing to the generic extension by forcing with S over a model of SA_{ω_1} . In later work, Todorćevic [15], introduced the strengthening of SA_{ω_1} (which can be seen as a Martin's Axiom like statement) to the PFA or proper poset version called PFA(S). The axiom statement PFA(S)[S] has come to mean a model obtained by forcing with with the coherent Souslin tree S over a model of PFA(S). The interested reader can consult [15] and [13] for a number of powerful consequences. It is generally interesting to determine the status of many of the consequences of PFA in these new PFA-like models. We are interested in these three consequences of PFA:

- (1) the non-existence of (compact) S-spaces (hereditarily separable but not hereditarily Lindelof) [12, 14],

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- (2) the Moore-Mrowka problem: every compact space of countable tightness is sequential [1],
- (3) every countably tight perfect pre-image of ω_1 contains a copy of ω_1 [7].

We mention (1) in part because of its historical close connection with (2). For example the first consistent examples of (2) failing (the classical Fedorchuk and Ostaszewski spaces) were primarily of interest because they were S-spaces. In models of PFA(S)[S] it remains an open problem if there are any S-spaces but it is shown in [15] that there are no compact S-spaces and that (2) holds. It is also shown in [5] that (3) holds in models of PFA(S)[S]. In models of PFA(S), there is, of course, a Souslin tree and so there are (compact) S-spaces (M. E. Rudin [10] constructed an S-space and M. Darough (unpublished 1979) modified the construction to make it compact). In this note we prove that (2) holds and (3) fails in a model of PFA(S). It is worth mentioning that in each of these models the statement (2) can be strengthened by asserting that such spaces also have a dense set of points with countable character. This was shown to hold under PFA in [6] and PFA(S)[S] [15].

The method of applying PFA(S) to prove results about either PFA(S), or the extension PFA(S)[S], is to produce a proper poset \mathbb{P} and prove that it preserves that the Souslin tree S remains Souslin. Such a poset is said to be S -preserving. For a downward closed sub-tree $S \subset \omega^{<\omega_1}$ and ordinal $\alpha \in \omega_1$ we let $S_\alpha = S \cap \omega^\alpha$. The homogeneous closure of a tree $S \subset \omega^{<\omega_1}$ will consist of all elements t of $\omega^{<\omega_1}$ that satisfy $s\Delta t$ is finite for each $s \in S_{\text{dom}(t)}$. If S is a coherent Souslin tree then its homogeneous closure is as well. Henceforth we assume that S is a coherent Souslin tree that is equal to its homogeneous closure. For $s, t \in S$ we let $s \oplus t$ denote the element of S that is equal to $s \cup (t \upharpoonright [\text{dom}(s), \text{dom}(t)))$. If g is any generic filter for S and $s \in S$, then $s \oplus g = \{s \oplus t : t \in g\}$ is also an S -generic filter.

Definition 1.1. *A tree $S \subset \omega^{<\omega_1}$ is coherent if $s\Delta t = \{\xi \in \text{dom}(s) \cap \text{dom}(t) : s(\xi) \neq t(\xi)\}$ is finite for all $s, t \in S$. The axiom PFA(S) is the statement that there is a coherent Souslin tree and for all S -preserving proper posets \mathbb{P} and for each family \mathfrak{D} of at most ω_1 dense subsets of \mathbb{P} there is a \mathfrak{D} -generic filter on \mathbb{P} .*

Miyamoto [9] characterized when a proper poset will preserve a Souslin tree, but for coherent Souslin tree the following simpler condition was established in [4].

Proposition 1.2. *For a proper poset \mathbb{P} the following are equivalent*

proper

- (1) \mathbb{P} preserves that S is Souslin,
- (2) $\mathbb{P} \times S$ is proper.

2. MOORE-MROWKA UNDER PFA(S)

In this section we will prove this theorem.

Theorem 2.1. *PFA(S) implies that compact spaces of countable tightness are sequential and have a dense set of points of countable character.* main1

When constructing proper posets which may be of the form $P * Q$ for some proper poset P or even when choosing the suitable family of ω_1 -many dense subsets of P it is common to pass to the forcing extension by P . In the case of working with PFA(S) we will see that it is also useful to pass to the forcing extension by S . We will let g denote an S -generic filter. Throughout this section we will be analyzing the structure of a compact space K of countable tightness in a model of PFA(S). We may assume that the base set for the space K is an ordinal. It is useful to recall forcing with S does not add any new countable sets of ordinals. Similarly a countably closed poset will not add any new countable sets of ordinals.

The poset $Col(\omega_1, \omega_2)$ is the standard countably closed poset that adds a function from ω_1 onto the ground model ordinal ω_2 . In a model in which CH fails, $Col(\omega_1, \omega_2)$ is forcing isomorphic to the tree $2^{<\omega_1}$.

We will speak of the space K (and some chosen subspace X) in such forcing extensions. The convention is that the topology from the ground model will be a base for a topology in the extension. If the forcing poset adds no new countable ordinals then the space K in the extension will still be countably compact. It was shown in [6] that if K has no points of countable character then it will no longer be compact in the forcing extension by $Col(\omega_1, \omega_2)$. However K will continue to have countable tightness in this extension.

Proposition 2.2 ([6]). *If K is a compact space of countable tightness, then K continues to have countable tightness after forcing with a countably closed poset.*

The standard proof from PFA for Theorem 2.1 could be used without much change if forcing by S preserved countable tightness in compact spaces. However we note the interesting fact that this is not the case.

Proposition 2.3. *Forcing with a Souslin tree does not (in general) preserve countable tightness in compact spaces (even sequential).*

Proof. Let S be a Souslin tree and let K denote the set $S \cup \{\infty\}$. We define a topology on K . For each $s < t$ both in S , the interval $(s, t] = \{u \in S : s < u \leq t\}$ is declared to be clopen. In particular, each set $t^\perp = \{u \in S : u \leq t\}$ is a compact open subset of S . Using this family as a base results in a locally countable, locally compact topology on S . The topology on K is simply the one-point compactification. Neighborhoods of ∞ are then simply complements of finite unions of intervals of the form t^\perp ($t \in S$). Clearly if $g \subset S$ is a generic filter, then ∞ is a limit point of g but not of any countable subset of g . On the other hand, K has countable tightness (in fact it is Frechet) in the ground model. To see this it suffices to suppose that every uncountable subset of S contains an infinite antichain. This is proven to hold for Aronszajn trees in [2]. \square

Lemma 2.4 ($\mathfrak{p} > \aleph_1$). *Suppose that S is a Souslin tree and that X is a countably compact dense subset of a compact space K of countable tightness. Then the following holds in the forcing extension by $S \times Col(\omega_1, \omega_2)$:*

if $z \in K$ and if \mathcal{Y} is a countable family of subsets of X satisfying that $t(z, Y) > \aleph_0$ for each $Y \in \mathcal{Y}$, then, for any countable $b \subset X$ with $z \in \bar{b}$, there is a countable set $a \subset X$ such that $x \in \bar{a} \subset \bar{b} \setminus \bigcup \mathcal{Y}$.

Proof. Let X and $z \in X$ be given. Let \mathcal{S}_z denote the collection of all countable sets $b \subset X$ that have x in their closure. The family \mathcal{S}_z is unchanged by the forcing $S \times Col(\omega_1, \omega_2)$. Now fix any $b \in \mathcal{S}_z$.

We first prove that the statement holds in the extension by S . Suppose that $\{\dot{Y}_\ell : \ell \in \omega\}$ is a set of S -names of subsets of X and that $1 \Vdash t(x, \dot{Y}_\ell) > \aleph_0$ for each $\ell \in \omega$. For each $s \in S$ and $\ell \in \omega$, let $\dot{Y}[b, s, \ell]$ denote the set of $\{y \in \bar{b} : s \Vdash y \in \dot{Y}_\ell\}$. Since X has countable tightness and $\dot{Y}[b, s, \ell]$ is a set in the ground model, it follows that x is not in the closure of $\dot{Y}[b, s, \ell]$ for each $s \in S$ and $\ell \in \omega$. For each $s \in S$ and $\ell \in \omega$, let $U(s, \ell)$ be an open neighborhood of x whose closure misses $\dot{Y}[b, s, \ell]$. For each neighborhood W of x , it follows from $\mathfrak{p} > \aleph_1$, that there is an infinite pseudointersection for the filter $\{b \cap U(s, \ell) \cap W : s \in S, \ell \in \omega\}$. Since X is countably compact it will contain limit points of each such pseudointersection. It follows then that there is a countable set $a \in \mathcal{S}_z$ such that $a \subset \bar{b} \cap U(s, \ell)$ for all $s \in S$ and $\ell \in \omega$. Since the closure of $U(s, \ell)$ is disjoint from $\dot{Y}[b, s, \ell]$, it follows that s does not force there is an element of \bar{a} that is also in \dot{Y}_ℓ .

Now we prove that this property continues to hold in the further forcing extension by the countably closed poset $P = Col(\omega_1, \omega_2)$. We let $g \subset S$ be a generic filter and we argue in the extension $V[g]$. Now we

let $\{\dot{Y}_\ell : \ell \in \omega\}$ be a sequence of $Col(\omega_1, \omega_2)$ -names and for each $p \in P$ and $\ell \in \omega$, let $\dot{Y}_\ell(p)$ denote the set $\{y \in X : (\exists q < p) (q \Vdash y \in \dot{Y}_\ell)\}$. Let $b_0 \in \mathcal{S}_z$ and $p_0 \in P$ be arbitrary and, by induction on ℓ , choose a sequence $\{b_\ell : \ell \in \omega\} \subset \mathcal{S}_z$ and a descending sequence $\{p_\ell : \ell \in \omega\} \subset P$ so that $b_{\ell+1} \subset \overline{b_\ell}$, and either $t(x, \dot{Y}_\ell(p_{\ell+1}) \cap \overline{b_\ell}) \neq \aleph_0$, or for all $q < p_\ell$ and $b_q \in \mathcal{S}_z$ with $b_q \subset \overline{b_\ell}$, $t(x, \dot{Y}_\ell(q) \cap \overline{b_q}) = \aleph_0$. Let $p_\omega \in P$ be any lower bound of the sequence $\{p_\ell : \ell \in \omega\}$ and let $b \in \mathcal{S}_z$ be any subset of $\bigcap \{\overline{b_\ell} : \ell \in \omega\}$. The fact that there is such a b follows as in the choice of a in the argument for just the extension by S . The family $\{\dot{Y}_\ell(p_\omega) : \ell \in \omega\}$ is in $V[g]$ and so either there is an a as required in the statement of the Lemma, or there is an ℓ such that $t(x, \dot{Y}_\ell(p_\omega) \cap \overline{b}) = \aleph_0$. We may as well assume the latter and we fix such an ℓ . We will finish the proof by showing that p_ω does not force that $t(x, \dot{Y}_\ell) > \aleph_0$.

Note that we then have that for all $q < p_\omega$ and $b_q \in \mathcal{S}_z$ contained in $\overline{b_{\ell+1}}$, $t(x, \dot{Y}_\ell(q) \cap \overline{b_q}) = \aleph_0$. Choose any countable elementary submodel M of a suitably large $H(\theta)$ such that $x, p_\omega, X, K, \dot{Y}_\ell$ are elements of M . Let $\{q_j : j \in \omega\}$ be an enumeration of $M \cap \{q \in P : q < p_\omega\}$. Recursively choose $\{a_j : j \in \omega\} \subset \mathcal{S}_z \cap M$ so that $a_0 = b$ and $a_{j+1} \subset \dot{Y}_\ell(q_j) \cap \overline{a_j}$. We may do so, by elementarity, and the fact that $t(x, \dot{Y}_\ell(q_j) \cap \overline{a_j})$ is countable. Now we briefly return to the model V , and set Z to be the compact subset of K that is equal to the intersection of the sequence $\{\overline{a_j} : j \in \omega\}$. Since Z has countable tightness, and therefore x has countable π -character in Z , we may fix a sequence $\{U_m : m \in \omega\}$ of open subsets of K with the property that the family $\{U_m \cap Z : m \in \omega\}$ forms a local π -base at x . That is, each neighborhood of x contains a set from $\{U_m \cap Z : m \in \omega\}$. Additionally, for each m , choose an open set W_m so that $W_m \cap Z \neq \emptyset$ and $\overline{W_m} \subset U_m$. We assume that the enumeration $\{W_m, U_m : m \in \omega\}$ is such that each element appears infinitely many times.

Returning to $V[g]$, we now we begin another recursive construction of a sequence $\{j_m, y_m : m \in \omega\}$ satisfying that $\{q_{j_m} : m \in \omega\}$ is a descending sequence and, for each m , $m \leq j_m$ and $q_{j_m} \Vdash y_m \in \dot{Y}_\ell \cap U_m \cap \overline{a_m}$. Assume that q_{j_m} has been chosen and recall that a_{j_m+1} is a subset of $\dot{Y}_\ell(q_{j_m})$. Since $a_{j_m+1} \in \mathcal{S}_z$ we have that $U_m \cap a_{j_m+1}$ is not empty. We may choose any $y_{m+1} \in W_m \cap a_{j_m+1}$ and $j_{m+1} > m$ so that $q_{j_{m+1}} < q_{j_m}$ forces that $y_{m+1} \in \dot{Y}_\ell$. Let q_ω be a lower bound of the sequence $\{q_{j_m} : m \in \omega\}$ and note that $q_\omega \Vdash \{y_m : m \in \omega\} \subset \dot{Y}_\ell$. We complete the proof by showing that $\{y_m : m \in \omega\} \in \mathcal{S}_z$. We prove this by showing, in V , that the closure of $\{y_m : m \in \omega\}$ meets $W_m \cap Z$ for each m . Fix any $m \in \omega$ and let $L_m = \{\ell : W_\ell = W_m \text{ and } U_\ell = U_m\}$.

Let $x_m \in X$ be any limit point of $\{y_\ell : \ell \in L_m\}$. It follows that $x_m \in \overline{W_m} \cap \bigcap_{\ell \in L_m} \overline{a_\ell} \subset U_m \cap Z$. \square

Let us now work in the forcing extension by $Col(\omega_1, \omega_2)$. We still have our compact space K of countable tightness. Since we are trying to prove that K is sequential, we may assume that it is not and note that then there is a point $x \in K$ together with a countably compact subset X of $K \setminus \{x\}$ having x in its closure. This is because we can first show that we may as well assume that K is sequentially compact (otherwise its cardinality is greater than \mathfrak{c} and we can choose a countably compact X and a maximal free filter which remains maximal after forcing with S).

This is a model of \diamond so we may choose a maximal filter \mathcal{F} of closed subsets of X with the additional property that the separable members of \mathcal{F} form a base. We must actually do more and so we use instead the forcing method from [3]. Consider the poset Q consisting of countable sequences $\langle a_\beta : \beta < \delta \rangle$ from \mathcal{S}_z that satisfy that $a_\gamma \subset \overline{a_\beta}$ for all $\beta < \gamma < \delta$, ordered simply by extension. It is shown in [3] that this is a countably closed poset. It is well-known that, since $\mathfrak{c} > \omega_1$, $Col(\omega_1, \omega_2)$ is forcing isomorphic to Q . Let $\langle a_\beta : \beta < \omega_1 \rangle$ denote a generic sequence for Q . We let \mathcal{F} be the filter of closed subsets of X generated by the family $\{\overline{a_\beta} : \beta < \omega_1\}$. Not only is \mathcal{F} a maximal filter on X but it also satisfies the following:

Lemma 2.5. *If $\langle a_\beta : \beta < \omega_1 \rangle$ is the generic sequence added by Q , then for each nice S -name \dot{H} of a subset of X and each $s \in S$, if for all $\beta < \omega_1$, $s \Vdash t(x, \dot{H} \cap \overline{a_\beta}) = \aleph_0$, then $s \Vdash (\forall \beta < \omega_1)(\exists \alpha > \beta)\dot{H} \supset a_\alpha$.*

Proof. Let $s \in S$ and \dot{H} be as in the Lemma, and assume that $s \Vdash t(x, \dot{H} \cap \overline{a_\beta}) = \aleph_0$ for all $\beta < \omega_1$. Choose any $s_1 \in S$ that is above S (i.e. a forcing extension). Now we work in the ground model before forcing with Q and we fix a countable elementary submodel M with Q, \dot{H}, s_1 all in M . We must be careful to realize that \dot{H} is actually a $Q \times S$ -name of a subset of X . Let $M \cap \omega_1 = \delta$ and let $\langle a_\beta : \beta < \delta \rangle$ be any (M, Q) -generic condition. We may also fix any $s_\delta \in S_\delta$ with $s_1 \subset s_\delta$. It follows that $(\langle a_\beta : \beta < \delta \rangle, s_\delta) \in Q \times S$ is an $(M, Q \times S)$ -generic condition. It then follows, by elementarity, that for each $\beta \in \delta$, there is an $\alpha \in \delta$ and a $b_\alpha \in \mathcal{S}_z \cap \overline{a_\beta}$ such that $s_\delta \upharpoonright \alpha \Vdash b_\alpha \subset \dot{H}$ (simply witnessing that $s_\delta \upharpoonright \alpha \Vdash t(x, \dot{H} \cap \overline{a_\beta}) = \aleph_0$). Choose any sequence $\{\alpha_n : n \in \omega\}$, increasing and cofinal in δ . Let $Z = X \cap \bigcap_{m \in \omega} \overline{\bigcup_{n > m} b_{\alpha_n}}$. Note that $Z \subset \overline{a_\beta}$ for all $\beta < \delta$. Similarly, if W is any closed neighborhood of x , then $Z \cap W \supset \overline{W \cap X \cap \bigcup_{n > m} b_{\alpha_n}}$ is not empty. This implies that there is a set $a_\delta \in \mathcal{S}_z$ such that $a_\delta \subset Z$. Now we have that $s_\delta \Vdash a_\delta \subset \dot{H}$

bytightness

and $(\langle a_\beta : \beta \leq \delta \rangle, s_\delta)$ forces that \dot{H} contains a member of the generic sequence $\langle a_\beta : \beta < \omega_1 \rangle$. Since s_1 was an arbitrary extension of s and $\langle a_\beta : \beta < \delta \rangle$ was an arbitrary (M, Q) -generic sequence, this proves in the extension by Q , we have that the set of extensions of s that force \dot{H} to contain an element of the sequence $\langle a_\beta : \beta < \omega_1 \rangle$ is dense above s . This completes the proof. \square

Now we continue the proof of Theorem. We let \mathcal{F} denote the filter generated by $\{\bar{a}_\beta : \beta < \omega_1\}$. Also, for each countable elementary submodel M with $\mathcal{F} \in M$, we let $Tr(\mathcal{F}, M) = \bigcap (\mathcal{F} \cap M)$. By construction $Tr(\mathcal{F}, M)$ is an element of \mathcal{F} . Let \mathcal{Y} denote the set of all nice S -names, \dot{Y} , of subsets of X satisfying that $1 \Vdash t(x, \dot{Y}) > \aleph_0$.

Lemma 2.6. *If z, \mathcal{F}, X are in $M \prec H(\theta)$ (a countable elementary submodel), then there is an $x_M \in Tr(\mathcal{F}, M)$ such that for all nice S -names $\dot{H} \in M$ for subsets of X and $s \in S \setminus M$, if $s \Vdash x_M \in \dot{H}$, then there is a countable $a \in M$ such that $s \Vdash \check{a} \subset \dot{H}$ and $\bar{a} \in \mathcal{F}$.* canpickx

Proof. We use Lemma 2.4 to pick our $x_M \in Tr(\mathcal{F}, M)$ so that for any S -generic filter g , $x_M \notin Y$ for all $Y \in M[g]$ satisfying that $t(x, Y) > \aleph_0$. Here is how. Recursively choose pairwise incomparable elements $\{s_\xi : \xi < \gamma\} \subset S$ together with a sequence $\{b_\xi : \xi < \gamma\} \subset \mathcal{S}_z$ so that

- (1) $\xi < \eta < \gamma$ implies that $b_\eta \subset \bar{b}_\xi \cap Tr(\mathcal{F}, m)$,
- (2) $s_\xi \Vdash \bar{b}_\xi \cap Y$ is empty for each $Y \in M[g]$ such that $t(z, Y) > \aleph_0$.

Suppose we have chosen $\{b_\xi : \xi < \gamma\}$. As we have seen before, we have that $\bigcap \{\bar{b}_\xi : \xi < \gamma\}$ will contain members of \mathcal{S}_z . If $\{s_\xi : \xi < \gamma\}$ is not a maximal antichain, then by Lemma 2.4 we can choose s_γ and b_γ as required. Once the set $\{s_\xi : \xi < \gamma\}$ is a maximal antichain, we can choose x_M to be any point in $X \cap \bigcap \{\bar{b}_\xi : \xi < \gamma\}$.

Let $M \cap \omega_1 = \delta$ and let $s \in S \setminus M$. Choose any $\dot{H} \in M$ and assume that $s \Vdash x_M \in \dot{H}$. It follows that $s \Vdash x_M \in \dot{H} \cap F$ for all $F \in \mathcal{F} \cap M$. By our choice of x_M , it follows that $s \Vdash t(z, \dot{H} \cap F) = \aleph_0$ for all $F \in \mathcal{F} \cap M$. By elementarity, there is a $\beta \in M$ such that $s \upharpoonright \beta$ also forces that $t(z, \dot{H} \cap F) = \aleph_0$ for all $F \in \mathcal{F}$. Now we apply Lemma 2.5 for the desired conclusion. \square

Now we are ready to define our final poset. For each $x \in X$ fix a regular pair of open neighborhoods of x , $W_x \subset \bar{W}_x \subset U_x$ such that z is not in the closure of U_x . It follows then that U_x is disjoint from some member of \mathcal{F} . Let \mathcal{W} denote this indexed sequence of neighborhood pairs, $\{W_x, U_x : x \in X\}$. Let κ denote the successor cardinal of $|\mathcal{P}(\mathcal{P}(X))|$.

Definition 2.7. Define the poset \mathbb{P} by the following: a condition $p \in \mathbb{P}$ is a function with domain \mathcal{M}_p satisfying

- (1) \mathcal{M}_p is a finite \in -chain of countable elementary submodels of $H(\kappa)$,
- (2) $\{z, \mathcal{F}, \mathcal{W}\} \in M$ for all $M \in \mathcal{M}_p$,
- (3) for each $M \in \mathcal{M}_p$, $p(M) \in Tr(\mathcal{F}, M)$, and, for each S -name \dot{H} in M and each $s \in S \setminus M$, if $s \Vdash p(M) \in \dot{H}$, then there is a countable $a \in M$ such that $\bar{a} \in \mathcal{F}$ and $s \Vdash a \subset \dot{H}$,
- (4) for each $M_1, M_2 \in \mathcal{M}_p$ with $M_1 \in M_2$, $p(M_1) \in M_2 \cap F$ for all $F \in \mathcal{F} \cap M_1$.

For a condition $p \in \mathbb{P}$ and element M of \mathcal{M}_p , we define $W_p(M)$ to be the intersection of the family $\{W_{p(M')} : M \subset M' \in \mathcal{M}_p \text{ and } p(M) \in W_{p(M')}\}$. Then the ordering on $\mathbb{P}(\mathcal{F}, \mathcal{W})$ is given by $q < p$ providing

- (3) $p \subset q$
- (4) for each $M_q \in \mathcal{M}_q \setminus \mathcal{M}_p$, if $\mathcal{M}_p \setminus M_q \neq \emptyset$ and M_p is the \in -minimum element of $\mathcal{M}_p \setminus M_q$, then $q(M_q)$ is an element of $W_p(M_p)$.

It is routine to check that for $p \in \mathbb{P}$ and $M \in \mathcal{M}_p$, $p_M = p \cap M = p \upharpoonright (\mathcal{M}_p \cap M)$ is also in \mathbb{P} and also that $p < p_M$. In addition, Lemma 2.6 shows that if $M \prec H(\kappa)$ and $p \in \mathbb{P} \cap M$, then there is an extension q of p with $M \in \mathcal{M}_q$. As usual \mathcal{F}^+ denotes the family of subsets of X that meet every member of \mathcal{F} . If $g \subset S$ is a generic filter, then let \mathcal{H}_g denote the set of $H \in \mathcal{F}^+$ satisfying that $t(z, H) = \aleph_0$.

The next lemma is the key step in proving that $S \times Col(\omega_1, \omega_2) * \mathbb{P}$ is proper. The poset \mathbb{P} is defined in the forcing extension by $Col(\omega_1, \omega_2)$ and we want to prove that a condition that we expect to be (M, \mathbb{P}) -generic condition will also be $(M[g], \mathbb{P})$ -generic when g is a generic filter for S .

Lemma 2.8. Let \mathbb{P} be an element of a countable elementary submodel M of $H(\theta)$ for some suitably large θ . Also let $g \subset S$ be a generic filter. If $p \in \mathbb{P}$ and $M \cap H(\kappa) \in \mathcal{M}_p$, then p is a $(M[g], \mathbb{P})$ -generic condition.

Proof. Let $D \in M[g]$ be a dense open subset of \mathbb{P} , which just means that there is an S -name for D in M . While D is not in M it is simply a subset of \mathbb{P} and so, since S is ccc, $D \cap M[g] = D \cap M$. By possibly extending p we may assume that $p \in D$. Let $M_0 = M \cap H(\kappa)$ and choose any countable elementary submodel M' of $H(\kappa)$ in M such that $p_{M_0} = p \upharpoonright (\mathcal{M}_p \cap M) \in M'$. Let ℓ be the cardinality of $\mathcal{M}_p \setminus M$ and let $\{M_0, M_1, \dots, M_{\ell-1}\}$ be the increasing enumeration of $\mathcal{M}_p \setminus M$.

In $M[g]$ we have the following set definable from D :

$$D(p_{M_0}, \ell) = \{q \in \mathbb{P} : q < p_{M_0}, \mathcal{M}_q \cap M' = \mathcal{M}_{p_{M_0}}, |\mathcal{M}_q \setminus M'| = \ell\}.$$

For each $q \in D(p_{M_0}, \ell)$ let $\{M_0^q, \dots, M_{\ell-1}^q\}$ denote the increasing enumeration of $\mathcal{M}_q \setminus M'$. Similarly, for each $q \in D(p_{M_0}, \ell)$ and $i < \ell$, let $x_i^q = q(M_i^q)$. Let \vec{x}^q denote the ℓ -tuple $\langle x_i^q : i < \ell \rangle$. Now we have an associated set that is in $M_0[g]$. Define $E_\ell^0 = \{\vec{x}^q : q \in D(p_{M_0}, \ell)\}$ and for each $i < \ell$, let $E_i^0 = \{\vec{x} \upharpoonright i : \vec{x} \in E_\ell^0\}$. For any $\vec{x} \in X^{<\ell}$ and $y \in X$, let $\vec{x} \frown \langle y \rangle$ denote the extension of \vec{x} in $X^{\leq \ell}$ obtained by putting y in the last coordinate. For each $i < \ell$ and $\vec{x} \in E_i^0$, let $E_{i+1}^0(\vec{x}) = \{y \in X : \vec{x} \frown \langle y \rangle \in E_{i+1}^0\}$.

Define $\mathcal{H} \in M[g]$ to be those members of \mathcal{F}^+ that have the property that they contain a countable set whose closure is in \mathcal{F} . Next, by a descending recursion on $i \leq \ell$, we define E_i^1 . First $E_\ell^1 = E_\ell^0$ and, for $i < \ell$,

$$E_i^1 = \{\vec{x} \in E_i^0 : \{y \in E_{i+1}^0(\vec{x}) : \vec{x} \frown \langle y \rangle \in E_{i+1}^1\} \in \mathcal{H}\}.$$

For convenience, we let $E_{i+1}^1(\vec{x})$ equal $\{y \in X : \vec{x} \frown \langle y \rangle \in E_{i+1}^1\}$ for all $\vec{x} \in E_i^1$. Of particular interest is the set E_0^1 which is either empty or has the empty sequence as its only possible element. We prove by induction on $i < \ell$ that $\vec{x}^p \upharpoonright i \in E_i^1$, which will show that the empty sequence is an element of E_0^1 .

For each $i < \ell$, \vec{x}_i^p is an element of M_i^p and so $E_{i+1}^1(\vec{x}_i^p \upharpoonright i)$ is also in $M_i^p[g]$. Using elementarity and the fact that $x_i^p \in E_{i+1}^1(\vec{x}_i^p)$, we deduce that $E_{i+1}^1(\vec{x}_i^p \upharpoonright i)$ is in \mathcal{H} . It then follows that $\vec{x}_i^p \upharpoonright i \in E_i^1$.

Now that we have that $E_1^1(\emptyset) \in \mathcal{H} \cap M[g]$, we select a countable $a_0 \in M$ that is contained in $E_1^1(\emptyset)$ and whose closure is an element of \mathcal{F} . Of course this implies that x_0^p is in the closure of a_0 and so there is a $y_0 \in a_0 \cap W_p(M_0)$. By recursion suppose $i < \ell$ and we have selected $\{y_j : j < i\} \subset W_p(M_0) \cap M$ such that $\langle y_j : j < i \rangle \in E_i^1$. We choose $a_i \subset E_{i+1}^1(\langle y_j : j < i \rangle)$ just as we did a_0 and similarly choose $y_i \in a_i \cap W_p(M_0)$. Once we have chosen $\langle y_i : i < \ell \rangle \in E_\ell^0$, we choose $q \in D(p_{M_0}, \ell) \cap M$ so that $\vec{x}^q = \langle y_i : i < \ell \rangle$ and it is routine to check that $q \in D \cap M$ is compatible with p . \square

3. PERFECT PREIMAGES OF ω_1 AND COPIES OF ω_1

It has been shown to be a consequence of each of PFA [7] and PFA(S)[S] [5, Theorem 3.8] that any perfect preimage of ω_1 that has countable tightness will itself contain a topological copy of ω_1 . On the other hand, the existence of a coherent Souslin tree implies that there is a 2-to-1 perfect preimage of ω_1 that contains no copy of ω_1 .

Theorem 3.1. *If there is a coherent Souslin tree S , then there is an uncountable locally compact locally countable \aleph_0 -bounded space X that contains no copy of ω_1 . In addition, there is a continuous closed mapping from X onto ω_1 such that each fiber contains exactly two points.*

Proof. Let Λ denote the set of limit ordinals in ω_1 . Let $\{s_\alpha : \alpha \in \Lambda\} \subset S$ be chosen so that $s_\alpha \in S_\alpha$ for each $\alpha \in \Lambda$. We define a coloring $e : S \rightarrow 2$ that we will use to define a topology on $\omega_1 \times \{0, 1\}$.

We define $e(s) = 0$ if s is on a successor level of S and also, if $s = s_\alpha$ for some $\alpha \in \Lambda$, then $e(s) = 0$. By recursion on $\alpha \in \Lambda$ we define $e(s)$ for $s \in S_\alpha$ as follows:

- (1) if $\alpha = \beta + \omega$ for some $\beta \in \Lambda$, then $e(s) = 0$ if and only if either
 - (a) $|(s\Delta s_\alpha) \setminus \beta|$ is even and $e(s \upharpoonright \beta) = e(s_\alpha \upharpoonright \beta)$, or
 - (b) $|(s\Delta s_\alpha) \setminus \beta|$ is odd and $e(s \upharpoonright \beta) \neq e(s_\alpha \upharpoonright \beta)$,
- (2) if $\Lambda \cap \alpha$ is cofinal in α and $\beta \in \Lambda \cap \alpha$ minimal so that $s\Delta s_\alpha \subset \beta$, then $e(s) = 0$ if and only if $e(s \upharpoonright \beta) = e(s_\alpha \upharpoonright \beta)$.

Fact 1. If $s, t \in S_\alpha$ for some $\alpha \in \Lambda$ and if $s\Delta t \subset \delta \in \alpha \cap \Lambda$, then $e(s) = e(t)$ if and only if $e(s \upharpoonright \delta) = e(t \upharpoonright \delta)$.

Proof of Fact 1. We prove this by induction on $\alpha \in \Lambda$. The base case when $\alpha = \omega$ is vacuous. If $\alpha = \beta + \omega$ for some $\beta \in \Lambda$, then the two sets $(s\Delta s_\alpha) \setminus \beta$ and $(t\Delta s_\alpha) \setminus \beta$ will be equal since s and t are assumed to agree on the set $[\delta, \alpha)$ for some limit $\delta \leq \beta$. By induction, $e(s \upharpoonright \beta) = e(t \upharpoonright \beta)$ if and only if $e(s \upharpoonright \delta) = e(t \upharpoonright \delta)$. Of course $|(s\Delta s_\alpha) \setminus \beta|$ is the same as $|(t\Delta s_\alpha) \setminus \beta|$. This means that if $e(s \upharpoonright \delta) = e(t \upharpoonright \delta)$, then $e(s) = 0$ iff $e(t) = 0$. Similarly, if $e(s \upharpoonright \delta) \neq e(t \upharpoonright \delta)$, then exactly one of $e(s)$ or $e(t)$ will be 0.

Now assume that $\Lambda \cap \alpha$ is cofinal in α . Let $\gamma \in \Lambda$ be minimal such that $s\Delta t \subset \gamma$. We proceed by induction on γ . Naturally $\gamma \leq \delta$. Next let $\beta_s \in \Lambda$ be minimal such that $s\Delta s_\alpha \subset \beta_s$ and similarly define $\beta_t \in \Lambda$. We may assume that $\beta_t \leq \beta_s$ and now note that $\beta_s \geq \gamma$ since as soon as s and t disagree with each other, one of them disagrees with s_α . If $\beta_s > \gamma$, then $\beta_s = \beta_t$, and in this case, each of $e(s)$ and $e(t)$ will equal 0 if and only if $e(s \upharpoonright \beta_s) = e(s_\alpha \upharpoonright \beta_s)$. If $\beta_s = \gamma$, then it follows that $e(s) = e(s_\alpha)$ if and only if $e(s \upharpoonright \gamma) = e(s_\alpha \upharpoonright \gamma)$. If β_t is also equal to γ , then the same is true of $e(t) = e(s_\alpha)$ and so we may assume that $\beta_t < \gamma$. We finish by applying the induction hypothesis to the pair t and s_α . Indeed, we have that $e(t) = e(s_\alpha)$ if and only if $e(t \upharpoonright \gamma) = e(s_\alpha \upharpoonright \gamma)$. \square

Fact 2. If $E \subset \omega_1$ is uncountable, then there is a cub $C \subset \omega_1$ such that for each $\delta \in C$ and each $\gamma < \delta$, there are $\alpha_0, \alpha_1 \in E \cap \delta \setminus \gamma$ so that $e(s_\delta \upharpoonright \alpha_0) = 0$ and $e(s_\delta \upharpoonright \alpha_1) = 1$.

Proof of Fact 2. Let B be the set of $s \in S$ such that there are no members of $\{s_\beta : \beta \in E\}$ above s . The minimal elements of B is an antichain, and is therefore countable. Choose any γ so that all the minimal elements of B are on a level less than γ . Let α be any element of $E \setminus \gamma$. Since no minimal member of B is above s_α , it follows that every member of S above s_α is below some member of $\{s_\beta : \beta \in E\}$. Again using that S is ccc, there is a cub $C \subset \omega_1$ such that for all $\beta \in E \setminus \delta$, $s = s_\beta \upharpoonright \delta$ satisfies that $\{\beta \in E \cap \delta : s_\beta < s\}$ is cofinal in δ .

Now choose any $\delta \in C$ and limit $\beta < \delta$ and any $s \in S_\delta$ such that there is some $\beta' \in E \setminus \delta$ with $s < s_{\beta'}$. Choose any $t \in S_{\beta+\omega}$ such that $s \upharpoonright \beta < t$ and $(t \Delta (s \upharpoonright (\beta + \omega))) \setminus \beta$ is odd. It follows from clause (1) in the definition of e that $e(t) \neq e(s \upharpoonright \beta + \omega)$. Since S is coherent, $t_1 = t \oplus s$ is in S_δ and by Fact 1, $e(s \upharpoonright \alpha) \neq e(t_1 \upharpoonright \alpha)$ for all $\beta + \omega \leq \alpha \leq \delta$. Fix $\beta_1 \in \Lambda \cap \delta$ so that $\beta + \omega \leq \beta_1$ and $s \Delta s_\delta \subset \beta_1$. By possibly switching the labelling of s and t_1 , we may assume that $e(s) = 0$ and $e(t_1) = 1$. Note that we then have, by Fact 1, that $e(s_\delta \upharpoonright \alpha) = e(s \upharpoonright \alpha)$ for all $\alpha \in \Lambda \cap \delta \setminus \beta_1$. Similarly $e(s_\delta \upharpoonright \alpha) \neq e(t_1 \upharpoonright \alpha)$ for all $\alpha \in \Lambda \cap \delta \setminus \beta_1$. Choose any $\alpha_0 \in E \cap \delta \setminus \beta_1$ so that $s_{\alpha_0} < s$ and similarly choose $\alpha_1 \in E \cap \delta \setminus \beta_1$ so that $s_{\alpha_1} < t_1$. Naturally, $e(s \upharpoonright \alpha_0) = e(s_{\alpha_0}) = 0$, and so $e(s_\delta \upharpoonright \alpha_0) = 0$. Similarly $e(t_1 \upharpoonright \alpha_1) = 0$ and so $e(s_\delta \upharpoonright \alpha_1) = 1$. \square

Now we are ready to construct our topology on $\omega_1 \times \{0, 1\}$. Each point in the set $(\omega_1 \setminus \Lambda) \times \{0, 1\}$ is isolated. For each limit δ we define a clopen partition, $\{W(\delta, 0), W(\delta, 1)\}$, of $(\delta + 1) \times \{0, 1\}$ where

$$W(\delta, 0) = \{(\alpha, e(s_\delta \upharpoonright \alpha)) : \alpha \in \Lambda \cap (\delta + 1)\} \cup \bigcup_{\beta \in \Lambda \cap \delta} (\beta, \beta + \omega] \times \{e(s_\delta \upharpoonright \beta + \omega)\}$$

and therefore $W(\delta, 1)$ is equal to

$$\{(\alpha, 1 - e(s_\delta \upharpoonright \alpha)) : \alpha \in \Lambda \cap (\delta + 1)\} \cup \bigcup_{\beta \in \Lambda \cap \delta} (\beta, \beta + \omega] \times \{1 - e(s_\delta \upharpoonright \beta + \omega)\}.$$

Clearly $W(\delta, 0) \setminus (W(\alpha, 0) \cup W(\alpha, 1))$ is clopen for all $\alpha < \delta$ in Λ ; and similarly, so is $W(\delta, 1) \setminus (W(\alpha, 0) \cup W(\alpha, 1))$.

Fact 3. For each $\omega < \alpha < \delta \in \Lambda$ and $i \in \{0, 1\}$, if $(\alpha, i) \in W(\delta, 0)$, then there is a $\beta < \alpha$ in Λ such that $W(\alpha, i) \setminus (W(\beta, 0) \cup W(\beta, 1)) \subset W(\delta, 0)$.

Proof of Fact 3. Since we are assuming that $(\alpha, i) \in W(\delta, 0)$, it follows that $e(s_\delta \upharpoonright \alpha) = i$. If $\alpha = \beta + \omega$ for some $\beta \in \Lambda$, then it is clear that $W(\alpha, i) \setminus (W(\beta, 0) \cup W(\beta, 1)) \subset W(\delta, 0)$ as required. Now suppose that $\Lambda \cap \alpha$ is cofinal in α and choose any $\beta \in \Lambda \cap \alpha$ so that $s_\alpha \Delta (s_\delta \upharpoonright \alpha)$ is contained in β . By Fact 2, we have that either $i = 0$ and $e(s_\alpha \upharpoonright \xi) = e(s_\delta \upharpoonright \xi)$ for all $\xi \in \Lambda \cap (\beta, \alpha)$, or $i = 1$ and $e(s_\alpha \upharpoonright \xi) \neq$

$e(s_\delta \upharpoonright \xi)$ for all $\xi \in \Lambda \cap (\beta, \alpha)$. In either case, we then have that $W(\alpha, i) \setminus (W(\beta, 0) \cup W(\beta, 1))$ is contained in $W(\delta, 0)$ as required. \square

Since the proof of Fact 4 is the same we omit the proof:

Fact 4. For each $\omega < \alpha < \delta \in \Lambda$ and $i \in \{0, 1\}$, if $(\alpha, i) \in W(\delta, 1)$, then there is a $\beta < \alpha$ in Λ such that $W(\alpha, i) \setminus (W(\beta, 0) \cup W(\beta, 1)) \subset W(\delta, 0)$.

Fact 3 and Fact 4 prove that for each $\delta \in \Lambda$ such that $\Lambda \cap \delta$ is cofinal in δ , and each $i \in \{0, 1\}$, the family $\{W(\delta, i) \setminus (W(\beta, 0) \cup W(\beta, 1)) : \beta \in \Lambda \cap \delta\}$ is a clopen neighborhood base for the point (δ, i) . This means that for all $\alpha \in \Lambda$ and $i \in \{0, 1\}$, the family $\{W(\alpha, i) \setminus ([0, \beta] \times \{0, 1\}) : \beta < \alpha\}$ is a clopen neighborhood base for (α, i) . Let X denote the space with this topology on $\omega_1 \times \{0, 1\}$. It is now a routine exercise to prove that the map sending each two element set $\{\alpha\} \times \{0, 1\}$ to the ordinal α is a continuous and closed mapping from X onto ω_1 .

We finish the proof by showing that there is no copy of ω_1 in the space X . Assume that X_0 is such a copy of ω_1 and let h be the homeomorphism from X_0 onto ω_1 . Choose $i \in \{0, 1\}$ so that $E = \{\beta \in \omega_1 : (\beta, i) \in X_0\}$ is uncountable. Let C be a cub as described in Fact 2. By simply choosing an unbounded closed subset of C we can assume that for any $\delta \in C$, $h^{-1}([0, \delta])$ is a subset of $[0, \delta] \times \{0, 1\}$ and that $h(\beta, i) < \delta$ for all $\beta \in E \cap \delta$. But now, let δ be any point of C such that $C \cap \delta$ is cofinal in δ . It follows from the definitions of $W(\delta, 0)$ and $W(\delta, 1)$ that they each meet $(E \cap (\beta, \delta)) \times \{i\}$ for all $\beta < \delta$. This means that each of $h[X_0 \cap W(\delta, 0)]$ and $h[X_0 \cap W(\delta, 1)]$ is cofinal in δ , which also implies that each $(\delta, 0)$ and $(\delta, 1)$ are in X_0 and are each mapped to δ . This contradicts that h is a homeomorphism. \square

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